An Introduction to the Incompressible Euler Equations

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We derive the incompressible Euler equations for the flow of an inviscid, incompressible fluid, describe some of their basic mathematical features, and provide a perspective on their physical applicability. We explain the importance of vorticity in understanding the dynamics of an inviscid, incompressible fluid. We also formulate boundary conditions at an impermeable boundary and a free surface.

1. EULER EQUATIONS

The incompressible Euler equations are the following PDEs for $(\vec{u}, p)$:

$$\vec{u}_t + \vec{u} \cdot \nabla \vec{u} + \nabla p = 0, \tag{1}$$

$$\nabla \cdot \vec{u} = 0. \tag{2}$$

This system models the flow of an inviscid, incompressible fluid with constant density. We have set the density equal to one without loss of generality. The vector-valued function $\vec{u}(\vec{x}, t)$ is the velocity of the fluid and the scalar-valued function $p(\vec{x}, t)$ is the pressure.

We denote the number of space-dimensions by $d = 2, 3$. Then (1)–(2) is a system of $(d+1)$-PDEs. In Cartesian coordinates, with $\vec{x} = (x_1, \dots, x_d)$, $\vec{u} = (u_1, \dots, u_d)$, the component form of the equations is

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{d} u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} = 0, \qquad 1 \le i \le d,$$

$$\sum_{j=1}^{d} \frac{\partial u_j}{\partial x_j} = 0.$$

In particular, we use the notation

$$\vec{u} \cdot \nabla = \sum_{j=1}^{d} u_j \frac{\partial}{\partial x_j}.$$

Equation (1) is Newton's second law, stating that the acceleration of a fluid particle is proportional to the pressure-force acting on it. Equation (2) is the incompressibility condition, stating that the volume of any part of the fluid does not change under the flow.

We briefly describe a few important qualitative properties of the Euler equations.

Nonlinear The advection of the velocity field by itself leads to the nonlinear term $\vec{u} \cdot \nabla \vec{u}$, which makes a general analysis of the equations very difficult. Although Euler first presented these equations in 1755,[1] many fundamental questions about them remain unanswered. In three space-dimensions it is not even known whether solutions of the equations are defined for all time or if they form singularities.

Conservative Smooth solutions of the Euler equations satisfy the of conservation of kinetic energy,

$$\left(\frac{1}{2}|\vec{u}|^2\right)_t + \nabla \cdot \left(\frac{1}{2}|\vec{u}|^2 \vec{u} + p\vec{u}\right) = 0,$$

which follows by taking the scalar product of the momentum equation (1) with $\vec{u}$, and using the incompressibility equation (2) to rewrite the result in divergence form. The Euler equations are invariant under the time-reversal $t \mapsto -t$, $\vec{u} \mapsto -\vec{u}$, and therefore may be solved equally well (or badly) forward and backward in time. Furthermore, they have a Hamiltonian structure, which is somewhat disguised by the use of spatial instead of material coordinates.[2]

Scale-invariant Every term in the Euler equations is proportional to a first-order derivative in space or time, so for any constant $\lambda > 0$ the equations are invariant under rescalings $\vec{x} \mapsto \lambda\vec{x}$, $t \mapsto \lambda t$. Physically, this invariance reflects the fact that a fluid modeled by the Euler equations behaves the same at all length-scales, however small or large. This scale-invariance is a mathematical idealization, but one may hope that it is applicable to flows that vary over length-scales that are much larger or much smaller than any other length-scales in the problem.

[1] Copies of Euler's papers at available online at http://www.eulerarchive.com.

[2] Arnol'd (1966) gave an elegant geometric Hamiltonian formulation of the incompressible Euler equations as an Euler-Poisson equation on the infinite-dimensional Lie group of volume-preserving diffeomorphisms.

1.1. Compressible Euler equations

The compressible Euler equations describe the flow of an inviscid compressible fluid. In addition to the velocity and pressure, the density of the fluid appears in these equations as a dependent variable. The controlling dimensionless parameter for compressible flows is the Mach number

$$M = \frac{U}{c_0},$$

where U is a typical flow speed and c_0 is a typical sound speed.

The incompressible equations can be derived from the compressible equations in the limit $M \to 0$.[3] Thus, the incompressible equations are generally applicable to fluid flows whose speed is much less than the speed of sound. For example, the speed of sound in air at standard temperature and pressure is $340\,\mathrm{m\,s^{-1}}$, and the incompressible equations can be used to model the flow past an aircraft that flies at speeds of less than about $100\,\mathrm{m\,s^{-1}}$. We will consider only incompressible flows.

2. NAVIER-STOKES EQUATIONS

The conservation of energy and scale-invariance of the Euler equations are the result of neglecting viscous effects. The inclusion of (Newtonian) viscosity, with a coefficient of kinematic viscosity $\nu > 0$, leads to the incompressible Navier-Stokes equations,

$$\begin{aligned}
&\vec{u}_t + \vec{u} \cdot \nabla \vec{u} + \nabla p = \nu \Delta \vec{u}, \\
&\nabla \cdot \vec{u} = 0.
\end{aligned}$$

Here, the Laplacian $\Delta \vec{u}$ of a vector is defined componentwise in Cartesian coordinates, meaning that[4]

$$\Delta (u_1, \ldots, u_d) = (\Delta u_1, \ldots, \Delta u_d).$$

Since the fundamental work of Leray (1934), it has been known that there is a global-forward-in-time weak solution of the initial value problem for the three-dimensional incompressible Navier-Stokes equations. It is not known, however, whether the Leray solutions are unique, or whether the solutions for suitable general smooth initial data remain smooth for

[3] The incompressible limit is somewhat subtle. For example, in the compressible Euler equations the pressure is a thermodynamic variable, and it is a function of any other pair of thermodynamic variables, such as density and temperature. In the incompressible equations the pressure is an arbitrary function that plays the role of a Lagrange multiplier enforcing the constant-volume constraint.

[4] This componentwise equation does not hold in general curvilinear coordinates.

all time.[5] This open problem is one of the Clay Mathematics Institute Millennium Problems

http://www.claymath.org/millennium/Navier-Stokes_Equations/

though perhaps you would be better off trying to solve one of the easier problems, such as the Riemann hypothesis.

The kinematic viscosity ν has dimensions L^2/T, where L = length and T = time, as can be seen by comparing the dimensions of $\vec{u}_t$ and $\nu\Delta\vec{u}$. Physically, ν is the diffusivity of velocity or momentum — in time T momentum diffuses a distance of the order $\sqrt{\nu T}$. For water at room temperature and pressure, we have $\nu \approx 1\,\mathrm{mm}^2\,\mathrm{s}^{-1}$, so direct viscous effects diffuse momentum a distance of the order 1 mm in 1 s.

According to kinetic theory, the kinematic viscosity ν of a gas is of the order ℓv, where ℓ is the mean-free path and v is the mean thermal speed of the molecules (of the same order of magnitude as the sound speed). Thus, viscosity is a vestige in the continuum limit of the molecular nature of the fluid. The Euler equations, corresponding to $\nu = 0$, set the molecular length-scales to zero, which accounts for their scale-invariance.

