CHAPTER 5

Product Measures

Given two measure spaces, we may construct a natural measure on their Cartesian product; the prototype is the construction of Lebesgue measure on $\mathbb{R}^2$ as the product of Lebesgue measures on $\mathbb{R}$. The integral of a measurable function on the product space may be evaluated as iterated integrals on the individual spaces provided that the function is positive or integrable (and the measure spaces are σ-finite). This result, called Fubini's theorem, is another one of the basic and most useful properties of the Lebesgue integral. We will not give complete proofs of all the results in this Chapter.

5.1. Product σ-algebras

We begin by describing product σ-algebras. If $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are measurable spaces, then a measurable rectangle is a subset $A \times B$ of $X \times Y$ where $A \in \mathcal{A}$ and $B \in \mathcal{B}$ are measurable subsets of X and Y, respectively. For example, if $\mathbb{R}$ is equipped with its Borel σ-algebra, then $\mathbb{Q} \times \mathbb{Q}$ is a measurable rectangle in $\mathbb{R} \times \mathbb{R}$. (Note that the 'sides' A, B of a measurable rectangle $A \times B \subset \mathbb{R} \times \mathbb{R}$ can be arbitrary measurable sets; they are not required to be intervals.)

Definition 5.1. Suppose that $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are measurable spaces. The product σ-algebra $\mathcal{A} \otimes \mathcal{B}$ is the σ-algebra on $X \times Y$ generated by the collection of all measurable rectangles,

$$\mathcal{A} \otimes \mathcal{B} = \sigma\left(\{A \times B : A \in \mathcal{A},\ B \in \mathcal{B}\}\right).$$

The product of $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ is the measurable space $(X \times Y, \mathcal{A} \otimes \mathcal{B})$.

Suppose that $E \subset X \times Y$. For any $x \in X$ and $y \in Y$, we define the x-section $E_x \subset Y$ and the y-section $E^y \subset X$ of E by

$$E_x = \{y \in Y : (x, y) \in E\}, \qquad E^y = \{x \in X : (x, y) \in E\}.$$

As stated in the next proposition, all sections of a measurable set are measurable.

Proposition 5.2. *If $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are measurable spaces and $E \in \mathcal{A} \otimes \mathcal{B}$, then $E_x \in \mathcal{B}$ for every $x \in X$ and $E^y \in \mathcal{A}$ for every $y \in Y$.*

Proof. Let

$$\mathcal{M} = \{E \subset X \times Y : E_x \in \mathcal{B} \text{ for every } x \in X \text{ and } E^y \in \mathcal{A} \text{ for every } y \in Y\}.$$

Then $\mathcal{M}$ contains all measurable rectangles, since the x-sections of $A \times B$ are either $\varnothing$ or B and the y-sections are either $\varnothing$ or A. Moreover, $\mathcal{M}$ is a σ-algebra since, for example, if $E, E_i \subset X \times Y$ and $x \in X$, then

$$(E^c)_x = (E_x)^c, \qquad \left(\bigcup_{i=1}^{\infty} E_i\right)_x = \bigcup_{i=1}^{\infty} (E_i)_x.$$

It follows that $\mathcal{M} \supset \mathcal{A} \otimes \mathcal{B}$, which proves the proposition. $\square$

As an example, we consider the product of Borel σ-algebras on $\mathbb{R}^n$.

Proposition 5.3. *Suppose that $\mathbb{R}^m$, $\mathbb{R}^n$ are equipped with their Borel σ-algebras $\mathcal{B}(\mathbb{R}^m)$, $\mathcal{B}(\mathbb{R}^n)$ and let $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$. Then*

$$\mathcal{B}(\mathbb{R}^{m+n}) = \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^n).$$

Proof. Every $(m+n)$-dimensional rectangle, in the sense of Definition 2.1, is a product of an m-dimensional and an n-dimensional rectangle. Therefore

$$\mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^n) \supset \mathcal{R}(\mathbb{R}^{m+n})$$

where $\mathcal{R}(\mathbb{R}^{m+n})$ denotes the collection of rectangles in $\mathbb{R}^{m+n}$. From Proposition 2.21, the rectangles generate the Borel σ-algebra, and therefore

$$\mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^n) \supset \mathcal{B}(\mathbb{R}^{m+n}).$$

