HAMILTONIAN FLUID MECHANICS

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1. KINEMATICS

Consider the deformation of a continuous medium. We label material points by their position $\mathbf{X} \in \mathcal{B} \subset \mathbb{R}^n$ in a reference configuration. The deformation of the medium is defined by a function

$$\phi : \mathcal{B} \times \mathbb{R} \to \mathcal{B}_T \subset \mathbb{R}^n$$

where $\mathbf{x} = \phi(\mathbf{X}, T)$ is the position of the material point $\mathbf{X}$ at time T.

We will refer to $\mathbf{X}$ as a material, or Lagrangian point, and $\mathbf{x}$ as a spatial or Eulerian point. If $\Omega \subset \mathcal{B}$ is a material region, we denote by $\Omega_T = \phi(\Omega, T)$ the spatial region occupied by the material points in Ω at time T.

We assume that the deformation ϕ is smooth, with a smooth inverse $\mathbf{X} = \Phi(\mathbf{x}, t)$. Here, we write $T = t$ for the time; the partial derivative ∂_T will denote the material time-derivative, taken holding $\mathbf{X}$ fixed, while ∂_t will denote the spatial time-derivative, taken holding $\mathbf{x}$ fixed. Abusing notation, we will usually denote the deformation and its inverse by $\mathbf{x}(\mathbf{X}, T)$ and $\mathbf{X}(\mathbf{x}, t)$, respectively.

We denote the Cartesian coordinates of the position vector of a point, chosen with respect to some convenient origin, in the spatial and reference configurations by

$$\mathbf{x} = \left(x^1, \dots, x^a, \dots, x^n\right), \qquad \mathbf{X} = \left(X^1, \dots, X^A, \dots, X^n\right),$$

respectively, with a corresponding notation for the components of other vectors and tensors. We will use the summation convention.

Following the usual procedure in Cartesian tensor analysis, we may identify vectors and one-forms and ignore the distinction between upper and lower indices, while maintaining the distinction between spatial indices (such as a) and material indices (such as A).

Alternatively, in order to maintain a distinction between vectors and one forms without introducing the full apparatus of tensor analysis, let $\mathbf{g}$ and $\mathbf{G}$ be the metric tensors in the spatial and reference configurations, respectively. The component matrix of each metric with respect to the corresponding Cartesian coordinate system is the identity matrix

$$g_{ab} = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{if } a \neq b, \end{cases} \qquad G_{AB} = \begin{cases} 1 & \text{if } A = B, \\ 0 & \text{if } A \neq B. \end{cases}$$

We denote the inner product of two spatial vectors $\mathbf{v}$, $\mathbf{w}$ with components v^a, w^a by

$$\mathbf{v} \cdot \mathbf{w} = g_{ab} v^a w^b = v^1 w^1 + v^2 w^2 + \cdots + v^n w^n,$$

and write $v_a = g_{ab} v^b$ for the (numerically equal) components of the covector associated with the vector $\mathbf{v}$. We write $g^{ab} = g_{ab}$ for the contravariant metric components, and use similar notation to 'raise' and 'lower' indices of other tensors. We will generally regard the resulting tensors as being equivalent. The notation for material vectors and tensors, is analogous, with g_{ab} replaced by G_{AB}.

The trace of a second-order spatial tensor $\mathbf{T}$ with components T^a_b is the scalar $\operatorname{tr}\mathbf{T} = T^a_a$.

We denote the material and spatial covariant derivatives by $\mathbf{D}$ and ∇, respectively, and the material and spatial divergences by Div and div, respectively. For example, if $\mathbf{v}$ is a spatial vector with components v^a and $\mathbf{S}$ is a mixed second-order tensor with components S^A_a, then

$$(\mathbf{D}\mathbf{v})^a_A = \frac{\partial v^a}{\partial X^A}, \qquad \operatorname{div}\mathbf{v} = \frac{\partial v^a}{\partial x^a}, \qquad (\operatorname{Div}\mathbf{S})_a = \frac{\partial S^A_a}{\partial X^A}.$$

1.1. **The deformation gradient.** The deformation gradient $\mathbf{F}(\mathbf{X}, T) : \mathbb{R}^n \to \mathbb{R}^n$ is the linear map that approximates the deformation $\mathbf{x}(\cdot, T)$ near $\mathbf{X}$. It is defined by

