CHAPTER 4

Integration

In this Chapter, we define the integral of real-valued functions on an arbitrary measure space and derive some of its basic properties. We refer to this integral as the Lebesgue integral, whether or not the domain of the functions is subset of $\mathbb{R}^n$ equipped with Lebesgue measure. The Lebesgue integral applies to a much wider class of functions than the Riemann integral and is better behaved with respect to pointwise convergence. We carry out the definition in three steps: first for positive simple functions, then for positive measurable functions, and finally for extended real-valued measurable functions.

4.1. Simple functions

Suppose that $(X, \mathcal{A}, \mu)$ is a measure space.

Definition 4.1. If $\phi : X \to [0, \infty)$ is a positive simple function, given by

$$\phi = \sum_{i=1}^{N} c_i \chi_{E_i}$$

where $c_i \geq 0$ and $E_i \in \mathcal{A}$, then the integral of ϕ with respect to μ is

$$\int \phi \, d\mu = \sum_{i=1}^{N} c_i \mu\left(E_i\right). \tag{4.1}$$

In (4.1), we use the convention that if $c_i = 0$ and $\mu(E_i) = \infty$, then $0 \cdot \infty = 0$, meaning that the integral of 0 over a set of measure ∞ is equal to 0. The integral may take the value ∞ (if $c_i > 0$ and $\mu(E_i) = \infty$ for some $1 \leq i \leq N$). One can verify that the value of the integral in (4.1) is independent of how the simple function is represented as a linear combination of characteristic functions.

Example 4.2. The characteristic function $\chi_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$ of the rationals is not Riemann integrable on any compact interval of non-zero length, but it is Lebesgue integrable with

$$\int \chi_{\mathbb{Q}} \, d\mu = 1 \cdot \mu(\mathbb{Q}) = 0.$$

The integral of simple functions has the usual properties of an integral. In particular, it is linear, positive, and monotone.

Proposition 4.3. *If $\phi, \psi : X \to [0, \infty)$ are positive simple functions on a measure space X, then:*

$$\int k\phi \, d\mu = k \int \phi \, d\mu \qquad \text{if } k \in [0, \infty);$$
$$\int (\phi + \psi) \, d\mu = \int \phi \, d\mu + \int \psi \, d\mu;$$
$$0 \leq \int \phi \, d\mu \leq \int \psi \, d\mu \qquad \text{if } 0 \leq \phi \leq \psi.$$

Proof. These follow immediately from the definition. □

4.2. Positive functions

We define the integral of a measurable function by splitting it into positive and negative parts, so we begin by defining the integral of a positive function.

Definition 4.4. If $f : X \to [0, \infty]$ is a positive, measurable, extended real-valued function on a measure space X, then

$$\int f \, d\mu = \sup \left\{ \int \phi \, d\mu : 0 \leq \phi \leq f, \, \phi \text{ simple} \right\}.$$

A positive function $f : X \to [0, \infty]$ is integrable if it is measurable and

$$\int f \, d\mu < \infty.$$

In this definition, we approximate the function f from below by simple functions. In contrast with the definition of the Riemann integral, it is not necessary to approximate a measurable function from both above and below in order to define its integral.

If $A \subset X$ is a measurable set and $f : X \to [0, \infty]$ is measurable, we define

$$\int_A f \, d\mu = \int f \chi_A \, d\mu.$$

Unlike the Riemann integral, where the definition of the integral over non-rectangular subsets of $\mathbb{R}^2$ already presents problems, it is trivial to define the Lebesgue integral over arbitrary measurable subsets of a set on which it is already defined.

The following properties are an immediate consequence of the definition and the corresponding properties of simple functions.

Proposition 4.5. *If $f, g : X \to [0, \infty]$ are positive, measurable, extended real-valued function on a measure space X, then:*

$$\int kf \, d\mu = k \int f \, d\mu \qquad \text{if } k \in [0, \infty);$$
$$0 \leq \int f \, d\mu \leq \int g \, d\mu \qquad \text{if } 0 \leq f \leq g.$$

The integral is also linear, but this is not immediately obvious from the definition and it depends on the measurability of the functions. To show the linearity, we will first derive one of the fundamental convergence theorem for the Lebesgue integral, the monotone convergence theorem. We discuss this theorem and its applications in greater detail in Section 4.5.

