A model KdV problem: Ion acoustic waves

John K. Hunter

November 7, 2006

1. THE KORTEWEG-DE VRIES EQUATION

The Korteweg-de Vries (KdV) equation is the following nonlinear PDE for $u(x,t)$:

$$u_t + uu_x + u_{xxx} = 0. \tag{1}$$

This equation was first derived by Korteweg and de Vries (1895) for shallow water waves, and it is a generic asymptotic equation that describes weakly nonlinear waves with weak long-wave dispersion.

The term u_t is the rate of change of the wave profile u in a reference frame moving with the linearized phase velocity of the wave. The term uu_x is an advective nonlinearity, and u_{xxx} is a linear dispersive term.

Water waves are described by a relatively complicated system of equations which involve a free boundary. Here, we derive the KdV equation from a simpler system of PDEs that describes ion acoustic waves in a plasma. This derivation illustrates the universal nature of the KdV equation, which applies to any wave motion with weak advective nonlinearity and weak long wave dispersion. Specifically, the linearized dispersion relation should have a Taylor expansion as $k \to 0$ of the form $\omega = c_0 k + \alpha k^3 + \dots$.

2. PLASMAS

A plasma is an ionized fluid consisting of positively charged ions and negatively charged electrons which interact through the electro-magnetic field they generate. Plasmas support waves analogous to sound waves in a simple compressible fluid, but the presence of ion and electron oscillations in plasmas makes these waves dispersive.

Here, we consider a simple 'two-fluid' model of a plasma in which the ions and electrons are treated as separate fluids.[1] The full system of equations

[1] More detailed models use a kinetic description of the plasma.

follows from the fluid equations for the motion of the ions and electrons, and Maxwell's equations for the electro-magnetic field generated by the charged fluids. We will consider relatively low frequency waves that involve the motion of the ions, and we assume that there are no magnetic fields. After simplification and nondimensionalization, we get (7) below.

Let n^i, n^e denote the number density of the ions and electrons, respectively, u^i, u^e their velocities, p^i, p^e their pressures, and E the electric field.

In one space dimension, the equations of conservation of mass and momentum for the ion fluid are

$$\begin{aligned} &n_t^i + (n^i u^i)_x = 0, \\ &m^i n^i \left(u_t^i + u^i u_{ix}\right) + p_x^i = en^i E. \end{aligned}$$

Here, m^i is the mass of an ion and e is its charge. For simplicity, we assume that this is the same as the charge of an electron. We suppose that the ion-fluid is 'cold', meaning that we neglect its pressure. Setting $p^i = 0$, we get

$$\begin{aligned} &n_t^i + (n^i u^i)_x = 0, \\ &m^i \left(u_t^i + u^i u_{ix}\right) = eE. \end{aligned}$$

The equations of conservation of mass and momentum for the electron fluid are

$$\begin{aligned} &n_t^e + (n^e u^e)_x = 0, \\ &m^e n^e \left(u_t^e + u^e u_x^e\right) + p_x^e = -en^e E, \end{aligned}$$

where m^e is the mass of an electron and $-e$ is its charge. The electrons are much lighter than the ions, and we neglect their inertia. Setting $m^e = 0$, we get

$$p_x^e = -en^e E. \tag{2}$$

As we will see, this equation provides an equation for the electron density n^e. The electron velocity u^e is then determined from the equation of conservation of mass, and is uncoupled from the remaining variables, so we do not need to consider it further.

We assume an isothermal equation of state for the electron fluid, meaning that

$$p^e = kTn^e, \tag{3}$$

where k is Boltzmann's constant and T is the temperature. Using (3) in (2) and writing $E = -\varphi_x$ in terms of an electrostatic potential φ, we get

$$kTn_x^e = en^e \varphi_x.$$

This equation implies that n^e is given in terms of φ by

$$n^e = n_0 \exp\left(\frac{e\varphi}{kT}\right), \tag{4}$$

where the constant n_0 is the electron number density at $\varphi = 0$.

Maxwell's equation for the electrostatic field E generated by a charge density σ is $\epsilon_0 \nabla \cdot E = \sigma$, where ϵ_0 is a dielectric constant. This equation implies that

$$\epsilon_0 E_x = e(n^i - n^e). \tag{5}$$

In terms of the potential φ, equation (5) becomes

$$-\varphi_{xx} = \frac{e}{\epsilon_0}\left(n^i - n^e\right). \tag{6}$$

We may then use (4) to eliminate n^e from (6).

