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ORIENTATION WAVES IN A DIRECTOR FIELD WITH ROTATIONAL INERTIA

Giuseppe Alì

Dipartimento di Matematica, Università della Calabria and INFN-Gruppo c. Cosenza
I-87036 Arcavacata di Rende (CS), Italy

John K. Hunter

Department of Mathematics, University of California at Davis
Davis CA 95616, USA

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Abstract. We study a variational system of nonlinear hyperbolic partial differential equations that describes the propagation of orientation waves in a director field with rotational inertia and potential energy given by the Oseen-Frank energy from the continuum theory of nematic liquid crystals. There are two types of waves, which we call splay and twist waves, respectively. Weakly nonlinear splay waves are described by the quadratically nonlinear Hunter-Saxton equation. In this paper, we derive a new cubically nonlinear asymptotic equation that describes weakly nonlinear twist waves. This equation provides a surprising representation of the Hunter-Saxton equation, and like the Hunter-Saxton equation it is completely integrable. There are analogous cubically nonlinear representations of the Camassa-Holm and Degasperis-Procesi equations. Moreover, two different, but compatible, variational principles for the Hunter-Saxton equation arise from a single variational principle for the primitive director field equations in the two different limits for splay and twist waves. We also use the asymptotic equation to analyze a one-dimensional initial value problem for the director-field equations with twist-wave initial data.

1. Introduction.

In this paper, we analyze a system of nonlinear hyperbolic partial differential equations that is obtained from a variational principle of the form

$$\delta \int_{\mathbb{R}^3\times\mathbb{R}} \left\{ \frac{1}{2}|\mathbf{n}_t|^2 - W\left(\mathbf{n}, \nabla\mathbf{n}\right) \right\} d\mathbf{x}dt = 0, \qquad \mathbf{n}\cdot\mathbf{n} = 1. \tag{1}$$

Here, $\mathbf{n}(\mathbf{x},t) \in \mathbb{S}^2$ is a field of unit vectors depending on a spatial variable $\mathbf{x} \in \mathbb{R}^3$ and time $t \in \mathbb{R}$. We call $\mathbf{n}$ a director field. The potential energy density W in (1) is given by the Oseen-Frank energy from the continuum theory of nematic liquid crystals ([13], Ch. 3.),

$$W\left(\mathbf{n}, \nabla\mathbf{n}\right) = \frac{1}{2}\alpha\left(\operatorname{div}\mathbf{n}\right)^2 + \frac{1}{2}\beta\left(\mathbf{n}\cdot\operatorname{curl}\mathbf{n}\right)^2 + \frac{1}{2}\gamma\left|\mathbf{n}\times\operatorname{curl}\mathbf{n}\right|^2 \tag{2}$$

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where α, β, γ are distinct, positive constants. As we explain in Section 2, this variational principle models the propagation of orientation waves in a massive director field.

The Euler-Lagrange equation associated with (1)–(2) is

$$\begin{aligned}\mathbf{n}_{tt} = \alpha\nabla(\operatorname{div}\mathbf{n}) - \beta\left\{A\operatorname{curl}\mathbf{n} + \operatorname{curl}(A\mathbf{n})\right\}\\ +\gamma\left\{\mathbf{B}\times\operatorname{curl}\mathbf{n} - \operatorname{curl}(\mathbf{B}\times\mathbf{n})\right\} + \lambda\mathbf{n},\end{aligned} \tag{3}$$

where

$$A = \mathbf{n}\cdot\operatorname{curl}\mathbf{n}, \qquad \mathbf{B} = \mathbf{n}\times\operatorname{curl}\mathbf{n}. \tag{4}$$

The Lagrange multiplier $\lambda(\mathbf{x}, t)$ in (3) is chosen so that $\mathbf{n}\cdot\mathbf{n} = 1$, and is given explicitly in terms of $\mathbf{n}$ by

$$\lambda = -|\mathbf{n}_t|^2 + \alpha\left[|\nabla\mathbf{n}|^2 - |\operatorname{curl}\mathbf{n}|^2\right] + 2\left[\beta A^2 + \gamma\,|\mathbf{B}|^2\right] + (\alpha-\gamma)\operatorname{div}\mathbf{B}.$$

In Section 3, we show that (3) is a hyperbolic system of wave equations. This system has two types of waves, which we call 'splay' and 'twist' waves. The splay waves carry perturbations of the director field that are in the same plane as the unperturbed director field and the direction of propagation of the wave, whereas the twist waves carry perturbations that are orthogonal to this plane.

The system (3) is invariant under space-time rescalings $\mathbf{x} \to r\mathbf{x}$, $t \mapsto rt$ for any $r \neq 0$. As a result, the speed of a wave propagating in a given direction is independent of its frequency and these waves are non-dispersive. The wave speeds, however, depend on the direction of wave propagation relative to the director field, and this leads to nonlinear effects since the waves themselves carry rotations of the director field.

Splay waves were investigated by Saxton [23] and Hunter and Saxton [17]. The simplest case is that of planar deformations of a director field depending on a single space variable, with $\mathbf{x} = (x, 0, 0)$ and $\mathbf{n}(x, t) = (\cos\varphi(x, t), \sin\varphi(x, t), 0)$. In that case, equation (1) reduces to a scalar variational principle for the angle $\varphi : \mathbb{R}\times\mathbb{R} \to \mathbb{T}$,

$$\delta\int \frac{1}{2}\left\{\varphi_t^2 - a^2(\varphi)\varphi_x^2\right\}\,dxdt = 0 \tag{5}$$

where $a^2(\varphi) = \alpha^2\cos^2\varphi + \gamma^2\sin^2\varphi$. The corresponding Euler-Lagrange equation is

$$\varphi_{tt} - \left[a^2(\varphi)\varphi_x\right]_x + a(\varphi)a'(\varphi)\varphi_x^2 = 0. \tag{6}$$

Here, and below, a prime denotes the derivative with respect to φ.

The PDE (6) is one of the simplest nonlinear, variational generalizations of the linear wave equation one can imagine. Nevertheless, the effects of this nonlinearity are remarkable. The derivative φ_x of smooth solutions of (6) typically blows up in finite time due to the formation of cusp-type singularities [14]. The PDE has global weak solutions, which are continuous but not smooth. Furthermore, in sharp contrast with the more familiar case of hyperbolic conservation laws [8], equation (6) has two natural classes of weak solutions: dissipative weak solutions; and conservative weak solutions that are compatible with the variational and Hamiltonian structure of the equation. The global well-posedness of the initial value problem for (6) for conservative weak solutions is proved in [5].

The use of weakly nonlinear asymptotics to study solutions of (6) that consist of small, localized perturbations $u(x, t)$ of a smooth solution φ_0 with $a'(\varphi_0) \neq 0$, leads, after a suitable normalization, to the Hunter-Saxton (HS) equation [17],

$$(u_t + uu_x)_x - \frac{1}{2}u_x^2 = 0. \tag{7}$$

The variational principle for (7) corresponding to (1) is

$$\delta \int \frac{1}{2}\left\{u_x u_t + u u_x^2\right\}\, dxdt = 0. \tag{8}$$

Equation (7) is completely integrable [18, 2, 20], and possesses global dissipative and conservative weak solutions [19, 26, 3, 9].

In this paper, we study the full system (3) of director-field equations, where qualitatively new phenomena arise that are not present for the scalar wave equation (6). The structure of the full system is most easily seen in the case of three-dimensional deformations of the director field that depend on a single space variable. We may then write $\mathbf{x} = (x, 0, 0)$ and

$$\mathbf{n}(x,t) = (\cos\varphi(x,t), \sin\varphi(x,t)\cos\psi(x,t), \sin\varphi(x,t)\sin\psi(x,t)) \tag{9}$$

where φ, ψ are spherical polar angles. Then, for director fields of the form (9), the variational principle (1) becomes

$$\delta \int \frac{1}{2}\left\{\varphi_t^2 - a^2(\varphi)\varphi_x^2 + q^2(\varphi)\left[\psi_t^2 - b^2(\varphi)\psi_x^2\right]\right\}\, dxdt = 0, \tag{10}$$

where

$$\begin{aligned} a^2(\varphi) &= \alpha\sin^2\varphi + \gamma\cos^2\varphi,\\ b^2(\varphi) &= \beta\sin^2\varphi + \gamma\cos^2\varphi,\\ q^2(\varphi) &= \sin^2\varphi. \end{aligned} \tag{11}$$

The Euler-Lagrange equation associated with (10) is a coupled system of wave equations,

$$\varphi_{tt} - \left(a^2\varphi_x\right)_x + aa'\varphi_x^2 + q^2bb'\psi_x^2 - qq'\left(\psi_t^2 - b^2\psi_x^2\right) = 0, \tag{12}$$

$$\psi_{tt} - \left(b^2\psi_x\right)_x + \frac{2q'}{q}\left(\varphi_t\psi_t - b^2\varphi_x\psi_x\right) = 0. \tag{13}$$

Variations of the angle φ correspond to splay waves, and variations of the angle ψ to twist waves. The splay-wave speed a and the twist-wave speed b depend on φ only, and the wave equation for φ is forced by terms that are proportional to quadratic functions of derivatives of ψ. When ψ is constant, the system (12)–(13) reduces to (6).

The main result of this paper is an asymptotic equation for localized, weakly-nonlinear twist waves. Denoting the perturbation of φ by u and the perturbation of ψ by v, we find, after a suitable normalization, that $u(x,t)$, $v(x,t)$ satisfy the PDE

$$\begin{aligned} (v_t + uv_x)_x &= 0,\\ u_{xx} &= v_x^2. \end{aligned} \tag{14}$$

The variational principle for (14) corresponding to (1) is

$$\delta \int \frac{1}{2}\left\{v_x\left(v_t + uv_x\right) + \frac{1}{2}u_x^2\right\}\, dxdt = 0. \tag{15}$$

Equation (14) is a cubically-nonlinear, scale-invariant system that consists of an advection equation for v in which the advection velocity u is reconstructed non-locally and quadratically from v. We may interpret (14) as follows: a twist wave with amplitude v nonlinearly generates a splay wave with with amplitude u (second equation); and the splay wave then affects the propagation velocity of the twist wave (first equation).

Remarkably, the elimination of v from (14) implies that u satisfies the derivative of the HS-equation,

$$\left[(u_t + uu_x)_x - \frac{1}{2}u_x^2\right]_x = 0. \tag{16}$$

Thus, the transformation

$$u \mapsto (u, v) \qquad \text{where } v_x^2 = u_{xx} \tag{17}$$

relates a solution u of (16) with a solution (u, v) of (14). Moreover, under this transformation, the variational principle (15) gives a different variational principle for (16) than (8), and the Hamiltonian structures associated with these two variational principles are compatible. The resulting bi-Hamiltonian structure of the HS-equation explains its complete integrability [18], and it follows that (14) is also completely integrable (see Section 5).

The transformation (17) from solutions u of (16) to solutions (u, v) of (14) is neither one-to-one nor onto. Only convex solutions of (16) with $u_{xx} \geq 0$ can be obtained from solutions of (14), while multiple solutions (u, v) of (14) for which v_x has the same magnitude but different signs (which may depend on x) correspond to the same solution of (16). Moreover, because of the nonlinear nature of the transformation, distributional solutions of (16) do not necessarily transform into distributional solutions of (14). Nevertheless, equation (14) provides an interesting, and new, representation of the HS-equation (16) for u as an advection equation for another variable v. As we show in Section 5, there are similar representations of the Camassa-Holm [7] and Degasperis-Procesi [10] equations.

We now summarize the contents of this paper. In Section 2, we describe the system of PDEs (3) for the director-field, and compare it with some other variational wave equations. In Section 3, we study the linearized equations. In Section 4, we derive asymptotic equations for weakly nonlinear, non-planar splay and twist waves. In Section 5, we show that (14) leads to the HS-equation, and describe its bi-Hamiltonian structure and integrability. In Section 6, we solve (14) explicitly by the method of characteristics, and use the result to solve an initial-boundary value problem for (14) that arises in Section 7. In Section 7, we construct an asymptotic solution of the one-dimensional initial value problem for the director-field equations with initial data corresponding to a small-amplitude, compactly supported twist wave. Finally, in Section 8, we derive asymptotic equations for weakly nonlinear, spatially periodic twist waves. The algebraic details of the derivations are summarized in the Appendix.

2. **Director fields and variational principles.** The system of wave equations we study here is motivated, in part, by continuum models of nematic liquid crystals. We consider a medium composed of rods that are free to rotate about their center of mass but do not, on average, translate (meaning that the medium does not 'flow'). We specify the mean orientation of the rods at a spatial location $\mathbf{x} \in \mathbb{R}^3$ and time $t \in \mathbb{R}$ by a director field $\mathbf{n}(\mathbf{x}, t) \in \mathbb{S}^2$.

We suppose that, in the absence of boundary constraints, the energy of the director field is minimized when it is in a uniform state, and that work is required to deform it to a spatially nonuniform state. A natural choice for the potential energy $\mathcal{W}$ of a director field $\mathbf{n}$ is then

$$\mathcal{W}(\mathbf{n}) = \int_{\mathbb{R}^3} W(\mathbf{n}, \nabla\mathbf{n})\, d\mathbf{x}, \tag{18}$$

where the potential energy density W is a quadratic function of $\nabla\mathbf{n}$ with coefficients depending on $\mathbf{n}$.

We suppose that the potential energy is invariant under reflections $\mathbf{n} \mapsto -\mathbf{n}$, as is the case for uniaxial nematic liquid crystals,[1] and also under simultaneous rotations and reflections of the spatial variable and the director field,

$$\mathbf{x} \mapsto R\mathbf{x}, \qquad \mathbf{n} \mapsto R\mathbf{n} \qquad \text{for all orthogonal maps } R : \mathbb{R}^3 \to \mathbb{R}^3. \tag{19}$$

Then, up to a null-Lagrangian, the potential energy is given by the Oseen-Frank energy (2). Following the theory of nematic liquid crystals, we call α, β, γ the elastic constants of splay, twist, and bend, respectively. For typical nematics they satisfy $0 < \beta < \alpha < \gamma$.

To model the dynamical behavior of the director field, we suppose that the rods have rotational inertia and neglect any damping of their rotational motion. In that case, the rotational kinetic energy of the director field is proportional to $|\mathbf{n}_t|^2$, and (normalizing the moment of inertia per unit volume of the director field to one) the principle of stationary action implies that the motion of the director field is given by the variational principle (1), with Euler-Lagrange equation (3).

This model is not applicable to standard liquid crystal hydrodynamics in which the motion of the director field is dominated by viscosity and the effects of rotational inertia are negligible. Nevertheless, it is conceivable that these equations could model the high-frequency excitation of liquid crystals composed of molecules, or rods, with a large moment of inertia.

When the elastic constants are distinct, the potential energy density $W(\mathbf{n}, \nabla\mathbf{n})$ in (2) depends explicitly on $\mathbf{n}$. In the one-constant approximation $\alpha = \beta = \gamma$, the function W reduces, up to a null-Lagrangian, to the harmonic map energy density,

$$W(\nabla\mathbf{n}) = \frac{1}{2}\alpha \left|\nabla\mathbf{n}\right|^2, \tag{20}$$

and the explicit dependence on $\mathbf{n}$ drops out. The corresponding PDE (3) reduces to the wave-map equation (or nonlinear sigma model) for maps from Minkowski space-time into the sphere [25],

$$\mathbf{n}_{tt} = \Delta\mathbf{n} + \lambda\mathbf{n}, \qquad \lambda = -\left|\mathbf{n}_t\right|^2 + \left|\nabla\mathbf{n}\right|^2. \tag{21}$$

The energy density in (20) is invariant under a larger group of transformations than those in (19), namely

$$\mathbf{x} \mapsto R\mathbf{x}, \qquad \mathbf{n} \mapsto S\mathbf{n} \qquad \text{for all orthogonal maps } R, S : \mathbb{R}^3 \to \mathbb{R}^3. \tag{22}$$

All of the nonlinear phenomena we study here vanish in this case. Thus, the qualitative behavior of (3) with distinct elastic constants is very different from that of the wave map equation (21). We assume throughout this paper that α, β, γ are distinct positive constants.

It is interesting to compare the director-field equation with the Einstein field equation in general relativity. The Einstein-Hilbert action density for the vacuum Einstein equation may be written, after an integration by parts, as a quadratic

[1] If the directions $\mathbf{n}$ and $-\mathbf{n}$ are identified, then we get a director field $\mathbf{n} : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}P^2$. The global topological difference between $\mathbb{S}^2$ and $\mathbb{R}P^2$ is important for the study of defects, but here we are concerned with small perturbation of the director field and this difference is not relevant.

function of derivatives of the metric g with coefficients depending on g. Moreover, after a suitable choice of gauge, the Einstein equation is hyperbolic and the wave operator acting on the metric has coefficients that depend on the metric. In this respect, the Einstein equation for the metric tensor field g is analogous to the director equation (3) for the director field $\mathbf{n}$. Like the director-field equation, but unlike the wave-map equation, the Einstein equation is invariant under suitable simultaneous transformations of the domain and range spaces, but not under a larger group of separate transformations of the domain and range. Despite this similarity, the nonlinearity of the Einstein equation is more degenerate than that of director-field equation. Weakly nonlinear asymptotic expansions for small-amplitude, localized gravitational waves in general relativity, analogous to the ones used here for director fields, leads to linear equations. One can, however, derive strongly nonlinear asymptotic equations for large-amplitude, localized gravitational waves [1].

