Hamiltonian Equations for Scale-Invariant Waves

G. Alì
Institute for Applied Mathematics
Consiglio Nazionale delle Ricerche, Napoli *

J. K. Hunter
Department of Mathematics and
Institute of Theoretical Dynamics
University of California at Davis †

D. F. Parker
Department of Mathematics
and Statistics
University of Edinburgh

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Abstract

We use a Hamiltonian formalism to derive equations for weakly nonlinear scale-invariant waves. We apply the results to nonlinear surface waves in elasticity and magnetohydrodynamics that satisfy nonlocal generalizations of the inviscid Burgers equation.

1 Introduction

A wave motion is scale-invariant if it is governed by equations of motion and boundary conditions which are invariant under the space-time rescaling

$$x \to \lambda x, \qquad t \to \lambda t \tag{1.1}$$

for any $\lambda > 0$. Scale-invariance implies that the linearized dispersion relation between the wavenumber k and the frequency $\omega(k)$ of a unidirectional wave is given by

$$\omega(k) = ck, \tag{1.2}$$

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where the constant c is the linearized wave speed. Thus, scale-invariant wave motions are nondispersive. Since different Fourier components propagate at the same velocity, a linear scale-invariant wave can have an arbitrary profile.

Three waves with wavenumbers $\{k_1, k_2, k_3\}$ satisfy the three-wave resonance condition for the dispersion relation (1.2) when

$$k_1 = k_2 + k_3,$$

since the corresponding equation for the frequencies $\omega(k_j)$ holds automatically. Nonlinear effects are therefore particularly strong in scale-invariant waves, and the resonant interaction of the harmonics within the wave leads to a progressive distortion of the spatial wave profile.

Examples of nonlinear scale-invariant waves include bulk waves, such as sound waves in compressible fluids, elastic solids, and magnetohydrodynamic plasmas. These waves are governed by hyperbolic conservation laws, and a suitable spatial waveform typically satisfies an inviscid Burgers equation.

Hyperbolic surface waves are scale-invariant when the geometry of the surface along which they propagate and the boundary conditions on the surface do not define a length scale. This is the case for nonlinear Rayleigh waves on a half-space in elasticity ([1]–[10]), nonlinear elastic waves on wedges [11, 12], and nonlinear surface waves on a planar discontinuity, such as a compressible vortex sheet [13] or a tangential discontinuity in magnetohydrodynamics [14]. If the speed of the surface wave is less than the speed of a bulk wave, then the self-interaction of the wave is nonlocal in the "subsonic" reference frame moving with the wave. A suitable spatial waveform then typically satisfies a nonlocal generalization of the inviscid Burger's equation [15].

In this paper, we study the general form of asymptotic equations for weakly nonlinear, scale-invariant Hamiltonian waves. We illustrate the results with a discussion of nonlinear sound waves, Rayleigh waves, surface waves, and wedge waves. The possibility of spatially nonlocal interactions makes the structure of the equations richer than one might expect. The Hamiltonian structure of the equations for nonlinear Rayleigh waves was derived explicitly by Hamilton *et. al.* [3], and the present paper puts their work in a general context. In particular, we use dimensional analysis to show that the form of the Hamiltonian depends on the dimension of the space of wavenumber vectors relative to the dimension of the space in which the waves propagate.

2 Scale-invariant Hamiltonian waves

We may describe a Hamiltonian wave field by a set of complex canonical variables [16, 17],

$$\left\{a(\mathbf{k}), a^*(\mathbf{k}) : \mathbf{k} \in \mathbb{R}^d\right\}.$$

We denote the dimension of wavenumber space by d and the dimension of physical space by n. For example, in three-dimensional space ($n = 3$), we have $d = 3$ for bulk waves, $d = 2$ for surface waves, and $d = 1$ for wedge waves.

The Hamiltonian $\mathcal{H}\left(\{a(\mathbf{k}), a^*(\mathbf{k})\}\right)$ is a real-valued functional of the canonical variables. The complex canonical form of Hamilton's equations is

$$ia_t(\mathbf{k}) = \frac{\delta \mathcal{H}}{\delta a^*(\mathbf{k})}, \tag{2.1}$$

where $\delta\mathcal{H}/\delta a^*(\mathbf{k})$ denotes the functional derivative of $\mathcal{H}$ with respect to $a^*(\mathbf{k})$, the t-subscript denotes a derivative with respect to time t, and we do not explicitly show the time dependence of a.

For weakly nonlinear waves, the Hamiltonian $\mathcal{H}$ may be expanded as

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3 + \dots,$$

where $\mathcal{H}_m$ is a homogeneous function of degree m in the canonical variables. Diagonalization of the quadratic part of the Hamiltonian by means of a canonical transformation gives

$$\mathcal{H}_2 = \int \omega(\mathbf{k}) a(\mathbf{k}) a^*(\mathbf{k})\, d\mathbf{k}, \tag{2.2}$$

where $\omega(\mathbf{k})$ is the frequency of the wave with wavenumber $\mathbf{k}$.