Theorem 4.6 (Monotone Convergence Theorem)**.** *If $\{f_n : n \in \mathbb{N}\}$ is a monotone increasing sequence*

$$0 \leq f_1 \leq f_2 \leq \dots \leq f_n \leq f_{n+1} \leq \dots$$

of positive, measurable, extended real-valued functions $f_n : X \to [0, \infty]$ and

$$f = \lim_{n\to\infty} f_n,$$

then

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu.$$

Proof. The pointwise limit $f : X \to [0, \infty]$ exists since the sequence $\{f_n\}$ is increasing. Moreover, by the monotonicity of the integral, the integrals are increasing, and

$$\int f_n \, d\mu \leq \int f_{n+1} \, d\mu \leq \int f \, d\mu,$$

so the limit of the integrals exists, and

$$\lim_{n\to\infty} \int f_n \, d\mu \leq \int f \, d\mu.$$

To prove the reverse inequality, let $\phi : X \to [0, \infty)$ be a simple function with $0 \leq \phi \leq f$. Fix $0 < t < 1$. Then

$$A_n = \{x \in X : f_n(x) \geq t\phi(x)\}$$

is an increasing sequence of measurable sets $A_1 \subset A_2 \subset \dots \subset A_n \subset \dots$ whose union is X. It follows that

$$\int f_n \, d\mu \geq \int_{A_n} f_n \, d\mu \geq t \int_{A_n} \phi \, d\mu. \tag{4.2}$$

Moreover, if

$$\phi = \sum_{i=1}^{N} c_i \chi_{E_i}$$

we have from the monotonicity of μ in Proposition 1.12 that

$$\int_{A_n} \phi \, d\mu = \sum_{i=1}^{N} c_i \mu(E_i \cap A_n) \to \sum_{i=1}^{N} c_i \mu(E_i) = \int \phi \, d\mu$$

as $n \to \infty$. Taking the limit as $n \to \infty$ in (4.2), we therefore get

$$\lim_{n\to\infty} \int f_n \, d\mu \geq t \int \phi \, d\mu.$$

Since $0 < t < 1$ is arbitrary, we conclude that

$$\lim_{n\to\infty} \int f_n \, d\mu \geq \int \phi \, d\mu,$$

and since $\phi \leq f$ is an arbitrary simple function, we get by taking the supremum over ϕ that

$$\lim_{n\to\infty} \int f_n \, d\mu \geq \int f \, d\mu.$$

This proves the theorem. □

In particular, this theorem implies that we can obtain the integral of a positive measurable function f as a limit of integrals of an increasing sequence of simple functions, not just as a supremum over all simple functions dominated by f as in Definition 4.4. As shown in Theorem 3.12, such a sequence of simple functions always exists.

Proposition 4.7. *If $f, g : X \to [0, \infty]$ are positive, measurable functions on a measure space X, then*

$$\int (f+g)\, d\mu = \int f\, d\mu + \int g\, d\mu.$$

Proof. Let $\{\phi_n : n \in \mathbb{N}\}$ and $\{\psi_n : n \in \mathbb{N}\}$ be increasing sequences of positive simple functions such that $\phi_n \to f$ and $\psi_n \to g$ pointwise as $n \to \infty$. Then $\phi_n + \psi_n$ is an increasing sequence of positive simple functions such that $\phi_n + \psi_n \to f + g$. It follows from the monotone convergence theorem (Theorem 4.6) and the linearity of the integral on simple functions that

$$\begin{aligned}\int (f+g)\, d\mu &= \lim_{n\to\infty} \int (\phi_n + \psi_n)\; d\mu \\ &= \lim_{n\to\infty} \left(\int \phi_n\, d\mu + \int \psi_n\, d\mu \right) \\ &= \lim_{n\to\infty} \int \phi_n\, d\mu + \lim_{n\to\infty} \int \psi_n\, d\mu \\ &= \int f\, d\mu + \int g\, d\mu,\end{aligned}$$

which proves the result. □

4.3. Measurable functions

If $f : X \to \overline{\mathbb{R}}$ is an extended real-valued function, we define the positive and negative parts $f^+, f^- : X \to [0, \infty]$ of f by

$$f = f^+ - f^-, \qquad f^+ = \max\{f, 0\}, \quad f^- = \max\{-f, 0\}. \tag{4.3}$$

For this decomposition,

$$|f| = f^+ + f^-.$$

Note that f is measurable if and only if f^+ and f^- are measurable.

Definition 4.8. If $f : X \to \overline{\mathbb{R}}$ is a measurable function, then

$$\int f\, d\mu = \int f^+\, d\mu - \int f^-\, d\mu,$$

provided that at least one of the integrals $\int f^+\, d\mu$, $\int f^-\, d\mu$ is finite. The function f is integrable if both $\int f^+\, d\mu$, $\int f^-\, d\mu$ are finite, which is the case if and only if

$$\int |f|\, d\mu < \infty.$$

Note that, according to Definition 4.8, the integral may take the values $-\infty$ or ∞, but it is not defined if both $\int f^+\, d\mu$, $\int f^-\, d\mu$ are infinite. Thus, although the integral of a positive measurable function always exists as an extended real number, the integral of a general, non-integrable real-valued measurable function may not exist.