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}, \qquad F^a_A = \frac{\partial x^a}{\partial X^A}. \tag{1}$$

We will also use the notation $\mathbf{F} = \mathbf{D}\mathbf{x}$. The deformation gradient maps vectors in the reference frame to vectors in the spatial frame. If $\mathbf{w} = \mathbf{F}\mathbf{V}$, where $\mathbf{w}$ has components w^a and $\mathbf{V}$ has components V^A, then

$$w^a = F^a_A V^A.$$

The inverse of $\mathbf{F}$ is given by

$$\mathbf{F}^{-1} = \frac{\partial \mathbf{X}}{\partial \mathbf{x}}, \qquad \left(\mathbf{F}^{-1}\right)^A_a = \frac{\partial X^A}{\partial x^a}.$$

We have

$$\mathbf{F}^a_A \left(\mathbf{F}^{-1}\right)^A_b = \delta^a_b, \qquad \left(\mathbf{F}^{-1}\right)^A_a F^a_B = \delta^A_B$$

where δ^a_b is the Kronecker-delta,

$$\delta^a_b = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{if } a \neq b. \end{cases}$$

We define the transpose $\mathbf{F}^t$ of $\mathbf{F}$ by

$$\left(\mathbf{F}^t\mathbf{v}\right) \cdot \mathbf{V} = \mathbf{v} \cdot (\mathbf{F}\mathbf{V}) \qquad \text{for all } \mathbf{v}, \mathbf{V} \in \mathbb{R}^n.$$

The components of $\mathbf{F}^t$ are obtained by raising and lowering the components of $\mathbf{F}$

$$\left(\mathbf{F}^t\right)^A_a = g_{ab} G^{AB} F^b_B.$$

Similarly, the inverse transpose has components

$$\left(\mathbf{F}^{-t}\right)^a_A = g^{ab} G_{AB} \left(\mathbf{F}^{-1}\right)^B_b$$

1.2. **Velocity and acceleration.** The velocity $\mathbf{U}(\mathbf{X}, T)$ of a material point is given by

$$\mathbf{U}(\mathbf{X}, T) = \mathbf{x}_T(\mathbf{X}, T).$$

We denote the velocity expressed as a function of spatial coordinates by $\mathbf{u}(\mathbf{x}, t)$, where

$$\mathbf{u}\left(\mathbf{x}(\mathbf{X}, t), t\right) = \mathbf{U}(\mathbf{X}, t).$$

The chain rule implies that the material time-derivative is related to the spatial time-derivative by

$$\frac{\partial}{\partial T} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla,$$

where

$$\mathbf{u}\cdot\nabla = u^a \frac{\partial}{\partial x^a}$$

is the covariant derivative in the direction $\mathbf{u}$. In particular, the acceleration of a material point is given by

$$\mathbf{U}_T = \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}.$$

1.3. **Density and the Jacobian.** Let $\rho_0(\mathbf{X})$ be the density of the medium in the reference configuration $\mathcal{B}$. The mass of a material region $\Omega \subset \mathcal{B}$ is conserved, and is given by

$$\int_\Omega \rho_0\, d\mathbf{X} = \int_{\Omega_t} \frac{\rho_0}{J}\, d\mathbf{x}$$

where the Jacobian $J(\mathbf{X}, T)$ is defined by

$$J = \det \mathbf{F}. \tag{2}$$

We assume throughout that $\det \mathbf{F} > 0$. Thus, the density $\rho(\mathbf{x}, t)$ in the spatial configuration is given by

$$\rho = \frac{\rho_0}{J}. \tag{3}$$

The derivative of the Jacobian $J(\mathbf{F})$ with respect to the deformation gradient is

$$\frac{\partial J}{\partial \mathbf{F}}(\mathbf{F})\mathbf{H} = \lim_{\varepsilon\to 0}\left[\frac{\det(\mathbf{F}+\varepsilon\mathbf{H}) - \det\mathbf{F}}{\varepsilon}\right] = (\det\mathbf{F})\lim_{\varepsilon\to 0}\left[\frac{\det(\mathbf{I}+\varepsilon\mathbf{F}^{-1}\mathbf{H}) - 1}{\varepsilon}\right].$$