Liquid crystalline phases are the result of a continuous breaking of rotational symmetry, and the orientation waves we analyze here may be regarded as associated Goldstone modes [22]. Similar phenomena would occur for other non-dispersive Goldstone modes in classical fields with continuously broken symmetry in which the energy density W associated with a set of order parameters Ψ^a has the anisotropic form

$$W(\Psi^a) = A^{ij}_{ab}(\Psi^c)\frac{\partial \Psi^a}{\partial x^i}\frac{\partial \Psi^b}{\partial x^j},$$

rather than the more commonly assumed isotropic Ginzburg-Landau density proportional to $|\nabla \Psi^a|^2$, such as the harmonic map energy (20). As the example of liquid crystals illustrates, the Ginzburg-Landau energy density is not dictated by general symmetry arguments in anisotropic media. For long-wave variations, it is natural to retain as leading-order terms in the energy those that are quadratic in the spatial derivatives of the order parameters, but the order parameters themselves may vary by a large amount, and one should retain any dependence of the coefficients of the spatial derivatives on the order parameters. The analysis in this paper shows that this dependence leads to qualitatively new nonlinear effects in comparison with the isotropic case when the field equations are hyperbolic PDEs.

3. **Orientation waves.**

In this section, we show that (3) forms a hyperbolic system of PDEs, and describe the corresponding waves.

We consider solutions of (3) of the form

$$\mathbf{n}(\mathbf{x}, t) = \mathbf{n}_0 + \mathbf{n}'(\mathbf{x}, t),$$

where $\mathbf{n}'$ is a small perturbation of a constant director field $\mathbf{n}_0$. We linearize the resulting equations for $\mathbf{n}'$, and look for Fourier solutions of the linearized equations of the form

$$\mathbf{n}'(\mathbf{x}, t) = \widehat{\mathbf{n}}\, e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}.$$

Here, $\mathbf{k} \in \mathbb{R}^3$ is the wavenumber vector, $\omega \in \mathbb{R}$ is the frequency, and $\widehat{\mathbf{n}} \in \mathbb{R}^3$ is a constant vector.

We find that (see Appendix A.1) ω satisfies the linearized dispersion relation

$$\left[\omega^2 - a^2(\mathbf{k}; \mathbf{n}_0)\right]\left[\omega^2 - b^2(\mathbf{k}; \mathbf{n}_0)\right] = 0, \tag{23}$$

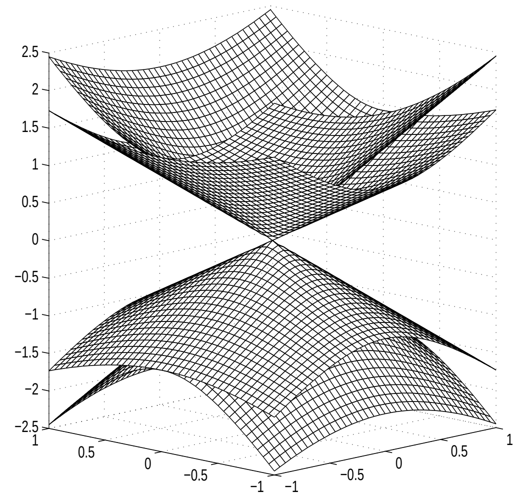

FIGURE 1. The characteristic variety of (3) in two space dimensions. The x, y axes of the plot correspond to k_1, k_2 and the z-axis to ω.

where, with $k = |\mathbf{k}|$,

$$a^2(\mathbf{k};\mathbf{n}_0) = \alpha\left[k^2 - (\mathbf{k}\cdot\mathbf{n}_0)^2\right] + \gamma\,(\mathbf{k}\cdot\mathbf{n}_0)^2\,, \tag{24}$$

$$b^2(\mathbf{k};\mathbf{n}_0) = \beta\left[k^2 - (\mathbf{k}\cdot\mathbf{n}_0)^2\right] + \gamma\,(\mathbf{k}\cdot\mathbf{n}_0)^2\,. \tag{25}$$

Thus, if $\alpha \neq \beta$, the characteristic variety of (3) consists of two nested elliptical cones (see Figure 1). There is a loss of strict hyperbolicity when $\mathbf{k}$ is parallel to $\mathbf{n}_0$, corresponding to a wave that propagates in the same direction as the unperturbed director field.

The waves associated with the branch $\omega^2 = a^2(\mathbf{k};\mathbf{n}_0)$ carry perturbations of the director field with $\widehat{\mathbf{n}} = \mathbf{R}$, where

$$\mathbf{R}\,(\mathbf{k};\mathbf{n}_0) = \mathbf{n}_0 \times (\mathbf{k}\times\mathbf{n}_0) \equiv \mathbf{k} - (\mathbf{k}\cdot\mathbf{n}_0)\,\mathbf{n}_0. \tag{26}$$

We call these waves *splay* waves. The perturbations of $\mathbf{n}$ are in the plane spanned by $\{\mathbf{k},\mathbf{n}_0\}$ and are orthogonal to $\mathbf{n}_0$ (as required by the constraint that $\mathbf{n}$ is a unit vector).

The waves associated with the branch $\omega^2 = b^2(\mathbf{k};\mathbf{n}_0)$ carry perturbations of the director field orthogonal with $\widehat{\mathbf{n}} = \mathbf{S}$, where

$$\mathbf{S}\,(\mathbf{k};\mathbf{n}_0) = \mathbf{k}\times\mathbf{n}_0. \tag{27}$$

We call these waves *twist* waves. The perturbations of $\mathbf{n}$ are orthogonal to the plane spanned by $\{\mathbf{k},\mathbf{n}_0\}$.

There is a fundamental difference in how the splay and twist wave speeds depend on $\mathbf{n}_0$. From (24), (26) we find that the derivative of the splay-wave speed in the direction of the perturbation carried by the wave is given by

$$\nabla_{\mathbf{n}_0} a(\mathbf{k};\mathbf{n}_0)\cdot\mathbf{R}(\mathbf{k};\mathbf{n}_0) = -\frac{(\alpha-\gamma)}{a(\mathbf{k};\mathbf{n}_0)}(\mathbf{k}\cdot\mathbf{n}_0)\left[k^2-(\mathbf{k}\cdot\mathbf{n}_0)^2\right]. \tag{28}$$

If $\alpha \neq \gamma$, this quantity is nonzero provided that $\mathbf{k}$ is not parallel or orthogonal to $\mathbf{n}_0$. On the other hand, from (25), (27) we find that the derivative of the twist-wave speed in the direction of the wave is identically zero,

$$\nabla_{\mathbf{n}_0} b(\mathbf{k};\mathbf{n}_0)\cdot\mathbf{S}(\mathbf{k};\mathbf{n}_0) = 0. \tag{29}$$

By analogy with the terms introduced by Lax in the context of first-order hyperbolic systems of conservation laws [8], we say that a wave-mode satisfying a second-order hyperbolic system of variational wave equations, whose wave speeds are functions of the dependent variables, is *genuinely nonlinear* if the derivative of the wave speed with respect to the dependent variables in the direction of the wave is nonzero, and *linearly degenerate* if the derivative is identically zero. Thus, the splay waves are genuinely nonlinear when they do not propagate in directions parallel or orthogonal to the director field, and the twist waves are linearly degenerate. This linear degeneracy may be seen in the asymptotic system (14), where the advection velocity u of v is independent of v. It may also be seen in the one-dimensional system of wave equations (12)–(13), where the wave speed $b(\varphi)$ of ψ is independent of ψ. The splay waves are genuinely nonlinear, except at values of φ such that $a'(\varphi) = 0$.

4. **Weakly nonlinear waves.** In this section, we derive asymptotic equations for weakly nonlinear, non-planar splay and twist waves. First, we show that the amplitude $u(x,t)$ of a propagating splay wave satisfies the Hunter-Saxton equation (7). Then, using a different ansatz, we show that the the amplitude $v(x,t)$ of a propagating twist wave satisfies the system (14), where $u(x,t)$ is the amplitude of a higher-order forced splay wave that is generated nonlinearly by the twist wave.

In the last part of this section, we analyze the exceptional case of waves that propagate in the same direction as the unperturbed director field, when there is a loss of strict hyperbolicity and the orientation waves are polarized.

The algebraic details of the derivations are summarized in Appendix A.

4.1. **Splay waves.** Weakly nonlinear asymptotics for genuinely nonlinear splay waves leads to the quadratically nonlinear Hunter-Saxton (HS) equation [17]. We summarize the expansion here.

Let ε be a small positive parameter. We look for a high-frequency asymptotic solution $\mathbf{n}^\varepsilon(\mathbf{x},t)$ of (3) as $\varepsilon \to 0^+$, with phase $\Phi(\mathbf{x},t)$, of the form

$$\mathbf{n}^\varepsilon(\mathbf{x},t) = \mathbf{n}\left(\frac{\Phi(\mathbf{x},t)}{\varepsilon},\mathbf{x},t;\varepsilon\right), \tag{30}$$

$$\mathbf{n}(\theta,\mathbf{x},t;\varepsilon) = \mathbf{n}_0(\mathbf{x},t) + \varepsilon\mathbf{n}_1(\theta,\mathbf{x},t) + O(\varepsilon^2). \tag{31}$$

Here, $\mathbf{n}_0(\mathbf{x},t)$ is a given smooth solution of (3).

We consider localized waves, such as pulses or fronts, rather than periodic waves. In that case, the expansion (30)–(31) provides an asymptotic solution of (3) near the wavefront $\Phi(\mathbf{x},t) = 0$, where $\theta = O(1)$, but it need not be uniformly valid away from the wavefront. We may obtain a global solution by matching the resulting 'inner' solution for a localized wave, valid when $\Phi = O(\varepsilon)$, with a suitable 'outer' solution, valid when $\Phi = O(1)$. For example, we use this procedure to obtain a global asymptotic solution of a one-dimensional Cauchy problem with localized initial data in Section 7. If the waves are spatially periodic, then additional mean-field interactions arise (see Section 8).

We define the local frequency $\omega(\mathbf{x}, t)$ and wavenumber $\mathbf{k}(\mathbf{x}, t)$ by

$$\omega = -\Phi_t, \qquad \mathbf{k} = \nabla\Phi. \tag{32}$$

We assume that $\mathbf{k} \neq 0$. It follows from the expansion that ω, $\mathbf{k}$ satisfy the linearized dispersion relation (23), and we suppose that they satisfy the splay-wave dispersion relation, $\omega^2 = a^2(\mathbf{k}; \mathbf{n}_0)$. The phase Φ then satisfies the linearized eikonal equation

$$\Phi_t^2 - a^2(\nabla\Phi; \mathbf{n}_0) = 0.$$

We restrict our attention to regions of space-time without caustics, and assume that $\Phi(\mathbf{x}, t)$ is smooth and single-valued.

We find that (see Appendix A.2)

$$\mathbf{n}_1(\theta, \mathbf{x}, t) = u(\theta, \mathbf{x}, t)\, \mathbf{R}(\mathbf{x}, t),$$

where $\mathbf{R}(\mathbf{x}, t)$ is defined by (26). The scalar wave-amplitude function $u(\theta, \mathbf{x}, t)$ satisfies the HS-equation

$$(u_t + \mathbf{a} \cdot \nabla u + \Gamma u u_\theta + P u)_\theta = \frac{1}{2}\Gamma u_\theta^2. \tag{33}$$

In this equation, $\mathbf{a}(\mathbf{x}, t)$ is the linearized group velocity vector ($\mathbf{a} = \nabla_{\mathbf{k}}\omega$),

$$\mathbf{a} = \frac{1}{\omega}\left[\alpha\mathbf{k} - (\alpha - \gamma)(\mathbf{k} \cdot \mathbf{n}_0)\,\mathbf{n}_0\right],$$

$\Gamma(\mathbf{x}, t)$ is the genuine-nonlinearity coefficient ($\Gamma = \nabla_{\mathbf{n}_0}\omega \cdot \mathbf{R}$),

$$\Gamma = -\left(\frac{\alpha - \gamma}{\omega}\right)(\mathbf{k} \cdot \mathbf{n}_0)\left[k^2 - (\mathbf{k} \cdot \mathbf{n}_0)^2\right],$$

and $P(\mathbf{x}, t)$ is given by

$$P = \frac{\left(\omega R^2\right)_t + \operatorname{div}\left(\omega R^2 \mathbf{a}\right)}{2\omega R^2},$$

where $R^2 \equiv |\mathbf{R}|^2 = k^2 - (\mathbf{k} \cdot \mathbf{n}_0)^2$. Equation (33) follows from the variational principle

$$\delta \int \frac{1}{2}\omega R^2 \left[u_\theta\left(u_t + \mathbf{a} \cdot \nabla u\right) + \Gamma u u_\theta^2\right] d\theta d\mathbf{x} dt = 0.$$

Introducing a derivative along the rays associated with Φ,

$$\partial_s = \partial_t + \mathbf{a} \cdot \nabla,$$

we may write (33) as an evolution equation for u along a ray,

$$(u_s + \Gamma u u_\theta + P u)_\theta = \frac{1}{2}\Gamma u_\theta^2. \tag{34}$$

If $J(\mathbf{x}, t)$ is a non-zero ray-density function such that $J_t + \operatorname{div}(J\mathbf{a}) = 0$, then we have

$$P = \frac{\left(\omega R^2\right)_s}{2\omega R^2} - \frac{J_s}{2J},$$

and the change of variables

$$\sqrt{\frac{\omega R^2}{J}}\, u \mapsto u, \qquad \partial_\theta \mapsto \partial_x, \qquad \sqrt{\frac{\omega R^2}{J}}\frac{1}{\Gamma}\,\partial_s \mapsto \partial_t$$

reduces (34) to the Hunter-Saxton equation (7).

4.2. **Twist waves.** We consider the propagation of weakly nonlinear twist waves with non-zero wavenumber vector $\mathbf{k}$ through an unperturbed director field $\mathbf{n}_0$. We assume that $\alpha \neq \beta$ and $\mathbf{k}$ is not parallel to $\mathbf{n}_0$. These conditions ensure the strict hyperbolicity of the system.

As a result of their linear degeneracy, there are no quadratically nonlinear effects on small amplitude twist waves, and the use of an expansion of the form (30)-(31) for twist waves leads to linear equations. There are, however, cubically nolinear effects, and to retain these we modify the ansatz to allow for perturbations with higher magnitude, of the order $\varepsilon^{1/2}$. We therefore look for an asymptotic solution of (3) of the form

$$\mathbf{n}^{\varepsilon}(\mathbf{x},t) = \mathbf{n}\left(\frac{\Phi(\mathbf{x},t)}{\varepsilon}, t; \varepsilon\right), \tag{35}$$

$$\mathbf{n}(\theta,\mathbf{x},t;\varepsilon) = \mathbf{n}_0(\mathbf{x},t) + \varepsilon^{1/2}\mathbf{n}_1(\theta,\mathbf{x},t) + \varepsilon\mathbf{n}_2(\theta,\mathbf{x},t) + O(\varepsilon^{3/2}) \tag{36}$$

as $\varepsilon \to 0^+$, where $\mathbf{n}_0$ is a solution of (3).

