Journal of Hyperbolic Differential Equations
Vol. 8, No. 4 (2011) 691–726
© World Scientific Publishing Company
DOI: 10.1142/S0219891611002548

ON THE WEAKLY NONLINEAR KELVIN–HELMHOLTZ INSTABILITY OF TANGENTIAL DISCONTINUITIES IN MHD

JOHN K. HUNTER

Department of Mathematics
University of California at Davis
Davis, California 95616, USA

J. B. THOO

Department of Mathematics, Yuba College
Marysville, California 95901, USA

Received 20 Sep. 2010
Accepted 20 Dec. 2010

Communicated by Philippe G. LeFloch

Abstract. We derive a quadratically nonlinear equation that describes the motion of a tangential discontinuity in incompressible magnetohydrodynamics near the onset of a Kelvin–Helmholtz instability. We show that nonlinear effects enhance the instability and give a criterion for it to occur in terms of a longitudinal strain of the fluid motion along the discontinuity.

Keywords: Nonlinear surface waves; hyperbolic conservation laws; MHD; tangential discontinuities; Kelvin–Helmholtz instability.

Mathematics Subject Classification 2010: 35L65, 74J15, 74J30

1. Introduction

A vortex sheet in an incompressible, inviscid fluid flow is subject to the Kelvin–Helmholtz instability in which, according to linearized stability theory, the growth rate of perturbations is proportional to their wave number. This proportionality follows from dimensional analysis since neither the incompressible Euler equations nor the planar geometry of an undisturbed vortex sheet define any length or time parameters; the only relevant parameter is the jump in the tangential velocity across the vortex sheet.

The Hadamard instability associated with the unbounded growth rate of short-wavelength perturbations means that the linearized vortex-sheet equations are ill-posed in any function space whose norms depend on only finitely many derivatives. This fact raises doubts about the use of the incompressible Euler equations in this problem, since the equations neglect all small-scale regularizing effects. The same difficulty arises when scale-invariant conservation laws are used to model other instabilities, such as the Rayleigh–Taylor instability.

One would like to understand what this linearized ill-posedness means for the fully nonlinear problem. For example, can nonlinear effects help control a Hadamard instability and reduce the influence of unstable small-scales on the large-scale motion or do they enhance the instability? Motivated in part by these issues, Beale and Schaeffer [3] considered some model equations that are linearly ill-posed but nonlinearly well-posed. These equations, however, were not derived from any physical problem and bear no relation to the Kelvin–Helmholtz instability.

The purpose of this paper is to derive equations for a weakly nonlinear Kelvin–Helmholtz instability with the aim of studying the effects of nonlinearity in as simple a setting as possible. The resulting equations are analogous to Ginzbug–Landau equations, which have been derived in many bifurcation problems, except that in the scale-invariant vortex-sheet problem all Fourier modes become unstable at the same parameter value, rather than a narrow band of modes around a wavenumber with the maximum growth rate.

The Kelvin–Helmholtz instability of a vortex sheet in the incompressible Euler equations is strong and the equations do not contain any parameter that can act as a bifurcation parameter. It is therefore not clear how one can derive approximate equations by systematic asymptotics (see, however, the model equations in Ambrose [2]).

We therefore study tangential discontinuities in ideal incompressible magnetohydrodynamics (MHD). In general, both the tangential fluid velocity and tangential magnetic field jump across a tangential discontinuity, while the normal components are continuous. The vortex-sheet solution of the incompressible Euler equations is a special case of a tangential discontinuity in which the magnetic field vanishes. If the magnetic field is sufficiently large compared with the jump in the velocity, then the tangential discontinuity is linearly stable, and there is a critical value of the magnetic field below which the discontinuity is unstable. Thus, the magnetic field provides a bifurcation parameter, and we can study the dynamics of the discontinuity close to the onset of a Kelvin–Helmholtz instability.

In addition to the above motivation, the nonlinear dynamics of tangential discontinuities is of interest in connection with various astrophysical problems; for example, the magnetopause and heliopause may be modeled locally as tangential discontinuities.

The main result of this paper is an asymptotic equation for the motion of a tangential discontinuity near a critical value of the magnetic field where the discontinuity loses stability. We show that, after a suitable normalization, the location $y = \phi(x,t)$ of the discontinuity satisfies the following singular integro-differential equation:

$$\phi_{tt} - \mu\phi_{xx} = \left(\frac{1}{2}\mathbf{H}\left[\psi^2\right]_{xx} + \psi\phi_{xx}\right)_x, \quad \psi = \mathbf{H}\left[\phi\right]. \tag{1.1}$$

Here, the constant μ is a bifurcation parameter and $\mathbf{H}$ denotes the Hilbert transform with respect to x, defined by

$$\mathbf{H}\left[e^{ikx}\right] = -i\,\mathrm{sgn}(k)e^{ikx}, \quad \mathbf{H}\left[f\right](x) = \frac{1}{\pi}\,\mathrm{p.v.}\int_{-\infty}^{+\infty} \frac{f(y)}{x-y}\,dy.$$

As μ decreases through zero, the linearization of (1.1) at $\phi = 0$ changes type from a hyperbolic wave equation to an elliptic evolution equation, corresponding to the onset of Hadamard instability in the linearized theory. The linear terms are modified by a nonlocal, quadratically nonlinear term.

As we will show, the nonlinearity in (1.1) does not regularize the Hadamard instability of the linearized equation; instead, it has a destabilizing effect. Specifically (1.1) is locally linearly unstable if

$$2\psi_x > \mu, \tag{1.2}$$

and an initially linearly stable solution typically evolves into the unstable regime. The validity of (1.1) is suspect once this happens. In particular, this result does not support the hope that the nonlinear conservation laws might be better behaved than their Hadamard-unstable linearizations.

In Sec. 2 we summarize the MHD equations and the asymptotic solution for the motion of a tangential discontinuity. In Sec. 3, we discuss some properties of the asymptotic equation (1.1) and derive the instability condition (1.2). In Sec. 4, we present numerical solutions of (1.1) which show the nonlinear development of a Kelvin–Helmholtz instability. In Sec. 5, we derive the asymptotic equation; some details of the lengthy algebra are given in Appendix A. Finally, in Appendix B we show that the asymptotic equation is consistent with a previously derived equation for unidirectional surface waves on a stable tangential discontinuity.

2. Tangential Discontinuities in MHD

In suitably nondimensionalized variables, the velocity $\mathbf{u}$, magnetic field $\mathbf{B}$, and total pressure p of an incompressible, inviscid, conducting fluid satisfy the incompressible MHD equations, whose nonconservative form is

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{B}\cdot\nabla\mathbf{B} + \nabla p = 0, \tag{2.1}$$

$$\mathbf{B}_t + \mathbf{u}\cdot\nabla\mathbf{B} - \mathbf{B}\cdot\nabla\mathbf{u} = 0, \tag{2.2}$$

$$\mathrm{div}\,\mathbf{u} = 0. \tag{2.3}$$

In addition, the magnetic field satisfies

$$\mathrm{div}\,\mathbf{B} = 0.$$

This constraint is preserved by the evolution equation (2.2), and we do not need to impose it explicitly in carrying out the expansion.

We consider two-dimensional flows with Cartesian spatial coordinates $\mathbf{x} = (x, y)$. In a planar tangential discontinuity located at $y = 0$, we have

$$\mathbf{u} = \begin{cases} \mathbf{u}^+ \text{ if } y > 0 \\ \mathbf{u}^- \text{ if } y < 0 \end{cases}, \quad \mathbf{B} = \begin{cases} \mathbf{B}^+ \text{ if } y > 0 \\ \mathbf{B}^- \text{ if } y < 0 \end{cases},$$

where the constant vectors

$$\mathbf{u}^\pm = (U^\pm, 0), \quad \mathbf{B}^\pm = (B^\pm, 0)$$

are in the x-direction, and the total pressure p is continuous across the discontinuity.

Let $U = (U^+, U^-)$, $B = (B^+, B^-)$ and define

$$\Delta(U, B) = \frac{1}{2}\left[(B^+)^2 + (B^-)^2\right] - \frac{1}{4}(U^+ - U^-)^2. \tag{2.4}$$

A planar tangential discontinuity is linearly stable if $\Delta(U, B) > 0$ and subject to a Kelvin–Helmholtz instability if $\Delta(U, B) < 0$ [8]. Thus, a vortex sheet is unstable in the absence of a magnetic field, but it is stabilized by a magnetic field that is sufficiently strong compared with the jump in velocity. This may be understood as a result of the opposition of growth in the perturbations by the "tension" in the magnetic field lines.

Although the discontinuity is stable if $\Delta > 0$, it is not uniformly stable [6, 10]. Surface waves propagate along it with distinct linearized speeds λ_1, λ_2 given by

$$\lambda_1 = \frac{1}{2}(U^+ + U^-) - \sqrt{\Delta}, \quad \lambda_2 = \frac{1}{2}(U^+ + U^-) + \sqrt{\Delta}. \tag{2.5}$$

These surface waves have finite energy and carry disturbances that decay exponentially away from the discontinuity into the interior of the fluid. As Δ decreases through zero, the wave speeds λ_1, λ_2 coalesce and become complex, corresponding to the onset of a Kelvin–Helmholtz instability.

Here, we investigate the effects of weak nonlinearity near the critical case when $\Delta(U, B) = 0$. Suppose that (U, B) depends on a small positive parameter ϵ and

$$U^\pm = U_0^\pm + \epsilon U_1^\pm + O(\epsilon^2), \quad B^\pm = B_0^\pm + \epsilon B_1^\pm + O(\epsilon^2) \tag{2.6}$$

as $\epsilon \to 0^+$ where $\Delta(U_0, B_0) = 0$, meaning that

$$(B_0^+)^2 + (B_0^-)^2 = \frac{1}{2}(U_0^+ - U_0^-)^2. \tag{2.7}$$

Then

$$\Delta(U, B) = \epsilon\mu + O(\epsilon^2) \tag{2.8}$$

as $\epsilon \to 0^+$ where

$$\mu = B_0^+ B_1^+ + B_0^- B_1^- - \frac{1}{2}(U_0^+ - U_0^-)(U_1^+ - U_1^-). \tag{2.9}$$

The parameter μ plays the role of a scaled bifurcation parameter. The planar tangential discontinuity is linearly stable for small positive values of ϵ if $\mu > 0$ and linearly unstable if $\mu < 0$. The surface waves speeds λ_1, λ_2 coalesce at $\epsilon = 0$, when they have the value

$$\lambda_0 = \frac{1}{2}(U_0^+ + U_0^-). \tag{2.10}$$

We suppose that the perturbed location of the tangential discontinuity is given by

$$y = \zeta(x, t; \epsilon),$$

where ζ has an asymptotic expansion of the form

$$\zeta(x, t; \epsilon) = \epsilon\zeta_1(x - \lambda_0 t, \epsilon^{1/2} t) + O(\epsilon^{3/2}) \quad \text{as } \epsilon \to 0^+.$$

This ansatz describes the evolution of a small-amplitude perturbation in the location of the discontinuity of the order ϵ in a reference frame moving with the critical linearized wave speed λ_0 over long times of the order $\epsilon^{-1/2}$. This time-scale is the one over which quadratically nonlinear effects become significant when a pair of real linearized wave-speeds coalesce and become complex [5].