Using the fact that

$$\det\left(\mathbf{I}+\varepsilon\mathbf{T}\right) = 1 + \varepsilon\,\mathrm{tr}\,\mathbf{T} + O\left(\varepsilon^2\right) \qquad \text{as } \varepsilon \to 0,$$

we get

$$\frac{\partial J}{\partial \mathbf{F}}(\mathbf{F})\mathbf{H} = J\,\mathrm{tr}\left(\mathbf{F}^{-1}\mathbf{H}\right), \qquad \frac{\partial J}{\partial F^a_A} H^a_A = J\frac{\partial X^A}{\partial x^a} H^a_A. \tag{4}$$

In particular, it follows that

$$J_T = J\,\mathrm{tr}\left(\mathbf{F}^{-1}\mathbf{F}_T\right) = J\,\mathrm{tr}\left(\frac{\partial \mathbf{X}}{\partial \mathbf{x}}\frac{\partial \mathbf{U}}{\partial \mathbf{X}}\right) = J\,\mathrm{tr}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right) = J\,\mathrm{div}\,\mathbf{u},$$

where $\mathrm{div}\,\mathbf{u} = \partial_{x^a} u^a$ denotes the spatial divergence of $\mathbf{u}$. Using this equation and (3), we get the spatial form of conservation of mass

$$\rho_t + \mathrm{div}\left(\rho\mathbf{u}\right) = 0.$$

2. Compressible fluids

2.1. **Thermodynamics.** A unit mass of a simple fluid in thermodynamic equilibrium is characterized by two quantities. Any thermodynamic quantity is a function of any other independent pair of thermodynamic quantities.

Let e, $v = 1/\rho$, p, T, s denote the specific internal energy, specific volume, pressure, temperature, specific entropy, respectively, of the fluid. The functional dependence of these quantities satisfies the basic thermodynamic identity

$$de = -pdv + Tds.$$

Let $h = e + pv$ denote the specific enthalpy. Then

$$dh = vdp + Tds.$$

It follows that

$$\begin{aligned} d\left(\frac{e}{v}\right) &= \frac{1}{v}de - \frac{e}{v^2}dv \\ &= -\frac{h}{v^2}dv + \frac{T}{v}ds. \end{aligned}$$

Hence,

$$d\left(\rho e\right) = hd\rho + \rho T ds.$$

Denoting a derivative with respect to ρ keeping s constant by a prime, we therefore have

$$e' = \frac{p}{\rho^2}, \qquad (\rho e)' = h, \qquad h' = \frac{p'}{\rho}. \tag{5}$$

2.2. **Lagrangian formulation: barotropic fluid.** We consider a barotropic fluid whose specific internal energy e is a function of the spatial density ρ only. For example, this is the case in isentropic flows. It follows that, in Lagrangian coordinates, the internal energy depends on the deformation only through the Jacobian J defined in (2).

The Lagrangian density for the fluid motion is the difference between its kinetic and internal energy densities in the reference configuration,

$$L\left(\mathbf{x}_T, \mathbf{D}\mathbf{x}\right) = \frac{1}{2}\rho_0 \mathbf{x}_T^2 - \rho_0 e\left(\frac{\rho_0}{J}\right), \qquad J = \det \mathbf{D}\mathbf{x}.$$

If ρ_0 depends on $\mathbf{X}$, then so does L, but, to simplify the notation, we do not show this dependence explicitly. The corresponding variational principle is

$$\delta \int \left\{\frac{1}{2}\rho_0 \mathbf{x}_T^2 - \rho_0 e\left(\frac{\rho_0}{J}\right)\right\} d\mathbf{X}dT = 0. \tag{6}$$