We find that (see Appendix A.3) the local frequency and wavenumber (32) satisfy the linearized dispersion relation (23), and we suppose that $\omega^2 = b^2(\mathbf{k};\mathbf{n}_0)$. The phase Φ satisfies the eikonal equation

$$\Phi_t^2 - b^2(\nabla\Phi;\mathbf{n}_0) = 0.$$

Moreover, we get

$$\mathbf{n}_1 = v\mathbf{S}, \tag{37}$$

$$\mathbf{n}_2 = -\left(\frac{\beta-\gamma}{\alpha-\beta}\right)(\mathbf{k}\cdot\mathbf{n}_0)\, u\mathbf{R} + \frac{1}{2}v^2(\mathbf{k}\times\mathbf{S}), \tag{38}$$

where $\mathbf{R}(\mathbf{x},t)$, $\mathbf{S}(x,t)$ are defined in (26), (27) and the scalar amplitude-functions $u(\theta,\mathbf{x},t)$, $v(\theta,\mathbf{x},t)$ satisfy

$$(v_t + \mathbf{b}\cdot\nabla v + \Lambda u v_\theta + Qv)_\theta = 0, \tag{39}$$

$$u_{\theta\theta} = v_\theta^2. \tag{40}$$

Here, $\mathbf{b}(\mathbf{x},t)$ is the group velocity vector,

$$\mathbf{b} = \frac{1}{\omega}\left[\beta\mathbf{k} - (\beta-\gamma)(\mathbf{k}\cdot\mathbf{n}_0)\,\mathbf{n}_0\right],$$

$\Lambda(\mathbf{x},t)$ is given by

$$\Lambda = \frac{(\beta-\gamma)^2}{\omega(\alpha-\beta)}(\mathbf{k}\cdot\mathbf{n}_0)^2\left[k^2 - (\mathbf{k}\cdot\mathbf{n}_0)^2\right],$$

and $Q(\mathbf{x},t)$ is given by

$$Q = \frac{\left(\omega S^2\right)_t + \operatorname{div}\left(\omega S^2\mathbf{b}\right)}{2\omega S^2},$$

where $S^2 \equiv |\mathbf{S}|^2 = k^2 - (\mathbf{k}\cdot\mathbf{n}_0)^2$. (With the normalization we adopt for $\mathbf{R}$ and $\mathbf{S}$, we have $R^2 = S^2$.) Equations (39)–(40) follow from the variational principle

$$\delta\int \frac{1}{2}\omega S^2\left[v_\theta\left(v_t + \mathbf{b}\cdot\nabla v\right) + \Lambda\left(uv_\theta^2 + \frac{1}{2}u_\theta^2\right)\right] d\theta d\mathbf{x}dt = 0.$$

The solution (35)–(38) consists of a leading-order twist wave with amplitude v and a higher-order forced splay wave with amplitude u that propagates at the twist-wave velocity. The amplitude and frequency of the forced splay wave are $O(\varepsilon)$ and $O(1/\varepsilon)$, respectively, which is the same scaling as in the weakly nonlinear solution for a free splay wave given in Section 4.1. The term proportional to v^2 in $\mathbf{n}_2$ ensures that $\mathbf{n}$ is a unit vector up to the first order in ε.

The coefficient Λ of the nonlinear term in (39) is non-zero if $\beta \neq \gamma$ and $\mathbf{k}$ is not parallel or orthogonal to $\mathbf{n}_0$. If $\mathbf{k}$, $\mathbf{n}_0$ are constant and $\mathbf{k}$ is orthogonal to $\mathbf{n}_0$, then there are exact large-amplitude traveling twist-wave solutions [11, 12, 24] and no weakly nonlinear effects arise. If $\mathbf{k}$ is parallel to $\mathbf{n}_0$, then there is a loss of strict hyperbolicity and one obtains a system of asymptotic equations instead of a scalar equation. We consider this case in Section 4.3.

Introducing a derivative along the rays associated with Φ,

$$\partial_s = \partial_t + \mathbf{b} \cdot \nabla,$$

we may write (39)–(40) as

$$(v_s + \Lambda u v_\theta + Qv)_\theta = 0, \qquad u_{\theta\theta} = v_\theta^2. \tag{41}$$

If $K(\mathbf{x}, t)$ is a non-zero ray-density function such that $K_t + \operatorname{div}(K\mathbf{b}) = 0$, then we have

$$Q = \frac{\left(\omega S^2\right)_s}{2\omega S^2} - \frac{K_s}{2K},$$

and the change of variables

$$\frac{\omega S^2}{K} u \mapsto u, \quad \sqrt{\frac{\omega S^2}{K}}\, v \mapsto v, \qquad \partial_\theta \mapsto \partial_x, \qquad \frac{\omega S^2}{K\Lambda} \partial_s \mapsto \partial_t$$

reduces (41) to (14).

Equation (14) differs from another cubically-nonlinear, scale-invariant modification of the HS-equation [17, 4],

$$\left(u_t + u^2 u_x\right)_x = u u_x^2, \tag{42}$$

given by the variational principle

$$\delta \int \frac{1}{2} \left\{u_x u_t + u^2 u_x^2\right\} dx dt = 0.$$

Unlike (14), equation (42) is an asymptotic limit of the scalar wave equation (6). It arises when there is a loss of genuine nonlinearity at the unperturbed state φ_0, meaning that $a'(\varphi_0) = 0$ but $a''(\varphi_0) \neq 0$. Next, we derive a generalization of (42) for non-planar deformations, given in (46)–(47), which describes the propagation of polarized orientation waves in the same direction as the unperturbed director field.

4.3. **Polarized waves.** The equations of motion (3) are invariant under spatial rotations and reflections that leave $\mathbf{n}$ fixed. As a consequence of this invariance, there is a loss of strict hyperbolicity and genuine nonlinearity for waves that propagate in the same direction as $\mathbf{n}$. The resulting polarized orientation waves are described by a cubically nonlinear, rotationally invariant asymptotic equation, which we derive in this section. An analogous phenomenon occurs for rotationally invariant waves in first-order hyperbolic systems of conservation laws [6].

To analyze such waves, we suppose that the unperturbed director field $\mathbf{n}_0$ is constant, and the wavenumber vector $\mathbf{k} = k\mathbf{n}_0$ is constant and parallel to $\mathbf{n}_0$. We look for an asymptotic solution $\mathbf{n}^\varepsilon$ of (3) as $\varepsilon \to 0^+$ of a similar form to the one used for twist waves,

$$\mathbf{n}^\varepsilon = \mathbf{n}\left(\frac{\mathbf{k} \cdot \mathbf{x} - \omega t}{\varepsilon}, \mathbf{x}, t; \varepsilon\right),$$

$$\mathbf{n}(\theta, \mathbf{x}, t; \varepsilon) = \mathbf{n}_0 + \varepsilon^{1/2}\mathbf{n}_1(\theta, \mathbf{x}, t) + \varepsilon \mathbf{n}_2(\theta, \mathbf{x}, t) + O(\varepsilon^{3/2}).$$

We find that $\omega^2 = \gamma k^2$ (see Appendix A.4), and

$$\mathbf{n}_1 = \mathbf{u} \qquad \text{where } \mathbf{n}_0 \cdot \mathbf{u} = 0,$$
$$\mathbf{n}_2 = -\frac{1}{2}\left(\mathbf{u}\cdot\mathbf{u}\right)\mathbf{n}_0.$$

The leading-order perturbation $\mathbf{u}(\theta, \mathbf{x}, t)$ satisfies the equation

$$\begin{aligned}\mathbf{u}_{\theta t} + \frac{\omega}{k}\mathbf{n}_0\cdot\nabla\mathbf{u}_\theta + \frac{(\alpha-\beta)k^2}{2\omega}\left(\mathbf{u}\cdot\mathbf{u}_\theta\right)_\theta\mathbf{u}\\ + \frac{(\beta-\gamma)k^2}{2\omega}\Big\{\left[\left(\mathbf{u}\cdot\mathbf{u}\right)\mathbf{u}_\theta\right]_\theta - \left(\mathbf{u}_\theta\cdot\mathbf{u}_\theta\right)\mathbf{u}\Big\} = 0,\end{aligned} \tag{43}$$

which is derived from the variational principle

$$\begin{aligned}\delta\int\frac{1}{2}\Big\{\mathbf{u}_\theta\cdot\Big(\mathbf{u}_t + \frac{\omega}{k}\mathbf{n}_0\cdot\nabla\mathbf{u}\Big) + \frac{(\alpha-\beta)k^2}{2\omega}\left(\mathbf{u}\cdot\mathbf{u}_\theta\right)^2\\ + \frac{(\beta-\gamma)k^2}{2\omega}(\mathbf{u}\cdot\mathbf{u})\left(\mathbf{u}_\theta\cdot\mathbf{u}_\theta\right)\Big\}\,d\theta d\mathbf{x}dt = 0.\end{aligned}$$

Making the change of variables

$$\partial_t + \frac{\omega}{k}\mathbf{n}_0\cdot\nabla \mapsto \partial_t, \qquad \partial_\theta \mapsto \partial_x,$$

and rescaling $\mathbf{u}$, we can write (43) in a normalized form for $\mathbf{u}(x,t)\in\mathbb{R}^2$ as

$$\mathbf{u}_{xt} + (\mu-\nu)\left(\mathbf{u}\cdot\mathbf{u}_x\right)_x\mathbf{u} + \nu\left[\left(\mathbf{u}\cdot\mathbf{u}\right)\mathbf{u}_x\right]_x - \nu\left(\mathbf{u}_x\cdot\mathbf{u}_x\right)\mathbf{u} = 0, \tag{44}$$

where

$$\mu = \frac{\alpha-\gamma}{\gamma}, \qquad \nu = \frac{\beta-\gamma}{\gamma}.$$

The corresponding variational principle is

$$\delta\int\frac{1}{2}\Big\{\mathbf{u}_x\cdot\mathbf{u}_t + (\mu-\nu)\left(\mathbf{u}\cdot\mathbf{u}_x\right)^2 + \nu(\mathbf{u}\cdot\mathbf{u})\left(\mathbf{u}_x\cdot\mathbf{u}_x\right)\Big\}\,dxdt = 0.$$

Writing $\mathbf{u} = (u\cos v, u\sin v)$, we find that this variational principle becomes

$$\delta\int\frac{1}{2}\left\{u_x u_t + \mu u^2 u_x^2 + u^2\left(v_x v_t + \nu u^2 v_x^2\right)\right\}\,dxdt = 0. \tag{45}$$

This result is consistent with what we obtain by expanding the one-dimensional variational principle (10)–(11) as $\varphi\to 0$, when

$$a^2 \sim a_0^2 + (\alpha-\gamma)\varphi^2, \quad b^2 \sim a_0^2 + (\beta-\gamma)\varphi^2, \quad q^2 \sim \varphi^2,$$

and making a unidirectional approximation $\partial_t \sim -a_0\partial_x$ in the resulting Lagrangian.

The Euler-Lagrange equations for (45) are

$$\left(u_t + \mu u^2 u_x\right)_x - \mu u u_x^2 - u v_x\left(v_t + 2\nu u^2 v_x\right) = 0, \tag{46}$$

$$\left(v_t + \nu u^2 v_x\right)_x + 2\nu u u_x v_x + \frac{1}{u}\left(u_x v_t + u_t v_x\right) = 0. \tag{47}$$

This system is a coupled pair of wave equations for u and v. The radial mode u has velocity μu^2, so it is genuinely nonlinear when $u\neq 0$, while the angular mode v has velocity νu^2, so it is linearly degenerate. If v is constant, corresponding to a plane-polarized wave, we recover the scalar cubic equation (42) for u.

5. **Integrability and Hamiltonian structure.** In this section, we show that the twist-wave equation (14) is a completely integrable, bi-Hamiltonian PDE, and that if (u, v) satisfies (14), then u satisfies the derivative HS-equation (16). This relation may be not completely unexpected, since the amplitude of a free splay-wave satisfies the HS-equation, and u represents the amplitude of a forced splay wave in the asymptotic solution (30)-(31) for twist waves, as can be seen from (38). Nevertheless, it is striking that two independent asymptotic expansions lead, in different ways, to two distinct forms of the same asymptotic equation.

We begin by describing the relation between (14) and the derivative HS-equation (16). In order to obtain the corresponding relation for the Camassa-Holm (CH) equation at the same time, we consider the following generalization of (14):

$$(v_t + uv_x)_x = 0, \tag{48}$$

$$Mu = v_x^2, \tag{49}$$

where M is a self-adjoint linear operator acting on functions of x that commutes with ∂_x. If $M = \partial_x^2$, then (48)–(49) is (14).

We suppose that u, v are smooth solutions of (48)–(49). Differentiating (49) with respect to t, using (48) to write v_{xt} in terms of u, v and their spatial derivatives, then using (49) to eliminate v from the result, we find that u satisfies

$$m_t + mu_x + (mu)_x = 0 \qquad \text{with } m = Mu. \tag{50}$$

If $M = \partial_x^2$, then (50) is the HS-equation (16); if $M = \partial_x^2 - 1$, then (50) is the CH-equation [7],

$$\left[(u_t + uu_x)_x - \frac{1}{2}u_x^2\right]_x = u_t + 3uu_x.$$

Conversely, if u is a smooth solution of (50) with $m > 0$, and we define $v_x = \sqrt{m}$, then (u, v) satisfies (48)–(49).

Because of the nonlinearity of this transformation, difficulties arise in its application to distributional solutions. For example, the function

$$u(x,t) = \begin{cases} 0 & \text{if } x \leq 0 \\ 2x/t & \text{if } x > 0 \end{cases}$$

is a weak solution of the HS-equation (16) in $t > 0$, and

$$u_{xx}(x,t) = \frac{2}{t}\delta(x)$$

is non-negative in the sense of distributions. There is, however, no standard way to define a distribution v such that $u_{xx} = v_x^2$.

An interesting further generalization of (14) is given by [21]

$$(v_t + uv_x)_x = 0, \qquad Mu = v_x^p, \tag{51}$$

where p is some exponent (the previous equations correspond to $p = 2$). Then, as in the above discussion, we find that u satisfies

$$m_t + pmu_x + mu_x = 0 \qquad \text{with } m = Mu. \tag{52}$$

Conversely, if u is a smooth solution of (52) with $m > 0$, and $v_x = m^{1/p}$, then (u, v) satisfies (51). If $p = 3$ and $M = \partial_x^2$, then (52) is the second x-derivative of the inviscid Burgers equation, while if $p = 3$ and $M = \partial_x^2 - 1$, then (52) is the Degasperis-Procesi equation [10],

$$(u_t + uu_x)_{xx} = u_t + 4uu_x.$$

In the following we will only consider the case $p = 2$.

The HS-equation (16) is bi-Hamiltonian and completely integrable [18], so (14) is also. Next, we describe these Hamiltonian structures.

The system (48)–(49) is obtained from the variational principle

$$\delta \int \frac{1}{2}\left\{-v_t v_x - u v_x^2 + \frac{1}{2} u M u\right\} dxdt = 0.$$

Variations with respect to u yield (49), and variations with respect to v yield (48). We may eliminate u by means of the constraint equation $u = M^{-1}\left(v_x^2\right)$ to obtain a variational principle for v alone,

$$\delta \int \frac{1}{2}\left\{-v_t v_x - \frac{1}{2} v_x^2 M^{-1}\left(v_x^2\right)\right\} dxdt = 0. \tag{53}$$

The Euler-Lagrange equation for (53),

$$\left[v_t + M^{-1}\left(v_x^2\right) v_x\right]_x = 0, \tag{54}$$

is equivalent to (48)–(49). Here, and below, we assume that operators such as M and ∂_x are invertible; this requires the addition of suitable boundary conditions which we do not specify explicitly.

Making a Legendre transform of the Lagrangian in (53), we get the corresponding Hamiltonian form of (54),

$$v_t = \partial_x^{-1}\left(\frac{\delta \mathcal{H}}{\delta v}\right), \qquad \mathcal{H} = \frac{1}{4}\int v_x^2 M^{-1}\left(v_x^2\right)\, dx, \tag{55}$$

where ∂_x^{-1} is the constant Hamiltonian operator associated canonically with the variational principle (53).

Proposition 1. *Let $\{\mathcal{F}, \mathcal{G}\}$ denote the Poisson bracket of functionals $\mathcal{F}$, $\mathcal{G}$ of v associated with the constant Hamiltonian operator ∂_x^{-1},*

$$\{\mathcal{F}, \mathcal{G}\} = \int \frac{\delta \mathcal{F}}{\delta v}\, \partial_x^{-1}\left(\frac{\delta \mathcal{G}}{\delta v}\right) dx. \tag{56}$$

Under the change of variables $u = M^{-1}\left(v_x^2\right)$, where M is a self-adjoint linear operator, the bracket (56) transforms formally into a Lie-Poisson bracket

$$\{\mathcal{F}, \mathcal{G}\} = \int \frac{\delta \mathcal{F}}{\delta u}\, J(u)\left(\frac{\delta \mathcal{G}}{\delta u}\right) dx, \tag{57}$$

$$J = -2M^{-1}\left(m\partial_x + \partial_x m\right) M^{-1}, \tag{58}$$

where $m = Mu$.

Proof. First, we consider the nonlinear change of variables $m = v_x^2$. Variations of v of the form $v^\varepsilon = v + \varepsilon k + \dots$ lead to variations $m^\varepsilon = m + \varepsilon h + \dots$ of m where $h = 2v_x k_x$. For any functional $\mathcal{F}$ of m, with $\mathcal{F}^\varepsilon = \mathcal{F}(m^\varepsilon)$, we have

$$\left.\frac{d}{d\varepsilon}\mathcal{F}^\varepsilon\right|_{\varepsilon=0} = \int \frac{\delta \mathcal{F}}{\delta m}\, h\, dx.$$

Writing h in terms of k and using the skew-adjointness of ∂_x, we compute that

$$\left.\frac{d}{d\varepsilon}\mathcal{F}^\varepsilon\right|_{\varepsilon=0} = -2\int \left(\frac{\delta \mathcal{F}}{\delta m} v_x\right)_x k\, dx.$$

Since

$$\left.\frac{d}{d\varepsilon}\mathcal{F}^{\varepsilon}\right|_{\varepsilon=0} = \int \frac{\delta \mathcal{F}}{\delta v}\, k\, dx,$$

we conclude that

$$\frac{\delta \mathcal{F}}{\delta v} = -2\left(\frac{\delta \mathcal{F}}{\delta m}\sqrt{m}\right)_x .$$

Using this equation in (56) and integrating by parts, we get

$$\begin{aligned}\{\mathcal{F},\mathcal{G}\} &= 4\int \left(\frac{\delta \mathcal{F}}{\delta m}\sqrt{m}\right)_x \frac{\delta \mathcal{G}}{\delta m}\sqrt{m}\, dx\\ &= -2\int \frac{\delta \mathcal{F}}{\delta m}\left(m\partial_x + \partial_x m\right)\frac{\delta \mathcal{G}}{\delta m}\, dx.\end{aligned}$$

Making the linear change of variables $u = M^{-1}m$ in this expression, and using the self-adjointness of M, we get (57)–(58). □

The Hamiltonian form of (50) for u that corresponds to (55) for v is therefore

$$u_t = J\left(\frac{\partial \mathcal{H}}{\partial u}\right), \qquad \mathcal{H} = \frac{1}{4}\int uMu\, dx,$$

where J is given in (58).