This Lebesgue integral has all the usual properties of an integral. We restrict attention to integrable functions to avoid undefined expressions involving extended real numbers such as $\infty - \infty$.

Proposition 4.9. *If $f, g : X \to \mathbb{R}$ are integrable functions, then:*

$$\int kf\, d\mu = k \int f\, d\mu \qquad \text{if } k \in \mathbb{R};$$
$$\int (f+g)\, d\mu = \int f\, d\mu + \int g\, d\mu;$$
$$\int f\, d\mu \leq \int g\, d\mu \qquad \text{if } f \leq g;$$
$$\left| \int f\, d\mu \right| \leq \int |f|\, d\mu.$$

Proof. These results follow by writing functions into their positive and negative parts, as in (4.3), and using the results for positive functions.

If $f = f^+ - f^-$ and $k \geq 0$, then $(kf)^+ = kf^+$ and $(kf)^- = kf^-$, so

$$\int kf\, d\mu = \int kf^+\, d\mu - \int kf^-\, d\mu = k \int f^+\, d\mu - k \int f^-\, d\mu = k \int f\, d\mu.$$

Similarly, $(-f)^+ = f^-$ and $(-f)^- = f^+$, so

$$\int (-f)\, d\mu = \int f^-\, d\mu - \int f^+\, d\mu = -\int f\, d\mu.$$

If $h = f + g$ and

$$f = f^+ - f^-, \qquad g = g^+ - g^-, \qquad h = h^+ - h^-$$

are the decompositions of f, g, h into their positive and negative parts, then

$$h^+ - h^- = f^+ - f^- + g^+ - g^-.$$

It need not be true that $h^+ = f^+ + g^+$, but we have

$$f^- + g^- + h^+ = f^+ + g^+ + h^-.$$

The linearity of the integral on positive functions gives

$$\int f^-\, d\mu + \int g^-\, d\mu + \int h^+\, d\mu = \int f^+\, d\mu + \int g^+\, d\mu + \int h^-\, d\mu,$$

which implies that

$$\int h^+\, d\mu - \int h^-\, d\mu = \int f^+\, d\mu - \int f^-\, d\mu + \int g^+\, d\mu - \int g^-\, d\mu,$$

or $\int (f+g)\, d\mu = \int f\, d\mu + \int g\, d\mu$.

It follows that if $f \leq g$, then

$$0 \leq \int (g-f)\, d\mu = \int g\, d\mu - \int f\, d\mu,$$

so $\int f\, d\mu \leq \int g\, d\mu$. The last result is then a consequence of the previous results and $-|f| \leq f \leq |f|$. $\square$

Let us give two basic examples of the Lebesgue integral.

Example 4.10. Suppose that $X = \mathbb{N}$ and $\nu : \mathcal{P}(\mathbb{N}) \to [0, \infty]$ is counting measure on $\mathbb{N}$. If $f : \mathbb{N} \to \mathbb{R}$ and $f(n) = x_n$, then

$$\int_{\mathbb{N}} f\, d\nu = \sum_{n=1}^{\infty} x_n,$$

where the integral is finite if and only if the series is absolutely convergent. Thus, the theory of absolutely convergent series is a special case of the Lebesgue integral. Note that a conditionally convergent series, such as the alternating harmonic series, does not correspond to a Lebesgue integral, since both its positive and negative parts diverge.

Example 4.11. Suppose that $X = [a, b]$ is a compact interval and $\mu : \mathcal{L}([a, b]) \to \mathbb{R}$ is Lesbegue measure on $[a, b]$. We note in Section 4.8 that any Riemann integrable function $f : [a, b] \to \mathbb{R}$ is integrable with respect to Lebesgue measure μ, and its Riemann integral is equal to the Lebesgue integral,

$$\int_a^b f(x)\, dx = \int_{[a,b]} f\, d\mu.$$

Thus, all of the usual integrals from elementary calculus remain valid for the Lebesgue integral on $\mathbb{R}$. We will write an integral with respect to Lebesgue measure on $\mathbb{R}$, or $\mathbb{R}^n$, as

$$\int f\, dx.$$

Even though the class of Lebesgue integrable functions on an interval is wider than the class of Riemann integrable functions, some improper Riemann integrals may exist even though the Lebesegue integral does not.

Example 4.12. The integral

$$\int_0^1 \left(\frac{1}{x} \sin\frac{1}{x} + \cos\frac{1}{x} \right) dx$$

is not defined as a Lebesgue integral, although the improper Riemann integral

$$\lim_{\epsilon \to 0^+} \int_\epsilon^1 \left(\frac{1}{x} \sin\frac{1}{x} + \cos\frac{1}{x} \right) dx = \lim_{\epsilon \to 0^+} \int_\epsilon^1 \frac{d}{dx}\left[x \cos\frac{1}{x} \right] dx = \cos 1$$

exists.