Taking variations in (6) with respect to $\mathbf{x}$, using (4) to compute the variation of the Jacobian and (5) to rewrite e', we get

$$\begin{aligned}
\delta \int L\, d\mathbf{X}dT &= \int \left\{ \rho_0 \mathbf{x}_T \cdot \delta \mathbf{x}_T + \frac{\rho_0^2}{J^2} e' \delta J \right\} d\mathbf{X}dT \\
&= \int \left\{ \rho_0 \mathbf{x}_T \cdot \delta \mathbf{x}_T + pJ \operatorname{tr}\left(\mathbf{F}^{-1} \delta \mathbf{F} \right) \right\} d\mathbf{X}dT \\
&= \int \left\{ \rho_0 x^{aT} \delta x_{aT} + pJ \left(\mathbf{F}^{-1} \right)^A_b \frac{\partial \delta x^b}{\partial X^A} \right\} d\mathbf{X}dT \\
&= -\int \left\{ \rho_0 x^a_{TT} + \frac{\partial}{\partial X^A} \left[pJ g^{ab} \left(\mathbf{F}^{-1} \right)^A_b \right] \right\} \delta x_a \, d\mathbf{X}dT \\
&= -\int \left\{ \rho_0 \mathbf{x}_{TT} + \operatorname{Div}\left[pJ\mathbf{F}^{-1} \right] \right\} \cdot \delta \mathbf{x} \, d\mathbf{X}dT.
\end{aligned}$$

It follows that

$$\rho_0 \mathbf{x}_{TT} = \operatorname{Div} \mathbf{S}\left(\mathbf{D}\mathbf{x} \right),$$

where $\mathbf{S}(\mathbf{F})$ is the (first) Piola-Kirchhoff stress tensor

$$\mathbf{S} = -p\left(\frac{\rho_0}{J} \right) J\mathbf{F}^{-1}, \qquad S^A_a = -p\left(\frac{\rho_0}{J} \right) \frac{\partial X^A}{\partial x^a}.$$

The component form of the equation is

$$\rho_0 x^a_{TT} = \frac{\partial S^{aA}}{\partial X^A}.$$

Using the Piola identity

$$\operatorname{Div}\left(J\mathbf{F}^{-1} \right) = 0, \qquad \frac{\partial}{\partial X^A} \left(J \frac{\partial X^A}{\partial x^a} \right) = 0 \tag{7}$$

and the fact that $\mathbf{F}^{-1}\mathbf{D} = \nabla$, we get the spatial form of the momentum equation,

$$\rho\left(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} \right) + \nabla p = 0.$$

2.3. **Stress tensors.** The Piola-Kirchhoff stress tensor is the derivative of the internal energy density

$$W(\mathbf{F}) = \rho_0 e \left(\frac{\rho_0}{\det \mathbf{F}} \right)$$

with respect to the deformation gradient. In components,

$$S^A_a = \frac{\partial W}{\partial F^a_A}.$$

The Piola-Kirchhoff stress tensor $\mathbf{S}$ gives the force per unit area in the reference configuration exerted across a surface by one part of the continuum on another part. By contrast, the Cauchy stress tensor $\mathbf{T}$ gives the force per unit area in the spatial configuration.

The Piola-Kirchhoff stress tensor $\mathbf{S}$ is related to the Cauchy stress tensor $\mathbf{T}$ by

$$\mathbf{S} = J\mathbf{T}\mathbf{F}^{-t}, \qquad S^a_A = JT^a_b \left(\mathbf{F}^{-1} \right)^b_A, \tag{8}$$

where $S^a_A = g^{ab} G_{AB} S^B_b$.

To see this, we linearize the deformation, and consider the case of three space dimensions for simplicity. A material parallelogram bounded by vectors $\mathbf{Y}$, $\mathbf{Z}$ has a non-unit normal $\tilde{\mathbf{N}} = \mathbf{Y} \times \mathbf{Z}$ whose magnitude $\|\tilde{\mathbf{N}}\|$ is equal to the area. The deformation tensor $\mathbf{F}$ maps this to spatial parallelogram bounded by vectors $\mathbf{y} = \mathbf{FY}$, $\mathbf{z} = \mathbf{FZ}$. The corresponding non-unit normal vector is $\tilde{\mathbf{n}} = \mathbf{x} \times \mathbf{y}$.