To give a second Hamiltonian structure for (48)–(49), we define a skew-adjoint operator K, depending on v, by

$$K = v_x \partial_x^{-1} M^{-1} v_x, \tag{59}$$

where M is a self-adjoint linear operator commuting with ∂_x, as before.

We find that the operator (59) satisfies the Jacobi identity if the quantity

$$\left(gh_x - hg_x\right)Mf_x + \left(hf_x - fh_x\right)Mg_x + \left(fg_x - gf_x\right)Mh_x$$

is an exact x-derivative for arbitrary functions f, g, h. This condition holds for $M = \partial_x^2$, since

$$\left(gh_x - hg_x\right)f_{xxx} = \left[\left(gh_x - hg_x\right)f_{xx}\right]_x + hf_{xx}g_{xx} - gh_{xx}f_{xx},$$

and the terms that are not exact derivatives cancel under a cyclic summation. The condition also holds for $M = \partial_x^2 - 1$. Moreover, in those cases, $\left(c_1\partial_x^{-1} + c_2 K\right)$ satisfies the Jacobi identity for arbitrary real constants c_1, c_2, so that ∂_x^{-1} and K define compatible Hamiltonian structures.

The Hamiltonian form of (54) with respect to K is

$$v_t = K\left(\frac{\delta \mathcal{P}}{\delta v}\right), \qquad \mathcal{P} = \int v_x^2\, dx. \tag{60}$$

If $M = \partial_x^2$, then $K = v_x\partial_x^{-3}v_x$, and (60) is equivalent to (14).

Under the transformation $m = v_x^2$, equation (60) becomes

$$\begin{aligned}&m_t = \widetilde{K}\left(\frac{\delta \mathcal{P}}{\delta m}\right), \qquad \mathcal{P} = \int m\, dx,\\ &\widetilde{K} = -\left(m\partial_x + \partial_x m\right)\left(\partial_x^{-1}M^{-1}\right)\left(m\partial_x + \partial_x m\right),\end{aligned}$$

which gives (50).

When $M = \partial_x^2$, equations (55) and (60) provide a bi-Hamiltonian structure for (14). One can then obtain an infinite sequence of commuting Hamiltonian flows by recursion. We will not write them out explicitly here, but we remark that among them is a Hamiltonian structure for v which maps to the Hamiltonian structure for u canonically associated with the variational principle in (8).

Lax pairs for (14) follow directly by transformation of the Lax pairs for the HS-equation [18]. We define

$$L = \partial_x^{-1} v_x^2 \partial_x^{-1}, \qquad A = \frac{1}{2}\left(u\partial_x + \partial_x u\right).$$

Then, using the identity $f\partial_x^{-1} - \partial_x^{-1} f = \partial_x^{-1} f_x \partial_x^{-1}$, we compute that the Lax equation $L_t = [L, A]$ is equivalent to

$$\left(v_x^2\right)_t + \left(uv_x^2 + \frac{1}{2}u_x^2\right)_x = 0, \qquad u_{xx} = v_x^2,$$

which may be rewritten as (14). Alternatively, we can set $u = \partial_x^{-2}\left(v_x^2\right)$ in the original Lax pair.

6. **Method of characteristics.** In this section, we solve (14) by the method of characteristics. The explicit nature of the solution is not surprising in view of the complete integrability of the equation discussed in the previous section.

Proposition 2. *Let (ξ, τ) be characteristic coordinates for the PDE (14), where $x = X(\xi, \tau)$, $t = \tau$, and write $U(\xi, \tau) = u\left(X(\xi, \tau), \tau\right)$, $V(\xi, \tau) = v\left(X(\xi, \tau), \tau\right)$. Then a formal solution of (14) is given by*

$$U = X_\tau, \qquad V = F + G, \qquad X = -\int_0^\xi \frac{F_\xi^2 (A+B)^2}{2A_\xi B_\tau}\, d\xi + H, \tag{61}$$

where $A(\xi)$, $B(\tau)$, $F(\xi)$, $G(\tau)$, $H(\tau)$ are arbitrary functions.

Proof. Writing (14) in terms of characteristic coordinates (ξ, τ) in which $\tau = t$ and $x_\tau = u$, we find that the PDE becomes

$$X_\tau = U, \qquad V_{\xi\tau} = 0, \qquad U_{\xi\xi} - \frac{J_\xi}{J} U_\xi = V_\xi^2,$$

where $J(\xi, \tau)$ is the Jacobian $J = X_\xi$. It follows that $V(\xi, \tau) = F(\xi) + G(\tau)$, where F, G are functions of integration, and

$$J_\tau = U_\xi, \qquad U_{\xi\xi} - \frac{J_\xi}{J} U_\xi = F_\xi^2.$$

The elimination of U from these equations yields a PDE for J,

$$J_{\xi\tau} - \frac{J_\xi J_\tau}{J} = F_\xi^2.$$

Making the change of variables $\eta = \eta(\xi)$ where $\eta_\xi = F_\xi^2$, and $J = -e^{-K}$, we find that this PDE transforms into an integrable Liouville equation,

$$K_{\eta\tau} = e^K.$$

The general solution is

$$e^K = \frac{2A_\eta B_\tau}{(A+B)^2},$$

where $A(\eta)$ and $B(\tau)$ are arbitrary functions. Integrating the equation $X_\xi = -e^{-K}$ with respect to ξ , we find that X is given by (61), which proves the result. □

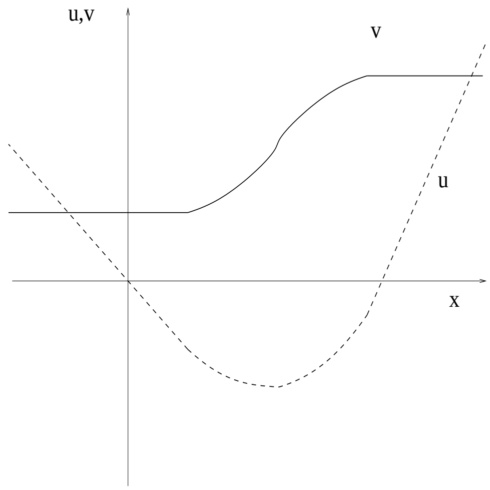

FIGURE 2. Schematic structure of the solution of the IBVP (62)–(65).

Next, we consider an IBVP for (14) in $-\infty < x < \infty$,

$$(v_t + uv_x)_x = 0, \tag{62}$$

$$u_{xx} = v_x^2, \tag{63}$$

with the initial condition

$$v(x,0) = F(x), \tag{64}$$

and the boundary conditions

$$u_x(\infty,t) = \sigma_+(t), \qquad u_x(-\infty,t) = \sigma_-(t). \tag{65}$$

Here, we assume that F is a smooth function such that F_x has compact support. Equation (62) then implies that $v_x(x,t)$ has compact support in x, whenever a smooth solution exists, and (63) implies that $u(x,t)$ is a linear function of x for sufficiently large positive and negative values of x. The boundary conditions (65) specify the corresponding values of u_x. We illustrate the structure of the solution schematically in Figure 2.

In order for this IBVP to have a smooth solution, the initial data $F(x)$ and the boundary data $\sigma_-(t)$, $\sigma_+(t)$ must satisfy certain compatibility conditions, which we derive next.

Proposition 3. *Suppose that u, v are smooth solutions of (62)–(63) such that $v_x(\cdot,t)$ has compact support. Then*

$$\frac{d}{dt}[u_x] + \frac{1}{2}\left[u_x^2\right] = 0, \tag{66}$$

where $[\cdot]$ denotes the jump from $x = -\infty$ to $x = \infty$.

Proof. Since v_x has compact support, it follows from (63) that v is a linear function of x for large negative and positive values of x. Moreover,

$$[u_x] = \int_{-\infty}^{\infty} v_x^2\,dx. \tag{67}$$

Multiplying (62) by v_x and using (63) to rewrite the result, we get

$$\left(v_x^2\right)_t + \left(uv_x^2 + \frac{1}{2}u_x^2\right)_x = 0$$

Integrating this equation with respect to x, and using the fact that v_x has compact support, we find that

$$\frac{d}{dt}\int_{-\infty}^{\infty} v_x^2\,dx + \left[\frac{1}{2}u_x^2\right] = 0.$$

Using (67) to eliminate v_x^2 from this equation, we get (66). □

It follows from this proposition that the data $\sigma_\pm$ must satisfy

$$\frac{d\sigma_+}{dt} + \frac{1}{2}\sigma_+^2 = \frac{d\sigma_-}{dt} + \frac{1}{2}\sigma_-^2. \tag{68}$$

Moreover, integrating (63), evaluated at $t = 0$, with respect to x, and using (64)–(65), we find that

$$\sigma_+(0) - \sigma_-(0) = \int_{-\infty}^{\infty} F_x^2\,dx. \tag{69}$$

The next proposition establishes the existence of smooth solutions of the IBVP (62)–(65) for compatible data. The solutions are not unique, since we may add an arbitrary function of time to v, and an arbitrary function of time to u (together with an appropriate time-dependent translation of the spatial coordinate x). We can remove this non-uniqueness by specifying, for example, u, v as functions of time at some value of x.

Proposition 4. *Suppose that $F : \mathbb{R} \to \mathbb{R}$ is a smooth function and that F_x has compact support. Also suppose that $\sigma+, \sigma_- : [0, t_*) \to \mathbb{R}$ are smooth functions defined in some time interval $0 \le t < t_*$, where $0 < t_* \le \infty$, that satisfy (68), (69). Then there is a smooth solution of (62)–(65), defined in $-\infty < x < \infty$, $0 \le t < t_*$.*

Proof. If $F(x) = F_0$ is constant, then (68)–(69) imply that $\sigma_+(t) = \sigma_-(t)$, and a solution is $u = x\sigma_+(t)$, $v = F_0$. We therefore assume that F is not constant.

We then have, from (69), that $\sigma_-(0) < \sigma_+(0)$. Equation (68) implies that

$$\frac{d}{dt}\left(\sigma_+ - \sigma_-\right) + \frac{1}{2}\left(\sigma_+ + \sigma_-\right)\left(\sigma_+ - \sigma_-\right) = 0,$$

so $\sigma_-(t) < \sigma_+(t)$ for all $0 \le t < t_*$.

We choose constants $\eta_- < \eta_+ < 0$ such that

$$\eta_+ - \eta_- = \int_{-\infty}^{\infty} F_\xi^2(\xi)\,d\xi,$$

and define the function $\eta(\xi)$ by

$$\begin{aligned}\eta(\xi) &= \eta_- + \int_{-\infty}^{\xi} F_{\xi'}^2(\xi')\,d\xi' \\ &= \eta_+ - \int_{\xi}^{\infty} F_{\xi'}^2(\xi')\,d\xi'.\end{aligned}$$

We also define a Jacobian $J(\xi, \tau)$ by

$$J = \frac{\left[\left(\eta_+ - \eta\right)E_+ + \left(\eta - \eta_-\right)E_-\right]^2}{\left(\eta_+ - \eta_-\right)\left(\sigma_+ - \sigma_-\right)}, \tag{70}$$

where

$$E_+(\tau) = \exp\left\{-\frac{1}{4}\int_0^\tau [\sigma_+(\tau') - \sigma_-(\tau')]\, d\tau'\right\}, \tag{71}$$

$$E_-(\tau) = \exp\left\{+\frac{1}{4}\int_0^\tau [\sigma_+(\tau') - \sigma_-(\tau')]\, d\tau'\right\}. \tag{72}$$

One can verify that $J = X_\xi$ is obtained from (61) with

$$A(\xi) = \frac{1}{\eta(\xi)}, \quad B(\tau) = -\left[\frac{E_+(\tau) - E_-(\tau)}{\eta_+ E_+(\tau) - \eta_- E_-(\tau)}\right].$$

We then let

$$X(\xi,\tau) = \int_0^\xi J(\xi',\tau)\, d\xi',$$
$$U(\xi,\tau) = X_\tau(\xi,\tau), \quad V(\xi) = F(\xi).$$

Since $E_-, E_+ > 0$, $\eta_- < \eta_+$, and $\eta_- \le \eta \le \eta_+$, we see from (70) that $J > 0$. It follows that the transformation $x = X(\xi,\tau)$, $t = \tau$ between spatial and characteristic coordinates is smoothly invertible, and, according to Proposition 2, these expressions define a smooth solution of (62)–(63), as may be verified directly. We show that this solution satisfies the required initial and boundary conditions.

First, at $\tau = 0$, we have $E_+ = E_- = 1$ and $\sigma_+ - \sigma_- = \eta_+ - \eta_-$. It follows from (70) that $J = 1$ at $\tau = 0$, so $x = \xi$, and $v(x,0) = F(x)$.

Second, using the equation

$$u_x = \frac{U_\xi}{X_\xi} = \frac{J_\tau}{J},$$

we compute from (70) that

$$u_x = 2\frac{(\eta_+ - \eta)\frac{dE_+}{d\tau} + (\eta - \eta_-)\frac{dE_-}{d\tau}}{(\eta_+ - \eta)E_+ + (\eta - \eta_-)E_-} - \frac{\frac{d}{d\tau}(\sigma_+ - \sigma_-)}{\sigma_+ - \sigma_-}.$$

From (71)–(72), we have

$$\frac{dE_+}{d\tau} = -\frac{1}{4}(\sigma_+ - \sigma_-)E_+, \quad \frac{dE_-}{d\tau} = \frac{1}{4}(\sigma_+ - \sigma_-)E_-,$$

and from the jump condition (68), we have

$$\frac{d}{d\tau}(\sigma_+ - \sigma_-) = -\frac{1}{2}\left(\sigma_+^2 - \sigma_-^2\right).$$

Using these equations to eliminate τ-derivatives from the expression for u_x and simplifying the result, we get

$$u_x = \frac{(\eta_+ - \eta)\sigma_- E_+ + (\eta - \eta_-)\sigma_+ E_-}{(\eta_+ - \eta)E_+ + (\eta - \eta_-)E_-}.$$

It follows that $u_x = \sigma_-$ at $x = -\infty$, when $\eta = \eta_-$, and $u_x = \sigma_+$ at $x = \infty$, when $\eta = \eta_+$. $\square$

For example, let us consider what happens when the derivative u_x vanishes at $x = -\infty$ or $x = \infty$. If $\sigma_- = 0$, then $\sigma_+ > 0$ satisfies the equation

$$\frac{d\sigma_+}{dt} + \frac{1}{2}\sigma_+^2 = 0,$$

which has a global smooth solution forward in time,

$$\sigma_+(t) = \frac{\sigma_+(0)}{1+\sigma_+(0)t/2}.$$

It follows that (62)–(65) has a global smooth solution forward in time, which may be specified uniquely by the requirement that $u = v = 0$ at $x = -\infty$. This case corresponds to the one that arises for the fast twist waves analyzed in Section 7.4.

On the other hand, if $\sigma_+ = 0$, then $\sigma_- < 0$ satisfies the equation

$$\frac{d\sigma_-}{dt} + \frac{1}{2}\sigma_-^2 = 0,$$

whose solution

$$\sigma_-(t) = \frac{\sigma_-(0)}{1+\sigma_-(0)t/2}$$

blows up as $t \uparrow t_*$, where $t_* = -2/\sigma_-(0) > 0$. Thus, a smooth solution of (62)–(65) exists only in the finite time-interval $0 \le t < t_*$. The derivative u_x blows up simultaneously in the entire semi-infinite spatial interval to the left of the support of v_x, so it does not appear possible to continue the smooth solution by a distributional solution after the singularity forms.

7. **One-dimensional equations.**

In the previous sections, we have shown that the propagation of a twist wave is described by (62)-(63), where v is the amplitude of a twist wave and u is the amplitude of a forced splay wave. In this section we analyze the propagation of a twist wave, and the consequent generation of a splay wave, for deformations of the director field that depend on a single space variable. Rather than use the general equations, we start from a one-dimensional form of (3). This permits a considerable simplification and provides additional insight into the structure of the equations.

The structure of the solution depends on whether the twist waves are faster or slower than the splay waves (see Figure 3). When the twist waves are faster, they move at a constant velocity into an unperturbed director field, and we obtain a Cauchy problem for the splay-wave equation (6) with data for φ, φ_x given on the space-like twist-wave trajectories. When the twist waves are slower, they are embedded inside the splay wave. We then obtain a free-boundary problem for (6). in which the twist-wave trajectories are not known *a priori*, and φ_x satisfies a jump condition across the trajectories that is derived from the asymptotic equation (14).