We show that $\zeta_1(\theta, \tau)$ satisfies the equation

$$\zeta_{1\tau\tau} - \mu\zeta_{1\theta\theta} = \nu\left(\frac{1}{2}\mathbf{H}\left[\eta_1^2\right]_{\theta\theta} + \eta_1\zeta_{1\theta\theta}\right)_\theta, \quad \eta_1 = \mathbf{H}\left[\zeta_1\right], \tag{2.11}$$

where $\mathbf{H}$ denotes the Hilbert transform with respect to θ, μ is given by (2.9), and

$$\nu = -\frac{1}{2}[\![B_0^2]\!] = -\frac{1}{2}[(B_0^+)^2 - (B_0^-)^2]. \tag{2.12}$$

The coefficient ν in (2.11) vanishes if $|B_0^+| = |B_0^-|$, in which case the critical surface wave speed λ_0 is equal to one of the the bulk Alfvén speeds $U_0^\pm \pm B_0^\pm$ on either side of the discontinuity, with an appropriate choice of signs. This resonance leads to a degeneracy in the expansion, and we assume that $\nu \neq 0$. Writing $(x, t) = (\theta, \tau)$ and $\phi(x, t) = \nu\zeta_1(\theta, \tau)$ in (2.11), we get the normalized asymptotic equation (1.1).

We may give a heuristic, physical explanation of the nonlinear instability condition (1.2) as follows: when $|B_0^+| \neq |B_0^-|$, so that $\nu \neq 0$ and (1.1) applies, the discontinuity can lose stability by moving toward the less stable side with the weaker magnetic field.

To explain this in more detail, we first write (1.2) in terms of the original physical variables (x, y, t). Let $y = \zeta(x, t)$ denote the location of the tangential discontinuity and $\eta(x, t)$ the Hilbert transform of ζ with respect to x. Since

$$\zeta(x, t) \sim \frac{\epsilon}{\nu}\phi\big(x - \lambda_0 t, \epsilon^{1/2} t\big), \quad \eta(x, t) \sim \frac{\epsilon}{\nu}\psi\big(x - \lambda_0 t, \epsilon^{1/2} t\big)$$

it follows from (1.2), (2.8) and (2.12) that the tangential discontinuity is unstable near the onset of Kelvin–Helmholtz instability if

$$-[\![B^2]\!]\eta_x > \Delta(U, B) \tag{2.13}$$

where $[\![B^2]\!]$ is defined as in (2.12) and Δ is defined in (2.4). As we will show, up to a linearized approximation, the Hilbert transform η of the vertical displacement

ζ is the perturbation in the tangential displacement of material particles along the discontinuity. Thus, (2.13) states that the discontinuity is locally unstable for $\Delta > 0$ if the perturbation in the longitudinal strain is sufficiently large and has an appropriate sign.

To see that η is a transverse displacement, we note that the velocity perturbations induced by the motion of the discontinuity are irrotational. The velocity potential Φ is harmonic and decays as $|y| \to \infty$. Hence its Fourier–Laplace transform in $y > 0$ has the form $\widehat{\Phi}e^{ikx-|k|y}$, and the transforms of the x, y perturbation velocities $u = \Phi_x$, $v = \Phi_y$ have the forms

$$ik\widehat{\Phi}e^{ikx-|k|y}, \quad -|k|\widehat{\Phi}e^{ikx-|k|y},$$

respectively. It follows that

$$\widehat{u} = -i(\operatorname{sgn} k)\widehat{v} \quad \text{on} \quad y = 0^+,$$

so $u = \mathbf{H}[v]$. Similarly, in $y < 0$ the transform of the velocity potential has the form $\widehat{\Phi}e^{ikx+|k|y}$ and $u = -\mathbf{H}[v]$ on $y = 0^-$. Since the normal velocity v is continuous across the discontinuity, the tangential velocity perturbations u have opposite signs on either side of the discontinuity. According to linearized theory, these relations between the velocity perturbations u, v imply the same relations between the perturbations in the material particle locations x, y relative to the unperturbed flows with velocity $U^{\pm}$ in the x-direction. In the linearized theory, we may also neglect the difference between the Eulerian coordinate x and a corresponding Lagrangian coordinate X in the spatial dependence of the perturbations. Thus, the location of material particles on $y = 0^{\pm}$ relative to the unperturbed flows is given approximately by

$$x \sim X \pm \eta(X,t), \quad y \sim \zeta(X,t), \quad \eta = \mathbf{H}[\zeta]\,. \tag{2.14}$$

The Hilbert transform η of the y-displacement ζ of the discontinuity may therefore be interpreted as the perturbation in the tangential x-displacement of material particles on the discontinuity.

To interpret the sign condition on η_x, let us assume that $\Delta > 0$, so that the planar discontinuity is linearly stable, and suppose for definiteness that $|B^+| < |B^-|$ so that $[\![B^2]\!] < 0$. Then (2.13) implies that the instability occurs when

$$\eta_X > -\frac{\Delta}{[\![B^2]\!]}$$

is sufficiently large and positive. In that case, it follows from (2.14) that $x_X > 1$ on $y = 0^+$ and $x_X < 1$ on $y = 0^-$, meaning that there is compression of material particles on the lower part of the discontinuity where the magnitude of the magnetic field is larger, and expansion of material particles on the upper part of the discontinuity where the magnitude of the magnetic field is smaller. This typically corresponds to formation of a "finger" in the discontinuity that extends into the upper half-space with the weaker magnetic field. Thus, the loss of stability of the discontinuity is associated with its movement toward the side with the less-stabilizing magnetic field. The numerical solutions of (1.1) in Sec. 4 illustrate this phenomenon.

3. The Asymptotic Equation

To examine the local linearized well-posedness of (1.1), we write the equation as

$$\phi_{tt} = \mu\phi_{xx} + \left(\mathbf{H}\left[\psi_x^2\right] - [\psi, \mathbf{H}]\psi_{xx}\right)_x, \tag{3.1}$$

where

$$[\psi, \mathbf{H}] = \psi\mathbf{H} - \mathbf{H}\psi$$

denotes the commutator of the multiplication operator by ψ and the Hilbert transform $\mathbf{H}$. Here, we have used the identity $\mathbf{H}^2 = -\mathbf{I}$, which implies that $\phi_{xx} = -\mathbf{H}\left[\psi_{xx}\right]$.

Let

$$\phi = \phi_0 + \phi', \quad \psi = \psi_0 + \psi', \quad \psi_0 = \mathbf{H}\left[\phi_0\right], \quad \psi' = \mathbf{H}\left[\phi'\right], \tag{3.2}$$

where $\phi_0(x, t)$ is a smooth solution of (3.1) and $\phi'(x, t)$ is a small perturbation. Using (3.2) in (3.1) and linearizing the resulting equation for ϕ', we get

$$\phi'_{tt} = \mu\phi'_{xx} + \left(2\mathbf{H}\left[\psi_{0x}\psi'_x\right] - [\psi_0, \mathbf{H}]\psi'_{xx} - [\psi', \mathbf{H}]\psi_{0xx}\right)_x.$$

Since ψ_0 is smooth, the commutator $[\psi_0, \mathbf{H}]$ is a smoothing pseudo-differential operator of order $-\infty$. Thus, neglecting smoothing terms, which amounts to "freezing" coefficients that depend on ψ_0 and its derivatives, commuting multiplication operators that depend on ψ_0 with $\mathbf{H}$, and writing $\mathbf{H}\left[\psi'\right] = -\phi'$, we find that rapidly varying perturbations ϕ' satisfy

$$\phi'_{tt} = (\mu - 2\psi_{0x})\phi'_{xx} - 3\psi_{0xx}\phi'_x - \psi_{0xxx}\phi'.$$

This equation is an elliptic evolution equation when $2\psi_{0x} > \mu$. Thus, (1.1) is locally linearly ill-posed in any Sobolev space when (1.2) holds. This condition agrees with the results of numerical simulations shown in Sec. 4.

Equation (1.1) inherits a variational and Hamiltonian structure from MHD. The variational principle for (1.1) is

$$\delta \int L(\phi, \phi_t)\, dxdt = 0,$$

where

$$L(\phi, \phi_t) = \frac{1}{2}\phi_t|\partial_x|^{-1}\left[\phi_t\right] - \frac{1}{2}\mu\phi_x|\partial_x|^{-1}\left[\phi_x\right] - \psi\phi_x\psi_x, \quad \psi = \mathbf{H}\left[\phi\right].$$

Here, $|\partial_x| = \mathbf{H}\partial_x$ denotes the positive, self-adjoint operator with symbol $|k|$ and $|\partial_x|^{-1}$ its formal inverse. The corresponding Hamiltonian form is

$$\begin{pmatrix}\phi\\ p\end{pmatrix}_t = \begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\begin{pmatrix}\delta\mathcal{H}/\delta\phi\\ \delta\mathcal{H}/\delta p\end{pmatrix},$$

where $p = |\partial_x|^{-1}[\phi_t]$ and

$$\mathcal{H}(\phi, p) = \int \left\{ \frac{1}{2} p |\partial_x| [p] + \frac{1}{2} \mu \phi_x |\partial_x|^{-1} [\phi_x] + \psi \phi_x \psi_x \right\} dx.$$

We may also write (1.1) in spectral form. Denoting the spatial Fourier transform of ϕ by $\widehat{\phi}$, where

$$\phi(x,t) = \int \widehat{\phi}(k,t) e^{ikx} dx,$$

and taking the Fourier transform of (1.1), we find that

$$\widehat{\phi}_{tt}(k,t) + \mu k^2 \widehat{\phi}(k,t) = |k| \int_{-\infty}^{+\infty} \Lambda(-k, k-\xi, \xi) \widehat{\phi}(k-\xi, t) \widehat{\phi}(\xi, t) d\xi, \tag{3.3}$$

where

$$\Lambda(k_1, k_2, k_3) = \frac{2|k_1||k_2||k_3|}{|k_1| + |k_2| + |k_3|}. \tag{3.4}$$

The coefficient $\Lambda(k_1, k_2, k_3)$ is an interaction coefficient for the three-wave resonant interaction $k_2 + k_3 \mapsto -k_1$ between Fourier coefficients k_1, k_2, k_3 such that $k_1 + k_2 + k_3 = 0$. The symmetry of $\Lambda(k_1, k_2, k_3)$ as a function of (k_1, k_2, k_3) is a consequence of the Hamiltonian structure of the equation. (We could omit the k_1-argument from Λ, but we include it to make the symmetry explicit.)