For the nondispersive, unidirectional waves that we study here, all three-wave resonant interactions are of the "decay" type,

$$\omega(\mathbf{k}_1) = \omega(\mathbf{k}_2) + \omega(\mathbf{k}_3), \qquad \mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3. \tag{2.3}$$

After the elimination of nonresonant terms by a near identity canonical transformation [17], the cubic Hamiltonian $\mathcal{H}_3$ may be written in the form

$$\begin{aligned}\mathcal{H}_3 \;=\; & \int\int\int \delta\left(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3\right)\left[T\left(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3\right) a^*\left(\mathbf{k}_1\right) a\left(\mathbf{k}_2\right) a\left(\mathbf{k}_3\right)\right. \\ & \left. + T^*\left(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3\right) a\left(\mathbf{k}_1\right) a^*\left(\mathbf{k}_2\right) a^*\left(\mathbf{k}_3\right)\right]\, d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3,\end{aligned} \tag{2.4}$$

where the three-wave interaction coefficient T satisfies the symmetry condition

$$T(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = T(\mathbf{k}_1, \mathbf{k}_3, \mathbf{k}_2). \tag{2.5}$$

The frequency $\omega(\mathbf{k})$ has dimension

$$[\omega(\mathbf{k})] = \frac{V}{L},$$

where V denotes a velocity dimension and L denotes a length dimension. For a scale-invariant wave motion, in which the system has one or more characteristic velocities but no characteristic length or time scales, the wavenumber $\mathbf{k}$ provides the only available length-scale, so we must have

$$\omega(\mathbf{k}) = c(\hat{\mathbf{k}}) |\mathbf{k}|,$$

where $c(\hat{\mathbf{k}})$ is the wave speed in the direction $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$.

The Hamiltonian function is an energy, and therefore has dimension

$$[\mathcal{H}] = MV^2,$$

where M denotes a mass dimension. Since $\mathbf{k} \in \mathbb{R}^d$, integration with respect to $\mathbf{k}$ has dimension L^{-d}. It follows from (2.2) that the dimension of the canonical variable $a(\mathbf{k})$ is

$$[a(\mathbf{k})] = (MV)^{1/2} L^{(d+1)/2}.$$

Using this result in (2.4), we find that the dimension of the three-wave interaction coefficient T is given by

$$[T] = \left(\frac{V}{M}\right)^{1/2} L^{(d-3)/2}.$$

We suppose that the mass dimension M of the medium is set by a density $R = ML^{-n}$. We then have

$$[T] = \left(\frac{V}{R}\right)^{1/2} L^{-\nu}, \tag{2.6}$$

where

$$\nu = \frac{n-d+3}{2}. \tag{2.7}$$

For scale-invariant waves, Eq. (2.6) implies that the interaction coefficient T is a homogeneous function of $\mathbf{k}$ of degree ν, meaning that for all $\lambda > 0$ we have

$$T(\lambda\mathbf{k}_1, \lambda\mathbf{k}_2, \lambda\mathbf{k}_3) = \lambda^{\nu} T(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3). \tag{2.8}$$

For example, in the case of bulk waves, we have $d = n$ and $\nu = 3/2$. In the case of waves that propagate along a surface of codimension one, such as surface waves in three-dimensional space, we have $d = n - 1$ and $\nu = 2$.

The inverse Fourier transform $\varphi(\mathbf{x})$ of $a(\mathbf{k})$,

$$\varphi(\mathbf{x}) = \int a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k},$$

is a spatial waveform. This waveform has dimension

$$[\varphi(\mathbf{x})] = (RV)^{1/2}\, L^{\nu-1}.$$

A related scale-invariant spatial wave field $u(\mathbf{x})$ is given by

$$u(\mathbf{x}) = \int |\mathbf{k}|^{\nu-1}\, a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k}, \tag{2.9}$$

which has the length-independent dimension $(RV)^{1/2}$. Other scale-invariant spatial waveforms may be constructed from Hilbert or Riesz transforms of $u(\mathbf{x})$.

3 Canonical equations

We now consider a single, unidirectional wave mode, in which case the wavenumber k is restricted to positive real values. The variables

$$\{a(k) : k > 0\}$$

are complex amplitudes of the positive frequency components of the wave field, and the conjugate variables

$$\{a^*(k) : k > 0\}$$

are complex amplitudes of the negative frequency components.