Example 4.13. The integral

$$\int_{-1}^1 \frac{1}{x}\, dx$$

is not defined as a Lebesgue integral, although the principal value integral

$$\text{p.v.} \int_{-1}^1 \frac{1}{x}\, dx = \lim_{\epsilon \to 0^+} \left\{ \int_{-1}^{-\epsilon} \frac{1}{x}\, dx + \int_\epsilon^1 \frac{1}{x}\, dx \right\} = 0$$

exists. Note, however, that the Lebesgue integrals

$$\int_0^1 \frac{1}{x}\, dx = \infty, \qquad \int_{-1}^0 \frac{1}{x}\, dx = -\infty$$

are well-defined as extended real numbers.

The inability of the Lebesgue integral to deal directly with the cancelation between large positive and negative parts in oscillatory or singular integrals, such as the ones in the previous examples, is sometimes viewed as a defect (although the integrals above can still be defined as an appropriate limit of Lebesgue integrals). Other definitions of the integral such as the Henstock-Kurzweil integral, which is a generalization of the Riemann integral, avoid this defect but they have not proved to be as useful as the Lebesgue integral. Similar issues arise in connection with Feynman path integrals in quantum theory, where one would like to define the integral of highly oscillatory functionals on an infinite-dimensional function-space.

4.4. Absolute continuity

The following results show that a function with finite integral is finite a.e. and that the integral depends only on the pointwise a.e. values of a function.

Proposition 4.14. *If $f : X \to \overline{\mathbb{R}}$ is an integrable function, meaning that $\int |f|\, d\mu < \infty$, then f is finite μ-a.e. on X.*

Proof. We may assume that f is positive without loss of generality. Suppose that

$$E = \{x \in X : f = \infty\}$$

has nonzero measure. Then for any $t > 0$, we have $f > t\chi_E$, so

$$\int f\, d\mu \geq \int t\chi_E\, d\mu = t\mu(E),$$

which implies that $\int f\, d\mu = \infty$. □

Proposition 4.15. *Suppose that $f : X \to \overline{\mathbb{R}}$ is an extended real-valued measurable function. Then $\int |f|\, d\mu = 0$ if and only if $f = 0$ μ-a.e.*

Proof. By replacing f with $|f|$, we can assume that f is positive without loss of generality. Suppose that $f = 0$ a.e. If $0 \leq \phi \leq f$ is a simple function,

$$\phi = \sum_{i=1}^{N} c_i \chi_{E_i},$$

then $\phi = 0$ a.e., so $c_i = 0$ unless $\mu(E_i) = 0$. It follows that

$$\int \phi\, d\mu = \sum_{i=1}^{N} c_i \mu(E_i) = 0,$$

and Definition 4.4 implies that $\int f\, d\mu = 0$.

Conversely, suppose that $\int f\, d\mu = 0$. For $n \in \mathbb{N}$, let

$$E_n = \{x \in X : f(x) \geq 1/n\}.$$

Then $0 \leq (1/n)\chi_{E_n} \leq f$, so that

$$0 \leq \frac{1}{n}\mu(E_n) = \int \frac{1}{n}\chi_{E_n}\, d\mu \leq \int f\, d\mu = 0,$$

and hence $\mu(E_n) = 0$. Now observe that

$$\{x \in X : f(x) > 0\} = \bigcup_{n=1}^{\infty} E_n,$$

so it follows from the countable additivity of μ that $f = 0$ a.e. □

In particular, it follows that if $f : X \to \overline{\mathbb{R}}$ is any measurable function, then

$$\int_A f\,d\mu = 0 \qquad \text{if } \mu(A) = 0. \tag{4.4}$$

For integrable functions we can strengthen the previous result to get the following property, which is called the absolute continuity of the integral.

Proposition 4.16. *Suppose that $f : X \to \overline{\mathbb{R}}$ is an integrable function, meaning that $\int |f|\,d\mu < \infty$. Then, given any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$0 \le \int_A |f|\,d\mu < \epsilon \tag{4.5}$$

whenever A is a measurable set with $\mu(A) < \delta$.

Proof. Again, we can assume that f is positive. For $n \in \mathbb{N}$, define $f_n : X \to [0,\infty]$ by

$$f_n(x) = \begin{cases} n & \text{if } f(x) \ge n, \\ f(x) & \text{if } 0 \le f(x) < n. \end{cases}$$

Then $\{f_n\}$ is an increasing sequence of positive measurable functions that converges pointwise to f. We estimate the integral of f over A as follows:

$$\begin{aligned} \int_A f\,d\mu &= \int_A (f - f_n)\,d\mu + \int_A f_n\,d\mu \\ &\le \int_X (f - f_n)\,d\mu + n\mu(A). \end{aligned}$$

By the monotone convergence theorem,

$$\int_X f_n\,d\mu \to \int_X f\,d\mu < \infty$$

as $n \to \infty$. Therefore, given $\epsilon > 0$, we can choose n such that

$$0 \le \int_X (f - f_n)\,d\mu < \frac{\epsilon}{2},$$

and then choose

$$\delta = \frac{\epsilon}{2n}.$$

If $\mu(A) < \delta$, we get (4.5), which proves the result. □

Proposition 4.16 may fail if f is not integrable.