To prove the the reverse inclusion, let

$$\mathcal{M} = \left\{A \subset \mathbb{R}^m : A \times \mathbb{R}^n \in \mathcal{B}(\mathbb{R}^{m+n})\right\}.$$

Then $\mathcal{M}$ is a σ-algebra, since $\mathcal{B}(\mathbb{R}^{m+n})$ is a σ-algebra and

$$A^c \times \mathbb{R}^n = (A \times \mathbb{R}^n)^c, \qquad \left(\bigcup_{i=1}^{\infty} A_i\right) \times \mathbb{R}^n = \bigcup_{i=1}^{\infty} (A_i \times \mathbb{R}^n).$$

Moreover, $\mathcal{M}$ contains all open sets, since $G \times \mathbb{R}^n$ is open in $\mathbb{R}^{m+n}$ if G is open in $\mathbb{R}^m$. It follows that $\mathcal{M} \supset \mathcal{B}(\mathbb{R}^m)$, so $A \times \mathbb{R}^n \in \mathcal{B}(\mathbb{R}^{m+n})$ for every $A \in \mathcal{B}(\mathbb{R}^m)$, meaning that

$$\mathcal{B}(\mathbb{R}^{m+n}) \supset \left\{A \times \mathbb{R}^n : A \in \mathcal{B}(\mathbb{R}^m)\right\}.$$

Similarly, we have

$$\mathcal{B}(\mathbb{R}^{m+n}) \supset \left\{\mathbb{R}^n \times B : B \in \mathcal{B}(\mathbb{R}^n)\right\}.$$

Therefore, since $\mathcal{B}(\mathbb{R}^{m+n})$ is closed under intersections,

$$\mathcal{B}(\mathbb{R}^{m+n}) \supset \left\{A \times B : A \in \mathcal{B}(\mathbb{R}^m),\, B \in \mathcal{B}(\mathbb{R}^n)\right\},$$

which implies that

$$\mathcal{B}(\mathbb{R}^{m+n}) \supset \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^n).$$

$\square$

By the repeated application of this result, we see that the Borel σ-algebra on $\mathbb{R}^n$ is the n-fold product of the Borel σ-algebra on $\mathbb{R}$. This leads to an alternative method of constructing Lebesgue measure on $\mathbb{R}^n$ as a product of Lebesgue measures on $\mathbb{R}$, instead of the direct construction we gave earlier.

5.2. Premeasures

Premeasures provide a useful way to generate outer measures and measures, and we will use them to construct product measures. In this section, we derive some general results about premeasures and their associated measures that we use below. Premeasures are defined on algebras, rather than σ-algebras, but they are consistent with countable additivity.

Definition 5.4. An algebra on a set X is a collection of subsets of X that contains $\varnothing$ and X and is closed under complements, finite unions, and finite intersections.

If $\mathcal{F} \subset \mathcal{P}(X)$ is a family of subsets of a set X, then the algebra generated by $\mathcal{F}$ is the smallest algebra that contains $\mathcal{F}$. It is much easier to give an explicit description of the algebra generated by a family of sets $\mathcal{F}$ than the σ-algebra generated by $\mathcal{F}$. For example, if $\mathcal{F}$ has the property that for $A, B \in \mathcal{F}$, the intersection $A \cap B \in \mathcal{F}$ and the complement A^c is a finite union of sets belonging to $\mathcal{F}$, then the algebra generated by $\mathcal{F}$ is the collection of all finite unions of sets in $\mathcal{F}$.