Approximating the Hamiltonian $\mathcal{H}$ up to cubic terms, and carrying out one integration in (2.4), we have

$$\begin{aligned}\mathcal{H} = \int_0^\infty cka(k)a^*(k)\, dk + \int_0^\infty \int_0^\infty T(k+\xi, k, \xi) a^*(k+\xi) a(k) a(\xi)\, dk d\xi \\ + \int_0^\infty \int_0^\infty T^*(k+\xi, k, \xi) a(k+\xi) a^*(k) a^*(\xi)\, dk d\xi.\end{aligned} \tag{3.1}$$

Computing the functional derivative of $\mathcal{H}$ with respect to $a^*(k)$, where $k > 0$, we find that Hamilton's equation (2.1) is

$$\begin{aligned} i a_t(k) = cka(k) + \int_0^k T(k, k-\xi, \xi) a(k-\xi) a(\xi)\, d\xi \\ +2 \int_0^\infty T^*(k+\xi, k, \xi) a(k+\xi) a^*(\xi)\, d\xi. \end{aligned} \tag{3.2}$$

It is convenient to rewrite Eq. (3.2) as a convolution equation for

$$\{a(k) : -\infty < k < \infty\},$$

where $a(-k) = a^*(k)$, of the form

$$i a_t(k) = cka(k) + \operatorname{sgn} k \int_{-\infty}^{\infty} \Lambda(k-\xi, \xi) a(k-\xi) a(\xi)\, d\xi. \tag{3.3}$$

The factor

$$\operatorname{sgn} k = \begin{cases} 1 & k > 0, \\ 0 & k = 0, \\ -1 & k > 0, \end{cases}$$

is included to simplify the subsequent equations.

We assume, without loss of generality, that

$$\Lambda(k, \xi) = \Lambda(\xi, k). \qquad \text{(symmetry)} \tag{3.4}$$

The condition $a(-k) = a^*(k)$ implies that

$$\Lambda(-k, -\xi) = \Lambda^*(k, \xi). \qquad \text{(reality)} \tag{3.5}$$

We will show that Eq. (3.3) can be written in the canonical Hamiltonian form (3.2) if and only if Λ satisfies, in addition to (3.4) and (3.5), the condition

$$\Lambda(k+\xi, -\xi) = \Lambda^*(k, \xi). \qquad \text{(Hamiltonian)} \tag{3.6}$$

This general symmetry property of the interaction coefficients in Hamiltonian systems is well-known [16], and was noted previously for nonlinear Rayleigh waves in Refs. [1, 2, 10].

Replacing $a(k)$ by $a^*(-k)$ for $k < 0$ and using the symmetry of Λ, we find that for $k > 0$ Eq. (3.3) implies

$$\begin{aligned} i a_t(k) = cka(k) + \int_0^k \Lambda(k-\xi, \xi) a(k-\xi) a(\xi)\, d\xi \\ +2 \int_0^\infty \Lambda(k+\xi, -\xi) a(k+\xi) a^*(\xi)\, d\xi. \end{aligned} \tag{3.7}$$

A comparison of (3.7) with (3.2) shows that Λ is related to T by

$$\Lambda(k,\xi) = T(k+\xi,k,\xi), \quad \Lambda(k+\xi,-\xi) = T^*(k+\xi,k,\xi) \quad \text{for } k,\xi > 0. \tag{3.8}$$

It follows from (3.4), (3.5), and (3.8) that Λ satisfies (3.6) for all $(k,\xi) \in \mathbb{R}^2$.

Finally, Eqs. (2.8) and (3.8) imply that, for scale-invariant wave motions, the kernel Λ satisfies the scaling condition

$$\Lambda(\lambda k, \lambda\xi) = \lambda^\nu \Lambda(k,\xi), \qquad \text{(scale-invariance)} \tag{3.9}$$

for all $\lambda > 0$. We discuss the transformation properties of Λ in the (k,ξ)-plane in greater detail in Section 4.

The Hamiltonian $\mathcal{H}$ in (3.1) may be written in terms of Λ as

$$\begin{aligned} \mathcal{H} \;=\; & \frac{1}{2}\int_{-\infty}^{\infty} c|k|a(k)a(-k)\,dk \\ & +\frac{1}{3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Lambda(k,\xi)a(-k-\xi)a(k)a(\xi)\,dkd\xi. \end{aligned} \tag{3.10}$$

For later use, we note a simple transformation of (3.3) that leads to an equivalent equation of the same form. Let

$$b(k) = [\alpha + \beta\sigma(k)]\,a(k), \tag{3.11}$$

where α, β are arbitrary real constants, not both of which are zero, and

$$\sigma(k) = i\operatorname{sgn} k. \tag{3.12}$$

Equations (3.3) and (3.11) imply that $b(k)$ satisfies

$$ib_t(k) = ckb(k) + \operatorname{sgn} k \int_{-\infty}^{\infty} \Gamma(k-\xi,\xi)b(k-\xi)b(\xi)\,d\xi, \tag{3.13}$$

where

$$\Gamma(k,\xi) = \left[\frac{\alpha + \beta\sigma(k+\xi)\sigma(k)\sigma(\xi)}{\alpha^2+\beta^2}\right]\Lambda(k,\xi). \tag{3.14}$$

The kernel Γ has the same transformation properties (3.4)–(3.6) and (3.9) as Λ. Thus, every member of the two-parameter family of kernels (3.14) leads to an equivalent evolution equation.