Example 4.17. Define $\nu : \mathcal{B}((0,1)) \to [0,\infty]$ by

$$\nu(A) = \int_A \frac{1}{x}\,dx,$$

where the integral is taken with respect to Lebesgue measure μ. Then $\nu(A) = 0$ if $\mu(A) = 0$, but (4.5) does not hold.

There is a converse to this theorem concerning the representation of absolutely continuous measures as integrals (the Radon-Nikodym theorem, stated in Theorem 6.27).

4.5. Convergence theorems

One of the most basic questions in integration theory is the following: If $f_n \to f$ pointwise, when can one say that

$$\int f_n \, d\mu \to \int f \, d\mu? \tag{4.6}$$

The Riemann integral is not sufficiently general to permit a satisfactory answer to this question.

Perhaps the simplest condition that guarantees the convergence of the integrals is that the functions $f_n : X \to \mathbb{R}$ converge uniformly to $f : X \to \mathbb{R}$ and X has finite measure. In that case

$$\left| \int f_n \, d\mu - \int f \, d\mu \right| \leq \int |f_n - f| \, d\mu \leq \mu(X) \sup_X |f_n - f| \to 0$$

as $n \to \infty$. The assumption of uniform convergence is too strong for many purposes, and the Lebesgue integral allows the formulation of simple and widely applicable theorems for the convergence of integrals. The most important of these are the monotone convergence theorem (Theorem 4.6) and the Lebesgue dominated convergence theorem (Theorem 4.24). The utility of these results accounts, in large part, for the success of the Lebesgue integral.

Some conditions on the functions f_n in (4.6) are, however, necessary to ensure the convergence of the integrals, as can be seen from very simple examples. Roughly speaking, the convergence may fail because 'mass' can leak out to infinity in the limit.

Example 4.18. Define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} n & \text{if } 0 < x < 1/n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_n \to 0$ as $n \to \infty$ pointwise on $\mathbb{R}$, but

$$\int f_n \, dx = 1 \qquad \text{for every } n \in \mathbb{N}.$$

By modifying this example, and the following ones, we can obtain a sequence f_n that converges pointwise to zero but whose integrals converge to infinity; for example

$$f_n(x) = \begin{cases} n^2 & \text{if } 0 < x < 1/n, \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.19. Define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1/n & \text{if } 0 < x < n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_n \to 0$ as $n \to \infty$ pointwise on $\mathbb{R}$, and even uniformly, but

$$\int f_n \, dx = 1 \qquad \text{for every } n \in \mathbb{N}.$$

Example 4.20. Define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } n < x < n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_n \to 0$ as $n \to \infty$ pointwise on $\mathbb{R}$, but

$$\int f_n \, dx = 1 \qquad \text{for every } n \in \mathbb{N}.$$

The monotone convergence theorem implies that a similar failure of convergence of the integrals cannot occur in an increasing sequence of functions, even if the convergence is not uniform or the domain space does not have finite measure. Note that the monotone convergence theorem does not hold for the Riemann integral; indeed, the pointwise limit of a monotone increasing, bounded sequence of Riemann integrable functions need not even be Riemann integrable.

Example 4.21. Let $\{q_i : i \in \mathbb{N}\}$ be an enumeration of the rational numbers in the interval $[0, 1]$, and define $f_n : [0, 1] \to [0, \infty)$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x = q_i \text{ for some } 1 \leq i \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{f_n\}$ is a monotone increasing sequence of bounded, positive, Riemann integrable functions, each of which has zero integral. Nevertheless, as $n \to \infty$ the sequence converges pointwise to the characteristic function of the rationals in $[0, 1]$, which is not Riemann integrable.

A useful generalization of the monotone convergence theorem is the following result, called Fatou's lemma.

Theorem 4.22. *Suppose that $\{f_n : n \in \mathbb{N}\}$ is sequence of positive measurable functions $f_n : X \to [0, \infty]$. Then*

$$\int \liminf_{n \to \infty} f_n \, d\mu \leq \liminf_{n \to \infty} \int f_n \, d\mu. \tag{4.7}$$

Proof. For each $n \in \mathbb{N}$, let

$$g_n = \inf_{k \geq n} f_k.$$

Then $\{g_n\}$ is a monotone increasing sequence which converges pointwise to $\liminf f_n$ as $n \to \infty$, so by the monotone convergence theorem

$$\lim_{n \to \infty} \int g_n \, d\mu = \int \liminf_{n \to \infty} f_n \, d\mu. \tag{4.8}$$

Moreover, since $g_n \leq f_k$ for every $k \geq n$, we have

$$\int g_n \, d\mu \leq \inf_{k \geq n} \int f_k \, d\mu,$$

so that

$$\lim_{n \to \infty} \int g_n \, d\mu \leq \liminf_{n \to \infty} \int f_n \, d\mu.$$

Using (4.8) in this inequality, we get the result. □

We may have strict inequality in (4.7), as in the previous examples. The monotone convergence theorem and Fatou's Lemma enable us to determine the integrability of functions.