We consider a director field

$$\mathbf{n}(x,t) = (n_1(x,t), n_2(x,t), n_3(x,t))$$

that depends upon a single space variable $x = x_1$. Writing $\mathbf{n}$ as in (9) and using the result in the variational principle (1), we find that (1) becomes (10)–(11). The associated Euler-Lagrange equation is (12)–(13). This system may also be obtained directly by use of (9) in (3).

7.1. **The initial value problem.**

Hunter and Saxton [17] construct an asymptotic solution $\varphi(x,t;\varepsilon)$ of the initial value problem for the scalar wave equation (6) in $-\infty < x < \infty$ and $t > 0$ with weakly nonlinear splay-wave initial data,

$$\varphi(x,0;\varepsilon) = \varphi_0 + \varepsilon f\left(\frac{x}{\varepsilon}\right), \qquad \varphi_t(x,0;\varepsilon) = g\left(\frac{x}{\varepsilon}\right),$$

where f, g have compact support, and ε is a small parameter. The solution consists of a superposition of right and left moving weakly nonlinear splay waves that originate from $x = 0$, whose width in x is of the order ε. The splay waves are separated by a slowly-varying, small-amplitude perturbation of the constant state φ_0 that satisfies a linearized wave equation.

Here, we construct an asymptotic solution for $\varphi(x, t; \varepsilon)$, $\psi(x, t; \varepsilon)$ of the system (12)–(13) with weakly nonlinear twist-wave initial data,

$$\begin{aligned} &\varphi(x, 0; \varepsilon) = \varphi_0, && \varphi_t(x, 0; \varepsilon) = 0, \\ &\psi(x, 0; \varepsilon) = \varepsilon^{1/2} f\left(\frac{x}{\varepsilon}\right), && \psi_t(x, 0; \varepsilon) = \varepsilon^{-1/2} g\left(\frac{x}{\varepsilon}\right), \end{aligned} \tag{73}$$

where f and g are smooth, compactly supported functions. A similar analysis applies to general initial data, but this complicates the solution without displaying any essentially new phenomena.

In (73), we suppose, without loss of generality, that the unperturbed constant value of ψ is equal to zero. The amplitude of the twist wave (which is described by ψ) is of the order $\varepsilon^{1/2}$, consistent with the asymptotic expansion described in section 4.2, and the twist wave will generate a splay wave of order ε.

We use a 0-subscript on a function of φ to denote evaluation at $\varphi = \varphi_0$. We assume that $a_0 \neq b_0$, so that (12)–(13) is strictly hyperbolic at φ_0. The system will then remain strictly hyperbolic, at least in some time-interval of the order one. We further assume that $b_0' \neq 0$, otherwise the leading-order nonlinear effects studied below vanish. In the case of the director-field wave speeds (11), these assumptions mean that $\varphi_0 \neq n\pi/2$ for $n \in \mathbb{Z}$.

Although φ is initially constant, it will not remain so. As we show, the weakly nonlinear twist wave, whose initial energy

$$\frac{1}{2}\int_{-\infty}^{\infty} \left\{\psi_t^2 + b^2(\varphi)\psi_x^2\right\} \, dx = \frac{1}{2}\int_{-\infty}^{\infty} \left\{g^2(\theta) + b_0^2 f_\theta^2(\theta)\right\} \, d\theta$$

is of the order one, generates a slowly-varying 'outer' splay-wave whose amplitude is of the order one.

We will construct an asymptotic solution of this initial value problem as follows.

1. In a short initial layer, when $t = O(\varepsilon)$, we use linearized theory. The initial data splits up into right and left moving twist waves.
2. For $t = O(1)$, nonlinear effects become important, and we use the method of matched asymptotic expansions, with different expansions for the twist and splay waves.
   (a) The twist waves are small-amplitude, localized waves, which vary on a spatial scale of the order ε. We describe them by means of the weakly nonlinear asymptotic equations for twist waves derived above. We call these the 'inner' solutions.
   (b) Away from the twist waves, the leading-order solution is a large-amplitude splay wave, which varies on a spatial scale of the order 1. We call this the 'outer' solution.
   (c) We obtain jump conditions for the 'outer' splay-wave solution across the trajectories of the right and left moving twist waves by matching it with the 'inner' twist-wave solutions.
   (d) We obtain initial data for the nonlinear solution by matching it with the linearized solution as $t \to 0^+$.

The matching between the inner twist waves and the outer splay wave, and the structure of the nonlinear solution, depend on whether the twist waves are faster or slower than the splay waves (see Figure 3). In deriving the jump conditions for the

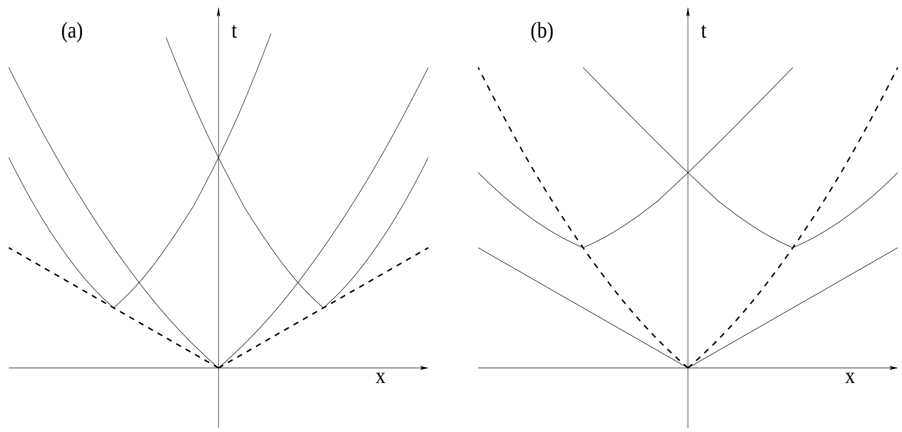

FIGURE 3. Characteristic structure for the solution of the initial value problem (12)–(73): (a) Fast twist waves; (b) Slow twist waves. The dashed lines are the trajectories of the twist waves, and the solid lines are the characteristic curves associated with the splay wave.

splay wave below, we will consider these cases separately. For liquid crystals, we have $\beta < \alpha$, meaning that twist deformations are not as 'stiff' as splay deformations, and the twist waves are slower.

7.2. **Initial layer.** First, we consider the solution of (12)–(13), (73) in a short initial layer when $t = O(\varepsilon)$. Because of the finite propagation speed of the system, the solution is constant, with $\varphi = \varphi_0$, $\psi = 0$, outside an interval of width of the order ε containing $x = 0$.

Near $x = 0$, we look for an asymptotic solution of the form

$$\varphi = \Phi_0(X,T) + O(\varepsilon), \quad \psi = \varepsilon^{1/2}\Phi_1(X,T) + O(\varepsilon), \quad X = \frac{x}{\varepsilon}, \quad T = \frac{t}{\varepsilon}.$$

Using this expansion in (12)–(73), we find that Φ_0, Ψ_1 satisfy

$$\begin{aligned}
&\Phi_{0TT} - \left(a^2(\Phi_0)\Phi_{0X}\right)_X + a(\Phi_0)a'(\Phi_0)\Phi_{0X}^2 = 0,\\
&\Psi_{1TT} - \left(b^2(\Phi_0)\Psi_{1X}\right)_X + \frac{2q'(\Phi_0)}{q(\Phi_0)}\left(\Phi_{0T}\Psi_{1T} - b^2(\Phi_0)\Phi_{0X}\Psi_{1X}\right) = 0,\\
&\Phi_0(X,0) = \varphi_0, \qquad \Phi_{0T}(X,0) = 0,\\
&\Psi_1(X,0) = f(X), \qquad \Psi_{1T}(X,0) = g(X).
\end{aligned}$$

The initial value problem for Φ_0 has the constant solution $\Phi_0 = \varphi_0$, and therefore Ψ_1 satisfies the linear wave equation

$$\begin{aligned}
&\Psi_{1TT} - b_0^2\Psi_{1XX} = 0,\\
&\Psi_1(X,0) = f(X), \quad \Psi_{1T}(X,0) = g(X).
\end{aligned}$$

The solution is

$$\Psi_1(X,T) = F_R\left(X - b_0T\right) + F_L\left(X + b_0T\right), \tag{74}$$

$$F_R(\theta) = \frac{1}{2}f(\theta) + \frac{1}{2b_0}\int_\theta^\infty g(\xi)\,d\xi, \tag{75}$$

$$F_L(\theta) = \frac{1}{2}f(\theta) - \frac{1}{2b_0}\int_\theta^\infty g(\xi)\,d\xi. \tag{76}$$

Since f, g have compact support, the functions $F_{R\theta}$, $F_{L\theta}$ have compact support.

7.3. **Twist waves.** Next, we consider the propagation of the twist-waves for $t = O(1)$ through a possibly non-uniform splay-wave field $\varphi(x,t)$. For definiteness, we consider the right-moving twist wave, which moves with velocity $b(\varphi)$. The trajectory $x = s_R(t)$ of the wave satisfies

$$\frac{ds_R}{dt} = b_R,$$

where an R-subscript on a function of φ denotes evaluation at $\varphi = \varphi_R(t)$, with $\varphi_R(t) = \varphi\left(s_R(t), t\right)$. For the initial value problem, we have $s_R(0) = 0$ and $\varphi_R(0) = \varphi_0$.

We introduce a stretched inner variable near this trajectory,

$$\theta = \frac{x - s_R(t)}{\varepsilon}.$$

We find that the weakly nonlinear twist-wave solution of (12)–(13) is

$$\varphi = \varphi_R(t) + \varepsilon\varphi_2(\theta,t) + O(\varepsilon^{3/2}), \tag{77}$$

$$\psi = \varepsilon^{1/2}\psi_1(\theta,t) + O(\varepsilon), \tag{78}$$

where φ_2, ψ_1 satisfy

$$\left[\psi_{1t} + b_R'\varphi_2\psi_{1\theta} + \left(\frac{b_{Rt}}{2b_R} + \frac{q_{Rt}}{q_R}\right)\psi_1\right]_\theta = 0, \tag{79}$$

$$\varphi_{2\theta\theta} = \left(\frac{q_R^2 b_R b_R'}{a_R^2 - b_R^2}\right)\psi_{1\theta}^2. \tag{80}$$

We may transform (79)–(80) into (14) by a suitable change of variables,

$$\frac{a_R^2 - b_R^2}{b_R'}\,\varphi_2 \mapsto u, \qquad \sqrt{q_R^2 b_R}\,\psi_1 \mapsto v, \qquad \frac{a_R^2 - b_R^2}{{b_R'}^2}\,\partial_t \mapsto \partial_t, \qquad \partial_\theta \mapsto \partial_x. \tag{81}$$

Matching the twist-wave solution (78) as $t \to 0^+$ with the linearized solution (74) as $T \to \infty$, we get the initial condition

$$\psi_1(\theta, 0) = F_R(\theta), \tag{82}$$

where F_R is given in (75). Equations (79)–(80) are supplemented with suitable boundary conditions for φ_2 and ψ_1 at $\theta = \pm\infty$, which we consider further below.

The main result we need in order to obtain equations for the 'outer' splay wave solution is the following jump condition for $\varphi_{2\theta}$ across the twist wave:

$$\frac{d}{dt}\left\{\left(\frac{a_R^2 - b_R^2}{b_R'}\right)[\varphi_{2\theta}]\right\} + \left(\frac{a_R^2 - b_R^2}{2}\right)[\varphi_{2\theta}^2] = 0, \tag{83}$$

which can be derived from Proposition 3, taking into account the change of variable (81) Moreover, we find

$$[\varphi_{2\theta}] = \left(\frac{q_R^2 b_R b_R'}{a_R^2 - b_R^2}\right) \int_{-\infty}^{\infty} \psi_{1\theta}^2 \, d\theta. \tag{84}$$

For the coefficients in (11), the jump condition (83) becomes

$$\frac{d}{dt}\left\{b_R \tan\varphi_R \left[\varphi_{2\theta}\right]\right\} + \frac{1}{2}(\beta - \gamma)\sin^2\varphi_R \left[\varphi_{2\theta}^2\right] = 0.$$

We have given a complete solution of the twist-wave equations in Section 6.

7.4. **Matching: fast twist waves.** This case corresponds to $0 < \alpha < \beta$, when $0 < a < b$. Since the twist waves are faster than the splay waves, they propagate into a constant state $\varphi = \varphi_0$, $\psi = 0$ ahead of them, and generate splay waves behind them. (See Figure 3(a).) It follows that the right and left moving twist waves move at a constant velocity along the trajectories $x = b_0 t$ and $x = -b_0 t$, respectively.

We consider the right-moving twist wave for definiteness. The appropriate inner variable is then

$$\theta = \frac{x - b_0 t}{\varepsilon}. \tag{85}$$

The weakly nonlinear solution inside the twist wave must match for large positive θ with the constant initial state ahead of the wave. This condition implies that $\varphi_2(\theta, t)$ and $\psi_1(\theta, t)$ in (77)–(78), with $\varphi_R = \varphi_0$, satisfy the boundary conditions

$$\varphi_2(\infty, t) = 0, \qquad \varphi_{2\theta}(\infty, t) = 0, \qquad \psi_1(\infty, t) = 0. \tag{86}$$

The derivative $\varphi_{2\theta}$ jumps from zero at $\theta = \infty$ to a value

$$\varphi_{2\theta}(-\infty, t) = \sigma_R(t)$$

at $\theta = -\infty$. It follows from the jump condition (83), with $a_R = a_0$, $b_R = b_0$, $b_R' = b_0'$ constants, that σ_R satisfies

$$\frac{d\sigma_R}{dt} + \frac{1}{2} b_0' \sigma_R^2 = 0.$$

From (80) and (82), we have $\sigma_R(0) = \sigma_{R0}$ where

$$\sigma_{R0} = -\left(\frac{q_0^2 b_0 b_0'}{a_0^2 - b_0^2}\right) \int_{-\infty}^{\infty} F_{R\theta}^2 \, d\theta.$$

The solution of this Riccati equation,

$$\sigma_R(t) = \frac{\sigma_{R0}}{1 + \sigma_{R0} b_0' t/2}, \tag{87}$$

is defined for all $t \geq 0$, since $\sigma_{R0} b_0' \geq 0$ when $0 < a_0 < b_0$.

In Section 6, we prove that when $a_0 < b_0$, equations (79)–(82), (86), with $a_R = a_0$ and so on, have a smooth solution defined for all $t \geq 0$. We note that the derivative $\varphi_{2\theta}(-\infty, t)$ decays as $t \to \infty$. This is a result of the fact that the twist wave radiates energy away from it in the form of splay waves. It also follows from the solution that the twist wave is a rarefaction, in the sense that its characteristics spread out with increasing time.

A similar analysis applies to the left-moving twist wave, in which

$$\theta = \frac{x + b_0 t}{\varepsilon},$$

and $\varphi_{2\theta}(\theta,t) = 0$ for θ sufficiently large and negative. We find that $\varphi_{2\theta} = \sigma_L$ for θ sufficiently large and positive, where

$$\sigma_L(t) = \frac{\sigma_{L0}}{1 - \sigma_{L0} b_0' t/2}, \tag{88}$$

with

$$\sigma_{L0} = \left(\frac{q_0^2 b_0 b_0'}{a_0^2 - b_0^2}\right) \int_{-\infty}^{\infty} F_{L\theta}^2 \, d\theta.$$

These 'inner' twist-wave solutions provide matching conditions for an 'outer' splay-wave solution $\varphi(x,t)$. Using (77) with $\varphi_R = \varphi_0$ to rewrite the condition for the right-moving twist wave,

$$\varphi_{2\theta}(\theta,t) \sim \sigma_R(t) \qquad \text{as } \theta \to -\infty$$

where θ is given by (85), in terms of the outer solution $\varphi(x,t)$, and equating the outer limit of the inner solution with the inner limit of the outer solution, we find that

$$\varphi_x(x,t) \sim \sigma_R(t) \qquad \text{as } x \to b_0 t^-.$$

Furthermore, the leading-order outer solution for φ is continuous across the twist-wave, and ψ is higher-order in ε. We obtain a condition for φ as $x \to -b_0 t^+$ in an analogous way.

Summarizing these results for the leading-order outer splay-wave solution $\varphi(x,t)$, we find that $\varphi = \varphi_0$ is constant if $x > b_0 t$ or $x < -b_0 t$. Inside the region $-b_0 t < x < b_0 t$, we find that φ satisfies (6), with data on the space-like lines $x = \pm b_0 t$ given by

$$\begin{aligned} \varphi(b_0 t, t) &= \varphi_0, & \varphi_x(b_0 t, t) &= \sigma_R(t), \\ \varphi(-b_0 t, t) &= \varphi_0, & \varphi_x(-b_0 t, t) &= \sigma_L(t), \end{aligned}$$

where σ_R, σ_L are given by (87), (88), respectively.

