CHAPTER 3

Measurable functions

Measurable functions in measure theory are analogous to continuous functions in topology. A continuous function pulls back open sets to open sets, while a measurable function pulls back measurable sets to measurable sets.

3.1. Measurability

Most of the theory of measurable functions and integration does not depend on the specific features of the measure space on which the functions are defined, so we consider general spaces, although one should keep in mind the case of functions defined on $\mathbb{R}$ or $\mathbb{R}^n$ equipped with Lebesgue measure.

Definition 3.1. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces. A function $f : X \to Y$ is measurable if $f^{-1}(B) \in \mathcal{A}$ for every $B \in \mathcal{B}$.

Note that the measurability of a function depends only on the σ-algebras; it is not necessary that any measures are defined.

In order to show that a function is measurable, it is sufficient to check the measurability of the inverse images of sets that generate the σ-algebra on the target space.

Proposition 3.2. *Suppose that $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are measurable spaces and $\mathcal{B} = \sigma(\mathcal{G})$ is generated by a family $\mathcal{G} \subset \mathcal{P}(Y)$. Then $f : X \to Y$ is measurable if and only if*

$$f^{-1}(G) \in \mathcal{A} \qquad \text{for every } G \in \mathcal{G}.$$

Proof. Set operations are natural under pull-backs, meaning that

$$f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$$

and

$$f^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(B_i), \quad f^{-1}\left(\bigcap_{i=1}^{\infty} B_i\right) = \bigcap_{i=1}^{\infty} f^{-1}(B_i).$$

It follows that

$$\mathcal{M} = \left\{B \subset Y : f^{-1}(B) \in \mathcal{A}\right\}$$

is a σ-algebra on Y. By assumption, $\mathcal{M} \supset \mathcal{G}$ and therefore $\mathcal{M} \supset \sigma(\mathcal{G}) = \mathcal{B}$, which implies that f is measurable. □

It is worth noting the indirect nature of the proof of containment of σ-algebras in the previous proposition; this is required because we typically cannot use an explicit representation of sets in a σ-algebra. For example, the proof does not characterize $\mathcal{M}$, which may be strictly larger than $\mathcal{B}$.

If the target space Y is a topological space, then we always equip it with the Borel σ-algebra $\mathcal{B}(Y)$ generated by the open sets (unless stated explicitly otherwise).

In that case, it follows from Proposition 3.2 that $f : X \to Y$ is measurable if and only if $f^{-1}(G) \in \mathcal{A}$ is a measurable subset of X for every set G that is open in Y. In particular, every continuous function between topological spaces that are equipped with their Borel σ-algebras is measurable. The class of measurable function is, however, typically much larger than the class of continuous functions, since we only require that the inverse image of an open set is Borel; it need not be open.

3.2. Real-valued functions

We specialize to the case of real-valued functions

$$f : X \to \mathbb{R}$$

or extended real-valued functions

$$f : X \to \overline{\mathbb{R}}.$$

We will consider one case or the other as convenient, and comment on any differences. A positive extended real-valued function is a function

$$f : X \to [0, \infty].$$

Note that we allow a positive function to take the value zero.

We equip $\mathbb{R}$ and $\overline{\mathbb{R}}$ with their Borel σ-algebras $\mathcal{B}(\mathbb{R})$ and $\mathcal{B}(\overline{\mathbb{R}})$. A Borel subset of $\overline{\mathbb{R}}$ has one of the forms

$$B, \qquad B \cup \{\infty\}, \qquad B \cup \{-\infty\}, \qquad B \cup \{-\infty, \infty\}$$

where B is a Borel subset of $\mathbb{R}$. As Example 2.22 shows, sets that are Lebesgue measurable but not Borel measurable need not be well-behaved under the inverse of even a monotone function, which helps explain why we do not include them in the range σ-algebra on $\mathbb{R}$ or $\overline{\mathbb{R}}$.