Example 4.23. For $\alpha \in \mathbb{R}$, consider the function $f : [0,1] \to [0,\infty]$ defined by

$$f(x) = \begin{cases} x^{-\alpha} & \text{if } 0 < x \leq 1, \\ \infty & \text{if } x = 0. \end{cases}$$

For $n \in \mathbb{N}$, let

$$f_n(x) = \begin{cases} x^{-\alpha} & \text{if } 1/n \leq x \leq 1, \\ n^{\alpha} & \text{if } 0 \leq x < 1/n. \end{cases}$$

Then $\{f_n\}$ is an increasing sequence of Lebesgue measurable functions (*e.g* since f_n is continuous) that converges pointwise to f. We denote the integral of f with respect to Lebesgue measure on $[0,1]$ by $\int_0^1 f(x)\,dx$. Then, by the monotone convergence theorem,

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \int_0^1 f_n(x)\,dx.$$

From elementary calculus,

$$\int_0^1 f_n(x)\,dx \to \frac{1}{1-\alpha}$$

as $n \to \infty$ if $\alpha < 1$, and to ∞ if $\alpha \geq 1$. Thus, f is integrable on $[0,1]$ if and only if $\alpha < 1$.

Perhaps the most frequently used convergence result is the following dominated convergence theorem, in which all the integrals are necessarily finite.

Theorem 4.24 (Lebesgue Dominated Convergence Theorem)**.** *If $\{f_n : n \in \mathbb{N}\}$ is a sequence of measurable functions $f_n : X \to \mathbb{R}$ such that $f_n \to f$ pointwise, and $|f_n| \leq g$ where $g : X \to [0,\infty]$ is an integrable function, meaning that $\int g\,d\mu < \infty$, then*

$$\int f_n\,d\mu \to \int f\,d\mu \qquad \text{as } n \to \infty.$$

Proof. Since $g + f_n \geq 0$ for every $n \in \mathbb{N}$, Fatou's lemma implies that

$$\int (g+f)\,d\mu \leq \liminf_{n\to\infty} \int (g+f_n)\,d\mu \leq \int g\,d\mu + \liminf_{n\to\infty} \int f_n\,d\mu,$$

which gives

$$\int f\,d\mu \leq \liminf_{n\to\infty} \int f_n\,d\mu.$$

Similarly, $g - f_n \geq 0$, so

$$\int (g-f)\,d\mu \leq \liminf_{n\to\infty} \int (g-f_n)\,d\mu \leq \int g\,d\mu - \limsup_{n\to\infty} \int f_n\,d\mu,$$

which gives

$$\int f\,d\mu \geq \limsup_{n\to\infty} \int f_n\,d\mu,$$

and the result follows. □

An alternative, and perhaps more illuminating, proof of the dominated convergence theorem may be obtained from Egoroff's theorem and the absolute continuity of the integral. Egoroff's theorem states that if a sequence $\{f_n\}$ of measurable functions, defined on a finite measure space $(X, \mathcal{A}, \mu)$, converges pointwise to a function f, then for every $\epsilon > 0$ there exists a measurable set $A \subset X$ such that $\{f_n\}$ converges uniformly to f on A and $\mu(X \setminus A) < \epsilon$. The uniform integrability of the functions and the absolute continuity of the integral imply that what happens off the set A may be made to have arbitrarily small effect on the integrals. Thus, the convergence theorems hold because of this 'almost' uniform convergence of pointwise-convergent sequences of measurable functions.

4.6. Complex-valued functions and a.e. convergence

In this section, we briefly indicate the generalization of the above results to complex-valued functions and sequences that converge pointwise almost everywhere. The required modifications are straightforward.

If $f : X \to \mathbb{C}$ is a complex valued function $f = g + ih$, then we say that f is measurable if and only if its real and imaginary parts $g, h : X \to \mathbb{R}$ are measurable, and integrable if and only if g, h are integrable. In that case, we define

$$\int f \, d\mu = \int g \, d\mu + i \int h \, d\mu.$$

Note that we do not allow extended real-valued functions or infinite integrals here. It follows from the discussion of product measures that $f : X \to \mathbb{C}$, where $\mathbb{C}$ is equipped with its Borel σ-algebra $\mathcal{B}(\mathbb{C})$, is measurable if and only if its real and imaginary parts are measurable, so this definition is consistent with our previous one.