Definition 5.5. Suppose that $\mathcal{E}$ is an algebra of subsets of a set X. A premeasure λ on $\mathcal{E}$, or on X if the algebra is understood, is a function $\lambda : \mathcal{E} \to [0, \infty]$ such that:

(a) $\lambda(\varnothing) = 0$;
(b) if $\{A_i \in \mathcal{E} : i \in \mathbb{N}\}$ is a countable collection of disjoint sets in $\mathcal{E}$ such that

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{E},$$

then

$$\lambda\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \lambda\left(A_i\right).$$

Note that a premeasure is finitely additive, since we may take $A_i = \varnothing$ for $i \geq N$, and monotone, since if $A \supset B$, then $\lambda(A) = \lambda(A \setminus B) + \lambda(B) \geq \lambda(B)$.

To define the outer measure associated with a premeasure, we use countable coverings by sets in the algebra.

Definition 5.6. Suppose that $\mathcal{E}$ is an algebra on a set X and $\lambda : \mathcal{E} \to [0, \infty]$ is a premeasure. The outer measure $\lambda^* : \mathcal{P}(X) \to [0, \infty]$ associated with λ is defined for $E \subset X$ by

$$\lambda^*(E) = \inf\left\{\sum_{i=1}^{\infty} \lambda(A_i) : E \subset \textstyle\bigcup_{i=1}^{\infty} A_i \text{ where } A_i \in \mathcal{E}\right\}.$$

As we observe next, the set-function λ^* is an outer measure. Moreover, every set belonging to $\mathcal{E}$ is λ^*-measurable and its outer measure is equal to its premeasure.

Proposition 5.7. *The set function $\lambda^* : \mathcal{P}(X) \to [0, \infty]$ given by Definition 5.6. is an outer measure on X. Every set $A \in \mathcal{E}$ is Carathéodory measurable and $\lambda^*(A) = \lambda(A)$.*

Proof. The proof that λ^* is an outer measure is identical to the proof of Theorem 2.4 for outer Lebesgue measure.

If $A \in \mathcal{E}$, then $\lambda^*(A) \leq \lambda(A)$ since A covers itself. To prove the reverse inequality, suppose that $\{A_i : i \in \mathbb{N}\}$ is a countable cover of A by sets $A_i \in \mathcal{E}$. Let $B_1 = A \cap A_1$ and

$$B_j = A \cap \left(A_j \setminus \bigcup_{i=1}^{j-1} A_i\right) \qquad \text{for } j \geq 2.$$

Then $B_j \in \mathcal{A}$ and A is the disjoint union of $\{B_j : j \in \mathbb{N}\}$. Since $B_j \subset A_j$, it follows that

$$\lambda(A) = \sum_{j=1}^{\infty} \lambda(B_j) \leq \sum_{j=1}^{\infty} \lambda(A_j),$$

which implies that $\lambda(A) \leq \lambda^*(A)$. Hence, $\lambda^*(A) = \lambda(A)$.

If $E \subset X$, $A \in \mathcal{E}$, and $\epsilon > 0$, then there is a cover $\{B_i \in \mathcal{E} : i \in \mathbb{N}\}$ of E such that

$$\lambda^*(E) + \epsilon \geq \sum_{i=1}^{\infty} \lambda(B_i).$$

Since λ is countably additive on $\mathcal{E}$,

$$\lambda^*(E) + \epsilon \geq \sum_{i=1}^{\infty} \lambda(B_i \cap A) + \sum_{i=1}^{\infty} \lambda(B_i \cap A^c) \geq \lambda^*(E \cap A) + \lambda^*(E \cap A^c),$$

and since $\epsilon > 0$ is arbitrary, it follows that $\lambda^*(E) \geq \lambda^*(E \cap A) + \lambda^*(E \cap A^c)$, which implies that A is measurable. □

Using Theorem 2.9, we see from the preceding results that every premeasure on an algebra $\mathcal{E}$ may be extended to a measure on $\sigma(\mathcal{E})$. A natural question is whether such an extension is unique. In general, the answer is no, but if the measure space is not 'too big,' in the following sense, then we do have uniqueness.

Definition 5.8. Let X be a set and λ a premeasure on an algebra $\mathcal{E} \subset \mathcal{P}(X)$. Then λ is σ-finite if $X = \bigcup_{i=1}^{\infty} A_i$ where $A_i \in \mathcal{E}$ and $\lambda(A_i) < \infty$.