It is useful to compare the two-way wave equation (1.1) with a corresponding one-way wave equation for unidirectional waves. The one-way wave equation can be derived from (1.1) in the limit $\mu \to +\infty$ by multiple-scale asymptotics, but it is simpler to give an informal derivation.

If μ is large and positive, then a solution of (1.1) is approximately a sum of decoupled right and left moving waves with speeds $\pm\sqrt{\mu}$. For a right-moving wave, we have

$$\phi_t + \sqrt{\mu}\phi_x \sim 0, \quad \phi_{tt} - \mu \phi_{xx} \sim -2\sqrt{\mu}(\phi_t + \sqrt{\mu}\phi_x)_x, \tag{3.5}$$

with an analogous approximation for a left-moving wave. Using the approximation (3.5) in (1.1) and integrating the result with respect to x, we get the unidirectional equation

$$\phi_t + \sqrt{\mu}\phi_x + \frac{1}{2\sqrt{\mu}} \left(\frac{1}{2} \mathbf{H} \left[\psi^2\right]_{xx} + \psi \phi_{xx} \right) = 0. \tag{3.6}$$

This equation was derived in [1] for unidirectional surface waves on a linearly stable tangential discontinuity away from values of (U, B) where it loses stability. We show in Appendix B that the coefficients of this one-way wave equation agree quantitatively with the ones found in [1] in the appropriate asymptotic limit. Related nonlocal one-way equations describe nonlinear Rayleigh waves (see e.g. [9]) and surface waves on phase boundaries [4].

As noted in [7], it follows from the distributional Hilbert transform pair

$$\mathbf{H}\left[|x|^\alpha\right] = -\tan\left(\frac{\alpha\pi}{2}\right) \operatorname{sgn} x |x|^\alpha, \quad \alpha \notin 2\mathbb{Z} + 1$$

that (3.6) has an exact steady distributional solution

$$\phi(x) = \frac{\sqrt{3}}{2} A \operatorname{sgn} x |x|^{2/3}, \quad \psi(x) = \frac{1}{2} A |x|^{2/3},$$

where A is an arbitrary constant. This solution describes the local structure of singularities observed in numerical solutions of (3.6). Similarly, if $\mu > 0$ equation (1.1) has exact distributional traveling wave solutions given by

$$\phi(x,t) = \frac{\sqrt{3}}{2} A \operatorname{sgn}(x - \lambda t) |x - \lambda t|^{2/3} \quad \psi(x,t) = \frac{1}{2} A |x - \lambda t|^{2/3},$$

where $\lambda = \pm\sqrt{\mu}$. These singularities are also observed to form in numerical solutions of (1.1), but when this happens it leads to instability since $|\psi_x| \to \infty$ as $x \to \lambda t$ and (1.2) holds near the singularity.

4. Numerical Solutions of the Asymptotic Equation

We computed spatially periodic solutions of (1.1) by use of a pseudo-spectral method. We normalized the spatial period to 2π without loss of generality. In most computations, we included a spectral viscosity term in the equation to get

$$\phi_{tt} - \mu\phi_{xx} = \left(\frac{1}{2}\mathbf{H}\left[\psi^2\right]_{xx} + \psi\phi_{xx}\right)_x + \varepsilon \mathbf{Q}_N \phi_{txx}, \quad \psi = \mathbf{H}[\phi],$$

where $\mathbf{Q}_N$ denotes the projection onto the Fourier modes with wavenumber greater than N and ε is a numerical viscosity. For numerical solutions with M Fourier modes, we took $N = \sqrt{M}$ and $\varepsilon = 2\pi/M$.

Figures 1–4 show a solution of (1.1) with $\mu = 4$ for linearly stable initial data corresponding roughly to a right-moving wave,

$$\phi(x,0) = \cos x, \quad \phi_t(x,0) = -\sqrt{\mu}\phi_x(x,0). \tag{4.1}$$

The solution is computed using $M = 2^{12}$ Fourier modes and spectral viscosity. The solution agrees well with the corresponding solution [7] of the unidirectional equation (3.6) until it develops a singularity at $t \approx 2.0$, which causes (1.1) to become locally ill-posed. The numerical solution then develops rapidly growing oscillations and cannot be continued past $t \approx 2.20$.

Figures 5–8 show a solution of (1.1) with $\mu = 2.1$ for initial data corresponding roughly to a standing wave,

$$\phi(x,0) = \cos x, \quad \phi_t(x,0) = 0. \tag{4.2}$$

The solution is computed with 2^{12} Fourier modes and spectral viscosity. The Hilbert transform of the initial data is $\psi(x,0) = \sin x$, and the initial data satisfies the stability condition $2\psi_x(x,0) < \mu$ for all $0 \leq x \leq 2\pi$. The solution undergoes a standing-wave oscillation, until singularities forms at $t \approx 2.5$ at $x \approx 2.0$ and $x \approx 4.3$. This leads to the violation of the stability condition, and the numerical solution cannot be continued past $t \approx 2.73$. (If no spectral viscosity is used, the numerical solution breaks down a little earlier at $t \approx 2.60$.)

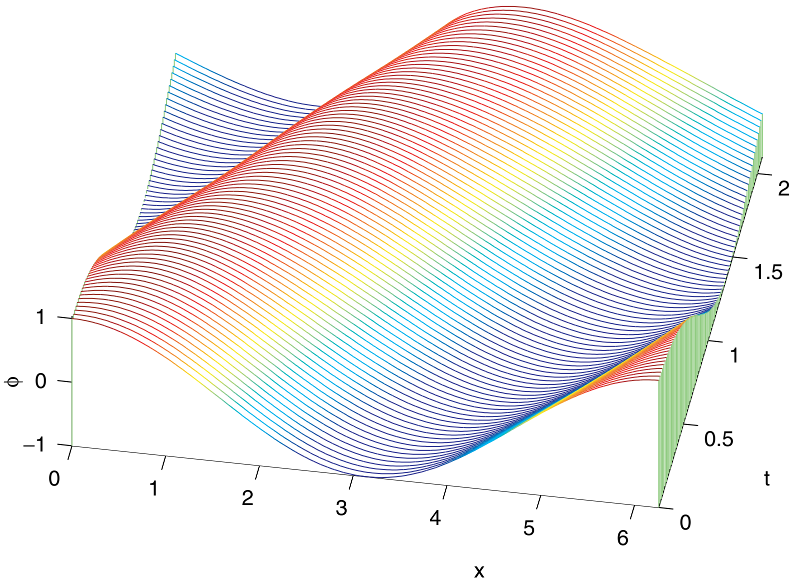

Fig. 1. Surface plot of the solution $\phi(x,t)$ of (1.1) with the traveling-wave initial data (4.1) for $\mu = 4$.

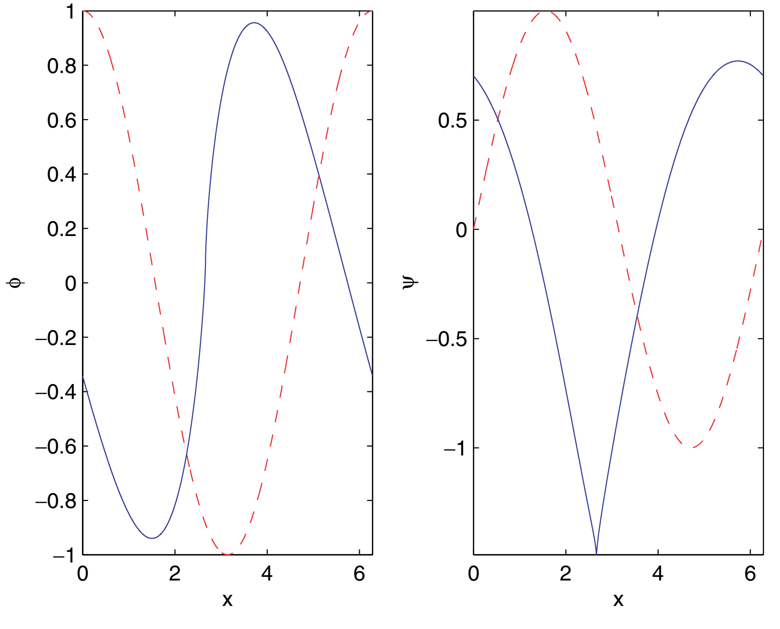

Fig. 2. Solutions for $\phi(x,t)$ and the Hilbert transform $\psi(x,t)$ of (1.1) with initial data (4.1) for $\mu = 4$ at times $t = 0$ (dashed lines) and $t = 2.1$ (solid lines).

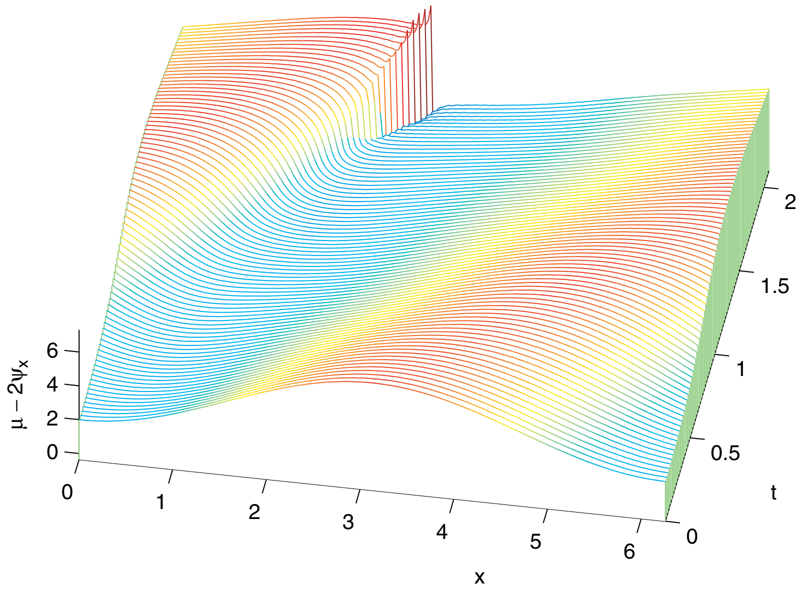

Fig. 3. Type indicator $\mu - 2\psi_x$ for the solution in Fig. 1 with $\mu = 4$.