Using the polar decomposition, we write $\mathbf{F} = \mathbf{RU}$ as the product of a rotation $\mathbf{R}$ and a symmetric, positive-definite tensor $\mathbf{U}$. Since the cross-product is invariant under rotations, we have

$$\mathbf{FY} \times \mathbf{FZ} = \mathbf{R}\left(\mathbf{UY} \times \mathbf{UZ}\right).$$

Let $\{\mathbf{E}_A : A = 1, 2, 3\}$ be an orthonormal basis of eigenvectors of $\mathbf{U}$ with eigenvalues $\{\lambda_A : A = 1, 2, 3\}$. Then, if $\mathbf{E}_A \times \mathbf{E}_B = \mathbf{E}_C$,

$$\mathbf{UE}_A \times \mathbf{UE}_B = \lambda_A \lambda_B \left(\mathbf{E}_A \times \mathbf{E}_B\right) = \frac{\lambda_A \lambda_B \lambda_C}{\lambda_C} \mathbf{E}_C = \left(\det \mathbf{F}\right) \mathbf{U}^{-1} \left(\mathbf{E}_A \times \mathbf{E}_B\right).$$

It follows that $\mathbf{UY} \times \mathbf{UZ} = J\mathbf{U}^{-1}\left(\mathbf{Y} \times \mathbf{Z}\right)$, and

$$\tilde{\mathbf{n}} = \mathbf{FY} \times \mathbf{FZ} = J\mathbf{RU}^{-1}\left(\mathbf{Y} \times \mathbf{Z}\right) = J\mathbf{F}^{-t}\tilde{\mathbf{N}}. \tag{9}$$

Introducing the unit normals $\mathbf{n} = \tilde{\mathbf{n}}/\|\tilde{\mathbf{n}}\|$, $\mathbf{N} = \tilde{\mathbf{N}}/\|\tilde{\mathbf{N}}\|$, the force $\mathbf{f}$ across the parallelogram is given by the Cauchy stress tensor acting on the unit spatial normal multiplied by the spatial area,

$$\mathbf{f} = \|\tilde{\mathbf{n}}\|\mathbf{Tn} = \mathbf{T}\tilde{\mathbf{n}},$$

and by the Piola-Kirchhoff stress tensor acting on the unit material normal multiplied by the material area,

$$\mathbf{f} = \|\tilde{\mathbf{N}}\|\mathbf{SN} = \mathbf{S}\tilde{\mathbf{N}}.$$

Equating these two expressions for $\mathbf{f}$, and using (9), we get (8).

For a barotropic fuid, we have $\mathbf{T} = -p\mathbf{I}$, which gives

$$\mathbf{S} = -pJ\mathbf{F}^{-t}.$$

2.4. **Hamiltonian formulation.** The canonically conjugate momentum for (6) is

$$\mathbf{p}\left(\mathbf{X}, T\right) = \rho_0\left(\mathbf{X}\right)\mathbf{x}_T\left(\mathbf{X}, T\right), \tag{10}$$

and the Hamiltonian is

$$\mathcal{H}\left(\mathbf{x}, \mathbf{p}\right) = \int \left\{\frac{1}{2\rho_0}\mathbf{p}^2 + \rho_0 e\left(\frac{\rho_0}{\det \mathbf{Dx}}\right)\right\} d\mathbf{X}. \tag{11}$$

The corresponding canonical Poisson bracket is

$$\{\mathcal{F}, \mathcal{G}\} = \int \left\{\frac{\delta \mathcal{F}}{\delta \mathbf{x}} \cdot \frac{\delta \mathcal{G}}{\delta \mathbf{p}} - \frac{\delta \mathcal{F}}{\delta \mathbf{p}} \cdot \frac{\delta \mathcal{G}}{\delta \mathbf{x}}\right\} d\mathbf{X}. \tag{12}$$

The Lagrangian equations of motion may be written in the Hamiltonian form

$$\mathbf{x}_T = \{\mathbf{x}, \mathcal{H}\}, \qquad \mathbf{p}_T = \{\mathbf{p}, \mathcal{H}\},$$

where $\mathbf{x}$, $\mathbf{p}$ are to be understood in the Poisson brackets as point-wise evaluation functionals.