If U is a typical flow speed and L is a typical length-scale over which a flow varies, then we may define a dimensionless parameter Re, called the Reynolds number, by

$$\mathrm{Re} = \frac{UL}{\nu}.$$

For example, if we consider the flow of water with a speed of the order $1\,\mathrm{m\,s}^{-1}$ in a tank that is 1 m long, then we find that the Reynolds number is $\mathrm{Re} = 10^6$.

The Reynolds number is inversely proportional to the kinematic viscosity ν, and one might hope that we can neglect the viscous term $\nu\Delta\vec{u}$ in the Navier-Stokes equations in comparison with the inertial term $\vec{u}\cdot\nabla\vec{u}$ when the Reynolds number is sufficiently large. The Navier-Stokes equations are, however, a singular perturbation of the Euler equations, since the viscosity ν multiplies the term that contains the highest-order spatial derivatives. As a result, the zero-viscosity, high-Reynolds' number limit of the Navier-Stokes equations is an extremely difficult one (turbulence, boundary layers,...).

As with the Euler equations, there is a compressible generalization of the Navier-Stokes equations. The compressible Navier-Stokes equations are more complicated than either the compressible Euler equations or the

[5]Presumably, if one could prove the global existence of suitable weak solutions of the Euler equations, then one could deduce the global existence and uniqueness of smooth solutions of the Navier-Stokes equations.

incompressible Navier-Stokes equations, and we confine our attention to the incompressible Euler equations.

3. THE INITIAL VALUE PROBLEM

Equation (1) provides an evolution equation for the velocity $\vec{u}$, and (2) provides an implicit equation for the pressure p. The lack of an evolution equation for p is a significant issue in the analysis and numerical solution of the incompressible Euler equations.

One way to obtain an explicit equation for the pressure is to take the divergence of (1), which eliminates the time-derivative $\vec{u}_t$. Using (2) to simplify the result, we find that p satisfies

$$-\Delta p = \mathrm{tr}\ (\nabla \vec{u})^2\,, \tag{3}$$

where tr denotes the trace. In components, we have

$$\mathrm{tr}\ (\nabla \vec{u})^2 = \sum_{i,j=1}^{d} \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}.$$

With the addition of suitable boundary conditions, equation (3) determines p in terms of the velocity $\vec{u}$ at the same instant of time.

As a result of the ellipticity of the PDE (3) for the pressure, a change in the velocity field near one spatial point instantly affects the pressure, and therefore the acceleration of the fluid, everywhere else. Physically, this fact is a result of the incompressible limit in which the sound speed becomes infinite relative to the flow speed, so that sound waves can carry pressure disturbances instantly across the entire fluid.

Perhaps the most basic problem for (1)–(2) is the initial-value problem on $\mathbb{R}^d$ with the initial condition

$$\vec{u}(\vec{x},0) = \vec{u}_0(\vec{x}), \tag{4}$$

where $\vec{u}_0 : \mathbb{R}^d \to \mathbb{R}^d$ is a given smooth velocity field that decays to zero sufficiently rapidly as $|\vec{x}| \to \infty$ and satisfies $\nabla \cdot \vec{u}_0 = 0$. We also require that

$$\nabla p \to 0 \qquad \text{as } |\vec{x}| \to \infty. \tag{5}$$

We do not impose an initial condition on the pressure, since there is no evolution equation for p.

Using a Green's function representation to solve equations (3), (5) for p and computing the gradient of the result, we find that

$$\nabla p(\vec{x},t) = -C_d \int_{\mathbb{R}^d} \frac{\vec{x}-\vec{y}}{|\vec{x}-\vec{y}|^d}\, \mathrm{tr}\ [\nabla \vec{u}(\vec{y},t)]^2\, d\vec{y},$$

where the constant C_d is given (in 2 or 3 space-dimensions) by

$$C_2 = \frac{1}{2\pi}, \qquad C_3 = \frac{1}{4\pi}.$$

Thus, the velocity $\vec{u}$ satisfies the integro-differential evolution equation

$$\vec{u}_t(\vec{x}, t) + \vec{u} \cdot \nabla \vec{u}(\vec{x}, t) = C_d \int_{\mathbb{R}^d} \frac{\vec{x} - \vec{y}}{|\vec{x} - \vec{y}|^d} \operatorname{tr} \left[\nabla \vec{u}(\vec{y}, t)\right]^2 \, d\vec{y},$$

which is supplemented with the initial condition (4).

We may also write this equation as

$$\vec{u}_t + P\left[\vec{u} \cdot \nabla \vec{u}\right] = 0, \tag{6}$$

where P is the Leray projection that maps a vector field to its zero-divergence part. Explicitly, if $\vec{v}$ is a smooth, square-integrable vector field on $\mathbb{R}^d$, then we may write $\vec{v}$ uniquely as the sum of a square-integrable divergence-free vector field $\vec{w}$ and a gradient (the Hodge decomposition):

$$\vec{v} = \vec{w} + \nabla \varphi, \qquad \text{where } \nabla \cdot \vec{w} = 0.$$

The Leray projection of $\vec{v}$ is then $P\vec{v} = \vec{w}$. Since $\vec{w}$ and $\nabla\varphi$ are smooth and square-integrable, the divergence theorem implies that

$$\int_{\mathbb{R}^d} \left(\vec{w} \cdot \nabla \varphi\right) \, d\vec{x} = 0.$$

Thus, P is an orthogonal projection on $L^2(\mathbb{R}^d)$.

In two space-dimensions there exist global smooth solutions of the initial-value problem (4), (6). It is not known to this day, however, whether or not there exist global-in-time smooth — or weak — solutions of the initial-value problem in three space-dimensions.[6]

The main obstacle to the existence of solutions in three space-dimensions is the formation of singularities in the vorticity $\vec{\omega} = \nabla \times \vec{u}$. Beale, Kato, and Majda (1984) proved that a smooth solution of the incompressible Euler equations breaks down on a time-interval $[0, T_*]$ if and only if

$$\int_0^T \|\omega\|_\infty(t) \, dt \to \infty \qquad \text{as } T \uparrow T_*,$$

[6]Weak solutions may be quite singular, and, in particular, they do not necessarily conserve kinetic energy. There are examples of weak solutions of the incompressible Euler equations, constructed by Scheffer (1993) and Schnirelman (1997), in which the velocity is initially zero, becomes nonzero in a compact spatial region, and then returns to zero.

where $\|\omega\|_\infty$ denotes the spatial L^∞-norm of the vorticity,

$$\|\omega\|_\infty(t) = \sup_{\vec{x}\in\mathbb{R}^3} |\vec{\omega}(\vec{x},t)| \, .$$

There have been a number of attempts to detect the formation of singularities numerically. While the results show a very rapid intensification of vorticity, there is — at the moment — no conclusive numerical evidence of a singularity. The physical mechanism for the increase and conceivable formation of singularities in the vorticity is vortex stretching, which we discuss further below.