By contrast, when the domain of a function is a measure space it is often convenient to use a complete space. For example, if the domain is $\mathbb{R}^n$ we typically equip it with the Lebesgue σ-algebra, although if completeness is not required we may use the Borel σ-algebra. With this understanding, we get the following definitions. We state them for real-valued functions; the definitions for extended real-valued functions are completely analogous

Definition 3.3. If $(X, \mathcal{A})$ is a measurable space, then $f : X \to \mathbb{R}$ is measurable if $f^{-1}(B) \in \mathcal{A}$ for every Borel set $B \in \mathcal{B}(\mathbb{R})$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is Lebesgue measurable if $f^{-1}(B)$ is a Lebesgue measurable subset of $\mathbb{R}^n$ for every Borel subset B of $\mathbb{R}$, and it is Borel measurable if $f^{-1}(B)$ is a Borel measurable subset of $\mathbb{R}^n$ for every Borel subset B of $\mathbb{R}$

This definition ensures that continuous functions $f : \mathbb{R}^n \to \mathbb{R}$ are Borel measurable and functions that are equal a.e. to Borel measurable functions are Lebesgue measurable. If $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable and $g : \mathbb{R}^n \to \mathbb{R}$ is Lebesgue (or Borel) measurable, then the composition $f \circ g$ is Lebesgue (or Borel) measurable since

$$(f \circ g)^{-1}(B) = g^{-1}\left(f^{-1}(B)\right).$$

Note that if f is Lebesgue measurable, then $f \circ g$ need not be measurable since $f^{-1}(B)$ need not be Borel even if B is Borel.

We can give more easily verifiable conditions for measurability in terms of generating families for Borel sets.

Proposition 3.4. *The Borel σ-algebra on $\mathbb{R}$ is generated by any of the following collections of intervals*

$$\{(-\infty, b) : b \in \mathbb{R}\}, \quad \{(-\infty, b] : b \in \mathbb{R}\}, \quad \{(a, \infty) : a \in \mathbb{R}\}, \quad \{[a, \infty) : a \in \mathbb{R}\}.$$

Proof. The σ-algebra generated by intervals of the form $(-\infty, b)$ is contained in the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ since the intervals are open sets. Conversely, the σ-algebra contains complementary closed intervals of the form $[a, \infty)$, half-open intersections $[a, b)$, and countable intersections

$$[a, b] = \bigcap_{n=1}^{\infty} [a, b + \frac{1}{n}).$$

From Proposition 2.20, the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ is generated by the collection of closed rectangles $[a, b]$, so

$$\sigma\left(\{(-\infty, b) : b \in \mathbb{R}\}\right) = \mathcal{B}(\mathbb{R}).$$

The proof for the other collections is similar. □

The properties given in the following proposition are sometimes taken as the definition of a measurable function.

Proposition 3.5. *If $(X, \mathcal{A})$ is a measurable space, then $f : X \to \mathbb{R}$ is measurable if and only if one of the following conditions holds:*

$$\begin{aligned}
&\{x \in X : f(x) < b\} \in \mathcal{A} && \textit{for every } b \in \mathbb{R};\\
&\{x \in X : f(x) \le b\} \in \mathcal{A} && \textit{for every } b \in \mathbb{R};\\
&\{x \in X : f(x) > a\} \in \mathcal{A} && \textit{for every } a \in \mathbb{R};\\
&\{x \in X : f(x) \ge a\} \in \mathcal{A} && \textit{for every } a \in \mathbb{R}.
\end{aligned}$$

Proof. Note that, for example,

$$\{x \in X : f(x) < b\} = f^{-1}\left((-\infty, b)\right)$$

and the result follows immediately from Propositions 3.2 and 3.4. □

If any one of these equivalent conditions holds, then $f^{-1}(B) \in \mathcal{A}$ for every set $B \in \mathcal{B}(\mathbb{R})$. We will often use a shorthand notation for sets, such as

$$\{f < b\} = \{x \in X : f(x) < b\}.$$

The Borel σ-algebra on $\overline{\mathbb{R}}$ is generated by intervals of the form $[-\infty, b)$, $[-\infty, b]$, $(a, \infty]$, or $[a, \infty]$ where $a, b \in \mathbb{R}$, and exactly the same conditions as the ones in Proposition 3.5 imply the measurability of an extended real-valued functions $f : X \to \overline{\mathbb{R}}$. In that case, we can allow $a, b \in \overline{\mathbb{R}}$ to be extended real numbers in Proposition 3.5, but it is not necessary to do so in order to imply that f is measurable.

Measurability is well-behaved with respect to algebraic operations.