Next, we determine how this bracket acts on functionals $\mathcal{F}(\rho, \mathbf{m})$ of the Eulerian variables, where $\rho(\mathbf{x})$ is the spatial density and $\mathbf{m}(\mathbf{x}) = \rho(\mathbf{x})\mathbf{u}(\mathbf{x})$ is the spatial momentum. From (3) and (10), we have

$$\rho(\mathbf{x}) = \frac{\rho_0}{J(\mathbf{X})}, \qquad \mathbf{m}(\mathbf{x}) = \frac{\mathbf{p}(\mathbf{X})}{J(\mathbf{X})}. \tag{13}$$

Suppose we vary

$$\mathbf{x}(\mathbf{X}) \mapsto \mathbf{x}(\mathbf{X}) + \delta\mathbf{x}(\mathbf{X}), \qquad \mathbf{p}(\mathbf{X}) = \mathbf{p}(\mathbf{X}) + \delta\mathbf{p}(\mathbf{X}),$$

where we omit an explicit indication of the time-dependence. Then, from (4),

$$J(\mathbf{X}) \mapsto J(\mathbf{X}) + \delta J(\mathbf{X}), \qquad \delta J = \operatorname{tr}\left(\mathbf{F}^{-1}\delta\mathbf{F}\right),$$

where $\mathbf{F} = \mathbf{D}\mathbf{x}$, $\delta\mathbf{F} = \mathbf{D}\delta\mathbf{x}$.

The corresponding variations of the Eulerian variables

$$\rho(\mathbf{x}) \mapsto \rho(\mathbf{x}) + \delta\rho(\mathbf{x}), \qquad \mathbf{m}(\mathbf{x}) \mapsto \mathbf{m}(\mathbf{x}) + \delta\mathbf{m}(\mathbf{x})$$

are taken keeping $\mathbf{x}$ fixed. Denoting by $\Delta\rho(\mathbf{X})$, the variation of ρ keeping $\mathbf{X}$ fixed, we have

$$\Delta\rho = \delta\rho + \delta\mathbf{x}\cdot\nabla\rho.$$

Also, from (13) and (4)

$$\Delta\rho = -\frac{\rho_0}{J^2}\delta J = -\frac{\rho_0}{J}\operatorname{tr}\left(\mathbf{F}^{-1}\delta\mathbf{F}\right)$$

Hence, using the equation $\operatorname{tr}\left(\mathbf{F}^{-1}\delta\mathbf{F}\right) = \operatorname{div}\delta\mathbf{x}$, we get

$$\delta\rho = -\rho\operatorname{div}\delta\mathbf{x} - \delta\mathbf{x}\cdot\nabla\rho = -\operatorname{div}\left(\rho\delta\mathbf{x}\right).$$

Similarly,

$$\delta\mathbf{m} = -\operatorname{div}\left(\mathbf{m}\otimes\delta\mathbf{x}\right) + \frac{1}{J}\delta\mathbf{p},$$

where

$$\operatorname{div}\left(\mathbf{m}\otimes\delta\mathbf{x}\right)^a = \frac{\partial}{\partial x^b}\left(m^a\delta x^b\right).$$

For a functional $\mathcal{F}(\rho,\mathbf{m})$, we therefore have on integrating by parts and changing the integration variable from $\mathbf{x}$ to $\mathbf{X}$ that

$$\begin{aligned}
\delta\mathcal{F} &= \int\left\{\frac{\delta\mathcal{F}}{\delta\rho}\delta\rho + \frac{\delta\mathcal{F}}{\delta\mathbf{m}}\cdot\delta\mathbf{m}\right\}d\mathbf{x}\\
&= \int\left\{-\frac{\delta\mathcal{F}}{\delta\rho}\operatorname{div}\left(\rho\delta\mathbf{x}\right) + \frac{\delta\mathcal{F}}{\delta\mathbf{m}}\cdot\left[-\operatorname{div}\left(\mathbf{m}\otimes\delta\mathbf{x}\right) + \frac{1}{J}\delta\mathbf{p}\right]\right\}d\mathbf{x}\\
&= \int\left\{\rho\nabla\left(\frac{\delta\mathcal{F}}{\delta\rho}\right)\cdot\delta\mathbf{x} + \delta\mathbf{x}\cdot\nabla\left(\frac{\delta\mathcal{F}}{\delta\mathbf{m}}\right)\cdot\mathbf{m} + \frac{1}{J}\frac{\delta\mathcal{F}}{\delta\mathbf{m}}\cdot\delta\mathbf{p}\right\}d\mathbf{x}\\
&= \int\left\{J\left[\rho\nabla\left(\frac{\delta\mathcal{F}}{\delta\rho}\right)\cdot\delta\mathbf{x} + \delta\mathbf{x}\cdot\nabla\left(\frac{\delta\mathcal{F}}{\delta\mathbf{m}}\right)\cdot\mathbf{m}\right] + \frac{\delta\mathcal{F}}{\delta\mathbf{m}}\cdot\delta\mathbf{p}\right\}d\mathbf{X}
\end{aligned}$$