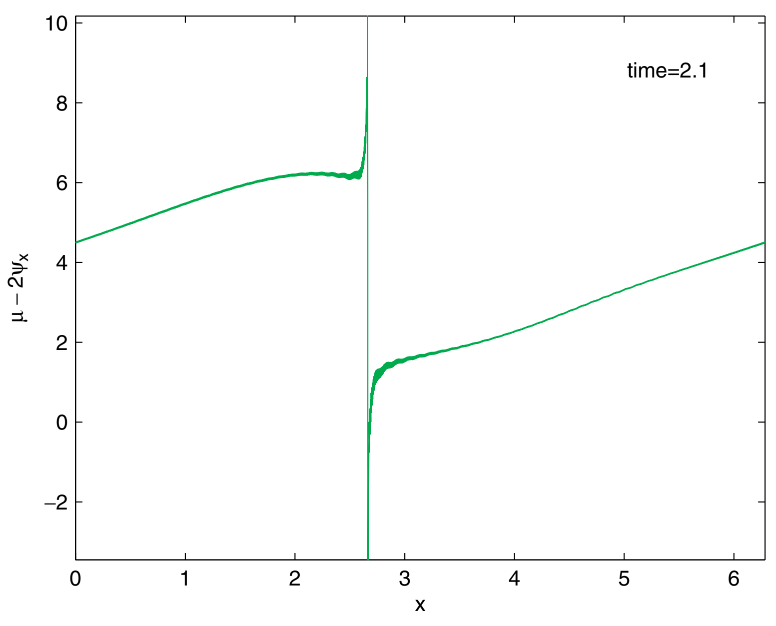

Fig. 4. Type indicator $\mu - 2\psi_x$ for the solution in Fig. 1 with $\mu = 4$ at $t = 2.1$. The solution is locally linearly ill-posed when $\mu - 2\psi_x < 0$.

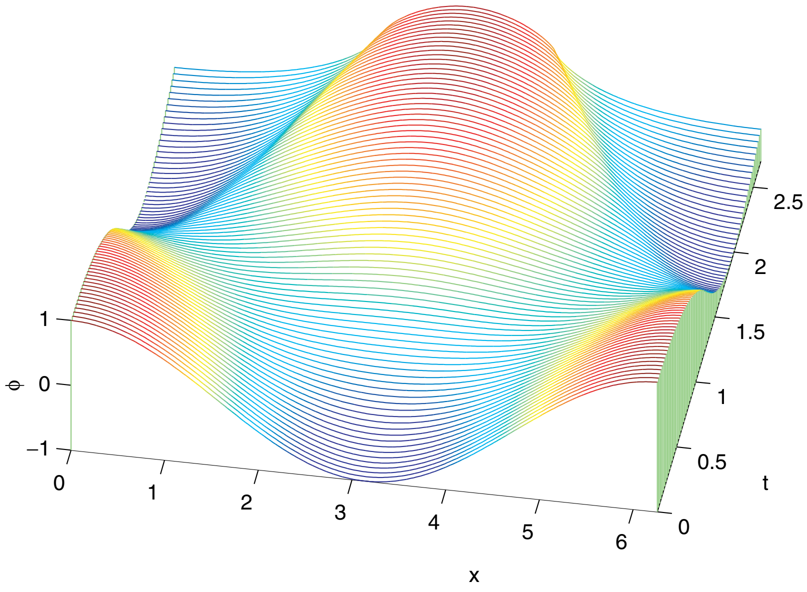

Fig. 5. Surface plot of the solution $\phi(x,t)$ of (1.1) with the standing-wave initial data (4.2) for $\mu = 2.1$.

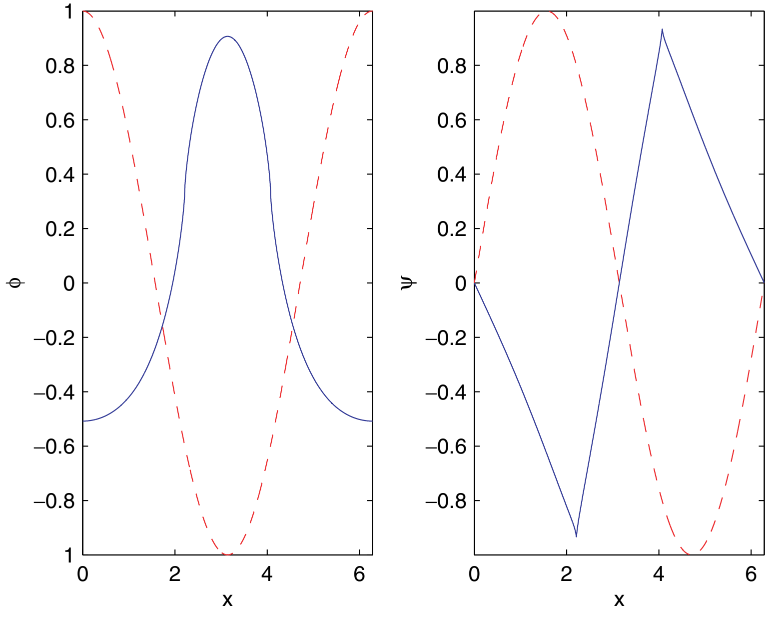

Fig. 6. Solutions for $\phi(x,t)$ and the Hilbert transform $\psi(x,t)$ of (1.1) with initial data (4.2) for $\mu = 2.1$ at times $t = 0$ (dashed lines) and $t = 2.7$ (solid lines).

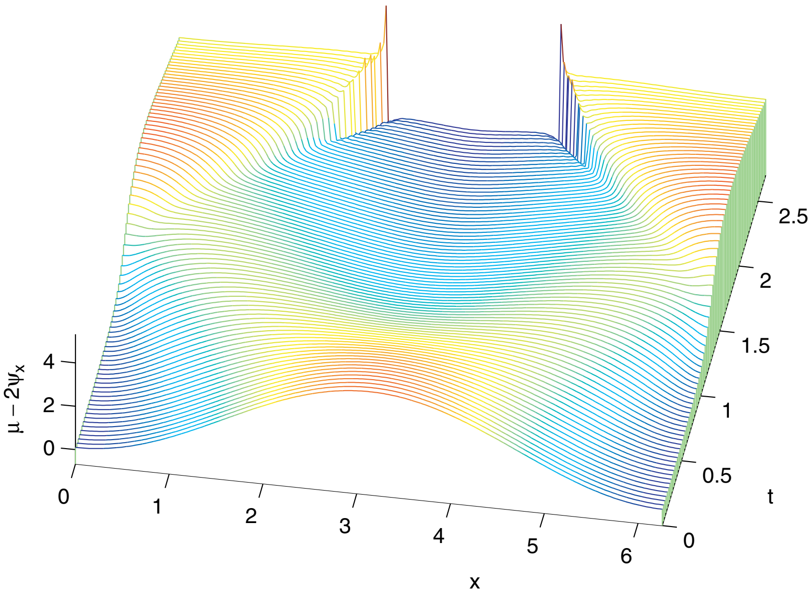

Fig. 7. Type indicator $\mu - 2\psi_x$ for the solution in Fig. 5 with $\mu = 2.1$.

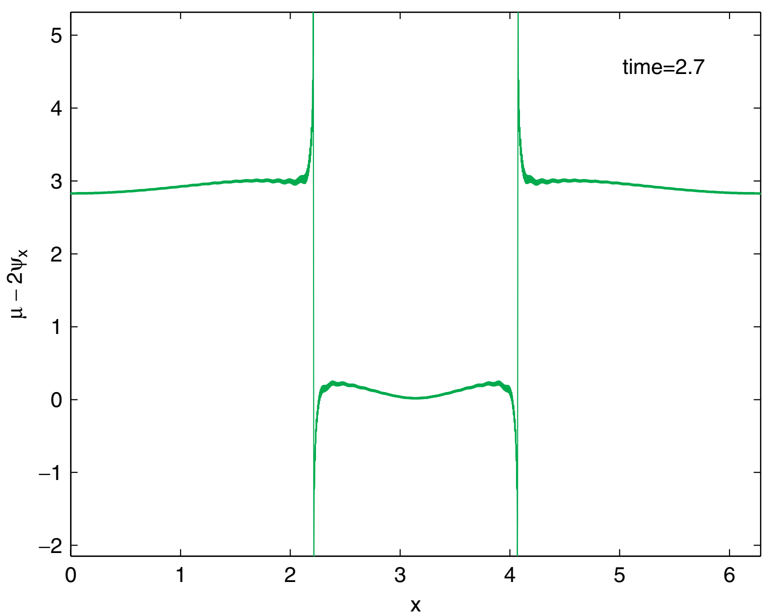

Fig. 8. Type indicator $\mu - 2\psi_x$ for the solution in Fig. 5 with $\mu = 2.1$ at $t = 2.7$.

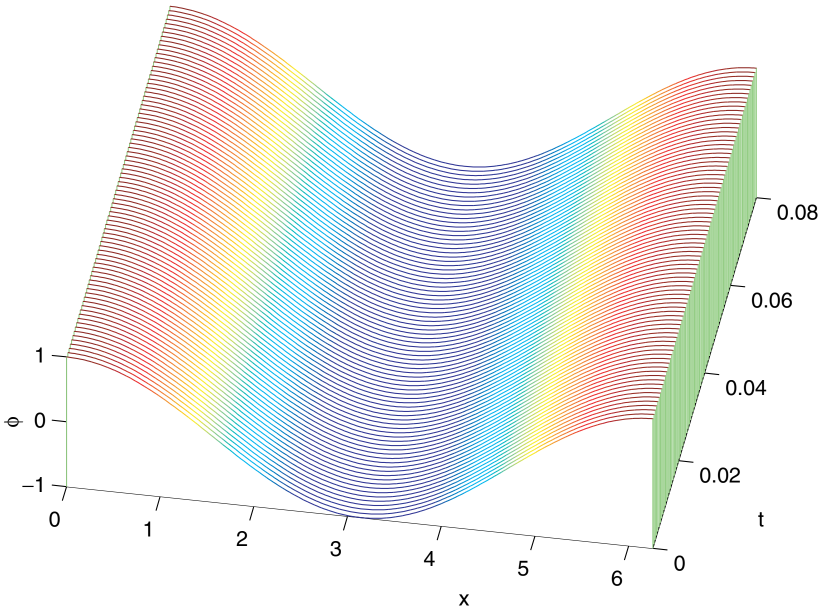

Fig. 9. Surface plot of the solution $\phi(x,t)$ of (1.1) with initial data (4.2) for $\mu = 1.9$.