Theorem 5.9. *If $\lambda : \mathcal{E} \to [0, \infty]$ is a σ-finite premeasure on an algebra $\mathcal{E}$ and $\mathcal{A}$ is the σ-algebra generated by $\mathcal{E}$, then there is a unique measure $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu(A) = \lambda(A)$ for every $A \in \mathcal{E}$.*

5.3. Product measures

Next, we construct a product measure on the product of measure spaces that satisfies the natural condition that the measure of a measurable rectangle is the product of the measures of its 'sides.' To do this, we will use the Carathéodory method and first define an outer measure on the product of the measure spaces in terms of the natural premeasure defined on measurable rectangles. The procedure is essentially the same as the one we used to construct Lebesgue measure on $\mathbb{R}^n$.

Suppose that $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are measurable spaces. The intersection of measurable rectangles is a measurable rectangle

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D),$$

and the complement of a measurable rectangle is a finite union of measurable rectangles

$$(A \times B)^c = (A^c \times B) \cup (A \times B^c) \cup (A^c \times B^c).$$

Thus, the collection of finite unions of measurable rectangles in $X \times Y$ forms an algebra, which we denote by $\mathcal{E}$. This algebra is not, in general, a σ-algebra, but obviously it generates the same product σ-algebra as the measurable rectangles.

Every set $E \in \mathcal{E}$ may be represented as a finite disjoint union of measurable rectangles, though not necessarily in a unique way. To define a premeasure on $\mathcal{E}$, we first define the premeasure of measurable rectangles.

Definition 5.10. If $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are measure spaces, then the product premeasure $\lambda(A \times B)$ of a measurable rectangle $A \times B \subset X \times Y$ is given by

$$\lambda(A \times B) = \mu(A)\nu(B)$$

where $0 \cdot \infty = 0$.

The premeasure λ is countably additive on rectangles. The simplest way to show this is to integrate the characteristic functions of the rectangles, which allows us to use the monotone convergence theorem.

Proposition 5.11. *If a measurable rectangle $A \times B$ is a countable disjoint union of measurable rectangles $\{A_i \times B_i : i \in \mathbb{N}\}$, then*

$$\lambda(A \times B) = \sum_{i=1}^{\infty} \lambda(A_i \times B_i).$$

PROOF. If

$$A \times B = \bigcup_{i=1}^{\infty} (A_i \times B_i)$$

is a disjoint union, then the characteristic function $\chi_{A\times B} : X \times Y \to [0, \infty)$ satisfies

$$\chi_{A\times B}(x, y) = \sum_{i=1}^{\infty} \chi_{A_i \times B_i}(x, y).$$

Therefore, since $\chi_{A\times B}(x, y) = \chi_A(x)\chi_B(y)$,

$$\chi_A(x)\chi_B(y) = \sum_{i=1}^{\infty} \chi_{A_i}(x)\chi_{B_i}(y).$$

Integrating this equation over Y for fixed $x \in X$ and using the monotone convergence theorem, we get

$$\chi_A(x)\nu(B) = \sum_{i=1}^{\infty} \chi_{A_i}(x)\nu(B_i).$$

Integrating again with respect to x, we get

$$\mu(A)\nu(B) = \sum_{i=1}^{\infty} \mu(A_i)\nu(B_i),$$

which proves the result. □

In particular, it follows that λ is finitely additive on rectangles and therefore may be extended to a well-defined function on $\mathcal{E}$. To see this, note that any two representations of the same set as a finite disjoint union of rectangles may be decomposed into a common refinement such that each of the original rectangles is a disjoint union of rectangles in the refinement. The following definition therefore makes sense.

Definition 5.12. Suppose that $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are measure spaces and $\mathcal{E}$ is the algebra generated by the measurable rectangles. The product premeasure $\lambda : \mathcal{E} \to [0, \infty]$ is given by

$$\lambda(E) = \sum_{i=1}^{N} \mu(A_i)\nu(B_i), \qquad E = \bigcup_{i=1}^{N} A_i \times B_i$$

where $E = \bigcup_{i=1}^{N} A_i \times B_i$ is any representation of $E \in \mathcal{E}$ as a disjoint union of measurable rectangles.