It follows that

$$\frac{\delta\mathcal{F}}{\delta\mathbf{x}} = J\left[\rho\nabla\left(\frac{\delta\mathcal{F}}{\delta\rho}\right) + \nabla\left(\frac{\delta\mathcal{F}}{\delta\mathbf{m}}\right)\cdot\mathbf{m}\right], \qquad \frac{\delta\mathcal{F}}{\delta\mathbf{p}} = \frac{\delta\mathcal{F}}{\delta\mathbf{m}}, \tag{14}$$

or, in components,

$$\frac{\delta\mathcal{F}}{\delta x^a} = J\left[\rho\frac{\partial}{\partial x^a}\left(\frac{\delta\mathcal{F}}{\delta\rho}\right) + m^b\frac{\partial}{\partial x^a}\left(\frac{\delta\mathcal{F}}{\delta m^b}\right)\right], \qquad \frac{\delta\mathcal{F}}{\delta p^a} = \frac{\delta\mathcal{F}}{\delta m^a}.$$

Using (14) in the canonical Poisson bracket (12), we get

$$\begin{aligned}
\{\mathcal{F},\mathcal{G}\} = \int\Bigg\{&\rho\left[\nabla\left(\frac{\delta\mathcal{F}}{\delta\rho}\right)\cdot\frac{\delta\mathcal{G}}{\delta\mathbf{m}} - \frac{\delta\mathcal{F}}{\delta\mathbf{m}}\cdot\nabla\left(\frac{\delta\mathcal{G}}{\delta\rho}\right)\right]\\
&+\left[\nabla\left(\frac{\delta\mathcal{F}}{\delta\mathbf{m}}\right)\cdot\mathbf{m}\right]\cdot\frac{\delta\mathcal{G}}{\delta\mathbf{m}} - \frac{\delta\mathcal{F}}{\delta\mathbf{m}}\cdot\left[\nabla\left(\frac{\delta\mathcal{G}}{\delta\mathbf{m}}\right)\cdot\mathbf{m}\right]\Bigg\}\,d\mathbf{x}.
\end{aligned} \tag{15}$$

In components,

$$\{\mathcal{F},\mathcal{G}\} = \int \left\{ \rho \left[\frac{\partial}{\partial x^a}\left(\frac{\delta \mathcal{F}}{\delta \rho}\right) \frac{\delta \mathcal{G}}{\delta m^a} - \frac{\delta \mathcal{F}}{\delta m^a}\frac{\partial}{\partial x^a}\left(\frac{\delta \mathcal{G}}{\delta \rho}\right)\right] \right.$$
$$\left. + m^a \frac{\partial}{\partial x^b}\left(\frac{\delta \mathcal{F}}{\delta m^a}\right)\frac{\delta \mathcal{G}}{\delta m^b} - m^b \frac{\delta \mathcal{F}}{\delta m^a}\frac{\partial}{\partial x^a}\left(\frac{\delta \mathcal{G}}{\delta m^b}\right)\right\} d\mathbf{x}.$$

Writing the Lagrangian Hamiltonian (11) in terms of the Eulerian variables, we get

$$\mathcal{H}(\rho, \mathbf{m}) = \int \left\{ \frac{1}{2}\frac{\mathbf{m}^2}{\rho} + \rho e(\rho) \right\} d\mathbf{x}.$$

The Hamiltonian equation

$$\rho_t = \{\rho, \mathcal{H}\}$$

then gives the Eulerian equation for conservation of mass,

$$\rho_t + \operatorname{div} \mathbf{m} = 0.$$

Using (5), we find that the equation

$$\mathbf{m}_t = \{\mathbf{m}, \mathcal{H}\}$$

gives

$$\mathbf{m}_t + \operatorname{div}\left(\frac{\mathbf{m}\otimes\mathbf{m}}{\rho}\right) + \rho\nabla h = 0,$$

which, for smooth solutions, is equivalent to the Eulerian equation for conservation of momentum

$$\mathbf{m}_t + \operatorname{div}\left(\frac{\mathbf{m}\otimes\mathbf{m}}{\rho}\right) + \nabla p = 0.$$