Dropping the i-superscript on the ion-variables (n^i, u^i), we may write the final system of equations for (n, u, φ) as

$$\begin{aligned} & n_t + (nu)_x = 0, \\ & u_t + uu_x + \frac{e}{m}\varphi_x = 0, \\ & -\varphi_{xx} + \frac{en_0}{\epsilon_0}\exp\left(\frac{e\varphi}{kT}\right) = \frac{e}{\epsilon_0} n. \end{aligned}$$

This system consists of a pair of evolution equations for (n, u) coupled with a semi-linear elliptic equation for φ.

To nondimensionalize these equations, we introduce the the Debye length λ_0 and the ion-acoustic sound speed c_0, defined by

$$\lambda_0^2 = \frac{\epsilon_0 kT}{n_0 e^2}, \qquad c_0^2 = \frac{kT}{m}.$$

These parameters vary by orders of magnitudes for plasmas in different conditions. For example, in a dense laboratory plasma, we may have $n_0 \approx 10^{20}\,\mathrm{m}^{-3}$, $T \approx 60,000\,\mathrm{K}$ and $\lambda_0 \approx 10^{-6}\,\mathrm{m}$; in the solar wind near the earth, we have $n_0 \approx 10^7\,\mathrm{m}^{-3}$, $T \approx 120,000\,\mathrm{K}$, and $\lambda_0 \approx 10\,\mathrm{m}$.

Introducing dimensionless variables

$$\bar{x} = \frac{x}{\lambda_0}, \quad \bar{t} = \frac{c_0 t}{\lambda_0}, \quad \bar{n} = \frac{n}{n_0}, \quad \bar{u} = \frac{u}{c_0}, \quad \bar{\varphi} = \frac{e\varphi}{kT},$$

and dropping the 'bars', we get the nondimensionalized equations

$$\begin{aligned} & n_t + (nu)_x = 0, \\ & u_t + uu_x + \varphi_x = 0, \\ & -\varphi_{xx} + e^{\varphi} = n. \end{aligned} \tag{7}$$

3. LINEARIZED EQUATIONS

To understand the dispersion of ion acoustic waves, we first consider the linearized equations. Linearizing the system (7) at $n = 1$, $\varphi = 0$ and $u = 0$, we get

$$\begin{aligned} n_t + u_x &= 0, \\ u_t + \varphi_x &= 0, \\ -\varphi_{xx} + \varphi &= n. \end{aligned}$$

We seek Fourier solutions

$$n(x,t) = \hat{n}e^{ikx-i\omega t}, \quad u(x,t) = \hat{u}e^{ikx-i\omega t}, \quad \varphi(x,t) = \hat{\varphi}e^{ikx-i\omega t}.$$

From the last equation, we find that

$$\hat{\varphi} = \frac{\hat{n}}{1+k^2}.$$

From the first and second equations, after eliminating $\hat{\varphi}$, we get

$$\begin{pmatrix} -i\omega & ik \\ ik/(1+k^2) & -i\omega \end{pmatrix} \begin{pmatrix} \hat{n} \\ \hat{u} \end{pmatrix} = 0$$

This linear system has a non-zero solution if the determinant of the matrix is zero, which implies that (ω, k) satisfies the dispersion relation

$$\omega^2 = \frac{k^2}{1+k^2}.$$

The corresponding solution is

$$\begin{pmatrix} \hat{n} \\ \hat{u} \end{pmatrix} = \hat{a} \begin{pmatrix} k \\ \omega \end{pmatrix},$$

where $\hat{a}$ is an arbitrary constant.

The phase velocity $c = \omega/k$ of these waves is given

$$c = \frac{1}{(1+k^2)^{1/2}},$$

so that $c \to 1$ as $k \to 0$ and $c \to 0$ as $k \to \infty$.

The group velocity $C = d\omega/dk$ is given by

$$C = \frac{1}{(1+k^2)^{3/2}}.$$

In this case, we have $C < c$ for all $k > 0$.

In the long-wave limit $k \to 0$, we get the leading order approximation $\omega = k$, corresponding to non-dispersive sound waves with phase speed $\omega/k = 1$. In the original dimensional variables, this speed is the ion-acoustic speed c_0, and the condition for long-wave dispersion to be weak is that $k\lambda_0 \ll 1$, meaning that the wavelength is much larger than the Debye length.[2] In these long waves, the electrons oscillate with the ions, and the fluid behaves essentially like a single fluid. The inertia of the wave is provided by the ions and the restoring pressure force by the electrons.