We may define spatial waveforms $\varphi(x)$ and $\psi(x)$ associated with $a(k)$ and $b(k)$, respectively, by

$$\varphi(x) = \int_{-\infty}^{\infty} a(k)e^{ikx}\,dk, \qquad \psi(x) = \int_{-\infty}^{\infty} b(k)e^{ikx}\,dk.$$

Then (3.11) implies that $\varphi(x)$, $\psi(x)$ are related by the nonlocal, scale-invariant transformation,

$$\psi(x) = \alpha\varphi(x) + \beta\mathbb{H}[\varphi(x)], \tag{3.15}$$

where $\mathbb{H}$ denotes the Hilbert transform (which corresponds, in Fourier space, to multiplication by $\sigma(k)$).

4 Symmetry group of the kernel

In this section, we study the symmetry group of the kernel $\Lambda(k,\xi)$. We consider the group generated by the symmetry, reality, and Hamiltonian conditions (3.4)–(3.6). The scaling condition (3.9) determines the kernel on an entire ray in the (k,ξ)-plane given its value at a single point on the ray.

We denote the (k,ξ)-plane by $\mathcal{P}$, and let J, B, C be the linear transformations on $\mathcal{P}$ associated with the matrices

$$J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}. \tag{4.1}$$

Thus, for example, $J : (k,\xi) \mapsto (-k,-\xi)$ is inversion through the origin, and $B : (k,\xi) \mapsto (\xi,k)$ is reflection in the line $k = \xi$. The conditions (3.4)–(3.6) imply that $\Lambda : \mathcal{P} \to \mathbb{C}$ satisfies

$$\Lambda \circ J = \Lambda^*, \quad \Lambda \circ B = \Lambda, \quad \Lambda \circ C = \Lambda^*. \tag{4.2}$$

The area-preserving linear transformations $\{J,B,C\}$ generate a representation of the order twelve dihedral group D_{12} on $\mathcal{P}$. A convenient choice of generators of the group is $\{A,B\}$, where the matrix of $A = CB$ is given by

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

These generators satisfy the defining relations

$$A^6 = I, \quad B^2 = I, \quad AB = BA^{-1},$$

where I is the identity transformation. The elements of the dihedral group are $\{I, A, \ldots, A^5, B, AB, \ldots, A^5B\}$.

A fundamental domain of the symmetry group is the region

$$\mathcal{D} = \{(k,\xi) : 0 \le k,\, 0 \le \xi \le k\}.$$

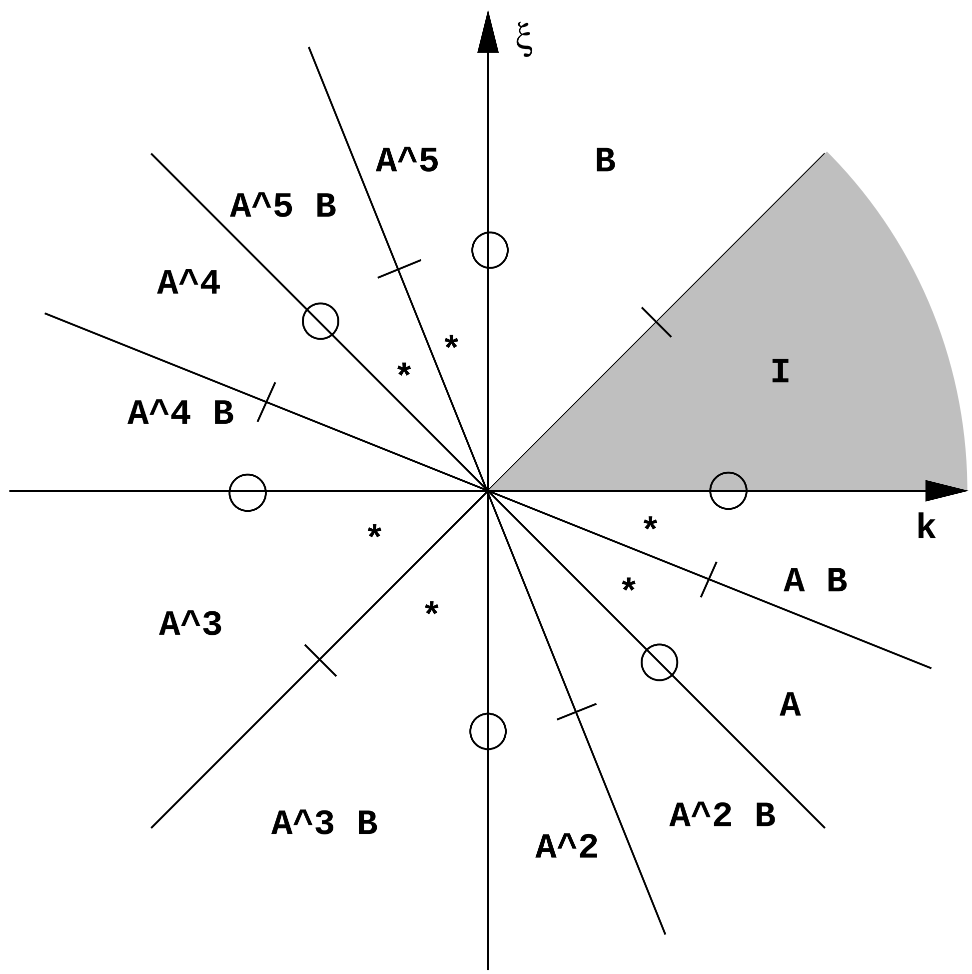

Figure 1: Symmetries of the kernel Λ in the Fourier plane. The fundamental domain $\mathcal{D}$ is shaded. The images of $\mathcal{D}$ under the symmetry group are labelled by the corresponding group element. The circles and lines identify images of corresponding edges. The regions in which the values of Λ are the complex conjugate of its values in $\mathcal{D}$ are indicated with a star.