Proposition 3.6. *If $f, g : X \to \mathbb{R}$ are real-valued measurable functions and $k \in \mathbb{R}$, then*

$$kf, f + g, \qquad fg, \qquad f/g$$

are measurable functions, where we assume that $g \neq 0$ in the case of f/g.

PROOF. If $k > 0$, then

$$\{kf < b\} = \{f < b/k\}$$

so kf is measurable, and similarly if $k < 0$ or $k = 0$. We have

$$\{f + g < b\} = \bigcup_{q+r<b;q,r\in\mathbb{Q}} \{f < q\} \cap \{g < r\}$$

so $f + g$ is measurable. The function f^2 is measurable since if $b \geq 0$

$$\left\{f^2 < b\right\} = \left\{-\sqrt{b} < f < \sqrt{b}\right\}.$$

It follows that

$$fg = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right]$$

is measurable. Finally, if $g \neq 0$

$$\{1/g < b\} = \begin{cases} \{1/b < g < 0\} & \text{if } b < 0, \\ \{-\infty < g < 0\} & \text{if } b = 0, \\ \{-\infty < g < 0\} \cup \{1/b < g < \infty\} & \text{if } b > 0, \end{cases}$$

so $1/g$ is measurable and therefore f/g is measurable. □

An analogous result applies to extended real-valued functions provided that they are well-defined. For example, $f + g$ is measurable provided that $f(x)$, $g(x)$ are not simultaneously equal to ∞ and $-\infty$, and fg is is measurable provided that $f(x)$, $g(x)$ are not simultaneously equal to 0 and $\pm\infty$.

Proposition 3.7. *If $f, g : X \to \overline{\mathbb{R}}$ are extended real-valued measurable functions, then*

$$|f|, \qquad \max(f,g), \qquad \min(f,g)$$

are measurable functions.

PROOF. We have

$$\{\max(f,g) < b\} = \{f < b\} \cap \{g < b\},$$
$$\{\min(f,g) < b\} = \{f < b\} \cup \{g < b\},$$

and $|f| = \max(f,0) - \min(f,0)$, from which the result follows. □

3.3. Pointwise convergence

Crucially, measurability is preserved by limiting operations on sequences of functions. Operations in the following theorem are understood in a pointwise sense; for example,

$$\left(\sup_{n\in\mathbb{N}} f_n\right)(x) = \sup_{n\in\mathbb{N}} \{f_n(x)\}.$$

Theorem 3.8. *If $\{f_n : n \in \mathbb{N}\}$ is a sequence of measurable functions $f_n : X \to \overline{\mathbb{R}}$, then*

$$\sup_{n\in\mathbb{N}} f_n, \qquad \inf_{n\in\mathbb{N}} f_n, \qquad \limsup_{n\to\infty} f_n, \qquad \liminf_{n\to\infty} f_n$$

are measurable extended real-valued functions on X.

Proof. We have for every $b \in \mathbb{R}$ that

$$\left\{\sup_{n\in\mathbb{N}} f_n \le b\right\} = \bigcap_{n=1}^{\infty} \{f_n \le b\},$$
$$\left\{\inf_{n\in\mathbb{N}} f_n < b\right\} = \bigcup_{n=1}^{\infty} \{f_n < b\}$$

so the supremum and infimum are measurable Moreover, since

$$\limsup_{n\to\infty} f_n = \inf_{n\in\mathbb{N}} \sup_{k\ge n} f_k,$$
$$\liminf_{n\to\infty} f_n = \sup_{n\in\mathbb{N}} \inf_{k\ge n} f_k$$

it follows that the limsup and liminf are measurable. □

Perhaps the most important way in which new functions arise from old ones is by pointwise convergence.

Definition 3.9. A sequence $\{f_n : n \in \mathbb{N}\}$ of functions $f_n : X \to \overline{\mathbb{R}}$ *converges pointwise* to a function $f : X \to \overline{\mathbb{R}}$ if $f_n(x) \to f(x)$ as $n \to \infty$ for every $x \in X$.

Pointwise convergence preserves measurability (unlike continuity, for example). This fact explains why the measurable functions form a sufficiently large class for the needs of analysis.