Figures 9–12 show a numerical solution of (1.1) for $\mu = 1.9$ with the standing-wave initial data (4.2). We use 2^{12} Fourier modes and no spectral viscosity. This initial data is not linearly stable near $x = 0, 2\pi$ where $2\psi_x(x,0) > \mu$, and an instability forms immediately in these regions. The numerical solution cannot be continued past $t \approx 0.0906$. (The inclusion of spectral viscosity allows the numerical solution to be continued to a longer, but still short time, of $t \approx 0.380$.)

The greatly decreased existence time for the initial data in (4.2) with $\mu = 1.9$ in comparison with $\mu = 2.1$ is a consequence of the fact that in the former case the initial data is locally unstable for values of x near 0 and 2π, whereas in the latter case the initial data is locally stable for all values of $0 \le x \le 2\pi$. For $\mu = 1.9$, the solution immediately develops a Hadamard instability near the endpoints $0, 2\pi$. For $\mu = 2.1$, the standing-wave solution remains locally stable over a full oscillation until the profile steepens and forms singularities in which $|\psi_x| \to \infty$ at two interior spatial locations. Near these points, $\mu - 2\psi_x$ evolves from an initially positive value to a negative value and the solution then develops a Hadamard instability.

5. Derivation of the Asymptotic Equation

In the rest of this paper, we describe the derivation of the asymptotic equation (1.1). We consider a tangential discontinuity in the flow of an incompressible, inviscid, conducting fluid in two space dimensions, described by the incompressible MHD

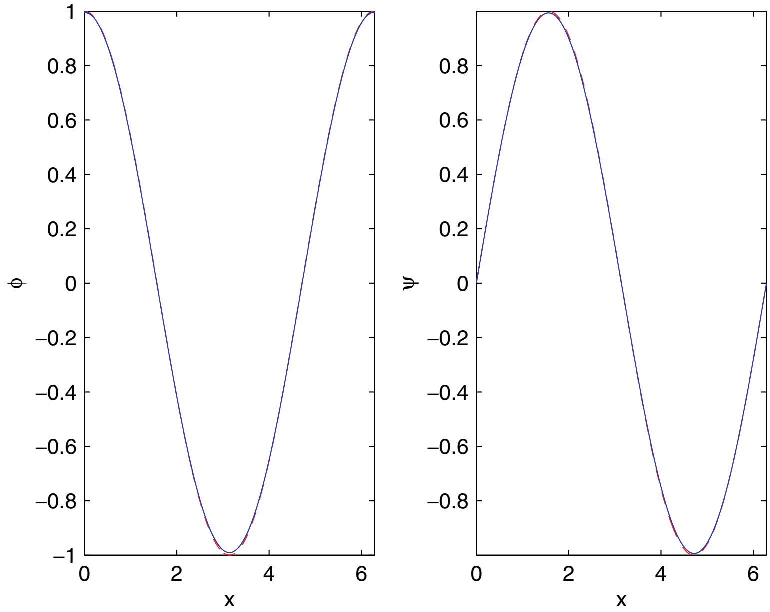

Fig. 10. Solutions for $\phi(x,t)$ and the Hilbert transform $\psi(x,t)$ of (1.1) with initial data (4.2) for $\mu = 1.9$ at $t = 0$ and $t = 0.08$ (the graphs almost coincide).

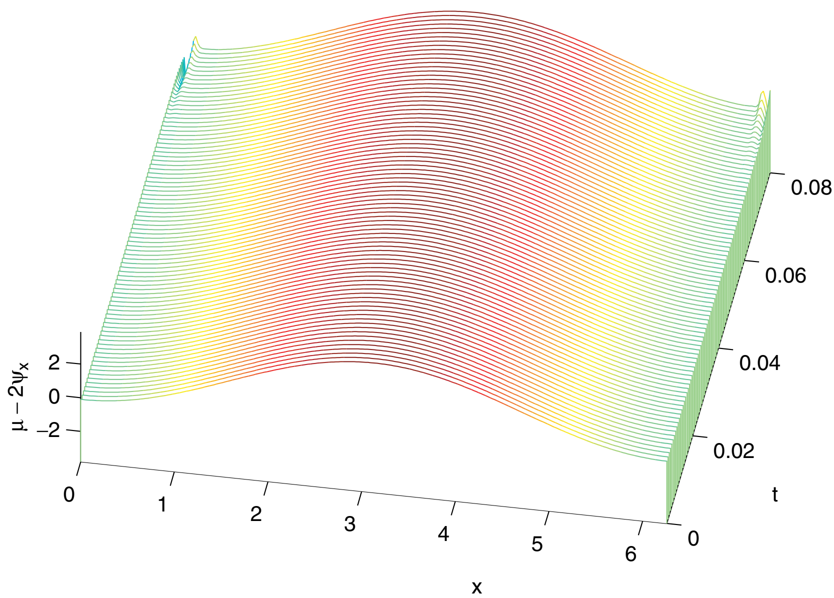

Fig. 11. Type indicator $\mu - 2\psi_x$ for the solution in Fig. 9 with $\mu = 1.9$.

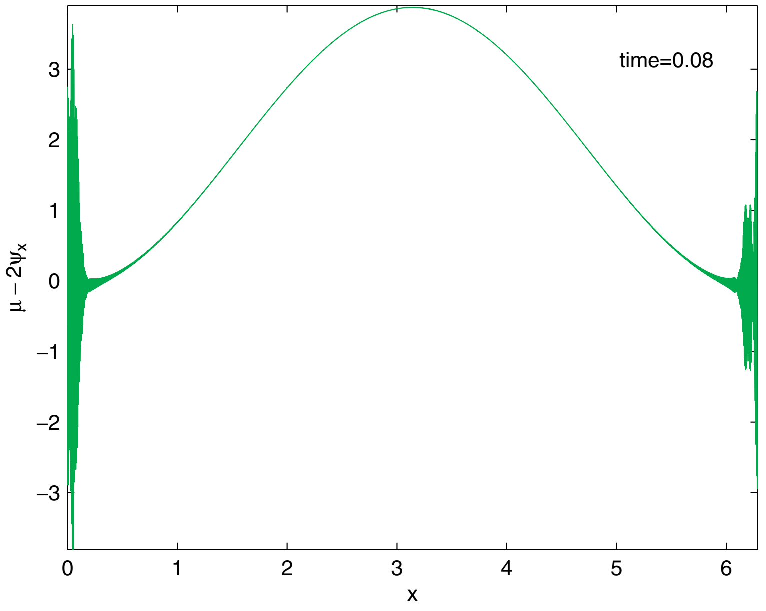

Fig. 12. Type indicator $\mu - 2\psi_x$ for the solution in Fig. 9 with $\mu = 1.9$ at $t = 0.08$.

equations (2.1)–(2.3). We denote the location of the discontinuity by

$$\Phi(x, y, t) = 0, \quad \text{where} \quad \Phi(x, y, t) = \zeta(x, t) - y, \tag{5.1}$$

the half-spaces above and below the discontinuity by

$$\Omega^+ = \{(x, y) \colon y > \zeta(x, t)\}, \quad \Omega^- = \{(x, y) \colon y < \zeta(x, t)\},$$

and the boundary by

$$\partial\Omega = \{(x, y) \colon y = \zeta(x, t)\}.$$

The jump conditions across a tangential discontinuity are [8]

$$\Phi_t + \mathbf{u}^\pm \cdot \nabla\Phi = 0 \quad \text{on } \partial\Omega, \tag{5.2}$$

$$[\![p]\!] = 0 \quad \text{on } \partial\Omega, \tag{5.3}$$

where $\mathbf{u}^\pm$ denotes the value of $\mathbf{u}$ in $\Omega^\pm$, and $[\![p]\!] = p^+ - p^-$ denotes the jump in p across $\partial\Omega$. Equation (5.2) states that the discontinuity moves with the fluid on either side, and implies that the normal component of the velocity is continuous across the discontinuity, while (5.3) states that the total pressure is continuous across the discontinuity. In addition, the magnetic field is tangent to the discontinuity on either side, meaning that

$$\mathbf{B}^\pm \cdot \nabla\Phi = 0 \quad \text{on } \partial\Omega.$$

This condition is the weak form of the divergence-free condition and follows from the other equations, as one may verify directly from the asymptotic solution, so we do not impose it explicitly in carrying out the expansion.

Finally, we impose the decay condition that

$$\mathbf{u} \to (U^{\pm}, 0), \quad \mathbf{B} \to (B^{\pm}, 0), \quad p \to 0 \quad \text{as } y \to \pm\infty, \tag{5.4}$$

where (U, B) have the expansions (2.6) as $\epsilon \to 0^+$ and (U_0, B_0) satisfies (2.7).

The full problem consists of the PDEs (2.1)–(2.3) in $\Omega^{\pm}$, the jump conditions (5.2) and (5.3) on $\partial\Omega$, and the decay condition (5.4). To find an asymptotic solution, we define a spatial variable θ in a reference frame moving with a linearized wave speed λ_0 and a "slow" time variable τ by

$$\theta = x - \lambda_0 t, \quad \tau = \epsilon^{1/2} t, \tag{5.5}$$

and look for a solution of the form

$$\mathbf{u} = \mathbf{u}_0^{\pm} + \epsilon \mathbf{u}_1^{\pm}(\theta, y, \tau) + \epsilon^{3/2}\mathbf{u}_2^{\pm}(\theta, y, \tau) + \epsilon^2 \mathbf{u}_3^{\pm}(\theta, y, \tau) + O(\epsilon^{5/2}), \tag{5.6}$$

$$\mathbf{B} = \mathbf{B}_0^{\pm} + \epsilon \mathbf{B}_1^{\pm}(\theta, y, \tau) + \epsilon^{3/2}\mathbf{B}_2^{\pm}(\theta, y, \tau) + \epsilon^2 \mathbf{B}_3^{\pm}(\theta, y, \tau) + O(\epsilon^{5/2}), \tag{5.7}$$

$$p = \epsilon p_1^{\pm}(\theta, y, \tau) + \epsilon^{3/2} p_2^{\pm}(\theta, y, \tau) + \epsilon^2 p_3^{\pm}(\theta, y, \tau) + O(\epsilon^{5/2}), \tag{5.8}$$

$$\zeta = \epsilon \zeta_1(\theta, \tau) + \epsilon^{3/2}\zeta_2(\theta, \tau) + \epsilon^2 \zeta_3(\theta, \tau) + O(\epsilon^{5/2}). \tag{5.9}$$