4. KINEMATICS

Kinematics refers to the description of the motion of a system, in our case an incompressible fluid. Dynamics, which we consider in the next section, refers to the effect of forces acting on the system.

A motion $\vec{X}$ of a continuous body moving in $\mathbb{R}^d$ is a function

$$\vec{X} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d.$$

We write $\vec{x} = \vec{X}(\vec{a}, t)$ for the location $\vec{x}$ of the particle $\vec{a}$ at time t. We call $\vec{x}$ the spatial or Eulerian coordinate of the particle and $\vec{a}$ the material or Lagrangian coordinate. (Both types of coordinates were, in fact, introduced by Euler.)

For fixed $\vec{a}$, the curve $\vec{x}(t) = X(\vec{a}, t)$ defines a particle path, which can be visualized by placing a spot of non-diffusing dye in the fluid.

We may specify the Lagrangian coordinate $\vec{a}$ of a material particle in any convenient way; for example, we can use the position of the particle at $t = 0$, in which case $\vec{X}(\vec{a}, 0) = \vec{a}$. In elasticity, one often uses the location of the particle in an unstressed configuration as the material coordinate, but since fluid particles can be rearranged without the creation of stress (the 'particle relabeling symmetry', realized by stirring a bucket of water), there is no unique unstressed reference configuration of a fluid, even modulo rigid motions.

We assume that $\vec{X}$ is a smooth function and that, for each $t \in \mathbb{R}$, the function $\vec{X}(\cdot, t)$ is a diffeomorphism of $\mathbb{R}^d$, meaning that it is invertible and differentiable as many times as we require and that the derivative $D_{\vec{a}}\vec{X}$, with components

$$\frac{\partial X_i}{\partial a_j},$$

is nonsingular. Roughly speaking, the last condition means that the motion does not crush a nonzero material volume to zero volume.[7]

The velocity $\vec{U}$ of a material particle $\vec{a}$ at time t is given by

$$\vec{U}(\vec{a},t) = \vec{X}_t(\vec{a},t),$$

where the partial derivative with respect to t is taken holding $\vec{a}$ fixed. The corresponding spatial velocity $\vec{u}(\vec{x},t)$ is defined by

$$\vec{u}\left(\vec{X}(\vec{a},t),t\right) = \vec{U}(\vec{a},t).$$

Conversely, given a smooth spatial velocity $\vec{u}(\vec{x},t)$, we may reconstruct the motion $\vec{X}(\vec{a},t)$ by solving the system of ODEs

$$\vec{X}_t(\vec{a},t) = \vec{u}\left(\vec{X}(\vec{a},t),t\right),$$

with, for example, the initial condition

$$\vec{X}(\vec{a},0) = \vec{a}.$$

This system of ODEs is, in general, nonlinear and is typically not explicitly solvable. For example, even very simple spatial velocity fields may lead to chaotic solutions for the particle paths, a phenomenon called 'chaotic advection'.

If $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is a function of spatial coordinates $(\vec{x},t)$, we define a corresponding function $F : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ of material coordinates $(\vec{a},t)$ by

$$F(\vec{a},t) = f(\vec{X}(\vec{a},t),t).$$

We denote the partial derivative of f with respect to t with $\vec{x}$ held fixed (the rate of change of f at a given spatial point) by f_t or $\partial_t f$, and the partial derivative of F with respect to t with $\vec{a}$ held fixed (the rate of change of f following a particle path) by F_t. We call this derivative the material time-derivative of f and denote it by Df/Dt.

According to the chain rule,

$$\begin{aligned}\frac{Df}{Dt}(\vec{X}(\vec{a},t),t) &= f_t(\vec{X}(\vec{a},t),t) + \vec{X}_t(\vec{a},t)\cdot\nabla f(\vec{X}(\vec{a},t),t)\\ &= f_t(\vec{X}(\vec{a},t),t) + \vec{u}(X(\vec{a},t),t)\cdot\nabla f(\vec{X}(\vec{a},t),t).\end{aligned}$$

[7]These assumptions may need to be reconsidered for weak solutions of the Euler equations in which the smoothness of the motion breaks down.

We may therefore express the material time-derivative in terms of spatial derivatives as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla.$$

For example, the Euler equations (1)–(2) may be written as

$$\frac{D\vec{u}}{Dt} + \nabla p = 0, \qquad \nabla \cdot \vec{u} = 0.$$

If $\mathcal{P} \subset \mathbb{R}^d$ is a bounded open set with smooth boundary $\partial\mathcal{P}$, we define

$$\mathcal{P}_t = \{X(\vec{a}, t) : \vec{a} \in \mathcal{P}\}$$

to be the spatial region occupied at time t by the material particles in $\mathcal{P}$. We call $\mathcal{P}_t$ a material volume.

We will need the following result from multi-variable calculus, which may be thought of as a generalization of the Leibnitz rule for differentiating one-dimensional integrals with variable endpoints.

THEOREM 4.1 (Reynolds' transport theorem). *Suppose that $\vec{X} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ is a smooth motion such that $\vec{X}(\cdot, t)$ is a diffeomeorphism of $\mathbb{R}^d$ for each $t \in \mathbb{R}$. Let $\mathcal{P} \subset \mathbb{R}^d$ be a smooth bounded open set, and define $\mathcal{P}_t = \vec{X}(\mathcal{P}, t)$. If $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is a smooth function, then*

$$\frac{d}{dt} \int_{\mathcal{P}_t} f \, d\vec{x} = \int_{\mathcal{P}_t} f_t \, d\vec{x} + \int_{\partial\mathcal{P}_t} f \vec{u} \cdot \vec{n} \, dS, \tag{7}$$

where $\vec{n}$ denotes the unit outward normal on $\partial\mathcal{P}_t$, and $\vec{u}$ is the spatial velocity field associated with X.

Proof. Making the change of variables $\vec{x} = X(\vec{a}, t)$ in the integral, we get

$$\int_{\mathcal{P}_t} f(\vec{x}, t) \, d\vec{x} = \int_{\mathcal{P}} f(\vec{X}(\vec{a}, t), t) J(\vec{a}, t) \, d\vec{a},$$

where

$$J = \det D_{\vec{a}} \vec{X} \tag{8}$$

is the Jacobian of the transformation.

Since $\mathcal{P}$ is independent of t, we may compute the derivative of an integral over $\mathcal{P}$ by differentiating the integrand with respect to t with $\vec{a}$ held fixed, so that

$$\frac{d}{dt} \int_{\mathcal{P}_t} f \, d\vec{x} = \int_{\mathcal{P}} \frac{D}{Dt} (fJ) \, d\vec{a}. \tag{9}$$

To simplify the right-hand side of this equation, we need to compute the derivative of a determinant. Let $F(t)$ be a non-singular second-order tensor depending smoothly on t. Then

$$\begin{aligned}
\frac{d}{dt}\det F(t) &= \lim_{h\to 0}\frac{\det F(t+h)-\det F(t)}{h} \\
&= \det F(t)\lim_{h\to 0}\frac{\det\left[F^{-1}(t)F(t+h)\right]-1}{h} \\
&= \det F(t)\lim_{h\to 0}\frac{1}{h}\left\{\det\left[I+hF^{-1}(t)\frac{dF}{dt}(t)\right]-1\right\} \\
&= \det F(t)\operatorname{tr}\left[F^{-1}(t)\frac{dF}{dt}(t)\right].
\end{aligned}$$

In the last equation tr denotes the trace, and we used the fact that

$$\det\left[I+hA\right] = 1 + h\operatorname{tr}A + O(h^2) \qquad \text{as } h\to 0,$$

as is clear from looking at the expansion of the determinant on the left-hand side. ('Derivative of determinant is trace.')