The integral of complex-valued functions satisfies the properties given in Proposition 4.9, where we allow $k \in \mathbb{C}$ and the condition $f \leq g$ is only relevant for real-valued functions. For example, to show that $|\int f \, d\mu| \leq \int |f| \, d\mu$, we let

$$\int f \, d\mu = \left| \int f \, d\mu \right| e^{i\theta}$$

for a suitable argument θ, and then

$$\left| \int f \, d\mu \right| = e^{-i\theta} \int f \, d\mu = \int \Re[e^{-i\theta} f] \, d\mu \leq \int |\Re[e^{-i\theta} f]| \, d\mu \leq \int |f| \, d\mu.$$

Complex-valued functions also satisfy the properties given in Section 4.4.

The monotone convergence theorem holds for extended real-valued functions if $f_n \uparrow f$ pointwise a.e., and the Lebesgue dominated convergence theorem holds for complex-valued functions if $f_n \to f$ pointwise a.e. and $|f_n| \leq g$ pointwise a.e. where g is an integrable extended real-valued function. If the measure space is not complete, then we also need to assume that f is measurable. To prove these results, we replace the functions f_n, for example, by $f_n \chi_{N^c}$ where N is a null set off which pointwise convergence holds, and apply the previous theorems; the values of any integrals are unaffected.

4.7. L^1 spaces

We introduce here the space $L^1(X)$ of integrable functions on a measure space X; we will study its properties, and the properties of the closely related L^p spaces, in more detail later on.

Definition 4.25. If $(X, \mathcal{A}, \mu)$ is a measure space, then the space $L^1(X)$ consists of the integrable functions $f : X \to \mathbb{R}$ with norm

$$\|f\|_{L^1} = \int |f| \, d\mu < \infty,$$

where we identify functions that are equal a.e. A sequence of functions

$$\{f_n \in L^1(X)\}$$

converges in L^1, or in mean, to $f \in L^1(X)$ if

$$\|f - f_n\|_{L^1} = \int |f - f_n| \, d\mu \to 0 \qquad \text{as } n \to \infty.$$

We also denote the space of integrable complex-valued functions $f : X \to \mathbb{C}$ by $L^1(X)$. For definiteness, we consider real-valued functions unless stated otherwise; in most cases, the results generalize in an obvious way to complex-valued functions

Convergence in mean is not equivalent to pointwise a.e.-convergence. The sequences in Examples 4.18–4.20 converges to zero pointwise, but they do not converge in mean. The following is an example of a sequence that converges in mean but not pointwise a.e.

Example 4.26. Define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_1(x) = 1, \quad f_2(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1/2, \\ 0 & \text{if } 1/2 < x \le 1, \end{cases} \qquad f_3(x) = \begin{cases} 0 & \text{if } 0 \le x < 1/2, \\ 1 & \text{if } 1/2 \le x \le 1, \end{cases}$$

$$f_4(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1/4, \\ 1 & \text{if } 1/4 < x \le 1, \end{cases} \qquad f_5(x) = \begin{cases} 0 & \text{if } 0 \le x < 1/4, \\ 1 & \text{if } 1/4 \le x \le 1/2, \\ 0 & \text{if } 1/2x < x \le 1, \end{cases}$$

and so on. That is, for $2^m \le n \le 2^m - 1$, the function f_n consists of a spike of height one and width 2^{-m} that sweeps across the interval $[0, 1]$ as n increases from 2^m to $2^m - 1$. This sequence converges to zero in mean, but it does not converge pointwise as any point $x \in [0, 1]$.

We will show, however, that a sequence which converges sufficiently rapidly in mean does converge pointwise a.e.; as a result, every sequence that converges in mean has a subsequence that converges pointwise a.e. (see Lemma 7.9 and Corollary 7.11).

Let us consider the particular case of $L^1(\mathbb{R}^n)$. As an application of the Borel regularity of Lebesgue measure, we prove that integrable functions on $\mathbb{R}^n$ may be approximated by continuous functions with compact support. This result means that $L^1(\mathbb{R}^n)$ is a concrete realization of the completion of $C_c(\mathbb{R}^n)$ with respect to the $L^1(\mathbb{R}^n)$-norm, where $C_c(\mathbb{R}^n)$ denotes the space of continuous functions $f : \mathbb{R}^n \to \mathbb{R}$ with compact support. The support of f is defined by

$$\text{supp} f = \overline{\{x \in \mathbb{R}^n : f(x) \ne 0\}}.$$

Thus, f has compact support if and only if it vanishes outside a bounded set.