We write

$$(\mathbf{u}, \mathbf{B}, p) = (U^{\pm}, 0, B^{\pm}, 0, 0) + \mathbf{U}^{\pm},$$

where

$$\mathbf{U}^{\pm} = (u^{\pm}, v^{\pm}, F^{\pm}, G^{\pm}, p^{\pm})$$

denotes the perturbation from the constant states on either side of the discontinuity; we denote the perturbation in $\mathbf{u}$ by $(u^{\pm}, v^{\pm})$ and the perturbation in $\mathbf{B}$ by $(F^{\pm}, G^{\pm})$. Thus, $\mathbf{U}^{\pm}$ satisfies the decay condition

$$\lim_{y \to \pm\infty} \mathbf{U}^{\pm}(\theta, y, \tau; \epsilon) = 0, \tag{5.10}$$

and in $\Omega^{\pm}$ we have

$$\begin{aligned}
\mathbf{U}^{\pm} &= \epsilon \mathbf{U}_1^{\pm} + \epsilon^{3/2}\mathbf{U}_2^{\pm} + \epsilon^2 \mathbf{U}_3^{\pm} + O(\epsilon^{5/2}), \\
(\mathbf{u}, \mathbf{B}, p) &= (U_0^{\pm}, 0, B_0^{\pm}, 0, 0) + \epsilon\{(U_1^{\pm}, 0, B_1^{\pm}, 0, 0) + \mathbf{U}_1^{\pm}\} \\
&\quad + \epsilon^{3/2}\mathbf{U}_2^{\pm} + \epsilon^2 \mathbf{U}_3^{\pm} + O(\epsilon^{5/2}).
\end{aligned}$$

Before carrying out the expansion in detail, we outline the results. At leading order, $O(\epsilon)$, we obtain the linearized PDEs and jump conditions. We solve the

PDE by Fourier-Laplace transforms. The flow is potential and the Fourier-Laplace modes are proportional to solutions of Laplace's equation of the form $e^{ikx\pm ky}$ where $k \in \mathbb{R}$. The decay condition implies that in $y > 0$ the coefficient of e^{ikx+ky} is zero if $k > 0$ and the coefficient of e^{ikx-ky} is zero if $k < 0$, and conversely in $y < 0$. Use of the solutions of the PDE in the jump conditions gives a homogeneous linear system for the leading order solution. The requirement that it has a nontrivial solution gives (2.10) for the surface wave speed λ_0, and the leading order solution may be expressed in terms of the displacement $\zeta_1(\theta, \tau)$ of the discontinuity, which is arbitrary at this order. This is the linearized surface wave solution. The equations at the next order, $O(\epsilon^{3/2})$, are solvable for the second-order corrections because (2.7) implies that λ_0 is a multiple wave speed associated with a two-by-two Jordan block. At $O(\epsilon^2)$, after solving the nonhomogeneous, linearized PDEs, using the result in the jump conditions, which are Taylor expanded about $y = 0$, and eliminating ζ_3, we obtain a singular nonhomogeneous linear system that is satisfied by the third order corrections $\mathbf{U}_3^{\pm}$ on the boundary. The solvability condition for this system gives the evolution equation for $\zeta_1(\theta, \tau)$.

Following this procedure, we substitute (5.5)–(5.9) into the MHD equations (2.1)–(2.3) and jump conditions (5.2) and (5.3), expand derivatives as

$$\partial_t = -\lambda_0 \partial_\theta + \epsilon^{1/2} \partial_\tau, \quad \partial_x = \partial_\theta, \quad \partial_y = \partial_y,$$

Taylor expand with respect to ϵ, and set the coefficients of powers of ϵ equal to zero in the resulting equations. This gives the following hierarchy of equations:

$O(\epsilon)$:

$$(A^{\pm} - \lambda_0 C)\mathbf{U}_{1\theta}^{\pm} + D\mathbf{U}_{1y}^{\pm} = 0, \tag{5.11a}$$

$$(U_0^{\pm} - \lambda_0)\zeta_{1\theta} - v_1^{\pm} = 0, \tag{5.11b}$$

$$[\![p_1]\!] = 0. \tag{5.11c}$$

$O(\epsilon^{3/2})$:

$$(A^{\pm} - \lambda_0 C)\mathbf{U}_{2\theta}^{\pm} + D\mathbf{U}_{2y}^{\pm} + C\mathbf{U}_{1\tau}^{\pm} = 0, \tag{5.12a}$$

$$\zeta_{1\tau} + (U_0^{\pm} - \lambda_0)\zeta_{2\theta} - v_2^{\pm} = 0, \tag{5.12b}$$

$$[\![p_2]\!] = 0. \tag{5.12c}$$

$O(\epsilon^2)$:

$$(A^{\pm} - \lambda_0 C)\mathbf{U}_{3\theta}^{\pm} + D\mathbf{U}_{3y}^{\pm} + C\mathbf{U}_{2\tau}^{\pm} + \mathbf{H}^{\pm} = 0, \tag{5.13a}$$

$$\zeta_{2\tau} + (U_0^{\pm} - \lambda_0)\zeta_{3\theta} + (U_1^{\pm} + u_1^{\pm})\zeta_{1\theta} - v_3^{\pm} - v_{1y}^{\pm}\zeta_1 = 0, \tag{5.13b}$$

$$[\![p_3 + p_{1y}\zeta_1]\!] = 0. \tag{5.13c}$$

Here,

$$A^{\pm} = \begin{bmatrix} U_0^{\pm} & 0 & -B_0^{\pm} & 0 & 1 \\ 0 & U_0^{\pm} & 0 & -B_0^{\pm} & 0 \\ -B_0^{\pm} & 0 & U_0^{\pm} & 0 & 0 \\ 0 & -B_0^{\pm} & 0 & U_0^{\pm} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

An explicit expression for $\mathbf{H}^{\pm}$, which depends only on $\mathbf{U}_1^{\pm}$, will be given when we solve (5.13).

To solve (5.11)–(5.13), we take the Fourier transform with respect to θ. We denote the Fourier transform of a function $f(\theta)$ by $\widehat{f}(k)$, where

$$\widehat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\theta) e^{-ik\theta}\, d\theta, \quad f(\theta) = \int_{-\infty}^{+\infty} \widehat{f}(k) e^{ik\theta}\, dk.$$

We will also drop the $\pm$ superscripts that indicate we are in $\Omega^{\pm}$ when their use is clear from the context.

5.1. *Order ϵ*

Taking the Fourier transform of (5.11), we get

$$ik(A - \lambda_0 C)\widehat{\mathbf{U}}_1 + D\widehat{\mathbf{U}}_{1y} = 0, \tag{5.14}$$

$$ik(U_0 - \lambda_0)\widehat{\zeta}_1 - \widehat{v}_1 = 0, \tag{5.15}$$

$$[\![\widehat{p}_1]\!] = 0, \tag{5.16}$$

where the jump conditions (5.15) and (5.16) are evaluated on $y = 0$. Equation (5.10) implies the decay condition

$$\lim_{y \to \pm\infty} \widehat{\mathbf{U}}_1(k, y, \tau) = 0. \tag{5.17}$$

As shown in Sec. A.1, a solution of (5.14) is given by

$$\widehat{\mathbf{U}}_1(k, y, \tau) = \alpha_1(k, \tau) e^{-ky}\mathbf{R} + \beta_1(k, \tau) e^{ky}\overline{\mathbf{R}}, \tag{5.18}$$

where α_1 and β_1 are arbitrary complex-valued functions, the bar denotes the complex conjugate, and the null vector $\mathbf{R}$ satisfies

$$[i(A - \lambda_0 C) - D]\mathbf{R} = 0. \tag{5.19}$$

Writing column vectors in row notation for brevity, we find that

$$\mathbf{R} = (c_0, ic_0, -B_0, -iB_0, c_0^2 - B_0^2), \tag{5.20}$$

where

$$c_0^{\pm} = \lambda_0 - U_0^{\pm}. \tag{5.21}$$

The function (5.18) satisfies the decay condition (5.17) if

- For $k > 0$:

$$\alpha_1^- = \beta_1^+ = 0. \tag{5.22}$$

- For $k < 0$:

$$\alpha_1^+ = \beta_1^- = 0. \tag{5.23}$$

Using (5.18) in the jump conditions (5.15) and (5.16), we obtain a system of equations for $(\alpha_1^+, \beta_1^-, \widehat{\zeta}_1)$ if $k > 0$, and a system for $(\beta_1^+, \alpha_1^-, \widehat{\zeta}_1)$ if $k < 0$. (See Sec. A.1.) We find that the systems have nontrivial solutions if and only if

$$(c_0^+)^2 - (B_0^+)^2 + (c_0^-)^2 - (B_0^-)^2 = 0, \tag{5.24}$$

or

$$\lambda_0 = \frac{1}{2}(U_0^+ + U_0^-) \pm \sqrt{\frac{1}{2}[(B_0^+)^2 + (B_0^-)^2] - \frac{1}{4}(U_0^+ - U_0^-)^2}. \tag{5.25}$$

The discriminant in (5.25) vanishes from (2.7), so λ_0 is equal to the critical wave speed (2.10). The solution is then

- For $k > 0$: $\alpha_1^+ = -k\widehat{\zeta}_1, \beta_1^- = k\widehat{\zeta}_1$.
- For $k < 0$: $\alpha_1^- = -k\widehat{\zeta}_1, \beta_1^+ = k\widehat{\zeta}_1$.

Here, $\widehat{\zeta}_1(k, \tau)$ is an arbitrary function.