Proposition 5.11 implies that λ is countably additive on $\mathcal{E}$, since we may decompose any countable disjoint union of sets in $\mathcal{E}$ into a countable common disjoint refinement of rectangles, so λ is a premeasure as claimed. The outer product measure associated with λ, which we write as $(\mu \otimes \nu)^*$, is defined in terms of countable coverings by measurable rectangles. This gives the following.

Definition 5.13. Suppose that $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are measure spaces. Then the product outer measure

$$(\mu \otimes \nu)^* : \mathcal{P}(X \times Y) \to [0, \infty]$$

on $X \times Y$ is defined for $E \subset X \times Y$ by

$$(\mu \otimes \nu)^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \mu(A_i)\nu(B_i) : E \subset \textstyle\bigcup_{i=1}^{\infty}(A_i \times B_i) \text{ where } A_i \in \mathcal{A},\ B_i \in \mathcal{B} \right\}.$$

The product measure

$$(\mu \otimes \nu) : \mathcal{A} \otimes \mathcal{B} \to [0, \infty], \qquad (\mu \otimes \nu) = (\mu \otimes \nu)^*|_{\mathcal{A} \otimes \mathcal{B}}$$

is the restriction of the product outer measure to the product σ-algebra.

It follows from Proposition 5.7 that $(\mu \otimes \nu)^*$ is an outer measure and every measurable rectangle is $(\mu \otimes \nu)^*$-measurable with measure equal to its product premeasure. We summarize the conclusions of the Carátheodory theorem and Theorem 5.9 in the case of product measures as the following result.

Theorem 5.14. *If $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are measure spaces, then*

$$(\mu \otimes \nu) : \mathcal{A} \otimes \mathcal{B} \to [0, \infty]$$

is a measure on $X \times Y$ such that

$$(\mu \otimes \nu)(A \times B) = \mu(A)\nu(B) \qquad \text{for every } A \in \mathcal{A},\ B \in \mathcal{B}.$$

Moreover, if $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite measure spaces, then $(\mu \otimes \nu)$ is the unique measure on $\mathcal{A} \otimes \mathcal{B}$ with this property.

Note that, in general, the σ-algebra of Carathéodory measurable sets associated with $(\mu \otimes \nu)^*$ is strictly larger than the product σ-algebra. For example, if $\mathbb{R}^m$ and $\mathbb{R}^n$ are equipped with Lebesgue measure defined on their Borel σ-algebras, then the Carathéodory σ-algebra on the product $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ is the Lebesgue σ-algebra $\mathcal{L}(\mathbb{R}^{m+n})$, whereas the product σ-algebra is the Borel σ-algebra $\mathcal{B}(\mathbb{R}^{m+n})$.

5.4. Measurable functions

If $f : X \times Y \to \mathbb{C}$ is a function of $(x, y) \in X \times Y$, then for each $x \in X$ we define the x-section $f_x : Y \to \mathbb{C}$ and for each $y \in Y$ we define the y-section $f^y : Y \to \mathbb{C}$ by

$$f_x(y) = f(x, y), \qquad f^y(x) = f(x, y).$$

Theorem 5.15. *If $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ are measure spaces and $f : X \times Y \to \mathbb{C}$ is a measurable function, then $f_x : Y \to \mathbb{C}$, $f^y : X \to \mathbb{C}$ are measurable for every $x \in X$, $y \in Y$. Moreover, if $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ are σ-finite, then the functions $g : X \to \mathbb{C}$, $h : Y \to \mathbb{C}$ defined by*

$$g(x) = \int f_x \, d\nu, \qquad h(y) = \int f^y \, d\mu$$

are measurable.

5.5. Monotone class theorem

We prove a general result about σ-algebras, called the monotone class theorem, which we will use in proving Fubini's theorem. A collection of subsets of a set is called a monotone class if it is closed under countable increasing unions and countable decreasing intersections.