Since the initial value problem for (6) is well-posed, this Cauchy problem is presumably solvable. The solution φ may form singularities, in which case it would have to be continued by a weak solution.

7.5. **Matching: slow twist waves.** This case corresponds to $\alpha > \beta > 0$, when $a > b > 0$. Since the twist waves are slower than the splay waves, they generate splay waves both in front and behind them. As a result, the twist waves are embedded inside a splay-wave field. (See Figure 3(b).) The speeds of the twist waves depend on the splay-wave field, leading to a free-boundary problem for the trajectories of the twist waves, coupled with a wave equation for the splay wave that is subject to jump conditions across the twist-wave trajectories.

We will not write out detailed asymptotic equations for the weakly nonlinear twist waves in this case, but we summarize the equations satisfied by the leading-order outer splay-wave solution $\varphi(x,t)$. The main point is the derivation of jump conditions for φ_x across the twist-wave trajectories.

The right and left moving twist waves are located at $x = s_R(t)$ and $x = s_L(t)$, respectively, where

$$\frac{ds_R}{dt}(t) = b\left(\varphi\left(s_R(t),t\right)\right), \quad \frac{ds_L}{dt}(t) = -b\left(\varphi\left(s_L(t),t\right)\right), \tag{89}$$

$$s_R(0) = 0, \qquad s_L(0) = 0. \tag{90}$$

The solution φ is continuous across $x = s_R(t)$ and $x = s_L(t)$, so that

$$[\varphi]_R = 0, \qquad [\varphi]_L = 0. \tag{91}$$

Here, and below, we use $[\cdot]_R$, $[\cdot]_L$ to denote the jumps across $x = s_R(t)$, $x = s_L(t)$, respectively, meaning that

$$\begin{aligned} [\varphi]_R(t) &= \lim_{x\to s_R(t)^+} \varphi(x,t) - \lim_{x\to s_R(t)^-} \varphi(x,t), \\ [\varphi]_L(t) &= \lim_{x\to s_L(t)^+} \varphi(x,t) - \lim_{x\to s_L(t)^-} \varphi(x,t). \end{aligned}$$

Considering the right-moving twist wave for definiteness, we have

$$\varphi_{2\theta}(\theta,t) \to \sigma_+(t) \qquad \text{as } \theta \to \infty, \tag{92}$$

$$\varphi_{2\theta}(\theta,t) \to \sigma_-(t) \qquad \text{as } \theta \to -\infty, \tag{93}$$

for some functions $\sigma_+(t)$, $\sigma_-(t)$. The corresponding matching conditions for $\varphi(x,t)$ are

$$\lim_{x\to s_R^+(t)} \varphi_x(x,t) = \sigma_+(t), \qquad \lim_{x\to s_R^-(t)} \varphi_x(x,t) = \sigma_-(t). \tag{94}$$

From (83), (92)–(93), (94), and the analogous equations for the left-moving twist wave, we find that φ_x satisfies the following jump conditions across the twist waves:

$$\frac{d}{dt}\left\{\left(\frac{a_R^2 - b_R^2}{b_R'}\right)[\varphi_x]_R\right\} + \left(\frac{a_R^2 - b_R^2}{2}\right)\left[\varphi_x^2\right]_R = 0, \tag{95}$$

$$\frac{d}{dt}\left\{\left(\frac{a_L^2 - b_L^2}{b_L'}\right)[\varphi_x]_L\right\} - \left(\frac{a_L^2 - b_L^2}{2}\right)\left[\varphi_x^2\right]_L = 0. \tag{96}$$

Furthermore, from (82) and (84), and their analogs for left-moving waves, we get the initial conditions

$$[\varphi_x]_R(0) = \left(\frac{q_0^2 b_0 b_0'}{a_0^2 - b_0^2}\right)\int_{-\infty}^{\infty} F_{R\theta}^2\, d\theta, \tag{97}$$

$$[\varphi_x]_L(0) = \left(\frac{q_0^2 b_0 b_0'}{a_0^2 - b_0^2}\right)\int_{-\infty}^{\infty} F_{L\theta}^2\, d\theta. \tag{98}$$

Finally, matching the outer solution with the initial, linearized solution, we find that $\varphi(x,t)$ satisfies the initial conditions

$$\varphi(x,0) = \varphi_0, \qquad \varphi_t(x,0) = 0, \qquad \text{for } x \neq 0. \tag{99}$$

Summarizing, we find that the free-boundary problem for φ, s_R, s_L consists of (6) for $\varphi(x,t)$ in $-\infty < x < \infty$, $t > 0$ with the initial condition (99). The functions $s_R(t)$, $s_L(t)$ satisfy (89)–(90), and $\varphi(x,t)$ satisfies the jump conditions (91), (95)–(98) across the curves $x = s_R(t)$, $x = s_L(t)$.

We will not investigate this problem here. We remark, however, that Proposition 4 in Section 6 implies that there is a smooth solution of the 'inner' asymptotic equations for the weakly nonlinear twist wave (79)–(82), (92)–(93) whenever the derivatives $\sigma_\pm(t) = \varphi_x\left(s_R^\pm(t), t\right)$ of the 'outer' splay-wave solution of the free-boundary problem on either side of the twist wave are smooth functions of time.

8. **Periodic twist waves.** Spatially periodic splay waves satisfy the following version of the HS-equation [15, 16]

$$(u_t + uu_x)_x = \frac{1}{2}\left(u_x^2 - \langle u_x^2\rangle\right). \tag{100}$$

Here, $u(x,t)$ is a periodic function of x, and angular brackets denote an average over a period. If the period is normalized to 2π, then, for a function $f(x,t)$, we have

$$\langle f\rangle(t) = \frac{1}{2\pi}\int_0^{2\pi} f(x,t)\, dx.$$

We note that differentiation of (100) with respect to x implies that u satisfies (16). A periodic splay wave drives a mean-field, which evolves on the same time-scale as the nonlinear time-scale of the wave. We do not write out the mean-field equations here.

In this section, we derive asymptotic equations for weakly nonlinear, spatially periodic twist waves. These waves also generate a mean field, but, unlike the case of splay waves, the mean-field evolves on a faster time-scale than the wave. This is a consequence of the fact that the nonlinear self-interaction of a weakly nonlinear twist wave is cubic, whereas the mean-field is driven by quadratic nonlinearities.

In this section, we derive the following generalization of the twist-wave asymptotic equation (14) that applies to periodic waves:

$$(v_t + uv_x)_x + \mu\langle v_x^2\rangle v = 0, \tag{101}$$

$$u_{xx} = v_x^2 - \langle v_x^2\rangle. \tag{102}$$

Here, $u(x,t)$, $v(x,t)$ are periodic functions of x, which we assume to have zero mean without loss generality, and μ is a constant that cannot be removed by rescaling. We will derive (101)–(102) from the one-dimensional wave equations (12)–(13), but a similar derivation would apply to more general systems.

It is interesting to note that the mean-field interaction introduces a dispersive term of Klein-Gordon type into the evolution equation (101) for v, despite the fact that the original system is scale-invariant and non-dispersive. The scale-invariance is preserved by the fact that the coefficient of the dispersive term is proportional to $\langle v_x^2\rangle$, which is constant for smooth solutions.

The mean-terms prevent the elimination of v from the system (101)–(102) by cross-differentiation, as is possible in the case of (14). Instead, one finds that

$$\left[(u_t + uu_x)_x - \frac{1}{2}u_x^2 - \langle v_x^2\rangle\left(2u + \mu v^2\right)\right]_x = 0,$$

which suggests that (101)–(102) may not be completely integrable when $\mu \neq 0$.

8.1. **Derivation of the periodic equation.** We consider spatially-periodic solutions of (12)–(13) with period of the order ε in which ψ has amplitude of the order $\varepsilon^{1/2}$, where ε is a small parameter. The corresponding time-scale for the nonlinear evolution of the ψ-wave is of the order 1. As we will see, the ψ-wave generates a mean φ-field which evolves on a time-scale of the order $\varepsilon^{1/2}$. This mean field modulates the speed of the ψ-wave on the same time-scale.

We therefore introduce multiple-scale variables

$$\begin{aligned}\theta &= \frac{x - \varepsilon^{1/2} s\left(t/\varepsilon^{1/2}\right)}{\varepsilon},\\ \tau &= \frac{t}{\varepsilon^{1/2}},\\ t &= t,\end{aligned} \tag{103}$$

where $s(\tau)$ is a function that we will determine as part of the expansion. We look for an asymptotic solution of (12)–(13) of the form

$$\begin{aligned}\varphi &= \varphi_0\left(\tau\right) + \varepsilon\varphi_2\left(\theta,\tau,t\right) + O(\varepsilon^{3/2}),\\ \psi &= \varepsilon^{1/2}\psi_1\left(\theta,\tau,t\right) + \varepsilon\psi_2\left(\theta,\tau,t\right) + O(\varepsilon^{3/2}),\end{aligned}$$

where all the terms are periodic functions of θ. We will also require below that the terms are periodic functions of τ.

We use this ansatz in (12)–(13), expand derivatives as

$$\partial_t \to -\frac{1}{\varepsilon}s_\tau\partial_\theta + \frac{1}{\varepsilon^{1/2}}\partial_\tau + \partial_t, \quad \partial_x \to \frac{1}{\varepsilon}\partial_\theta,$$

Taylor expand the result with respect to ε, and equate coefficients of powers of $\varepsilon^{1/2}$ to zero.

We consider first the ψ-equation (13). At the order $\varepsilon^{-3/2}$, we obtain that

$$\left(s_\tau^2 - b_0^2\right)\psi_{1\theta\theta} = 0,$$

where the 0-subscript on a function of φ denotes evaluation at $\varphi = \varphi_0$. It follows from this equation that $s_\tau^2 = b_0^2$. For definiteness, we consider a right-moving wave and assume that

$$s_\tau = b_0, \tag{104}$$

where $b_0 > 0$.

At the order ε^{-1}, we obtain that

$$2s_\tau\psi_{1\theta\tau} + \left(s_{\tau\tau} + \frac{2q_{0\tau}}{q_0}s_\tau\right)\psi_{1\theta} = 0. \tag{105}$$

We may choose ψ_1 so that it is a zero-mean periodic function of θ. It follows from (104) and (105) that

$$\psi_1(\theta,\tau,t) = \frac{v(\theta,t)}{q_0(\tau)b_0^{1/2}(\tau)} \tag{106}$$

where $v(\theta,t)$ is a zero-mean periodic function of θ which is independent of τ.

At the order $\varepsilon^{-1/2}$, we obtain that

$$2s_\tau\psi_{2\theta\tau} + \left(s_{\tau\tau} + \frac{2q_{0\tau}}{q_0}s_\tau\right)\psi_{2\theta} + 2s_\tau\psi_{1\theta t} + \left(2b_0b_0'\varphi_2\psi_{1\theta}\right)_\theta - \psi_{1\tau\tau} = 0.$$

Using (104), we may write this equation as

$$\left(q_0b_0^{1/2}\psi_{2\theta}\right)_\tau + q_0b_0^{1/2}\psi_{1\theta t} + \left(q_0b_0^{1/2}b_0'\varphi_2\psi_{1\theta}\right)_\theta - \frac{q_0}{2b_0^{1/2}}\psi_{1\tau\tau} = 0. \tag{107}$$

We will return to (107) after we expand the φ-equation.

The leading-order terms in the expansion of the φ-equation (12) are of the order ε^{-1}, and give

$$\left(s_\tau^2 - a_0^2\right)\varphi_{2\theta\theta} + q_0^2b_0b_0'\psi_{1\theta}^2 + \varphi_{0\tau\tau} = 0.$$

Using (104) and (106) in this equation, we get

$$\left(b_0^2 - a_0^2\right)\varphi_{2\theta\theta} + b_0' v_\theta^2 + \varphi_{0\tau\tau} = 0. \tag{108}$$

Averaging this equation with respect to θ, we find that

$$\varphi_{0\tau\tau} + \langle v_\theta^2 \rangle b_0' = 0, \tag{109}$$

where the angular brackets denote an average with respect to θ. As we will see, for smooth solutions, the quantity $\langle v_\theta^2\rangle$ is a constant independent of t, so (109) is consistent with the ansatz that φ_0 depends only on τ.

Equation (109) provides an ODE for φ_0, corresponding to motion in a potential proportional to the twist-wave speed $b_0 = b(\varphi_0)$. For definiteness, we assume that the solution of (109) for $\varphi_0(\tau)$ is a periodic function of τ. We then require that all other terms in the expansion are periodic functions of τ.

Subtracting (109) from (108), we find that

$$\left(b_0^2 - a_0^2\right)\varphi_{2\theta\theta} + b_0'\left(v_\theta^2 - \langle v_\theta^2\rangle\right) = 0.$$

Hence, we may write

$$\varphi_2(\theta,\tau,t) = \left[\frac{b_0'(\tau)}{a_0^2(\tau) - b_0^2(\tau)}\right] u(\theta,t), \tag{110}$$

where u satisfies

$$u_{\theta\theta} = v_\theta^2 - \langle v_\theta^2\rangle.$$

Using (106) and (110) in (107), we get

$$\left(q_0 b_0^{1/2}\psi_{2\theta}\right)_\tau + v_{\theta t} + \left[\frac{(b_0')^2}{a_0^2 - b_0^2}\right](u v_\theta)_\theta - \frac{q_0}{2 b_0^{1/2}}\left(\frac{1}{q_0 b_0^{1/2}}\right)_{\tau\tau} v = 0.$$

Averaging this equation with respect to τ, we get

$$\left(v_t + \Lambda u v_\theta\right)_\theta + N v = 0,$$

where

$$\begin{aligned}
\Lambda &= \oint \frac{(b_0')^2}{a_0^2 - b_0^2}\, d\tau,\\
N &= -\frac{1}{2}\oint \frac{q_0}{b_0^{1/2}}\left(\frac{1}{q_0 b_0^{1/2}}\right)_{\tau\tau} d\tau.
\end{aligned}$$

Here, $\oint$ denotes an average over a period in τ.

To make the dependence of φ_0 on v explicit, we introduce a new time variable

$$T = \langle v_\theta^2\rangle^{1/2}\tau.$$

We may then rewrite (109) as

$$\varphi_{0TT} + b_0' = 0. \tag{111}$$

Given a solution of this equation for $\varphi_0(T)$, we have $N = \langle v_\theta^2\rangle M$, where

$$M = -\frac{1}{2}\oint \frac{q_0}{b_0^{1/2}}\left(\frac{1}{q_0 b_0^{1/2}}\right)_{TT} dT$$

is a constant independent of v, and $\oint$ denotes an average with respect to T over a period. From (111), we have

$$\frac{1}{2}\varphi_{0T}^2 + b_0 = E$$

for some constant E. Using an integration by parts, we may also write M as

$$\begin{aligned} M &= \frac{1}{2}\oint \frac{1}{b_0}\left(\frac{b_{0T}^2}{4b_0^2} - \frac{q_{0T}^2}{q_0^2}\right) dT \\ &= \oint \frac{E-b_0}{b_0}\left[\frac{(b_0')^2}{4b_0^2} - \frac{(q_0')^2}{q_0^2}\right] dT. \end{aligned}$$

Thus, the final equations for $u(\theta,t)$, $v(\theta,t)$ are

$$(v_t + \Lambda u v_\theta)_\theta + M\langle v_\theta^2\rangle v = 0, \tag{112}$$

$$u_{\theta\theta} = v_\theta^2 - \langle v_\theta^2\rangle. \tag{113}$$

It follows from these equations that, for smooth solutions,

$$\left(v_\theta^2\right)_t + \left[\Lambda\left(u v_\theta^2 + \frac{1}{2}u_\theta^2 + \langle v_\theta^2\rangle u\right) + M\langle v_\theta^2\rangle v^2\right]_\theta = 0.$$

Taking the average of this equation with respect to θ, we find that $\langle v_\theta^2\rangle$ is constant in time, as stated earlier.

In summary, the asymptotic solution of (12)–(13) is given by

$$\begin{aligned} \varphi &= \varphi_0(\tau) + \frac{\varepsilon b_0'(\tau)}{a_0^2(\tau) - b_0^2(\tau)}\, u(\theta,t) + O(\varepsilon^{3/2}), \\ \psi &= \frac{\varepsilon^{1/2}}{q_0(\tau) b_0^{1/2}(\tau)}\, v(\theta,t) + O(\varepsilon), \end{aligned}$$

where the multiple-scale variables θ, τ are evaluated at (103), s satisfies (104), φ_0 satisfies (109), and u, v satisfy (112)–(113).

If $\Lambda \neq 0$ then we may rescale variables in (112)–(113) to get (101)–(102) with

$$\mu = \frac{M}{\Lambda}.$$

As an example, let us consider the wave speeds in (11) arising from the one-dimensional director field equations, where

$$b(\varphi) = \sqrt{\beta \sin^2\varphi + \gamma\cos^2\varphi}.$$

If $\beta < \gamma$, then b has a minimum at $\varphi = \pi/2$, and (109) has periodic solutions for the mean field φ_0 that oscillate around $\pi/2$. Our asymptotic solution applies in this case. If $\beta > \gamma$, then b has a minimum at $\varphi = 0$. Although (109) also has periodic solutions in this case, there is a loss of strict hyperbolicity at $\varphi = 0$, where $a = b$, and the asymptotic solution breaks down.