Applying this result to the Jacobian J in (8), exchanging the order of the material time-derivative and the derivative with respect to $\vec{a}=\vec{A}(\vec{x},t)$, using the fact that

$$\left(D_{\vec{x}}\vec{A}\right)(\vec{X}(\vec{a},t),t) = \left(D_{\vec{a}}\vec{X}\right)^{-1}(\vec{a},t)$$

and the chain rule, we get

$$\begin{aligned}
\frac{DJ}{Dt} &= J\operatorname{tr}\left[\left(D_{\vec{a}}\vec{X}\right)^{-1}\left(D_{\vec{a}}\vec{X}_t\right)\right] \\
&= J\operatorname{tr}\left[\left(D_{\vec{x}}\vec{A}\right)\left(D_{\vec{a}}\vec{U}\right)\right] \\
&= J\operatorname{tr}\left(D_{\vec{x}}\vec{u}\right).
\end{aligned}$$

Since $\operatorname{tr}\left(D_{\vec{x}}\vec{u}\right) = \nabla\cdot\vec{u}$, we conclude that

$$\frac{DJ}{Dt} = J\nabla\cdot\vec{u}.$$

Using this equation in (9), and transforming the integration variable back to $\vec{x}$ in the result, we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{P}_t} f\, d\vec{x} &= \int_{\mathcal{P}} \left\{ J\frac{Df}{Dt} + f\frac{DJ}{Dt}\right\} d\vec{a} \\ &= \int_{\mathcal{P}} \left\{ \frac{Df}{Dt} + f\nabla\cdot\vec{u}\right\} J d\vec{a} \\ &= \int_{\mathcal{P}_t} \left\{ \frac{Df}{Dt} + f\nabla\cdot\vec{u}\right\} d\vec{x}.\end{aligned}$$

Expressing the material time-derivative in terms of spatial derivatives, and using a vector identity, we have

$$\begin{aligned}\frac{Df}{Dt} + f\nabla\cdot\vec{u} &= f_t + \vec{u}\cdot\nabla f + f\nabla\cdot\vec{u} \\ &= f_t + \nabla\cdot(f\vec{u})\,.\end{aligned}$$

Hence,

$$\frac{d}{dt}\int_{\mathcal{P}_t} f\, d\vec{x} = \int_{\mathcal{P}_t} \left\{ f_t + \nabla\cdot(f\vec{u})\right\} d\vec{x}. \tag{10}$$

Using the divergence theorem in this equation, we get (7). ∎

5. DERIVATION OF THE EULER EQUATIONS

The incompressible Euler equations (1)–(2) follow from the conservation of mass and momentum, together with the assumption that the density of the fluid is constant

First, we derive the equation of conservation of mass. We consider a fluid with possibly non-constant spatial density $\rho(\vec{x},t)$. The mass contained in a material volume $\mathcal{P}_t$ is then given by

$$\int_{\mathcal{P}_t} \rho d\vec{x}.$$

If the mass of the material volume does not change as the fluid moves, then

$$\frac{d}{dt}\int_{\mathcal{P}_t} \rho d\vec{x} = 0.$$

Hence, using (10), we get that

$$\int_{\mathcal{P}_t} \left\{ \rho_t + \nabla\cdot(\rho\vec{u})\right\} d\vec{x} = 0.$$

Since this equation holds at any fixed time for an arbitrary smooth region $\mathcal{P}_t$, we conclude that[8]

$$\rho_t + \nabla \cdot (\rho \vec{u}) = 0. \tag{11}$$

Next, we derive the equation of conservation of momentum. We suppose that two types of forces act on the fluid: (a) a body force $\vec{F}$ per unit volume; (b) a surface pressure force.

The body force may be an external force acting of the fluid — for example, gravity or the Coriolis force in a rotating fluid. The surface force is the force exerted by one part of the fluid on another part, and arises from direct inter-molecular forces as well as the transport of momentum across surfaces by the thermal motion of molecules. In modeling an inviscid fluid, we assume that the surface force is a pressure force that acts across a surface in the inward normal direction with a magnitude per unit surface area equal to the pressure p.[9]

The total momentum of a material volume $\mathcal{P}_t$ is

$$\int_{\mathcal{P}_t} \rho \vec{u}\, d\vec{x}.$$

The requirement that the rate of change of momentum of a material volume is equal to the force acting on it (Newton's second law) then leads to the equation

$$\frac{d}{dt} \int_{\mathcal{P}_t} \rho \vec{u}\, d\vec{x} = -\int_{\partial \mathcal{P}_t} p \vec{n}\, dS + \int_{\mathcal{P}_t} \vec{F}\, d\vec{x}.$$

Using (10), which may be applied componentwise, and the divergence theorem, we find that

$$\int_{\mathcal{P}_t} \left\{ (\rho \vec{u})_t + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) + \nabla p - \vec{F} \right\} d\vec{x} = 0,$$

where the tensor product $\vec{u} \otimes \vec{u}$ is the second-order tensor with components $u_i u_j$. The component form of this equation is

$$\int_{\mathcal{P}_t} \left\{ \frac{\partial}{\partial t} (\rho u_i) + \sum_{j=1}^{d} \frac{\partial}{\partial x_j} (\rho u_i u_j) + \frac{\partial p}{\partial x_i} - F_i \right\} d\vec{x} = 0.$$

[8]In detail, suppose that the integrand were nonzero at some point $\vec{x}_0$. Since we assume smoothness, the integrand is continuous, so there would be a ball of nonzero radius containing $\vec{x}_0$ in which the integrand is bounded away from zero. The integral over this ball would be nonzero, proving the result.

[9]A force per unit surface area is called a stress.

Since this equation holds at any fixed time for an arbitrary smooth region $\mathcal{P}_t$, we conclude that

$$(\rho\vec{u})_t + \nabla\cdot(\rho\vec{u}\otimes\vec{u}) + \nabla p = \vec{F}. \tag{12}$$

We now assume that the fluid has constant density ρ_0. The conservation of mass equation (11) then reduces to (2), which expresses conservation of volume. Using (2), we may write the conservation of momentum equation (12) as

$$\rho_0\left(\vec{u}_t + \vec{u}\cdot\nabla\vec{u}\right) + \nabla p = \vec{F}.$$

Setting $\vec{F} = 0$ and normalizing $\rho_0 = 1$, we get (1).