The images of $\mathcal{D}$ under the group are shown in Figure 1. The group maps the fundamental domain into two wedges in each of the first and third quadrants, and four wedges in each of the second and fourth quadrants.

The kernel Λ is completely determined by its values Λ_0 on $\mathcal{D}$, since

$$\Lambda \circ A = \Lambda^*, \qquad \Lambda \circ B = \Lambda. \tag{4.3}$$

Explicitly, if $\Lambda_0(k,\xi)$ is a symmetric function of (k,ξ) that is defined on the first quadrant of the (k,ξ)-plane, then

$$\begin{aligned}
\Lambda(k,\xi) &= \Lambda_0(k,\xi), & k \geq 0, \xi \geq 0,\\
\Lambda(k,\xi) &= \Lambda_0^*(k+\xi,-k), & k+\xi \geq 0, k \leq 0,\\
\Lambda(k,\xi) &= \Lambda_0(-k-\xi,\xi), & k+\xi \leq 0, \xi \geq 0,\\
\Lambda(k,\xi) &= \Lambda_0^*(-k,-\xi), & k \leq 0, \xi \leq 0,\\
\Lambda(k,\xi) &= \Lambda_0(-k-\xi,k), & k \geq 0, k+\xi \leq 0,\\
\Lambda(k,\xi) &= \Lambda_0^*(k+\xi,-\xi), & k+\xi \geq 0, \xi \leq 0.
\end{aligned} \tag{4.4}$$

Any two images of $\mathcal{D}$ with nonempty intersection overlap on a common edge which is fixed by the symmetry group. Thus, if Λ is continuous on $\mathcal{D}\backslash\{(0,0)\}$ and real-valued on the edges of $\mathcal{D}$, then its extension to $\mathcal{P}\backslash\{(0,0)\}$ is continuous; but if Λ has nonzero imaginary part on the edges, then its extension is discontinuous.

The kernel $\Lambda(k,\xi)$ of Section 4 must be invariant under the action of the symmetry group given in (4.3), and a group-invariant function may be obtained by averaging an arbitrary function with respect to the action of the group. For example, if $F(k,\xi)$ is any symmetric function of (k,ξ) such that $F(-k,-\xi) = F^*(k,\xi)$, then

$$\Lambda(k,\xi) = \frac{1}{3}\left[F(k,\xi) + F^*(k+\xi,-k) + F(-k-\xi,\xi)\right]. \tag{4.5}$$

satisfies (3.4)–(3.6).

5 Sound waves

In canonical coordinates, the normalized three-wave interaction coefficients for sound waves in a compressible fluid are given by [17]

$$T(k+\xi,k,\xi) = \left([k+\xi]\,k\xi\right)^{1/2}, \qquad k,\xi \geq 0.$$

The corresponding kernel Λ is

$$\Lambda(k,\xi) = \left(|k+\xi||k||\xi|\right)^{1/2}. \tag{5.1}$$

This kernel scales according to (3.9) with $\nu = 3/2$, as required by the dimensional analysis in Section 2 with $d = n$.

We introduce noncanonical, scale-invariant variables

$$f(k) = \sqrt{|k|}a(k). \tag{5.2}$$

Use of (5.1) and (5.2) in (3.3) implies that $f(k)$ satisfies the equation

$$if_t(k) = ckf(k) + k\int_{-\infty}^{\infty} f(k-\xi)f(\xi)\,d\xi. \tag{5.3}$$

We define a spatial waveform $u(x)$ to be the inverse Fourier transform of $f(k)$,

$$u(x) = \int_{-\infty}^{\infty} f(k)e^{ikx}\,dk. \tag{5.4}$$

Taking the inverse Fourier transform of (5.3), we get an inviscid Burgers equation for u,

$$u_t + cu_x + \left(u^2\right)_x = 0, \tag{5.5}$$

which is well-known in nonlinear acoustics.