Theorem 4.27. *The space $C_c(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$. Explicitly, if $f \in L^1(\mathbb{R}^n)$, then for any $\epsilon > 0$ there exists a function $g \in C_c(\mathbb{R}^n)$ such that*

$$\|f - g\|_{L^1} < \epsilon.$$

Proof. Note first that by the dominated convergence theorem

$$\left\|f - f\chi_{B_R(0)}\right\|_{L^1} \to 0 \qquad \text{as } R \to \infty,$$

so we can assume that $f \in L^1(\mathbb{R}^n)$ has compact support. Decomposing $f = f^+ - f^-$ into positive and negative parts, we can also assume that f is positive. Then there is an increasing sequence of compactly supported simple functions that converges to f pointwise and hence, by the monotone (or dominated) convergence theorem, in mean. Since every simple function is a finite linear combination of characteristic functions, it is sufficient to prove the result for the characteristic function χ_A of a bounded, measurable set $A \subset \mathbb{R}^n$.

Given $\epsilon > 0$, by the Borel regularity of Lebesgue measure, there exists a bounded open set G and a compact set K such that $K \subset A \subset G$ and $\mu(G \setminus K) < \epsilon$. Let $g \in C_c(\mathbb{R}^n)$ be a Urysohn function such that $g = 1$ on K, $g = 0$ on G^c, and $0 \le g \le 1$. For example, we can define g explicitly by

$$g(x) = \frac{d(x, G^c)}{d(x, K) + d(x, G^c)}$$

where the distance function $d(\cdot, F) : \mathbb{R}^n \to \mathbb{R}$ from a subset $F \subset \mathbb{R}^n$ is defined by

$$d(x, F) = \inf \{|x - y| : y \in F\}.$$

If F is closed, then $d(\cdot, F)$ is continuous, so g is continuous.

We then have that

$$\|\chi_A - g\|_{L^1} = \int_{G \setminus K} |\chi_A - g| \, dx \le \mu(G \setminus K) < \epsilon,$$

which proves the result. □

4.8. Riemann integral

Any Riemann integrable function $f : [a, b] \to \mathbb{R}$ is Lebesgue measurable, and in fact integrable since it must be bounded, but a Lebesgue integrable function need not be Riemann integrable. Using Lebesgue measure, we can give a necessary and sufficient condition for Riemann integrability.

Theorem 4.28. *If $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then f is Lebesgue integrable on $[a, b]$ and its Riemann integral is equal to its Lebesgue integral. A Lebesgue measurable function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if it is bounded and the set of discontinuities $\{x \in [a, b] : f \text{ is discontinuous at } x\}$ has Lebesgue measure zero.*

For the proof, see *e.g.* Folland [**4**].

Example 4.29. The characteristic function of the rationals $\chi_{\mathbb{Q}\cap[0,1]}$ is discontinuous at every point and it is not Riemann integrable on $[0, 1]$. This function is, however, equal a.e. to the zero function which is continuous at every point and is Riemann integrable. (Note that being continuous a.e. is not the same thing as being equal a.e. to a continuous function.) Any function that is bounded and continuous except at countably many points is Riemann integrable, but these are not the only Riemann integrable functions. For example, the characteristic function of a Cantor set with zero measure is a Riemann integrable function that is discontinuous at uncountable many points.

4.9. Integrals of vector-valued functions

In Definition 4.4, we use the ordering properties of $\overline{\mathbb{R}}$ to define real-valued integrals as a supremum of integrals of simple functions. Finite integrals of complex-valued functions or vector-valued functions that take values in a finite-dimensional vector space are then defined componentwise.

An alternative method is to define the integral of a vector-valued function $f : X \to Y$ from a measure space X to a Banach space Y as a limit in norm of integrals of vector-valued simple functions. The integral of vector-valued simple functions is defined as in (4.1), assuming that $\mu(E_n) < \infty$; linear combinations of the values $c_n \in Y$ make sense since Y is a vector space. A function $f : X \to Y$ is integrable if there is a sequence of integrable simple functions $\{\phi_n : X \to Y\}$ such that $\phi_n \to f$ pointwise, where the convergence is with respect to the norm $\|\cdot\|$ on Y, and

$$\int \|f - \phi_n\| \, d\mu \to 0 \qquad \text{as } n \to \infty.$$

Then we define

$$\int f \, d\mu = \lim_{n\to\infty} \int \phi_n \, d\mu,$$

where the limit is the norm-limit in Y.

This definition of the integral agrees with the one used above for real-valued, integrable functions, and amounts to defining the integral of an integrable function by completion in the L^1-norm. We will not develop this definition here (see [**6**], for example, for a detailed account), but it is useful in defining the integral of functions that take values in an infinite-dimensional Banach space, when it leads to the Bochner integral. An alternative approach is to reduce vector-valued integrals to scalar-valued integrals by the use of continuous linear functionals belonging to the dual space of the Banach space.