Finally, we generalize the incompressible Euler equations to nonuniform fluids.[10] The conservation of mass equation (11) may be written as

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\vec{u} = 0.$$

Thus, the incompressibility condition of conservation of volume under the flow ($\nabla\cdot\vec{u} = 0$) is equivalent to the condition that the density of a material particle does not change in time ($D\rho/Dt = 0$). The corresponding incompressible Euler equations for $(\rho, \vec{u}, p)$ may then be written as

$$\rho\frac{D\vec{u}}{Dt} + \nabla p = 0, \quad \frac{D\rho}{Dt} = 0, \quad \nabla\cdot\vec{u} = 0.$$

6. LOCAL KINEMATICS

A differentiable velocity field may be approximated locally as a superposition of a constant velocity field and a linear velocity field,

$$\vec{u}(\vec{x}+\vec{h}, t) = \vec{u}(\vec{x}, t) + \nabla\vec{u}(\vec{x}, t)\vec{h} + o(\vec{h}) \qquad \text{as } \vec{h}\to 0.$$

The notation $D\vec{u}$ for the derivative of $\vec{u}$ would be more consistent with general mathematical use, but the notation $\nabla\vec{u}$ is standard in continuum mechanics. The component form of this equation is

$$u_i(\vec{x}+\vec{h}, t) = u_i(\vec{x}, t) + \sum_{j=1}^{d}\frac{\partial u_i}{\partial x_j}(\vec{x}, t)h_j + o(\vec{h}) \qquad \text{as } \vec{h}\to 0.$$

[10] For example, saline water with variable salt density. We assume that the diffusion of mass from one material particle to another is negligible, otherwise additional surface terms would enter into the conservation of mass equation.

We write the velocity-gradient $\nabla \vec{u}$ in terms of its symmetric and antisymmetric parts as $\nabla \vec{u} = D + W$, where

$$D = \frac{1}{2}\left(\nabla \vec{u} + \nabla \vec{u}^T\right), \qquad W = \frac{1}{2}\left(\nabla \vec{u} - \nabla \vec{u}^T\right),$$

and F^T denotes the transpose of F. In components,

$$D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad W_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right).$$

We then have

$$\vec{u}(\vec{x} + \vec{h}, t) = \vec{u}(\vec{x}, t) + D(\vec{x}, t)\vec{h} + W(\vec{x}, t)\vec{h} + o(\vec{h}) \qquad \text{as } \vec{h} \to 0.$$

This equation provides a local representation of the velocity field near a fixed point $\vec{x}$ at time t as a superposition of a constant field $\vec{u}(\vec{x}, t)$, a strain $D(\vec{x}, t)\vec{h}$, and a rotation $W(\vec{x}, t)\vec{h}$.

We consider the case of three space-dimensions for definiteness. The rate-of-strain, or stretching, tensor D is symmetric, so it has three orthogonal real eigenvectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$, with real (not necessarily distinct) eigenvalues λ_1, λ_2, λ_3. The origin $\vec{h} = 0$ is a stagnation point of the velocity field $D\vec{h}$, meaning that the velocity is zero there. The velocity in the direction $\vec{e}_i$ is directed radially outward from the origin if $\lambda_i > 0$, corresponding to stretching, and radially inward if $\lambda_i < 0$, corresponding to compression.

Note that $\operatorname{tr} D = \nabla \cdot \vec{u}$. For incompressible flow, we have $\nabla \cdot \vec{u} = 0$, so

$$\lambda_1 + \lambda_2 + \lambda_3 = 0,$$

meaning that stretching in one direction must be balanced by compression in another.

The spin tensor W is antisymmetric, and

$$W\vec{h} = \frac{1}{2}\vec{\omega} \times \vec{h},$$

where

$$\vec{\omega} = \nabla \times \vec{u} \tag{13}$$

is the vorticity. The velocity field $W\vec{h}$ therefore corresponds to a local rotation of the fluid with angular velocity equal to $\vec{\omega}/2$.

7. VORTICITY

The vorticity $\vec{\omega} = \nabla \times \vec{u}$ is the key quantity in an analysis of the motion of an incompressible, inviscid fluid. It is convenient to consider three-dimensional and two-dimensional flows separately.

7.1. Three-dimensional flows

We take the curl of the momentum equation (1), which eliminates the pressure, and use (2) to simplify the result. We find that $\vec{\omega}$ satisfies the vorticity equation

$$\vec{\omega}_t + \vec{u} \cdot \nabla \vec{\omega} = \vec{\omega} \cdot \nabla \vec{u}. \tag{14}$$

The left-hand side of this equation is the material time-derivative of the vorticity and describes the advection of vorticity by the fluid flow. The right-hand side is a vortex-stretching term. It describes the possible intensification of vorticity by stretching of the fluid, analogous to the way in which an ice-skater spins faster as she pulls in her arms.

Since $\vec{\omega} \cdot \nabla \vec{u} = \vec{\omega} \cdot D + \vec{\omega} \cdot W$, and

$$\vec{\omega} \cdot D = D\vec{\omega}, \qquad \vec{\omega} \cdot W = -W\vec{\omega} = -\frac{1}{2}\vec{\omega} \times \vec{\omega} = 0,$$

we may write (14), in somewhat unfortunate notation, as

$$\frac{D\vec{\omega}}{Dt} = D\vec{\omega}.$$

Thus, the vorticity $\vec{\omega}$ of a material particle increases when the vorticity is aligned with an eigenvector of the rate-of-strain tensor D with positive eigenvalue, and decreases when the vorticity is aligned with an eigenvector of the rate-of-strain tensor with negative eigenvalue.

This fact provides a possible mechanism ('geometric depletion') for the avoidance of blow-up in the vorticity. One might hope that an intensification of vorticity destroys the alignment of the vorticity with eigenvectors of D whose eigenvalues are positive. There is some numerical and analytical evidence that this is what occurs, but — at the moment — no proof that it prevents the vorticity from blowing up.

7.2. Two-dimensional flows

For two-dimensional flows, we have

$$\begin{aligned}
\vec{u} &= (u_1(x_1, x_2, t), u_2(x_1, x_2, t), 0)\,, \\
\vec{\omega} &= (0, 0, \omega(x_1, x_2, t))\,,
\end{aligned}$$

where the scalar vorticity ω is given by

$$\omega = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}.$$

It follows that $\vec{\omega} \cdot \nabla \vec{u} = 0$, and ω satisfies the transport equation

$$\omega_t + \vec{u} \cdot \nabla \omega = 0. \tag{15}$$

The absence of vortex-stretching is a crucial simplification for two-dimensional flows in comparison with three-dimensional flows, and explains in large part why they are much better understood.

8. STREAM FUNCTION-VORTICITY FORMULATION

A useful way of writing the incompressible Euler equations is in terms of a stream function and the vorticity.