In summary, (5.18) is a solution of (5.14)–(5.17) if λ_0 given by (2.10) and

- For $k > 0$:

$$\alpha_1^+ = -k\widehat{\zeta}_1, \quad \alpha_1^- = 0, \quad \beta_1^+ = 0, \quad \beta_1^- = k\widehat{\zeta}_1. \tag{5.26}$$

- For $k < 0$:

$$\alpha_1^+ = 0, \quad \alpha_1^- = -k\widehat{\zeta}_1, \quad \beta_1^+ = k\widehat{\zeta}_1, \quad \beta_1^- = 0. \tag{5.27}$$

Substituting α_1 and β_1 into (5.18), we have

$$u_1^{\pm}(\theta, y, \tau) = \mp c_0^{\pm} \int_{-\infty}^{+\infty} |k|\widehat{\zeta}_1(k, \tau) e^{\mp|k|y} e^{ik\theta}\, dk, \tag{5.28}$$

$$v_1^{\pm}(\theta, y, \tau) = -ic_0^{\pm} \int_{-\infty}^{+\infty} k\widehat{\zeta}_1(k, \tau) e^{\mp|k|y} e^{ik\theta}\, dk, \tag{5.29}$$

$$F_1^{\pm}(\theta, y, \tau) = \pm B_0^{\pm} \int_{-\infty}^{+\infty} |k|\widehat{\zeta}_1(k, \tau) e^{\mp|k|y} e^{ik\theta}\, dk, \tag{5.30}$$

$$G_1^{\pm}(\theta, y, \tau) = iB_0^{\pm} \int_{-\infty}^{+\infty} k\widehat{\zeta}_1(k, \tau) e^{\mp|k|y} e^{ik\theta}\, dk, \tag{5.31}$$

$$p_1^{\pm}(\theta, y, \tau) = \mp\big((c_0^{\pm})^2 - (B_0^{\pm})^2\big) \int_{-\infty}^{+\infty} |k|\widehat{\zeta}_1(k, \tau) e^{\mp|k|y} e^{ik\theta}\, dk. \tag{5.32}$$

5.2. *Order* $\epsilon^{3/2}$

Taking the Fourier transform of (5.12), we find that

$$ik(A - \lambda_0 C)\widehat{\mathbf{U}}_2 + D\widehat{\mathbf{U}}_{2y} + C\widehat{\mathbf{U}}_{1\tau} = 0, \tag{5.33}$$

$$\widehat{\zeta}_{1\tau} + ik(U_0 - \lambda_0)\widehat{\zeta}_2 - \widehat{v}_2 = 0, \tag{5.34}$$

$$[\![\widehat{p}_2]\!] = 0. \tag{5.35}$$

The decay condition (5.10) implies that

$$\lim_{y\to\pm\infty} \widehat{\mathbf{U}}_2(k, y, \tau) = 0. \tag{5.36}$$

We look for a solution of (5.33) of the form

$$\begin{aligned}\widehat{\mathbf{U}}_2(k, y, \tau) &= a_2(k, \tau)e^{-ky}\mathbf{S}_0 + b_2(k, \tau)e^{ky}\mathbf{S}_1 \\ &\quad + \alpha_2(k, \tau)e^{-ky}\mathbf{R} + \beta_2(k, \tau)e^{ky}\overline{\mathbf{R}},\end{aligned} \tag{5.37}$$

where a_2, b_2, α_2, and β_2 are complex-valued functions, $\mathbf{R}$ and $\overline{\mathbf{R}}$ are given by (5.20), and $\mathbf{S}_0$ and $\mathbf{S}_1$ satisfy

$$[i(A - \lambda_0 C) - D]\mathbf{S}_0 = C\mathbf{R}, \quad [i(A - \lambda_0 C) + D]\mathbf{S}_1 = C\overline{\mathbf{R}}.$$

We find that

$$\mathbf{S}_0 = (-i, 1, 0, 0, -2ic_0), \quad \mathbf{S}_1 = (-i, -1, 0, 0, -2ic_0). \tag{5.38}$$

To determine a_2 and b_2, we substitute (5.37) and (5.18) into (5.33) and solve the resulting equations. To determine α_2 and β_2, we impose the decay condition (5.36), which implies that $\alpha_2^- = \beta_2^+ = 0$ for $k > 0$ and $\alpha_2^+ = \beta_2^- = 0$ for $k < 0$. We then substitute (5.37) into the jump conditions (5.34) and (5.35) evaluated at $y = 0$. This gives a system of equations for $(\alpha_2^+, \beta_2^-, \widehat{\zeta}_2)$ if $k > 0$, and a system for $(\beta_2^+, \alpha_2^-, \widehat{\zeta}_2)$ if $k < 0$, which we solve. (See Sec. A.2.)

Summarizing the results, we find that (5.37) is a solution of (5.33)–(5.36) if

- For $k > 0$:

$$a_2^+ = \widehat{\zeta}_{1\tau}, \quad a_2^- = 0, \quad b_2^+ = 0, \quad b_2^- = -\widehat{\zeta}_{1\tau}, \tag{5.39}$$

$$\alpha_2^+ = -k\widehat{\zeta}_2, \quad \alpha_2^- = 0, \quad \beta_2^+ = 0, \quad \beta_2^- = k\widehat{\zeta}_2. \tag{5.40}$$

- For $k < 0$:

$$a_2^+ = 0, \quad a_2^- = \widehat{\zeta}_{1\tau}, \quad b_2^+ = -\widehat{\zeta}_{1\tau}, \quad b_2^- = 0, \tag{5.41}$$

$$\alpha_2^+ = 0, \quad \alpha_2^- = -k\widehat{\zeta}_2, \quad \beta_2^+ = k\widehat{\zeta}_2, \quad \beta_2^- = 0. \tag{5.42}$$

5.3. *Order* ϵ^2

Taking the Fourier transform of (5.13), we get

$$ik(A-\lambda_0 C)\widehat{\mathbf{U}}_3 + D\widehat{\mathbf{U}}_{3y} + C\widehat{\mathbf{U}}_{2\tau} + \widehat{\mathbf{H}} = 0, \tag{5.43}$$

$$\widehat{\zeta}_{2\tau} + ik(U_0-\lambda_0)\widehat{\zeta}_3 + \left[(U_1+u_1)\zeta_{1\theta}\right]^{\wedge} - \widehat{v}_3 - \widehat{v_{1y}\zeta_1} = 0, \tag{5.44}$$

$$\left[\!\left[\left[p_3 + p_{1y}\zeta_1\right]^{\wedge} \right]\!\right] = 0, \tag{5.45}$$

where

$$\mathbf{H} = \begin{bmatrix} (U_1+u_1)u_{1\theta} + v_1 u_{1y} - (B_1+F_1)F_{1\theta} - G_1 F_{1y} \\ (U_1+u_1)v_{1\theta} + v_1 v_{1y} - (B_1+F_1)G_{1\theta} - G_1 G_{1y} \\ (U_1+u_1)F_{1\theta} + v_1 F_{1y} - (B_1+F_1)u_{1\theta} - G_1 u_{1y} \\ (U_1+u_1)G_{1\theta} + v_1 G_{1y} - (B_1+F_1)v_{1\theta} - G_1 v_{1y} \\ 0 \end{bmatrix}.$$

The decay condition (5.10) implies that

$$\lim_{y\to\pm\infty} \widehat{\mathbf{U}}_3(k,y,\tau) = 0. \tag{5.46}$$

We look for a solution of (5.43) of the form

$$\widehat{\mathbf{U}}_3(k,y,\tau) = \mathbf{Q}(k,y,\tau) + a_3(k,y,\tau)\mathbf{R} + b_3(k,y,\tau)\overline{\mathbf{R}}, \tag{5.47}$$

where a_3 and b_3 are complex-valued functions, $\mathbf{R}$ and $\overline{\mathbf{R}}$ are given by (5.20), and $\mathbf{Q}$ belongs to a conveniently chosen complementary subspace to the one spanned by $\{\mathbf{R}, \overline{\mathbf{R}}\}$.

Substituting (5.47) into (5.43), and using (5.37) and (5.19) to simplify the result, we obtain the equation

$$\begin{aligned} &ik(A-\lambda_0 C)\mathbf{Q} + D\mathbf{Q}_y + (a_{3y} + ka_3)D\mathbf{R} + (b_{3y} - kb_3)D\overline{\mathbf{R}} \\ &\quad + a_{2\tau}(k,\tau)e^{-ky}C\mathbf{S}_0 + b_{2\tau}(k,\tau)e^{ky}C\mathbf{S}_1 \\ &\quad + \alpha_{2\tau}(k,\tau)e^{-ky}C\mathbf{R} + \beta_{2\tau}(k,\tau)e^{ky}C\overline{\mathbf{R}} + \widehat{\mathbf{H}} = 0. \end{aligned} \tag{5.48}$$

To determine a_3 and b_3, we introduce left null vectors $\mathbf{l}$ and $\bar{\mathbf{l}}$ such that

$$\mathbf{l}\cdot[i(A-\lambda_0 C) - D] = 0, \quad \bar{\mathbf{l}}\cdot[i(A-\lambda_0 C) + D] = 0,$$

with the normalizations

$$\mathbf{l}\cdot D\mathbf{R} = 1, \quad \mathbf{l}\cdot D\overline{\mathbf{R}} = 0, \quad \bar{\mathbf{l}}\cdot D\overline{\mathbf{R}} = 1, \quad \bar{\mathbf{l}}\cdot D\mathbf{R} = 0,$$
$$\mathbf{l}\cdot D\mathbf{Q} = 0, \quad \bar{\mathbf{l}}\cdot D\mathbf{Q} = 0.$$

These null vectors are given explicitly by

$$\mathbf{l} = \frac{1}{2c_0(c_0^2 - B_0^2)}\big(-ic_0, c_0, iB_0, -B_0, -i(c_0^2 - B_0^2)\big),$$

$$\bar{\mathbf{l}} = \frac{1}{2c_0(c_0^2 - B_0^2)}\big(ic_0, c_0, -iB_0, -B_0, i(c_0^2 - B_0^2)\big).$$

We may then choose $\mathbf{Q}$ to have the form

$$\mathbf{Q} = (q_1, 0, q_3, q_4, 0),$$

where q_1, q_3, q_4 are arbitrary.

Left-multiplying (5.48) by $\mathbf{l}$ and $\bar{\mathbf{l}}$ and solving the resulting equations for a_3 and b_3, we find that

$$a_3 = e^{-ky}\Bigg\{a_{3,0} + \int_0^y \Bigg(\frac{1}{c_0^2 - B_0^2} b_{2\tau} e^{2ky'} + i\frac{c_0^2 + B_0^2}{c_0(c_0^2 - B_0^2)}\beta_{2\tau} e^{2ky'} - e^{ky'}\mathbf{l}\cdot\widehat{\mathbf{H}}\Bigg)dy'\Bigg\},$$

$$b_3 = e^{ky}\Bigg\{b_{3,0} - \int_0^y \Bigg(\frac{1}{c_0^2 - B_0^2} a_{2\tau} e^{-2ky'} + i\frac{c_0^2 + B_0^2}{c_0(c_0^2 - B_0^2)}\alpha_{2\tau} e^{-2ky'} + e^{-ky'}\bar{\mathbf{l}}\cdot\widehat{\mathbf{H}}\Bigg)dy'\Bigg\},$$

where $a_{3,0}$ and $b_{3,0}$ are arbitrary functions of integration. Once these solvability conditions are satisfied, we may solve (5.48) for $\mathbf{Q}$.