More generally, a scale-invariant bulk wave in any system has $d = n$, so that the kernel Λ satisfies (3.9) with $\nu = 3/2$. We may define a new function $\Gamma(k,\xi)$ by

$$\Gamma(k,\xi) = \frac{\Lambda(k,\xi)}{\left(|k+\xi||k||\xi|\right)^{1/2}}.$$

The kernel Γ is homogeneous of degree zero, meaning that for $\lambda > 0$,

$$\Gamma(\lambda k, \lambda\xi) = \Gamma(k,\xi). \tag{5.6}$$

Moreover, if Λ satisfies (3.4)–(3.6), then so does Γ. The variable $f(k)$ defined in (5.2) satisfies

$$if_t(k,t) = ckf(k,t) + k\int_{-\infty}^{\infty} \Gamma(k-\xi,\xi)f(k-\xi,t)f(\xi,t)\,d\xi,$$

and the inverse Fourier transform $u(x)$ of $f(k)$, defined in (5.4), satisfies

$$u_t(x,t) + \left(cu(x,t) + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x-y,x-z)u(y,t)u(z,t)\,dydz\right)_x = 0, \tag{5.7}$$

where K is proportional to the inverse Fourier transform of Γ,

$$K(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma(k,\xi)e^{ikx+i\xi y}\,dkd\xi. \tag{5.8}$$

In general, the kernel K is a distribution, and the convolution in (5.7) is to be understood in the distributional sense. It follows from (3.4)–(3.6), (5.6), and (5.8) that K satisfies the dual symmetry conditions

$$\begin{aligned}
&K(x,y) = K(y,x), && \text{(symmetry)} && (5.9)\\
&K(x,y) \in \mathbb{R}, && \text{(reality)} && (5.10)\\
&K(-x,-x+y) = K(x,y), && \text{(Hamiltonian)} && (5.11)\\
&K(\lambda x,\lambda y) = \lambda^{-2}K(x,y), \quad \lambda > 0. && \text{(scale-invariance)} && (5.12)
\end{aligned}$$

If $\Gamma = 1$, then $K(x,y) = \delta(x)\delta(y)$, and (5.7) reduces to (5.5). For nonconstant $\Gamma(k,\xi)$, Eq. (5.7) is a nonlocal generalization of the inviscid Burgers equation (5.5). The piecewise constant kernel Γ given by

$$\Gamma(k,\xi) = \alpha + \beta\sigma(k+\xi)\sigma(k)\sigma(\xi),$$

where α, β are real constants, and $\sigma(k)$ is defined in (3.12), leads to the equation studied in Ref. [15],

$$u_t + \left(cu + \alpha u^2 + \beta\mathbb{H}\left[(\mathbb{H}[u])^2\right]\right)_x = 0. \tag{5.13}$$

This equation can be reduced to a local inviscid Burgers equation by a transformation of the form (3.15). Specifically, $v = \alpha u - \beta\mathbb{H}[u]$ satisfies

$$v_t + cv_x + \left(v^2\right)_x = 0.$$

Thus, Eq. (5.13) is a nonlocal transformation of a local equation, rather than an intrinsically nonlocal equation. See [18] and [19] for other examples of integro-differential evolution equations that may be reduced to a differential evolution equation by means of a nonlocal transformation.

Eq. (5.7) cannot be reduced to a local equation by a linear, scale-invariant transformation (3.15) when Γ is not piecewise constant.

The transformation

$$\{a(k), a^*(k) : k > 0\} \mapsto \{u(x) : -\infty < x < \infty\},$$

defined by (5.4) and (5.2) with $a^*(k) = a(-k)$, maps the canonical Poisson bracket

$$\{\mathcal{F},\mathcal{G}\} = -i\int_0^{\infty}\left(\frac{\delta\mathcal{F}}{\delta a(k)}\frac{\delta\mathcal{G}}{\delta a^*(k)} - \frac{\delta\mathcal{F}}{\delta a^*(k)}\frac{\delta\mathcal{G}}{\delta a(k)}\right)dk \tag{5.14}$$

to the bracket

$$\{\mathcal{F}, \mathcal{G}\} = -2\pi \int_{-\infty}^{\infty} \frac{\delta \mathcal{F}}{\delta u(x)} \frac{\partial}{\partial x} \left(\frac{\delta \mathcal{G}}{\delta u(x)} \right) dx,$$

with associated symplectic operator

$$\mathbb{J} = -2\pi \frac{\partial}{\partial x}. \tag{5.15}$$

Using (5.2) and (5.4) in (3.1), we find that $\mathcal{H}$ is given as a function of $u(x)$ by

$$\begin{aligned} \mathcal{H} &= \frac{c}{4\pi} \int_{-\infty}^{\infty} u^2(x)\, dx \\ &+ \frac{1}{6\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x-y, x-z) u(x) u(y) u(z)\, dx dy dz. \end{aligned} \tag{5.16}$$

Thus, Eq. (5.7) has the Hamiltonian form

$$u_t(x) = \mathbb{J} \left[\frac{\delta \mathcal{H}}{\delta u(x)} \right], \tag{5.17}$$

as may be verified directly by the computation of the functional derivative of $\mathcal{H}$ in (5.16) with respect to $u(x)$ and the simplification of the result using (5.11).