8.1. Two-dimensional flows

In $\mathbb{R}^2$, or any other simply connected two-dimensional domain, the incompressibility condition,

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0,$$

implies that there is a scalar-valued function $\psi(x_1, x_2, t)$ such that

$$u_1 = \frac{\partial \psi}{\partial x_2}, \qquad u_2 = -\frac{\partial \psi}{\partial x_1}.$$

We also write this equation as

$$\vec{u} = -\nabla^\perp \psi,$$

where

$$\nabla^\perp = \left(-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1} \right).$$

We call ψ a stream function because, at any instant of time, the velocity $\vec{u}$ is tangent to the streamlines $\psi =$ constant. Note that for time-dependent flows the streamlines do not coincide with the particle paths.

The two-dimensional incompressible Euler equations may be rewritten in terms of ψ and ω as

$$\begin{aligned} &\omega_t - \left(\nabla^\perp \psi\right) \cdot \nabla \omega = 0, \\ &-\Delta \psi = \omega. \end{aligned}$$

This formulation may be thought of as a transport equation for an 'active' scalar ω that is advected with a velocity $\vec{u}$, depending on ω, given by

$$\vec{u} = \nabla^\perp \left(\Delta^{-1}\omega\right).$$

For example, on $\mathbb{R}^2$, using the free-space Green's function of the Laplacian, we have

$$\psi(\vec{x}, t) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log|\vec{x}-\vec{y}|\omega(\vec{y}, t)\, d\vec{y},$$

where we assume that ω decays sufficiently rapidly at infinity. It follows that $\vec{u} = -\nabla^\perp\psi$ is given in terms of ω by the integral operator

$$\vec{u}(\vec{x}, t) = \int_{\mathbb{R}^2} \vec{K}_2(\vec{x}-\vec{y})\omega(\vec{y}, t)\, d\vec{y},$$

where

$$\vec{K}_2(\vec{x}) = \frac{1}{2\pi}\frac{(-x_2, x_1)}{|\vec{x}|^2}.$$

8.2. Three-dimensional flows

In three space-dimensions, the incompressibility condition $\nabla \cdot \vec{u} = 0$ implies that, in a simply connected domain, there exists a vector-valued function $\vec{\psi}$ such that $\vec{u} = \nabla \times \vec{\psi}$ and $\nabla \cdot \vec{\psi} = 0$. For two-dimensional flows, $\vec{\psi} = (0, 0, \psi)$ where ψ is the stream function. The corresponding stream function-vorticity equations are

$$\begin{aligned}
&\vec{\omega}_t + \vec{u}\cdot\nabla\vec{\omega} = \vec{\omega}\cdot\nabla\vec{u},\\
&-\Delta\vec{\psi} = \vec{\omega}, \quad \vec{u} = \nabla\times\vec{\psi}.
\end{aligned}$$

The fact that the vorticity and the stream function are vectors in three dimensions makes these equations less useful than in the two-dimensional case.

There is a similar representation in $\mathbb{R}^3$ to the one given above in $\mathbb{R}^2$ for the velocity as a singular integral operator acting on the vorticity. In $\mathbb{R}^3$, we have

$$\nabla \times \vec{u} = \vec{\omega}, \qquad \nabla\cdot\vec{u} = 0.$$

The solution of this linear system for $\vec{u}$ is

$$\vec{u}(\vec{x}, t) = \int_{\mathbb{R}^3} \vec{K}_3(\vec{x}-\vec{y})\vec{\omega}(\vec{y}, t)\, d\vec{y}, \tag{16}$$

where $\vec{K}_3(\vec{x}) : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$\vec{K}_3(\vec{x})\vec{h} = \frac{1}{4\pi}\frac{\vec{x}\times\vec{h}}{|\vec{x}|^3}.$$

The vorticity equation (14) and the representation (16) give a singular integro-differential equation for the vorticity $\vec{\omega}$.

Equation (16) is mathematically identical to the Biot-Savart law for the magnetic field $\vec{B}$ generated by a steady current $\vec{J}$. In suitable units, Maxwell's equations imply that

$$\nabla\times\vec{B} = \vec{J}, \qquad \nabla\cdot\vec{B} = 0,$$

which have exactly the same form as the equations that relate an incompressible velocity field $\vec{u}$ to its vorticity $\vec{\omega}$.

9. IRROTATIONAL FLOWS

One consequence of the vorticity equation (14) is that if a material particle has zero vorticity initially, then its vorticity remains zero for all time. (Normal pressure forces cannot create rotation.) In particular, if the initial data $\vec{u}_0$ in (4) satisfies $\nabla\times\vec{u}_0 = 0$, then the solution $\vec{u}$ of the Euler equations (1)–(2) satisfies $\nabla\times\vec{u} = 0$.

We call a flow whose vorticity is identically zero an irrotational flow. On $\mathbb{R}^d$, or any other simply-connected domain, it follows that there exists a scalar-valued function $\varphi(\vec{x}, t)$, called the velocity potential, such that $\vec{u} = \nabla\varphi$.

For irrotational flows, we may write the momentum equation (1) as

$$\nabla\left\{\varphi_t + \frac{1}{2}|\nabla\varphi|^2 + p\right\} = 0.$$

This equation is satisfied by the pressure

$$p = -\varphi_t - \frac{1}{2}|\nabla\varphi|^2 + P, \tag{17}$$

where $P(t)$ is an arbitrary function of integration that depends only on time. Equation (17) is called Bernoulli's law.

For steady (time-independent) flows with $\varphi_t = 0$ and $P = \text{constant}$, Bernoulli's law implies that high velocities lead to low pressures. This fact can have dramatic effects; for example, cavitation may occur in a liquid when the pressure drops below the vapor pressure.

An irrotational flow $\vec{u} = \nabla\varphi$ remains irrotational in the presence of a conservative body force $\vec{F} = -\nabla\Phi$, such as gravity, in which case Bernoulli's equation becomes

$$p = -\varphi_t - \frac{1}{2}|\nabla\varphi|^2 - \Phi + P. \qquad (18)$$

The Coriolis force in a rotating fluid is not conservative, and consequently generates vorticity. This accounts for the profound effect of the earth's rotation on flows in the atmosphere and the ocean.

The incompressibility condition (2) is satisfied for an irrotational flow if

$$\Delta\varphi = 0,$$

meaning that the velocity potential φ is a harmonic function. Thus, for irrotational flows, the nonlinear incompressible Euler equations reduce to a linear Laplace equation. Note also that the time-dependence of such flows is parametric, and the velocity field is determined simultaneously throughout the fluid at each instant of time.

9.1. Paradoxes of irrotational flow

The fact that vorticity cannot be generated in the interior of an inviscid, incompressible fluid with uniform density has a number of consequences that are contradicted by experience. For example, it implies that the fluid cannot exert any drag or lift forces on a solid body that moves through it starting from rest (d'Alembert's paradox[11]).

The resolution of this, and other, paradoxes depends on an understanding of the effects of viscosity. Prandtl (1905) observed that, for flows with large Reynolds number, a viscous boundary layer containing vorticity forms on the surface of a solid body. This boundary layer may 'separate', and then the vorticity it contains is advected into the interior of the fluid. Boundary layers, and the phenomenon of boundary layer separation, lead to difficult issues which are still far from being understood mathematically.