At the next order in k, we find that

$$\omega = k - \frac{1}{2}k^3 + O(k^5) \qquad \text{as } k \to 0. \tag{8}$$

The $O(k^3)$ correction corresponds to weak KdV-type long-wave dispersion.

In the short-wave limit $k \to \infty$, we get waves with constant frequency $\omega = 1$, corresponding in dimensional terms to the ion plasma frequency $\omega_0 = c_0/\lambda_0$. In these short waves, the ions oscillate in an essentially fixed background of electrons.

For very long waves, we may neglect φ_{xx} in comparison with e^φ in (7), which gives $n = e^\varphi$ and $n_x = n\varphi_x$. In that case, (n, u) satisfy the isothermal compressible Euler equations

$$\begin{aligned} &n_t + (nu)_x = 0, \\ &n\,(u_t + uu_x) + n_x = 0. \end{aligned}$$

These equations form a nondispersive hyperbolic system.[3] In general, solutions form shocks, but then the long-wave approximation breaks down and it is no longer self-consistent.

A weakly nonlinear expansion of these long wave equations, which is a limiting case of the KdV expansion given below,

$$\begin{pmatrix} n \\ u \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \epsilon a(x - t, \epsilon t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + O(\epsilon^2),$$

leads to an inviscid Burgers equation for $a(\xi, \tau)$,

$$a_\tau + aa_\xi = 0.$$

[2]Compare this with shallow water waves, where the condition for long-wave dispersion to be weak is that the wavelength is much larger than the depth of the fluid.

[3]The analogous system for water waves is the system of shallow water equations.

In the next section, we apply a similar expansion to (7) and include the effect of weak long wave dispersion, leading to a KdV equation.

4. KDV EXPANSION

We can see from the KdV equation (1) what orders of magnitude of the wave amplitude and the spatial and temporal scales lead to a balance between weak nonlinearity and long-wave dispersion. We need u to have the same order of maginitude as ∂_x^2 and ∂_t to have the same order of magnitude as ∂_x^3. Thus, we want

$$u = O(\epsilon), \qquad \partial_x = O(\epsilon^{1/2}), \qquad \partial_t = O(\epsilon^{3/2})$$

where ϵ is a small positive parameter.[4] Here, the time-derivative ∂_t is taken in a reference frame moving with the linearized wave velocity.

Motivated by this scaling, we seek an asymptotic solution of (7), depending on a small parameter ϵ of the form

$$\begin{aligned} n &= n\left(\epsilon^{1/2}(x - \lambda t), \epsilon^{3/2} t; \epsilon\right), \\ u &= u\left(\epsilon^{1/2}(x - \lambda t), \epsilon^{3/2} t; \epsilon\right), \\ \varphi &= \varphi\left(\epsilon^{1/2}(x - \lambda t), \epsilon^{3/2} t; \epsilon\right). \end{aligned}$$

We will determine the wave velocity λ as part of the solution. The parameter ϵ does not appear explicitly in the PDE (7), but it could appear in the initial conditions, for example.

We introduce multiple-scale variables

$$\xi = \epsilon^{1/2}(x - \lambda t), \qquad \tau = \epsilon^{3/2} t.$$

According to the chain rule, we may expand the original space-time derivatives as

$$\partial_x = \epsilon^{1/2}\partial_\xi, \qquad \partial_t = -\epsilon^{1/2}\lambda\partial_\xi + \epsilon^{3/2}\partial_\tau.$$

After including the small parameter ϵ explicitly in the new variables, we assume that derivatives with respect to ξ, τ are of the order 1 as $\epsilon \to 0^+$, which is not the case for derivatives with respect to the original variables x, t.

[4]We could replace ϵ by ϵ^2, or some other small parameter, provided that we retain the same relative scalings.