6 Rayleigh waves

The Hamiltonian structure of the equations for weakly nonlinear Rayleigh waves is given in Refs. [3, 10]. For hyperelastic, isotropic solids, the evolution equation may be written in the form (3.3) with the kernel [1, 8]

$$\begin{aligned} \Lambda(k,\xi) \quad = \quad & \frac{\alpha|k+\xi||k||\xi|}{|k+\xi|+r|k|+r|\xi|} + \frac{\alpha|k+\xi||k||\xi|}{r|k+\xi|+|k|+r|\xi|} \\ & + \frac{\alpha|k+\xi||k||\xi|}{r|k+\xi|+r|k|+|\xi|} + \frac{\beta|k+\xi||k||\xi|}{r|k+\xi|+|k|+|\xi|} \\ & + \frac{\beta|k+\xi||k||\xi|}{|k+\xi|+r|k|+|\xi|} + \frac{\beta|k+\xi||k||\xi|}{|k+\xi|+|k|+r|\xi|} \\ & + \frac{\gamma|k+\xi||k||\xi|}{|k+\xi|+|k|+|\xi|}. \end{aligned} \tag{6.1}$$

In (6.1), the constant $r = c_t/c_l$ is the ratio of the solid's transverse and longitudinal wave speeds, c_t and c_l, and α, β, and γ are real constants.

The kernel (6.1) satisfies the symmetry conditions (3.4)–(3.6) and the scaling condition (3.9) with $\nu = 2$. This value of the scaling exponent ν is consistent with the dimensional analysis in Section 2 for surface waves with $d = n - 1$.

Hamilton *et. al.* [2] obtained a model kernel for nonlinear Rayleigh waves by neglecting the terms proportional to α and β in (6.1). After a convenient normalization, their model kernel is

$$\Lambda(k,\xi) = \frac{2|k+\xi||k||\xi|}{|k+\xi| + |k| + |\xi|}. \tag{6.2}$$

The same kernel occurs in the equation for weakly nonlinear surface waves on a tangential discontinuity in magnetohydrodynamics [14]. Since this kernel provides a basic example of surface wave kernels, we consider it in more detail.

The expression (6.2) for Λ may be written in the form (4.4) as

$$\Lambda(k,\xi) = \begin{cases} k\xi & k \geq 0,\ \xi \geq 0, \text{ or } k \leq 0,\ \xi \leq 0, \\ -k(k+\xi) & k+\xi \geq 0,\ k \leq 0, \text{ or } k \geq 0,\ k+\xi \leq 0, \\ -(k+\xi)\xi & k+\xi \leq 0,\ \xi \geq 0, \text{ or } k+\xi \geq 0,\ \xi \leq 0, \end{cases}$$

and in the form (4.5) as

$$\Lambda(k,\xi) = \frac{1}{2}\left\{|k||\xi| + k\xi + |k+\xi||k| - (k+\xi)k + |k+\xi||\xi| - (k+\xi)\xi\right\}. \tag{6.3}$$

We may write the evolution equation (3.3) for the Fourier transform $a(k)$ of the surface wave profile as

$$a_t(k) + icka(k) + \sigma(k)\int_{-\infty}^{\infty} \Lambda(k-\xi,\xi)a(k-\xi)a(\xi)\,d\xi = 0, \tag{6.4}$$

where $\sigma(k)$ is defined in (3.12).

The surface wave profile $\varphi(x)$ is the inverse Fourier transform of $a(k)$,

$$\varphi(x) = \int_{-\infty}^{\infty} a(k)e^{ikx}\,dk. \tag{6.5}$$

To derive a spatial evolution equation for $\varphi(x)$, we use (6.3) to write

$$\begin{aligned} &\sigma(k)\int_{-\infty}^{\infty} \Lambda(k-\xi,\xi)a(k-\xi)a(\xi)\,d\xi = \\ &\quad \frac{1}{2}\sigma(k)\left\{[|k|a(k)] * [|k|a(k)] - [ika(k)] * [ika(k)]\right\} \\ &\qquad +ik[a(k)] * [|k|a(k)] - |k|[a(k)] * [ika(k)], \end{aligned} \tag{6.6}$$

where the star denotes convolution,

$$(f * g)(k) = \int_{-\infty}^{\infty} f(k - \xi) g(\xi)\, d\xi.$$

We take the inverse Fourier transform of (6.4), using (6.6) and the facts that if $u(x) \leftrightarrow f(k)$ and $v(x) \leftrightarrow g(k)$ are Fourier transform pairs, then

$$u_x(x) \leftrightarrow ikf(k), \qquad \mathbb{H}[u(x)] \leftrightarrow \sigma(k) f(k), \qquad u(x)v(x) \leftrightarrow (f * g)(k),$$

where $\mathbb{H}$ denotes the Hilbert transform with respect to x. The resulting equation for φ is

$$\begin{aligned} \varphi_t + c\varphi_x + \frac{1}{2}\mathbb{H}\left[(\mathbb{H}[\varphi_x])^2 - (\varphi_x)^2 \right] & \\ + (\mathbb{H}[\varphi\varphi_x] - \varphi\mathbb{H}[\varphi_x])_x = 0. & \qquad (6.7) \end{aligned}$$

By use of the Hilbert transform convolution theorem,

$$\mathbb{H}[uv] = u\mathbb{H}[v] + v\mathbb{H}[u] + \mathbb{H}[\mathbb{H}[u]\,\mathbb{H}[v]],$$

this equation may also be written as

$$\varphi_t + c\varphi_x + \frac{1}{2}\left(\mathbb{H}\left[(\mathbb{H}[\varphi])^2\right]\right)_{xx} + \mathbb{H}[\varphi]\,\varphi_{xx} = 0. \qquad (6.8)$$