Despite these difficulties, there are many physical problems (such as flow past a streamlined body) for which inviscid, irrotational flow theory provides a excellent mathematical model. On the other hand, many (but not all) inviscid flows that contain vorticity are highly unstable, leading in

[11] D'Alembert published his paradox in 1752, as a result of a competition held in 1750 by the Academy of Berlin on the theory of the resistance of fluids. According to René Dugas in *A History of Mechanics* (1955), the prize for the competition was not awarded because the Academy required participants to demonstrate that their calculations agreed with experiment, and d'Alembert was embittered by this decision. In the same paper, d'Alembert wrote down the two-dimensional equations for fluid motion several years before Euler, but his derivation of them was so long and tortuous that he recieved little credit.

practice to turbulent flows that are not well-understood, but which are presumably modeled by the Navier-Stokes equations.[12] A systematic analysis of turbulence is, perhaps, the major unsolved question in classical physics.

9.2. Two-dimensional irrotational flows

Incompressible, irrotational flow is particularly simple to analyze in two space dimensions because we can introduce both a velocity potential φ and a stream function ψ. Explicitly, we have

$$u_1 = \frac{\partial \varphi}{\partial x_1}, \quad u_2 = \frac{\partial \varphi}{\partial x_2}; \qquad u_1 = \frac{\partial \psi}{\partial x_2}, \quad u_2 = -\frac{\partial \psi}{\partial x_1}.$$

It follows that φ, ψ satisfy the Cauchy-Riemann equations. Therefore, if we introduce a complex spatial variable $z = x_1 + ix_2$ and a complex-valued function $F : \mathbb{C} \to \mathbb{C}$ given by

$$F(z) = \varphi(x_1, x_2) + i\psi(x_1, x_2),$$

then F is analytic, with derivative $F'(z) = u_1(x_1, x_2) - iu_2(x_1, x_2)$.

This formulation enables us to solve many two-dimensional incompressible irrotational flow problems by the use of complex variable techniques such as conformal mapping.

10. BOUNDARY CONDITIONS

So far, we have considered fluids that move in $\mathbb{R}^d$. If the fluid moves in a (possibly time-dependent) spatial region $\Omega_t \subset \mathbb{R}^d$, then we need to impose conditions on the boundary $\partial\Omega_t$. We consider here two types of boundaries: (a) impermeable boundaries; (b) free surfaces.

10.1. Impermeable boundaries

Perhaps the simplest type of boundary is an impermeable wall, such as the side of a wave-tank or the hull of a ship. If the boundary $\partial\Omega$ is stationary, then the appropriate boundary condition for an inviscid fluid is

$$\vec{u} \cdot \vec{n} = 0 \qquad \text{on } \partial\Omega,$$

where $\vec{n}$ is the unit outward normal to the boundary. This 'no-flow' condition states that the fluid does not flow through the boundary. An inviscid fluid can 'slide' over an impermeable boundary, and the tangential velocity

[12]It is conceivable that the infinite Reynolds number limit of turbulent flows leads to weak solutions of the Euler equations that conserve mass and momentum but dissipate kinetic energy.

is, in general, nonzero. An attempt to specify it would lead to an over-determined problem for the Euler equations.

The Navier-Stokes equation contain second-order spatial derivatives and therefore they require additional boundary conditions. On a solid stationary boundary $\partial\Omega$, the appropriate condition is the 'no-slip' condition that

$$\vec{u} = 0 \qquad \text{on } \partial\Omega,$$

meaning that a viscous fluid 'sticks' to the boundary.

Thus, the motion of an inviscid, incompressible fluid in a bounded, stationary, impermeable container $\Omega \subset \mathbb{R}^d$ is described by the following IBVP for $\vec{u} : \Omega \times \mathbb{R} \to \mathbb{R}^d$ and $p : \Omega \times \mathbb{R} \to \mathbb{R}$:

$$\begin{aligned}
&\vec{u}_t + \vec{u} \cdot \nabla \vec{u} + \nabla p = 0, \quad \nabla \cdot \vec{u} = 0, \qquad \text{in } \Omega, \\
&\vec{u} \cdot \vec{n} = 0 \qquad \text{on } \partial\Omega, \\
&\vec{u} = \vec{u}_0 \qquad \text{on } t = 0,
\end{aligned}$$

where $\vec{u}_0 : \Omega \to \mathbb{R}^d$ is a given divergence-free initial velocity. Note that the solution for $p(\vec{x}, t)$ is not unique. We may add an arbitrary function of time $P(t)$ to any solution for the pressure and obtain another solution. Physically, this corresponds to the fact that the motion of an incompressible fluid in a rigid container is not affected by an instantaneous compression or expansion.

If the boundary moves with velocity $\vec{V}$, then we replace these boundary conditions by $\vec{u} \cdot \vec{n} = \vec{V} \cdot \vec{n}$ for Euler, and $\vec{u} = \vec{V}$ for Navier-Stokes.

10.2. Free surfaces

A free surface is a boundary whose location is not known *a priori*. The solution of free-surface problems for PDEs is, in general, very difficult. One has to determine both the solution of the PDE — in an unknown domain which may vary in time — and the location of the free surface. Since the location of the free surface is not specified in advance, we usually require more boundary conditions than in the case of a fixed boundary. One can think of these as providing boundary conditions for the PDE, together with conditions that determine the location of the free surface.

Many types of free surface problems arise in fluid mechanics. For example, the Stefan problem for the melting of ice; the Hele-Shaw problem for flow in a porous medium; and the motion of a shock wave in a compressible fluid. Here, we consider only free surfaces problems of the type that arise in 'water waves'. The free surface is then an interface that separates one fluid from another, such as the surface of a swimming pool or the ocean that separates air from water

We distinguish between two types of free-surface boundary conditions: a kinematic condition which states that the free surface moves with the fluid;

and a dynamic condition that expresses a balance of forces across the free surface.

10.2.1. Kinematic condition

The kinematic condition on a free surface is the same as the boundary condition on a moving impermeable surface, and states that the fluid velocity $\vec{u}$ satisfies

$$\vec{u} \cdot \vec{n} = \vec{V} \cdot \vec{n}, \tag{19}$$

on the free surface, where $\vec{n}$ is a unit normal to the surface and $\vec{V}$ is the velocity of the surface.

To write this boundary condition in a more convenient form, we suppose that the free surface has the equation $F(\vec{x}, t) = 0$. We assume that F is a smooth function and $\nabla F \neq 0$, which ensures that this equation defines a smooth surface.

A unit normal and the normal velocity are given in terms of F by

$$\vec{n} = \frac{\nabla F}{|\nabla F|}, \qquad \vec{V} = -\frac{F_t}{|\nabla F|}\vec{n}.$$

Using these expressions in (19), we find that the kinematic boundary condition may be written as

$$\frac{DF}{Dt} = 0 \qquad \text{on } F = 0, \tag{20}$$

meaning that material particles on the surface remain on the surface.