Appendix A. Algebraic details.

A.1. Linearized equations.

The linearization of (3) at $\mathbf{n}_0$ is

$$\mathbf{n}_{tt}' = \alpha\nabla(\operatorname{div}\mathbf{n}') - \beta\operatorname{curl}\,(A'\mathbf{n}_0) - \gamma\operatorname{curl}\,(\mathbf{B}'\times\mathbf{n}_0) + \lambda'\mathbf{n}_0, \tag{114}$$

where

$$A' = \mathbf{n}_0\cdot\operatorname{curl}\mathbf{n}', \qquad \mathbf{B}' = \mathbf{n}_0\times\operatorname{curl}\mathbf{n}',$$

and the Lagrange multiplier λ' is chosen so that $\mathbf{n}_0\cdot\mathbf{n}' = 0$.

The Fourier mode

$$\mathbf{n}'(\mathbf{x},t) = \widehat{\mathbf{n}}\, e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}, \qquad \lambda'(\mathbf{x},t) = \widehat{\lambda}\, e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}$$

is a solution of the linearized equations (114) if

$$\mathcal{L}\widehat{\mathbf{n}} - \widehat{\lambda}\mathbf{n}_0 = 0, \qquad \mathbf{n}_0 \cdot \widehat{\mathbf{n}} = 0, \tag{115}$$

where the linear map $\mathcal{L} : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$\mathcal{L}\widehat{\mathbf{n}} = \omega^2\widehat{\mathbf{n}} - \alpha\left(\mathbf{k}\cdot\widehat{\mathbf{n}}\right)\mathbf{k} + \beta\widehat{A}\left(\mathbf{k}\times\mathbf{n}_0\right) + \gamma\mathbf{k}\times\left(\widehat{\mathbf{B}}\times\mathbf{n}_0\right), \tag{116}$$

$$\widehat{A} = \mathbf{n}_0\cdot\left(\mathbf{k}\times\widehat{\mathbf{n}}\right), \qquad \widehat{\mathbf{B}} = \mathbf{n}_0\times\left(\mathbf{k}\times\widehat{\mathbf{n}}\right).$$

We solve this eigenvalue problem in the next proposition.

Proposition 5. *Suppose that $\mathbf{n}_0, \mathbf{k} \in \mathbb{R}^3$ where $\mathbf{n}_0$ is a unit vector and $\mathbf{k}$ is non-zero, and $\alpha, \beta, \gamma \in \mathbb{R}$ are distinct positive constants. For $\omega \in \mathbb{R}$ let $\mathcal{L} : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined in (116). Then the linear system*

$$\mathcal{L}\mathbf{m} - \lambda\mathbf{n}_0 = \mathbf{F}, \qquad \mathbf{n}_0\cdot\mathbf{m} = G \tag{117}$$

has a unique solution for $\{\mathbf{m}, \lambda\} \in \mathbb{R}^3 \times \mathbb{R}$ for every $\{\mathbf{F}, G\} \in \mathbb{R}^3 \times \mathbb{R}$ unless $\omega^2 = a^2(\mathbf{k}; \mathbf{n}_0)$ or $\omega^2 = b^2(\mathbf{k}; \mathbf{n}_0)$, where a^2, b^2 are defined in (24)–(25).

(a) If $\mathbf{k}$ is not parallel to $\mathbf{n}_0$ and $\omega^2 = a^2(\mathbf{k}; \mathbf{n}_0)$, then the general solution of (117) when $\mathbf{F} = 0$, $G = 0$ is

$$\mathbf{m} = c\mathbf{R}, \qquad \lambda = -c\left(\alpha-\gamma\right)R^2\left(\mathbf{k}\cdot\mathbf{n}_0\right), \tag{118}$$

where c is an arbitrary constant, $\mathbf{R} = \mathbf{k} - \left(\mathbf{k}\cdot\mathbf{n}_0\right)\mathbf{n}_0$, and $R = |\mathbf{R}|$. Equation (117) is solvable for $\{\mathbf{m}, \lambda\}$ if and only if $\{\mathbf{F}, G\}$ satisfy

$$\mathbf{F}\cdot\mathbf{R} + (\alpha-\gamma)R^2(\mathbf{k}\cdot\mathbf{n}_0)G = 0. \tag{119}$$

(b) If $\mathbf{k}$ is not parallel to $\mathbf{n}_0$ and $\omega^2 = b^2(\mathbf{k}; \mathbf{n}_0)$, then the general solution of (117) when $\mathbf{F} = 0$, $G = 0$ is

$$\mathbf{m} = c\mathbf{S}, \qquad \lambda = 0, \tag{120}$$

where c is an arbitrary constant and $\mathbf{S} = \mathbf{k}\times\mathbf{n}_0$. Equation (117) is solvable for $\{\mathbf{m}, \lambda\}$ if and only if $\mathbf{F}$ satisfies

$$\mathbf{F}\cdot\mathbf{S} = 0. \tag{121}$$

(c) If $\mathbf{k}$ is parallel to $\mathbf{n}_0$ and $\omega^2 = \gamma(\mathbf{k}\cdot\mathbf{n}_0)^2$, then the general solution of (117) when $\mathbf{F} = 0$, $G = 0$ is $\{\mathbf{m}, \lambda\} = \{\mathbf{m}^\perp, 0\}$ where $\mathbf{m}^\perp$ is any vector orthogonal to $\mathbf{n}_0$. Equation (117) is solvable for $\{\mathbf{m}, \lambda\}$ if and only if $\mathbf{F}$ is parallel to $\mathbf{n}_0$.

Proof. First, we suppose that $\mathbf{k}$ is not parallel to $\mathbf{n}_0$. Expanding

$$\mathbf{m} = m_1\mathbf{n}_0 + m_2\mathbf{k} + m_3\mathbf{S},$$

we find, after some algebra, that (117) is equivalent to

$$\begin{aligned}
&[(\omega^2-\gamma k^2)m_1 - \lambda]\mathbf{n}_0 + [(\omega^2-\alpha k^2)m_2 - (\alpha-\gamma)(\mathbf{k}\cdot\mathbf{n}_0)m_1]\mathbf{k}\\
&\qquad +[(\omega^2-\gamma(\mathbf{k}\cdot\mathbf{n}_0)^2 - \beta(k^2-(\mathbf{k}\cdot\mathbf{n}_0)^2)m_3]\mathbf{S} = \mathbf{F},\\
&m_1 + (\mathbf{k}\cdot\mathbf{n}_0)m_2 = G,
\end{aligned}$$

with $k = |\mathbf{k}|$. The last equation gives:

$$m_1 = G - (\mathbf{k}\cdot\mathbf{n}_0)m_2.$$

Using it in the previous equation, we find

$$\begin{aligned}
&[(\omega^2-\gamma k^2)m_1 - \lambda]\mathbf{n}_0 + [(\omega^2 - a^2(\mathbf{k};\mathbf{n}_0))m_2]\mathbf{k}\\
&\qquad +[(\omega^2 - b^2(\mathbf{k};\mathbf{n}_0))m_3]\mathbf{S} = \mathbf{F} + (\alpha-\gamma)(\mathbf{k}\cdot\mathbf{n}_0)G\mathbf{k}.
\end{aligned}$$

Using the decomposition $\mathbf{F} = F_1\mathbf{n}_0 + F_2\mathbf{k} + F_3\mathbf{S}$, and taking the components of $\mathbf{n}_0$, we can determine λ:

$$\lambda = (\omega^2 - \gamma k^2)[G - (\mathbf{k}\cdot\mathbf{n}_0)m_2] - F_1.$$

Similarly, taking the components of $\mathbf{k}$ and $\mathbf{S}$, we find equations for m_2 and m_3:

$$(\omega^2 - a^2(\mathbf{k};\mathbf{n}_0))m_2 = F_2 + (\alpha-\gamma)(\mathbf{k}\cdot\mathbf{n}_0)G,$$
$$(\omega^2 - b^2(\mathbf{k};\mathbf{n}_0))m_3 = F_3.$$

It is possible to verify that

$$R^2F_1 = \mathbf{F}\cdot(\mathbf{S}\times\mathbf{k}), \quad R^2F_2 = \mathbf{F}\cdot\mathbf{R}, \quad R^2F_3 = \mathbf{F}\cdot\mathbf{S}.$$

If $\omega^2 = a^2$, then $\omega^2 \neq b^2$ (since $\mathbf{k}$ is not parallel to $\mathbf{n}_0$) and the second equation is solvable for m_3. The first equation reduces to the solvability condition (119). If $\mathbf{F} = 0$, $G = 0$, then we find that $m_3 = 0$ and $m_2 = c$, where c is an arbitrary constant. Computing the corresponding value of λ, and knowing that $m_1 = -(\mathbf{k}\cdot\mathbf{n}_0)c$, we get (118).

If $\omega^2 = b^2$, then the first equation is solvable for m_2. The second equation reduces to the solvability condition (121). If $\mathbf{F} = 0$, $G = 0$, then $m_2 = 0$, and $m_3 = c$, which gives $\lambda = 0$, $m_1 = 0$, and thus (120).

Finally, we suppose that $\mathbf{k} = k\mathbf{n}_0$ is parallel to $\mathbf{n}_0$, in which case $a^2 = b^2 = \gamma k^2$. Then

$$\mathcal{L}\mathbf{m} = \left(\omega^2 - \gamma k^2\right)\mathbf{m} - (\alpha-\gamma)k^2\left(\mathbf{n}_0\cdot\mathbf{m}\right)\mathbf{n}_0.$$

Equation (117) is therefore uniquely solvable unless $\omega^2 = \gamma k^2$, when

$$\mathcal{L}\mathbf{m} = -(\alpha-\gamma)k^2\left(\mathbf{n}_0\cdot\mathbf{m}\right)\mathbf{n}_0.$$

In that case, (117) is solvable if and only if $\mathbf{F} = F\mathbf{n}_0$ is parallel to $\mathbf{n}_0$, and the solution is

$$\mathbf{m} = G\mathbf{n}_0 + \mathbf{m}^\perp, \qquad \lambda = -\left[F + (\alpha-\gamma)k^2G\right],$$

where $\mathbf{m}^\perp$ is an arbitrary vector orthogonal to $\mathbf{n}_0$. □

From Proposition 5, the solutions of the eigenvalue problem (115) are given by (23)–(27).

A.2. **Weakly nonlinear splay waves.** We look for an asymptotic solution of (3) of the form (30)–(31). We expand derivatives as

$$\partial_t \to -\frac{\omega}{\varepsilon}\partial_\theta + \partial_t, \quad \nabla \to \frac{\mathbf{k}}{\varepsilon}\partial_\theta + \nabla, \tag{122}$$

where ω, $\mathbf{k}$ are defined in (32). The corresponding expansions of λ, A, $\mathbf{B}$ are

$$\lambda = \frac{1}{\varepsilon}\lambda_1 + \lambda_2 + \dots, \quad A = A_1 + \varepsilon A_2 + \dots, \quad \mathbf{B} = \mathbf{B}_1 + \varepsilon\mathbf{B}_2 + \dots.$$

where

$$A_1 = \mathbf{n}_0\cdot(\mathbf{k}\times\mathbf{n}_{1\theta}) + \mathbf{n}_0\cdot\operatorname{curl}\mathbf{n}_0,$$
$$\mathbf{B}_1 = \mathbf{n}_0\times(\mathbf{k}\times\mathbf{n}_{1\theta}) + \mathbf{n}_0\times\operatorname{curl}\mathbf{n}_0,$$
$$A_2 = \mathbf{n}_0\cdot(\mathbf{k}\times\mathbf{n}_{2\theta}) + \mathbf{n}_1\cdot(\mathbf{k}\times\mathbf{n}_{1\theta}) + \mathbf{n}_0\cdot\operatorname{curl}\mathbf{n}_1 + \mathbf{n}_1\cdot\operatorname{curl}\mathbf{n}_0,$$
$$\mathbf{B}_2 = \mathbf{n}_0\times(\mathbf{k}\times\mathbf{n}_{2\theta}) + \mathbf{n}_1\times(\mathbf{k}\times\mathbf{n}_{1\theta}) + \mathbf{n}_0\times\operatorname{curl}\mathbf{n}_1 + \mathbf{n}_1\times\operatorname{curl}\mathbf{n}_0,$$

with $\mathbf{n}_2$ term of order ε^2 in (31). We write these expressions as

$$\begin{aligned} A_i &= \widehat{A}_i + \widetilde{A}_i, & \mathbf{B}_i &= \widehat{\mathbf{B}}_i + \widetilde{\mathbf{B}}_i, \\ \widehat{A}_i &= \mathbf{n}_0 \cdot (\mathbf{k} \times \mathbf{n}_{i\theta}), & \widehat{\mathbf{B}}_i &= \mathbf{n}_0 \times (\mathbf{k} \times \mathbf{n}_{i\theta}). \end{aligned} \tag{123}$$

We use these expansions in (3), Taylor expand the result with respect to ε, and equate coefficients of powers of ε. At the order ε^{-1}, we obtain linearized equations for $\mathbf{n}_1$, λ_1, which have the form

$$\mathcal{L}\mathbf{n}_{1\theta\theta} - \lambda_1 \mathbf{n}_0 = 0, \qquad \mathbf{n}_0 \cdot \mathbf{n}_1 = 0, \tag{124}$$

where $\mathcal{L}$ is the linear map defined in (116).

The system (124) has the splay-wave eigenvalue $\omega^2 = a^2(\mathbf{k}; \mathbf{n}_0)$, where a is defined in (24). The corresponding solution for $\mathbf{n}_1$, after two integrations with respect to θ, is

$$\mathbf{n}_1(\theta, \mathbf{x}, t) = u(\theta, \mathbf{x}, t)\mathbf{R}(\mathbf{x}, t),$$

where u is an arbitrary scalar-valued function, and $\mathbf{R}$ is defined in (26). The solution for λ_1 is

$$\lambda_1 = -(\alpha - \gamma)\, u_{\theta\theta} R^2 (\mathbf{k} \cdot \mathbf{n}_0).$$

At the order ε^0, we obtain equations for $\mathbf{n}_2$, λ_2. Using the fact that $\mathbf{n}_0$ is a solution of (3), with $\lambda = \lambda_0$ say, we may write them as

$$\mathcal{L}\mathbf{n}_{2\theta\theta} - \widetilde{\lambda}_2 \mathbf{n}_0 = \mathbf{F}_1, \qquad \mathbf{n}_0 \cdot \mathbf{n}_2 = G_1, \tag{125}$$

where $\widetilde{\lambda}_2 = \lambda_2 - \lambda_0$ and

$$\begin{aligned} \mathbf{F}_1 = {} & 2\omega \mathbf{n}_{1\theta t} + \omega_t \mathbf{n}_{1\theta} + \alpha \left\{ (\operatorname{div} \mathbf{n}_{1\theta})\, \mathbf{k} + \nabla (\mathbf{k} \cdot \mathbf{n}_{1\theta}) \right\} \\ & -\beta \Big\{ \widetilde{A}_{2\theta} (\mathbf{k} \times \mathbf{n}_0) + A_1 (\mathbf{k} \times \mathbf{n}_{1\theta}) + \mathbf{k} \times (A_1 \mathbf{n}_1)_\theta \\ & \qquad + \widehat{A}_1 \operatorname{curl} \mathbf{n}_0 + \operatorname{curl}\left(\widehat{A}_1 \mathbf{n}_0\right) \Big\} \\ & +\gamma \Big\{ -\mathbf{k} \times \left(\widetilde{\mathbf{B}}_{2\theta} \times \mathbf{n}_0\right) + \mathbf{B}_1 \times (\mathbf{k} \times \mathbf{n}_{1\theta}) - \mathbf{k} \times (\mathbf{B}_1 \times \mathbf{n}_1)_\theta \\ & \qquad + \widehat{\mathbf{B}}_1 \times \operatorname{curl} \mathbf{n}_0 - \operatorname{curl}\left(\widehat{\mathbf{B}}_1 \times \mathbf{n}_0\right) \Big\} + \lambda_1 \mathbf{n}_1, \\ G_1 = {} & -\frac{1}{2} \mathbf{n}_1 \cdot \mathbf{n}_1. \end{aligned}$$

From Proposition 5, equation (125) is solvable for $\mathbf{n}_{2\theta\theta}$ and $\widetilde{\lambda}_2$ if and only if

$$\mathbf{R} \cdot \mathbf{F}_1 + (\alpha - \gamma)\, R^2 (\mathbf{k} \cdot \mathbf{n}_0)\, G_{1\theta\theta} = 0.$$

After some algebra, we find that this solvability condition gives (33).