The transformation

$$\{a(k), a^*(k) : k > 0\} \mapsto \{\varphi(x) : -\infty < x < \infty\},$$

defined by (6.5) with $a^*(k) = a(-k)$, maps the canonical Poisson bracket in (5.14) to the bracket

$$\{\mathcal{F}, \mathcal{G}\} = -2\pi \int_{-\infty}^{\infty} \frac{\delta\mathcal{F}}{\delta\varphi(x)} \mathbb{H}\left[\frac{\delta\mathcal{G}}{\delta\varphi(x)}\right] dx,$$

with associated symplectic operator

$$\mathbb{J} = -2\pi\mathbb{H}. \qquad (6.9)$$

Expressing the Hamiltonian (3.10) of (6.2) and (6.4) in terms of φ, we find that

$$\begin{aligned} \mathcal{H} = & \frac{c}{4\pi}\int_{-\infty}^{\infty} \mathbb{H}[\varphi(x)]\,\varphi_x(x)\, dx \\ & + \frac{1}{4\pi}\int_{-\infty}^{\infty} \varphi(x)\left\{\mathbb{H}[\varphi_x(x)]^2 - \varphi_x^2(x)\right\} dx. \qquad (6.10) \end{aligned}$$

Thus, Eq. (6.7) has the Hamiltonian form

$$\varphi_t(x) = \mathbb{J}\left[\frac{\delta\mathcal{H}}{\delta\varphi(x)}\right],$$

where $\mathbb{J}$ is given in (6.9), and $\mathcal{H}$ is given in (6.10).

We may use (3.11) with $\alpha = 0$ and $\beta = 1$ to obtain an equivalent equation for the Hilbert transform $\psi = \mathbb{H}[\varphi]$ of φ with respect to x. From (3.14) and (6.2), the Fourier transform b of ψ satisfies (3.13) with the kernel

$$\Gamma(k,\xi) = \frac{-2i(k+\xi)k\xi}{|k+\xi|+|k|+|\xi|}.$$

The spatial form of the equation for ψ is the equation given by Hamilton *et. al.* (see Eq. (38) in [2])

$$\psi_t + c\psi_x = \frac{1}{2}\left(\psi^2\right)_{xx} + \mathbb{H}\left[\psi\mathbb{H}\left[\psi_{xx}\right]\right]. \tag{6.11}$$

We may also obtain this equation directly by taking the Hilbert transform of (6.8)

The spatial form of the equation for general surface wave kernels appears to be complicated [1, 20] and is being investigated further.

7 Wedge waves

A nonlinear elastic wave travelling along the tip of a wedge having two plane traction-free surfaces [12] is scale-invariant with $d = 1$ and $n = 3$. For isotropic materials, the computational evidence suggests that for wide-angle wedges the displacement modes are symmetric about the mid-plane, while for slender wedges they are anti-symmetric [21]. In the first case, quadratically nonlinear interactions may occur, leading to the present theory. In the latter case, the three-wave interaction coefficients vanish, due to the anti-symmetry, so that nonlinear effects appear only when one takes account of four-wave interactions [11]. The Hamiltonian structure of the four-wave interaction equations for scale-invariant waves could be determined in a similar way to the structure of the three-wave interaction equations studied here.

Even in the symmetric case, the functional form of the kernel $\Lambda(k,\xi)$ is presently unknown, since there are no known analytical expressions for the displacement fields of wedge modes, even within an isotropic material. A recent approach for a model problem, exploiting the scale-invariance of the boundary-value problem defining wedge modes, is given in Ref. [22].

A model kernel that is homogeneous of degree ν and contains both the inviscid Burgers kernel (5.1) and the model surface wave kernel (6.2) as special cases is given by

$$\Lambda(k,\xi) = \frac{2^{2\nu-3}\left(|k+\xi||k||\xi|\right)^{\nu-1}}{\left(|k+\xi|+|k|+|\xi|\right)^{2\nu-3}}. \tag{7.1}$$

If $d = 1$ and $n = 3$, then (2.7) implies that $\nu = 5/2$, so (7.1) with $\nu = 5/2$ provides a simple model kernel for nonlinear wedge waves.

If a satisfies (3.3) with the model kernel (7.1), then the scale-invariant variable

$$f(k) = |k|^{\nu-1} a(k),$$

satisfies

$$if_t(k) = ckf(k) + k\int \Gamma(k-\xi,\xi) f(k-\xi) f(\xi)\, d\xi, \tag{7.2}$$

where the homogeneous degree zero kernel Γ is given by

$$\Gamma(k,\xi) = \left(\frac{2|k+\xi|}{|k+\xi|+|k|+|\xi|}\right)^{2\nu-3}. \tag{7.3}$$

The variable $f(k)$ is noncanonical, and Γ does not satisfy the Hamiltonian symmetry condition in (3.6), except when $\nu = 3/2$.

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