For a surface that is a graph with equation

$$x_3 = f(x_1, x_2, t), \tag{21}$$

the condition (20) becomes

$$f_t + u_1 f_{x_1} + u_2 f_{x_2} = u_3 \qquad \text{on } x_3 = f(x_1, x_2, t).$$

The assumption that the free surface is a graph can be unduly restrictive; for example, it fails when a surface wave ‘turns over’ and begins to break. Wave-breaking illustrates another fundamental difficulty in free surface problems: there may be very complex topological changes as drops of fluid pinch off.

10.2.2. Dynamic condition

The simplest dynamic condition on a free surface is that the stresses on either side of the surface are equal. For inviscid fluids, this means that the pressure is continuous across the free surface.[13] That is, we have $[p] = 0$, where $[p]$ denotes the jump in p across the surface.

In the case of an air-water interface, we can often neglect the motion of the air, because of its much smaller density, and treat it as a fluid with constant pressure p_0. The dynamic boundary condition for the water is then $p = p_0$. For irrotational flows, with the choice $P = p_0$ for the function of integration in Bernoulli's equation (18), we find that the velocity potential φ satisfies the condition

$$\varphi_t + \frac{1}{2}|\nabla\varphi|^2 + \Phi = 0$$

on the free surface.

If surface tension is taken into account, then according to the Young-Laplace law the jump in pressure across the free surface is proportional to the mean curvature H of the surface, so that

$$[p] = -2\sigma H, \tag{22}$$

where σ is the coefficient of surface tension. The sign of the mean curvature is chosen so that pressure forces tend to flatten the interface, meaning that the pressure is greater in a fluid whose surface is convex.

Surface tension has the dimensions of force/length or energy/area. It arises physically from the molecular energy of the interface. A surface of minimal area (for example, a soap film on a wire frame) has zero mean curvature, and cannot support a pressure difference. The Young-Laplace law expresses a balance between the pressure forces acting on the interface, which stretch it, and the contraction of the interface toward a surface with minimal local surface area and interfacial energy.

The assumption that σ is constant is reasonable in many cases, although variations in fluid composition and temperature, or the effect of surfactants, may lead to changes in the values of σ. For example, the Marangoni effect ('tears of wine') is the result of variations in the surface tension.

The surface tension of a clean air-water interface at standard temperature and pressure is $\sigma \approx 0.07\,\mathrm{N\,m^{-1}}$. In comparing the relative strength of gravitational and surface-tension forces for an air-water interface, we may form from the surface tension σ, the density of water $\rho_0 \approx 10^3\,\mathrm{Kg\,m^{-3}}$, and

[13] For viscous fluids, one must also consider the tangential stresses.

the acceleration due to gravity $g \approx 10\,\mathrm{m\,s^{-2}}$ a capillary lengthscale,

$$\sqrt{\frac{\sigma}{\rho_0 g}} \approx 3\,\mathrm{mm}.$$

This lengthscale gives an idea of the water waves for which surface tension plays an important role, namely small ripples whose wavelengths are of the order of millimeters. The propagation of longer water waves, like waves on a beach, is dominated by gravity, and the propagation of very short water waves — or waves in a gravity-free environment, like the space station — is dominated by surface tension.

The mean curvature

$$H = \frac{1}{2}\left(\kappa_1 + \kappa_2\right)$$

is the mean of the principal curvatures κ_1, κ_2 of the surface. The principal curvatures at a point on the surface are the maximum and minimum curvatures of curves on the surface through the point. The principal curvatures may have the same sign, as in the case of a sphere, or opposite signs, as in the case of a hyperboloid.

To define the principal curvatures more precisely, we consider a unit normal vector-field $\vec{n}$ on the surface, which is unique up to a sign.[14] For any point $\vec{x}$ on the surface, we define the shape operator $S(\vec{x})$ to be the linear map that takes a vector $\vec{v}$ tangent to the surface at $\vec{x}$ to the negative covariant derivative $-(\vec{v}\cdot\nabla\vec{n})(\vec{x})$ of the normal vector field in the direction $\vec{v}$ evaluated at $\vec{x}$.[15] Since $\vec{n}$ is a unit vector field, this derivative is orthogonal to $\vec{n}(\vec{x})$ and hence is tangent to the surface at $\vec{x}$. The shape operator $S(\vec{x})$ is thus a linear map on the two-dimensional tangent space of the surface at $\vec{x}$. One can show that this map is symmetric, so it has two real eigenvalues. These eigenvalues are the principal curvatures of the surface at $\vec{x}$, and

$$H = \frac{1}{2}\operatorname{tr} S.$$

In the case of a surface that is a graph with equation (21), we find that

$$H = \frac{1}{2}\nabla_h \cdot \left(\frac{\nabla_h f}{\sqrt{1 + |\nabla_h f|^2}}\right),$$

where $\nabla_h = (\partial_{x_1}, \partial_{x_2})$ denotes the horizontal gradient.

[14]We assume that the surface is orientable.

[15]The negative sign is not essential, but is conventional in the definition of the shape operator.

For example, if the fluid in $x_3 > f(x_1, x_2, t)$ has constant pressure p_0, then the pressure $p(\vec{x}, t)$ of the fluid in $x_3 < f(x_1, x_2, t)$ satisfies the boundary condition

$$p = p_0 - \sigma \nabla_h \cdot \left(\frac{\nabla_h f}{\sqrt{1 + |\nabla_h f|^2}} \right) \qquad \text{on } x_3 = f(x_1, x_2, t).$$

11. FURTHER READING

Chorin and Marsden [3] provides a quick introduction to fluid mechanics. Batchelor [1] gives an extensive, traditional discussion of incompressible fluid mechanics. Majda and Bertozzi [6] is a more mathematically oriented account. Landau and Lifshitz [5] describes fluid mechanics from the point of view of physics. Gurtin [4] gives a clear introduction to general continuum mechanics. For water waves and wave propagation, see Whitham [9].

Marchioro and Pulvirenti is an introduction to the mathematical theory of the incompressible Euler equations, also described in [6]. The lecture notes of Bressan and Donadello [2] provide a concise account.

Finally, no one should study fluid mechanics without looking at the wonderful pictures in [8].

REFERENCES

1. G. K. Batchelor, *An Introduction to Fluid Mechanics*, Cambridge University Press, Cambridge, 1967.
2. A. Bressan and C. Donadello, Lecture Notes on Fluid Dynamics. (Available at: http://www.math.psu.edu/bressan/)
3. A. J. Chorin, and J. E. Marsden, *A Mathematical Introduction to Fluid Mechanics*, 3rd ed., Springer-Verlag, New York, 1992.
4. M. E. Gurtin, *An Introduction to Continuum Mechanics*, Academic Press, 1981.
5. L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, 2nd ed., Pergamon Press, Oxford, 1987.
6. A. J. Majda and A. L. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge University Press, Cambridge, 2002.
7. C. Marchioro and M. Pulvirenti, *Mathematical Theory of Incompressible Nonviscous Fluids*, pringer-Verlag, New York, 1994.
8. M. Van Dyke, *Album of Fluid Motion*, Parabolic Press, Stanford, 1982.
9. G. B. Whitham, *Linear and Nonlinear Waves*, John Wiley & Sons, New York, 1974.