It follows that $n(\xi,\tau;\epsilon)$, $u(\xi,\tau;\epsilon)$, $\varphi(\xi,\tau;\epsilon)$ satisfy

$$\begin{aligned} &(nu)_\xi - \lambda n_\xi + \epsilon n_\tau = 0,\\ &\varphi_\xi - \lambda u_\xi + u u_\xi + \epsilon u_\tau = 0,\\ &e^\varphi - \epsilon \varphi_{\xi\xi} = n. \end{aligned} \tag{9}$$

We look for an asymptotic solution of (9) of the form

$$\begin{aligned} n &= 1 + \epsilon n_1 + \epsilon^2 n_2 + \epsilon^3 n_3 + O(\epsilon^4),\\ u &= \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + O(\epsilon^4),\\ \varphi &= \epsilon \varphi_1 + \epsilon^2 \varphi_2 + \epsilon^3 \varphi_3 + O(\epsilon^4). \end{aligned}$$

Using these expansions in (9), Taylor expanding the result with respect to ϵ, and equating coefficients of ϵ, we find that

$$u_{1\xi} - \lambda n_{1\xi} = 0,$$

$$\varphi_{1\xi} - \lambda u_{1\xi} = 0, \tag{10}$$

$$\varphi_1 - n_1 = 0. \tag{11}$$

Equating coefficients of ϵ^2, we find that

$$u_{2\xi} - \lambda n_{2\xi} + n_{1\tau} + (n_1 u_1)_\xi = 0,$$

$$\varphi_{2\xi} - \lambda u_{2\xi} + u_{1\tau} + u_1 u_{1\xi} = 0, \tag{12}$$

$$\varphi_2 - n_2 + \frac{1}{2}\varphi_1^2 - \varphi_{1\xi\xi} = 0. \tag{13}$$

Eliminating φ_1 from (10), we get a homogeneous linear system for (n_1, u_1),

$$\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} n_1 \\ u_1 \end{pmatrix}_\xi = 0.$$

This system has a nontrivial solution if $\lambda^2 = 1$. We suppose that $\lambda = 1$ for definiteness, corresponding to a right-moving wave. Then

$$\begin{pmatrix} n_1 \\ u_1 \end{pmatrix} = a(\xi,\tau) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \varphi_1 = a(\xi,\tau), \tag{14}$$

where $a(\xi,\tau)$ is an arbitrary scalar-valued function.

At the next order, after setting $\lambda = 1$ and eliminating φ_2 in (12), we obtain a nonhomogeneous linear system for (n_2, u_2),

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} n_2 \\ u_2 \end{pmatrix}_\xi + \begin{pmatrix} n_{1\tau} + (n_1 u_1)_\xi \\ u_{1\tau} + u_1 u_{1\xi} - \varphi_1 \varphi_{1\xi} + \varphi_{1\xi\xi\xi} \end{pmatrix} = 0. \tag{15}$$

This system is solvable for (n_2, u_2) if and only if the nonhomogeneous term is orthogonal to the null-vector $(1, 1)$. Using (14), we find that this condition implies that $a(\xi, \tau)$ satisfies a KdV equation

$$a_\tau + a a_\xi + \frac{1}{2} a_{\xi\xi\xi} = 0. \tag{16}$$

Note that the linearized dispersion relation of this equation agrees with the long wave expansion (8) of the linearized dispersion relation of the original system.

If a satisfies (16), then we may solve (15) for (n_2, u_2). The solution is the sum of a solution of the nonhomogeneous equations and an arbitrary multiple

$$a_2(\xi, \tau) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

of the solution of the homogeneous problem.

We may compute higher-order terms in the asymptotic solution in a similar way. At the order ϵ^k, we obtain a nonhomogeneous linear equation for (n_k, u_k) of the form

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} n_k \\ u_k \end{pmatrix}_\xi + \begin{pmatrix} f_{k-1} \\ g_{k-1} \end{pmatrix} = 0,$$

where f_{k-1}, g_{k-1} depend only on (n_1, u_1),..., (n_{k-1}, u_{k-1}), and φ_k may be expressed explicitly in terms of $n_1, \ldots, n_k$. The condition that this equation is solvable for (n_k, u_k) is $f_{k-1} + g_{k-1} = 0$, and this condition is satisfied if a_{k-1} satisfies a suitable equation. The solution for (n_k, u_k) then involves an arbitrary function of integration a_k. An equation for a_k follows from the solvability condition for the order $(k + 1)$-equations.

In summary, the leading-order asymptotic solution of (7) as $\epsilon \to 0^+$ is

$$\begin{pmatrix} n \\ u \\ \varphi \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \epsilon a \left(\epsilon^{1/2}(x - t), \epsilon^{3/2} t \right) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + O(\epsilon^2),$$

where $a(\xi, \tau)$ satisfies the KdV equation (16). We expect that this asymptotic solution is valid for long times of the order $\tau = O(1)$ or $t = O(\epsilon^{-3/2})$.