We write $u = (\rho, \mathbf{m})^t$, with components u^α given by $u^0 = \rho$, $u^a = m^a$, and define the functional derivative with respect to u by

$$\frac{\delta \mathcal{F}}{\delta u} = \begin{pmatrix} \delta \mathcal{F}/\delta \rho \\ \delta \mathcal{F}/\delta \mathbf{m} \end{pmatrix},$$

Then the Poisson bracket (15) can be written as

$$\{\mathcal{F},\mathcal{G}\} = -\int \left(\frac{\delta \mathcal{F}}{\delta u}\right)^t \cdot \mathbf{J}(u)\left[\frac{\delta \mathcal{G}}{\delta u}\right] d\mathbf{x}$$

where

$$\mathbf{J}(u) = \begin{pmatrix} 0 & \nabla^t \rho \\ \rho\nabla & \mathbf{m}^t\nabla + \nabla^t\mathbf{m} \end{pmatrix}.$$

In components,

$$\{\mathcal{F},\mathcal{G}\} = -\int \frac{\delta \mathcal{F}}{\delta u^\alpha} J^{\alpha\beta}(u) \frac{\delta \mathcal{G}}{\delta u^\beta}\, d\mathbf{x},$$

where we sum over $0 \le \alpha, \beta \le n$, and

$$J^{00} = 0, \quad J^{0b}(\rho) = \frac{\partial}{\partial x^b}\rho, \quad J^{a0}(\rho) = \rho\frac{\partial}{\partial x^a}, \quad J^{ab}(\mathbf{m}) = m^b\frac{\partial}{\partial x^a} + \frac{\partial}{\partial x^b}m^a.$$

Hamilton's equations are

$$u_t + \mathbf{J}(u)\left[\frac{\delta \mathcal{H}}{\delta u}\right] = 0.$$

2.5. **The inviscid Burgers equation.** Consider, as a special case of the above analysis, a pressureless gas ($e(\rho) = 0$) in one space dimension. Choosing the density ρ_0 in the reference configuration equal to one, the variational principle is

$$\delta \int \frac{1}{2} x_T^2 \, dX dT = 0.$$

The Euler-Lagrange equation is

$$x_{TT} = 0.$$

Introducing the velocity $U(X,T) = x_T(X,T)$, we may write this equation as

$$x_T = U, \qquad U_T = 0.$$

To write this equation in Eulerian form, we let

$$\rho(x,t) = \frac{1}{J(X,T)}, \qquad m(x,t) = \frac{x_T(X,T)}{J(X,T)}$$

where $t = T$ and $J = x_X$ is the Jacobian of the transformation from Lagrangian to Eulerian coordinates.

Then, from the preceding section, the Hamiltonian is

$$\mathcal{H}(\rho, m) = \int \frac{1}{2} \frac{m^2}{\rho} \, dx,$$

the Hamiltonian operator $\mathbf{J}$ is

$$\mathbf{J}(\rho, m) = \begin{pmatrix} 0 & \partial_x \rho \\ \rho \partial_x & m\partial_x + \partial_x m \end{pmatrix}.$$

and Hamilton's equations are

$$\rho_t + m_x = 0, \qquad m_t + \left(\frac{m^2}{\rho}\right)_x = 0. \tag{16}$$

If we write $m = \rho u$, these equations are

$$\rho_t + (\rho u)_x = 0, \qquad (\rho u)_t + \left(\rho u^2\right)_x = 0.$$

For smooth solutions, these equations are equivalent to

$$\rho_t + (\rho u)_x = 0, \qquad u_t + u u_x = 0.$$

Thus, we recover the inviscid Burgers equation together with an equation for the Jacobian of the transformation between spatial and characteristic coordinates.

Weak solutions of the conservation law (16) are, however, not equivalent to weak solutions of the usual conservative form of the inviscid Burgers equation,

$$u_t + \left(\frac{1}{2} u^2\right)_x = 0.$$