Definition 5.16. A monotone class on a set X is a collection $\mathcal{C} \subset \mathcal{P}(X)$ of subsets of X such that if $E_i, F_i \in \mathcal{C}$ and

$$E_1 \subset E_2 \subset \cdots \subset E_i \subset \ldots, \qquad F_1 \supset F_2 \supset \cdots \supset F_i \supset \ldots,$$

then

$$\bigcup_{i=1}^{\infty} E_i \in \mathcal{C}, \qquad \bigcap_{i=1}^{\infty} F_i \in \mathcal{C}.$$

Obviously, every σ-algebra is a monotone class, but not conversely. As with σ-algebras, every family $\mathcal{F} \subset \mathcal{P}(X)$ of subsets of a set X is contained in a smallest monotone class, called the monotone class generated by $\mathcal{F}$, which is the intersection of all monotone classes on X that contain $\mathcal{F}$. As stated in the next theorem, if $\mathcal{F}$ is an algebra, then this monotone class is, in fact, a σ-algebra.

Theorem 5.17 (Monotone Class Theorem)**.** *If $\mathcal{F}$ is an algebra of sets, the monotone class generated by $\mathcal{F}$ coincides with the σ-algebra generated by $\mathcal{F}$.*

5.6. Fubini's theorem

Theorem 5.18 (Fubini's Theorem)**.** *Suppose that $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite measure spaces. A measurable function $f : X \times Y \to \mathbb{C}$ is integrable if and only if either one of the iterated integrals*

$$\int \left(\int |f^y| \, d\mu \right) d\nu, \qquad \int \left(\int |f_x| \, d\nu \right) d\mu$$

is finite. In that case

$$\int f \, d\mu \otimes d\nu = \int \left(\int f^y \, d\mu \right) d\nu = \int \left(\int f_x \, d\nu \right) d\mu.$$

Example 5.19. An application of Fubini's theorem to counting measure on $\mathbb{N} \times \mathbb{N}$ implies that if $\{a_{mn} \in \mathbb{C} \mid m, n \in \mathbb{N}\}$ is a doubly-indexed sequence of complex numbers such that

$$\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} |a_{mn}| \right) < \infty$$

then

$$\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} a_{mn} \right) = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} a_{mn} \right).$$

5.7. Completion of product measures

The product of complete measure spaces is not necessarily complete.

Example 5.20. If $N \subset \mathbb{R}$ is a non-Lebesgue measurable subset of $\mathbb{R}$, then $\{0\} \times N$ does not belong to the product σ-algebra $\mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$ on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, since every section of a set in the product σ-algebra is measurable. It does, however, belong to $\mathcal{L}(\mathbb{R}^2)$, since it is a subset of the set $\{0\} \times \mathbb{R}$ of two-dimensional Lebesgue measure zero, and Lebesgue measure is complete. Instead one can show that the Lebesgue σ-algebra on $\mathbb{R}^{m+n}$ is the completion with respect to Lebesgue measure of the product of the Lebesgue σ-algebras on $\mathbb{R}^m$ and $\mathbb{R}^n$:

$$\mathcal{L}(\mathbb{R}^{m+n}) = \overline{\mathcal{L}(\mathbb{R}^m) \otimes \mathcal{L}(\mathbb{R}^n)}.$$

We state a version of Fubini's theorem for Lebesgue measurable functions on $\mathbb{R}^n$.

Theorem 5.21. *A Lebesgue measurable function $f : \mathbb{R}^{m+n} \to \mathbb{C}$ is integrable, meaning that*

$$\int_{\mathbb{R}^{m+n}} |f(x,y)|\, dxdy < \infty,$$

if and only if either one of the iterated integrals

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} |f(x,y)|\, dx \right) dy, \qquad \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} |f(x,y)|\, dy \right) dx$$

is finite. In that case,

$$\int_{\mathbb{R}^{m+n}} f(x,y)\, dxdy = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} f(x,y)\, dx \right) dy = \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} f(x,y)\, dy \right) dx,$$

where all of the integrals are well-defined and finite a.e.