A.3. **Weakly nonlinear twist waves.** We look for an asymptotic solution of (3) of the form (35)–(36), expanding derivatives as in (122). The corresponding expansions of λ, A, $\mathbf{B}$ are

$$\begin{aligned} \lambda &= \frac{1}{\varepsilon^{3/2}} \lambda_1 + \frac{1}{\varepsilon} \lambda_2 + \frac{1}{\varepsilon^{1/2}} \lambda_3 + \dots, \\ A &= \frac{1}{\varepsilon^{1/2}} A_1 + A_2 + \varepsilon^{1/2} A_3 + \dots, \\ \mathbf{B} &= \frac{1}{\varepsilon^{1/2}} \mathbf{B}_1 + \mathbf{B}_2 + \varepsilon^{1/2} \mathbf{B}_3 + \dots, \end{aligned}$$

where

$$\begin{aligned}
A_1 &= \mathbf{n}_0 \cdot (\mathbf{k} \times \mathbf{n}_{1\theta}), \\
\mathbf{B}_1 &= \mathbf{n}_0 \times (\mathbf{k} \times \mathbf{n}_{1\theta}), \\
A_2 &= \mathbf{n}_0 \cdot (\mathbf{k} \times \mathbf{n}_{2\theta}) + \mathbf{n}_1 \cdot (\mathbf{k} \times \mathbf{n}_{1\theta}) + \mathbf{n}_0 \cdot \operatorname{curl} \mathbf{n}_0, \\
\mathbf{B}_2 &= \mathbf{n}_0 \times (\mathbf{k} \times \mathbf{n}_{2\theta}) + \mathbf{n}_1 \times (\mathbf{k} \times \mathbf{n}_{1\theta}) + \mathbf{n}_0 \times \operatorname{curl} \mathbf{n}_0, \\
A_3 &= \mathbf{n}_0 \cdot (\mathbf{k} \times \mathbf{n}_{3\theta}) + \mathbf{n}_1 \cdot (\mathbf{k} \times \mathbf{n}_{2\theta}) + \mathbf{n}_2 \cdot (\mathbf{k} \times \mathbf{n}_{1\theta}) \\
&\qquad\qquad\qquad + \mathbf{n}_0 \cdot \operatorname{curl} \mathbf{n}_1 + \mathbf{n}_1 \cdot \operatorname{curl} \mathbf{n}_0, \\
\mathbf{B}_3 &= \mathbf{n}_0 \times (\mathbf{k} \times \mathbf{n}_{3\theta}) + \mathbf{n}_1 \times (\mathbf{k} \times \mathbf{n}_{2\theta}) + \mathbf{n}_2 \times (\mathbf{k} \times \mathbf{n}_{1\theta}) \\
&\qquad\qquad\qquad + \mathbf{n}_0 \times \operatorname{curl} \mathbf{n}_1 + \mathbf{n}_1 \times \operatorname{curl} \mathbf{n}_0,
\end{aligned}$$

with $\mathbf{n}_3$ term of order $\varepsilon^{3/2}$ in (36). We write these expressions as in (123).

Using these expansions in (3), Taylor expanding the result, and equating coefficients of powers of $\varepsilon^{1/2}$, we get at the order $\varepsilon^{-3/2}$ the linearized equations (124). From Proposition 5, this system has the twist-wave eigenvalue $\omega^2 = b^2(\mathbf{k}; \mathbf{n}_0)$, where b is given by (25). The corresponding solution for $\{\mathbf{n}_1, \lambda_1\}$ is

$$\mathbf{n}_1(\theta, \mathbf{x}, t) = v(\theta, \mathbf{x}, t)\mathbf{S}(\mathbf{x}, t), \qquad \lambda_1 = 0, \tag{126}$$

where v is an arbitrary scalar-valued function, and $\mathbf{S}$ is defined in (27).

Equating coefficients of the order ε^{-1}, we obtain that

$$\mathcal{L}\mathbf{n}_{2\theta\theta} - \lambda_2 \mathbf{n}_0 = \mathbf{F}_1, \qquad \mathbf{n}_0 \cdot \mathbf{n}_2 = G_1, \tag{127}$$

where $\mathcal{L}$ is defined in (116) and

$$\begin{aligned}
\mathbf{F}_1 &= -\beta \left[\widetilde{A}_{2\theta} (\mathbf{k} \times \mathbf{n}_0) + \mathbf{k} \times (A_1 \mathbf{n}_1)_\theta + A_1 (\mathbf{k} \times \mathbf{n}_{1\theta}) \right] \\
&\quad - \gamma \left[\mathbf{k} \times \left(\widetilde{\mathbf{B}}_{2\theta} \times \mathbf{n}_0 \right) + \mathbf{k} \times (\mathbf{B}_1 \times \mathbf{n}_1)_\theta - \mathbf{B}_1 \times (\mathbf{k} \times \mathbf{n}_{1\theta}) \right] + \lambda_1 \mathbf{n}_1, \\
G_1 &= -\frac{1}{2} \mathbf{n}_1 \cdot \mathbf{n}_1.
\end{aligned}$$

Using Proposition 5 to solve (127), we find that

$$\begin{aligned}
\mathbf{n}_2 &= -ru\,(\mathbf{k} \cdot \mathbf{n}_0)\,\mathbf{R} + \frac{1}{2} v^2 (\mathbf{k} \times \mathbf{S}), \\
\lambda_2 &= v_\theta^2 \left\{ \beta k^2 + r\,(\beta - \gamma)\,(\mathbf{k} \cdot \mathbf{n}_0)^2 \right\} \left[k^2 - (\mathbf{k} \cdot \mathbf{n}_0)^2 \right],
\end{aligned} \tag{128}$$

where $u(\theta, \mathbf{x}, t)$ satisfies (40), and

$$r = \frac{\beta - \gamma}{\alpha - \beta}.$$

We could add an arbitrary scalar multiple of the null-vector $\mathbf{S}$ to $\mathbf{n}_2$, but this would not alter our final result, so we omit it for simplicity.

Equating coefficients of the order $\varepsilon^{-1/2}$, we obtain that

$$\mathcal{L}\mathbf{n}_{3\theta\theta} - \lambda_3 \mathbf{n}_0 = \mathbf{F}_2, \qquad \mathbf{n}_0 \cdot \mathbf{n}_3 = G_2, \tag{129}$$

where

$$
\begin{aligned}
\mathbf{F}_2 = {} & 2\omega \mathbf{n}_{1\theta t} + \omega_t \mathbf{n}_{1\theta} + \lambda_2 \mathbf{n}_1 + \lambda_1 \mathbf{n}_2 + \alpha \left[\operatorname{div}\, (\mathbf{n}_{1\theta})\, \mathbf{k} + \nabla\, (\mathbf{k}\cdot \mathbf{n}_{1\theta})\right] \\
& -\beta \Big[\widetilde{A}_{3\theta}\, (\mathbf{k}\times \mathbf{n}_0) + \mathbf{k}\times (A_2 \mathbf{n}_1)_\theta + \mathbf{k}\times (A_1\mathbf{n}_2)_\theta + A_2\, (\mathbf{k}\times \mathbf{n}_{1\theta}) \\
& \qquad + A_1\, (\mathbf{k}\times \mathbf{n}_{2\theta}) + A_1 \operatorname{curl} \mathbf{n}_0 + \operatorname{curl}\, (A_1 \mathbf{n}_0)\Big] \\
& -\gamma \Big[\mathbf{k}\times \Big(\widetilde{\mathbf{B}}_{3\theta}\times \mathbf{n}_0\Big) + \mathbf{k}\times (\mathbf{B}_2\times \mathbf{n}_1)_\theta + \mathbf{k}\times (\mathbf{B}_1\times \mathbf{n}_2)_\theta \\
& \qquad - \mathbf{B}_2 \times (\mathbf{k}\times \mathbf{n}_{1\theta}) - \mathbf{B}_1\times (\mathbf{k}\times \mathbf{n}_{2\theta}) \\
& \qquad\quad - \mathbf{B}_1 \times \operatorname{curl} \mathbf{n}_0 + \operatorname{curl}\, (\mathbf{B}_1 \times \mathbf{n}_0)\Big], \\
G_2 = {} & -\mathbf{n}_1\cdot \mathbf{n}_2.
\end{aligned}
$$

From Proposition 5, this system is solvable if $\mathbf{S}\cdot\mathbf{F}_2 = 0$. After some algebra, we compute that this solvability condition gives (39).

A.4. **Weakly nonlinear polarized waves.** We assume that $\mathbf{n}_0$ and $\mathbf{k}$ are constant and $\mathbf{k} = k\mathbf{n}_0$ is parallel to $\mathbf{n}_0$. We look for an expansion of the same form as the one used in the previous section for twist waves, with a plane-wave phase $\Phi(x,t) = \mathbf{k}\cdot\mathbf{x} - \omega t$.

When $\mathbf{k}$ is parallel to $\mathbf{n}_0$, the linearized dispersion relation (23) reduces to $\left[\omega^2 - \gamma k^2\right]^2 = 0$, so $\omega^2 = \gamma k^2$. From Proposition 5, the solution of the leading-order $O(\varepsilon^{-3/2})$-equation (124) is then

$$
\mathbf{n}_1(\theta, \mathbf{x}, t) = \mathbf{u}(\theta,\mathbf{x},t), \qquad \lambda_1 = 0,
$$

where $\mathbf{u}$ is an arbitrary vector such that $\mathbf{n}_0\cdot\mathbf{u} = 0$.

The $O(\varepsilon^{-1})$-equation (127) simplifies to

$$
\begin{aligned}
\mathcal{L}\mathbf{n}_{2\theta\theta} - \lambda_2 \mathbf{n}_0 &= -\gamma k^2\, (\mathbf{u}_\theta\cdot\mathbf{u}_\theta)\, \mathbf{n}_0, \\
\mathbf{n}_0\cdot\mathbf{n}_2 &= -\frac{1}{2}(\mathbf{u}\cdot\mathbf{u}),
\end{aligned}
$$

whose solution is

$$
\begin{aligned}
\mathbf{n}_2 &= -\frac{1}{2}(\mathbf{u}\cdot\mathbf{u})\mathbf{n}_0, \\
\lambda_2 &= \frac{1}{2}(\alpha-\gamma)k^2\, (\mathbf{u}\cdot\mathbf{u})_{\theta\theta} + \gamma k^2\, (\mathbf{u}_\theta\cdot\mathbf{u}_\theta)\,.
\end{aligned}
$$

We could add to $\mathbf{n}_2$ an arbitrary vector orthogonal to $\mathbf{n}_0$, but this would not alter our final equations, so we omit it for simplicity.

Computing $\mathbf{F}_2$ in the $O(\varepsilon^{-1/2})$-equation (129) and imposing the solvability condition in Proposition 5 that $\mathbf{F}_2$ is parallel to $\mathbf{n}_0$, we get after some algebra that

$$
\begin{aligned}
& \mathbf{u}_{\theta t} + \frac{\omega}{k}\mathbf{n}_0\cdot\nabla\mathbf{u}_\theta + \frac{\alpha k^2}{2\omega}\, (\mathbf{u}\cdot\mathbf{u}_\theta)_\theta\, \mathbf{u} \\
& - \frac{\beta k^2}{2\omega}\left\{\left[\mathbf{u}\cdot(\mathbf{n}_0\times\mathbf{u}_{\theta\theta})\right](\mathbf{n}_0\times\mathbf{u}) + 2\left[\mathbf{u}\cdot(\mathbf{n}_0\times\mathbf{u}_\theta)\right](\mathbf{n}_0\times\mathbf{u}_\theta)\right\} \\
& - \frac{\gamma k^2}{2\omega}\left\{\left[(\mathbf{u}\cdot\mathbf{u})\,\mathbf{u}_\theta\right]_\theta - (\mathbf{u}_\theta\cdot\mathbf{u}_\theta)\,\mathbf{u}\right\} = 0.
\end{aligned} \tag{130}
$$

A computation in terms of components shows that if $\mathbf{u}$ is orthogonal to the constant unit vector $\mathbf{n}_0$, then

$$
\begin{aligned}
& \left[\mathbf{u}\cdot(\mathbf{n}_0\times\mathbf{u}_{\theta\theta})\right](\mathbf{n}_0\times\mathbf{u}) + 2\left[\mathbf{u}\cdot(\mathbf{n}_0\times\mathbf{u}_\theta)\right](\mathbf{n}_0\times\mathbf{u}_\theta) \\
& \qquad = (\mathbf{u}\cdot\mathbf{u}_\theta)_\theta\,\mathbf{u} - \left[(\mathbf{u}\cdot\mathbf{u})\,\mathbf{u}_\theta\right]_\theta + (\mathbf{u}_\theta\cdot\mathbf{u}_\theta)\,\mathbf{u}.
\end{aligned}
$$

Using this result in (130), we obtain (43).

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REFERENCES

[1] G. Alì and J. K. Hunter, *Diffractive nonlinear geometrical optics for variational wave equations and the Einstein equations*, Comm. Pure Appl. Math., **60** (2007), 1522–1557.

[2] R. Beals, D. Sattinger and J. Szmigielske, *Inverse scattering solutions of the Hunter-Saxton equation*, Applicable Analysis, **78** (2001), 255–269.

[3] A. Bressan and A. Constantin, *Global solutions of the Hunter-Saxton equation*, SIAM J. Math. Anal., **37** (2005), 996–1026.

[4] A. Bressan, P. Zhang and Y. Zheng, *Asymptotic variational wave equations*, Arch. Ration. Mech. Anal., **183** (2007), 163–185.

[5] A. Bressan and Y. Zheng, *Conservative solutions to a nonlinear variational wave equation*, Comm. Math. Phys., **266** (2006), 471–497.

[6] M. Brio and J. K. Hunter, *Rotationally invariant hyperbolic waves*, Comm. Pure Appl. Math., **43** (1990), 1037–1053.

[7] R. Camassa and D. Holm, *An integrable shallow water equation with peaked solitons*, Phys. Rev. Lett., **71** (1993), R51–R79.

[8] C. M. Dafermos, "Hyperbolic Conservation Laws in Continuum Physics," 2^{nd} edition, Springer-Verlag, Berlin, 2005.

[9] C. M. Dafermos, *Continuous solutions for balance laws*, *Ric. Mat.*, **55** (2006), 79–91.

[10] A. Degasperis and M. Procesi, *Asymptotic integrability*, in "Symmetry and Perturbation Theory" (eds. A. Degasperis and G. Gaeta), World Scientific (1999), 23–37.

[11] J. L. Ericksen, *Twisting of liquid crystals*, J. Fluid Mech., **27** (1967), 59–64.

[12] J. L. Ericksen, *Twist waves in liquid crystals*, Q. J. Mech. Appl. Math., (1972) **25**, 463–465.

[13] P. G. de Gennes and J. Prost, "The Physics of Liquid Crystals," 2^{nd} edition, Oxford University Press, Oxford, 1993.

[14] R. Glassey, J. K. Hunter and Y. Zheng, *Singularities of a variational wave equation*, J. Diff. Eq., **129** (1996), 49–78.

[15] R. Glassey, J. K. Hunter and Y. Zheng, *Singularities and oscillations in a nonlinear variational wave equation*, in "Singularities and Oscillations" (eds. J. Rauch and M. Taylor), IMA Volumes in Mathematics and its Applications, **91**, Springer-Verlag (1996), 37–60.

[16] J. K. Hunter, *Asymptotic equations for nonlinear hyperbolic waves*, in "Surveys in Applied Mathematics," Vol. **2** (eds. M. Freidlin et.al.), Plenum Press (1995), 167–276.

[17] J. K. Hunter and R. Saxton, *Dynamics of director fields*, SIAM J. Appl. Math., **51** (1991), 1498–1521.

[18] J. K. Hunter and Y. Zheng, *On a completely integrable nonlinear hyperbolic variational equation*, Physica D, **79** (1994), 361–386.

[19] J. K. Hunter and Y. Zheng, *On a nonlinear hyperbolic variational wave equation I. Global existence of weak solutions*, Arch. Rat. Mech. Anal., **129** (1995), 355–383.

[20] B. Khesin and G. Misiolek, *Euler equations on homogeneous spaces and Virasoro orbits*, Adv. Math., **176** (2003), 116–144.

[21] H. Lundmark, private communication.

[22] G. F. Mazenko, "Fluctuations, Order, and Defects," John Wiley & Sons, New Jersey, 2003.

[23] R. Saxton, *Dynamic instability of the liquid crystal director*, in "Contemp. Math.," Vol. **100**, "Current Progress in Hyperbolic Systems," (ed. W. B. Lindquist), AMS, Providence, RI, (1989), 325–330.

[24] M. Shahinpoor, *Finite twist waves in liquid crystals*, Q. J. Mech. Appl. Math., **28** (1975), 223–231.

[25] J. Shatah and M. Struwe, *Geometric Wave Equations*, Courant Lecture Notes in Mathematics, Vol. **2**, Courant Institute, New York, 1998.

[26] P. Zhang and Y. Zheng, *Existence and uniqueness of solutions of an asymptotic equation arising from a variational wave equation with general data*, Arch. Rat. Mech. Anal., **155** (2000), 49–83.

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E-mail address: `giuseppe.ali@unical.it;jkhunter@ucdavis.edu`