Theorem 3.10. *If $\{f_n : n \in \mathbb{N}\}$ is a sequence of measurable functions $f_n : X \to \overline{\mathbb{R}}$ and $f_n \to f$ pointwise as $n \to \infty$, then $f : X \to \overline{\mathbb{R}}$ is measurable.*

Proof. If $f_n \to f$ pointwise, then

$$f = \limsup_{n\to\infty} f_n = \liminf_{n\to\infty} f_n$$

so the result follows from the previous proposition. □

3.4. Simple functions

The *characteristic function* (or *indicator function*) of a subset $E \subset X$ is the function $\chi_E : X \to \mathbb{R}$ defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

The function χ_E is measurable if and only if E is a measurable set.

Definition 3.11. A *simple function* $\phi : X \to \mathbb{R}$ on a measurable space $(X, \mathcal{A})$ is a function of the form

$$\phi(x) = \sum_{n=1}^{N} c_n \chi_{E_n}(x) \tag{3.1}$$

where $c_1, \dots, c_N \in \mathbb{R}$ and $E_1, \dots, E_N \in \mathcal{A}$.

Note that, according to this definition, a simple function is measurable. The representation of ϕ in (3.1) is not unique; we call it a standard representation if the constants c_n are distinct and the sets E_n are disjoint.

Theorem 3.12. *If $f : X \to [0, \infty]$ is a positive measurable function, then there is a monotone increasing sequence of positive simple functions $\phi_n : X \to [0, \infty)$ with $\phi_1 \le \phi_2 \le \dots \le \phi_n \le \dots$ such that $\phi_n \to f$ pointwise as $n \to \infty$. If f is bounded, then $\phi_n \to f$ uniformly.*

Proof. For each $n \in \mathbb{N}$, we divide the interval $[0, 2^n]$ in the range of f into 2^{2n} subintervals of width 2^{-n},

$$I_{k,n} = (k2^{-n}, (k+1)2^{-n}], \qquad k = 0, 1, \dots, 2^{2n} - 1,$$

let $J_n = (2^n, \infty]$ be the remaining part of the range, and define

$$E_{k,n} = f^{-1}(I_{k,n}), \qquad F_n = f^{-1}(J_n).$$

Then the sequence of simple functions given by

$$\phi_n = \sum_{k=0}^{2^n-1} k2^{-n}\chi_{E_{k,n}} + 2^n \chi_{F_n}$$

has the required properties. □

In defining the Lebesgue integral of a measurable function, we will approximate it by simple functions. By contrast, in defining the Riemann integral of a function $f : [a, b] \to \mathbb{R}$, we partition the domain $[a, b]$ into subintervals and approximate f by step functions that are constant on these subintervals. This difference is sometime expressed by saying that in the Lebesgue integral we partition the range, and in the Riemann integral we partition the domain.

3.5. Properties that hold almost everywhere

Often, we want to consider functions or limits which are defined outside a set of measure zero. In that case, it is convenient to deal with complete measure spaces.

Proposition 3.13. *Let $(X, \mathcal{A}, \mu)$ be a complete measure space and $f, g : X \to \overline{\mathbb{R}}$. If $f = g$ pointwise μ-a.e. and f is measurable, then g is measurable.*

Proof. Suppose that $f = g$ on N^c where N is a set of measure zero. Then

$$\{g < b\} = (\{f < b\} \cap N^c) \cup (\{g < b\} \cap N).$$

Each of these sets is measurable: $\{f < b\}$ is measurable since f is measurable; and $\{g < b\} \cap N$ is measurable since it is a subset of a set of measure zero and X is complete. □

The completeness of X is essential in this proposition. For example, if X is not complete and $E \subset N$ is a non-measurable subset of a set N of measure zero, then the functions 0 and χ_E are equal almost everywhere, but 0 is measurable and χ_E is not.

Proposition 3.14. *Let $(X, \mathcal{A}, \mu)$ be a complete measure space. If $\{f_n : n \in \mathbb{N}\}$ is a sequence of measurable functions $f_n : X \to \overline{\mathbb{R}}$ and $f_n \to f$ as $n \to \infty$ pointwise μ-a.e., then f is measurable.*

Proof. Since f_n is measurable, $g = \limsup_{n \to \infty} f_n$ is measurable and $f = g$ pointwise a.e., so the result follows from the previous proposition. □