Using (5.28)–(5.31), we find that the components of $\widehat{\mathbf{H}}$ are (see Sec. A.3)

$$\begin{aligned}
h_1^{\pm} &= \mp i(c_0U_1 + B_0B_1)k|k|\widehat{\zeta_1}(l,\tau)e^{\mp|k|y} \\
&\quad - i(c_0^2 - B_0^2)\int_{-\infty}^{+\infty}(\sigma^2(k-\sigma) - \sigma|\sigma||k-\sigma|) \\
&\quad \times \widehat{\zeta_1}(k-\sigma,\tau)\widehat{\zeta_1}(\sigma,\tau)e^{\mp(|k-\sigma|+|\sigma|)y}\,d\sigma, \\
h_2^{\pm} &= (c_0U_1 + B_0B_1)k^2\widehat{\zeta_1}(l,\tau)e^{\mp|k|y} \\
&\quad \pm (c_0^2 - B_0^2)\int_{-\infty}^{+\infty}(\sigma|\sigma|(k-\sigma) - \sigma^2|k-\sigma|) \\
&\quad \times \widehat{\zeta_1}(k-\sigma,\tau)\widehat{\zeta_1}(\sigma,\tau)e^{\mp(|k-\sigma|+|\sigma|)y}\,d\sigma, \\
h_3^{\pm} &= \pm i(B_0U_1 + c_0B_1)k|k|\widehat{\zeta_1}(k,\tau)e^{\mp|k|y}, \\
h_4^{\pm} &= -(B_0U_1 + c_0B_1)k^2\widehat{\zeta_1}(k,\tau)e^{\mp|k|y}, \\
h_5^{\pm} &= 0,
\end{aligned}$$

so that

$$\mathbf{l}\cdot\widehat{\mathbf{H}^{\pm}} = \frac{(c_0^2+B_0^2)U_1+2c_0B_0B_1}{2c_0(c_0^2-B_0^2)}(k^2\mp k|k|)\widehat{\zeta_1}(k,\tau)e^{\mp|k|y}$$
$$-\frac{1}{2}\int_{-\infty}^{+\infty}(\sigma^2\mp\sigma|\sigma|)(k-\sigma\pm|k-\sigma|)\widehat{\zeta_1}(k-\sigma,\tau)\widehat{\zeta_1}(\sigma,\tau)e^{\mp(|k-\sigma|+|\sigma|)y}\,d\sigma,$$

$$\bar{\mathbf{l}}\cdot\widehat{\mathbf{H}^{\pm}} = \frac{(c_0^2+B_0^2)U_1+2c_0B_0B_1}{2c_0(c_0^2-B_0^2)}(k^2\pm k|k|)\widehat{\zeta_1}(k,\tau)e^{\mp|k|y}$$
$$+\frac{1}{2}\int_{-\infty}^{+\infty}(\sigma^2\pm\sigma|\sigma|)(k-\sigma\mp|k-\sigma|)\widehat{\zeta_1}(k-\sigma,\tau)\widehat{\zeta_1}(\sigma,\tau)e^{\mp(|k-\sigma|+|\sigma|)y}\,d\sigma.$$

Therefore, using (5.39)–(5.42), we find that

$$a_3^{\pm}(k,y,\tau) = e^{-ky}\Bigg\{a_{3,0}^{\pm}+\frac{((c_0^{\pm})^2+(B_0^{\pm})^2)U_1^{\pm}+2c_0^{\pm}B_0^{\pm}B_1^{\pm}}{2c_0^{\pm}((c_0^{\pm})^2-(B_0^{\pm})^2)}k\widehat{\zeta_1}(k,\tau)(e^{(k\mp|k|)y}-1)$$
$$-\frac{1}{2k((c_0^{\pm})^2-(B_0^{\pm})^2)}\widehat{\zeta_{1\tau\tau}}(k,\tau)(e^{(k\mp|k|)y}-1)$$
$$+i\frac{(c_0^{\pm})^2+(B_0^{\pm})^2}{2c_0^{\pm}((c_0^{\pm})^2-(B_0^{\pm})^2)}\widehat{\zeta_{2\tau}}(k,\tau)(e^{(k\mp|k|)y}-1)$$
$$+\frac{1}{2}\int_{-\infty}^{+\infty}(\sigma^2\mp\sigma|\sigma|)\frac{k-\sigma\pm|k-\sigma|}{k\mp|k-\sigma|\mp|\sigma|}$$
$$\times\widehat{\zeta_1}(k-\sigma,\tau)\widehat{\zeta_1}(\sigma,\tau)(e^{(k\mp|k-\sigma|\mp|\sigma|)y}-1)d\sigma\Bigg\}, \tag{5.49}$$

$$b_3^{\pm}(k,y,\tau) = e^{ky}\Bigg\{b_{3,0}^{\pm}-\frac{((c_0^{\pm})^2+(B_0^{\pm})^2)U_1^{\pm}+2c_0^{\pm}B_0^{\pm}B_1^{\pm}}{2c_0^{\pm}((c_0^{\pm})^2-(B_0^{\pm})^2)}k\widehat{\zeta_1}(k,\tau)(e^{-(k\pm|k|)y}-1)$$
$$+\frac{1}{2k((c_0^{\pm})^2-(B_0^{\pm})^2)}\widehat{\zeta_{1\tau\tau}}(k,\tau)(e^{-(k\pm|k|)y}-1)$$
$$-i\frac{(c_0^{\pm})^2+(B_0^{\pm})^2}{2c_0^{\pm}((c_0^{\pm})^2-(B_0^{\pm})^2)}\widehat{\zeta_{2\tau}}(k,\tau)(e^{-(k\pm|k|)y}-1)$$
$$+\frac{1}{2}\int_{-\infty}^{+\infty}(\sigma^2\pm\sigma|\sigma|)\frac{k-\sigma\mp|k-\sigma|}{k\pm|k-\sigma|\pm|\sigma|}$$
$$\times\widehat{\zeta_1}(k-\sigma,\tau)\widehat{\zeta_1}(\sigma,\tau)(e^{-(k\pm|k-\sigma|\pm|\sigma|)y}-1)d\sigma\Bigg\}. \tag{5.50}$$

The functions a_3^+ and b_3^- satisfy the decay condition (5.46) if $k>0$, and a_3^- and b_3^+ satisfy the decay condition if $k<0$. Imposing (5.46) for the opposite signs of k, we must have $a_3^-=b_3^+=0$ if $k<0$ and $a_3^+=b_3^-=0$ if $k>0$. It follows that

- For $k > 0$:

$$
\begin{aligned}
a^-_{3,0}(k,y,\tau) = {} & -\frac{((c^-_0)^2 + (B^-_0)^2)U^-_1 + 2c^-_0 B^-_0 B^-_1}{2c^-_0((c^-_0)^2 - (B^-_0)^2)} k\widehat{\zeta_1}(k,\tau)e^{ky} \\
& - \frac{1}{2k((c^-_0)^2 - (B^-_0)^2)}\widehat{\zeta_{1\tau\tau}}(k,\tau) + i\frac{(c^-_0)^2 + (B^-_0)^2}{2c^-_0((c^-_0)^2 - (B^-_0)^2)}\widehat{\zeta_{2\tau}}(k,\tau) \\
& + \frac{1}{2}\int_0^{+\infty} \sigma(k - \sigma - |k-\sigma|)\widehat{\zeta_1}(k-\sigma,\tau)\widehat{\zeta_1}(\sigma,\tau)d\sigma, \\
b^+_{3,0}(k,y,\tau) = {} & \frac{((c^+_0)^2 + (B^+_0)^2)U^+_1 + 2c^+_0 B^+_0 B^+_1}{2c^+_0((c^+_0)^2 - (B^+_0)^2)} k\widehat{\zeta_1}(k,\tau)e^{-ky} \\
& + \frac{1}{2k((c^+_0)^2 - (B^+_0)^2)}\widehat{\zeta_{1\tau\tau}}(k,\tau) - i\frac{(c^+_0)^2 + (B^+_0)^2}{2c^+_0((c^+_0)^2 - (B^+_0)^2)}\widehat{\zeta_{2\tau}}(k,\tau) \\
& + \frac{1}{2}\int_0^{+\infty} \sigma(k - \sigma - |k-\sigma|)\widehat{\zeta_1}(k-\sigma,\tau)\widehat{\zeta_1}(\sigma,\tau)d\sigma.
\end{aligned}
$$

- For $k < 0$:

$$
\begin{aligned}
a^+_{3,0}(k,y,\tau) = {} & -\frac{((c^+_0)^2 + (B^+_0)^2)U^+_1 + 2c^+_0 B^+_0 B^+_1}{2c^+_0((c^+_0)^2 - (B^+_0)^2)} k\widehat{\zeta_1}(k,\tau)e^{ky} \\
& - \frac{1}{2k((c^+_0)^2 - (B^+_0)^2)}\widehat{\zeta_{1\tau\tau}}(k,\tau) + i\frac{(c^+_0)^2 + (B^+_0)^2}{2c^+_0((c^+_0)^2 - (B^+_0)^2)}\widehat{\zeta_{2\tau}}(k,\tau) \\
& + \frac{1}{2}\int_{-\infty}^{0} \sigma(k - \sigma + |k-\sigma|)\widehat{\zeta_1}(k-\sigma,\tau)\widehat{\zeta_1}(\sigma,\tau)d\sigma, \\
b^-_{3,0}(k,y,\tau) = {} & \frac{((c^-_0)^2 + (B^-_0)^2)U^-_1 + 2c^-_0 B^-_0 B^-_1}{2c^-_0((c^-_0)^2 - (B^-_0)^2)} k\widehat{\zeta_1}(k,\tau)e^{-ky} \\
& + \frac{1}{2k((c^-_0)^2 - (B^-_0)^2)}\widehat{\zeta_{1\tau\tau}}(k,\tau) - i\frac{(c^-_0)^2 + (B^-_0)^2}{2c^-_0((c^-_0)^2 - (B^-_0)^2)}\widehat{\zeta_{2\tau}}(k,\tau) \\
& + \frac{1}{2}\int_{-\infty}^{0} \sigma(k - \sigma + |k-\sigma|)\widehat{\zeta_1}(k-\sigma,\tau)\widehat{\zeta_1}(\sigma,\tau)\, d\sigma.
\end{aligned}
$$

Therefore, substituting $a_{3,0}$ and $b_{3,0}$ into (5.49) and (5.50), we find that

- For $k > 0$:

$$
\begin{aligned}
a^-_3(k,y,\tau) = {} & -\frac{((c^-_0)^2 + (B^-_0)^2)U^-_1 + 2c^-_0 B^-_0 B^-_1}{2c^-_0((c^-_0)^2 - (B^-_0)^2)} k\widehat{\zeta_1}(k,\tau)e^{ky} \\
& - \frac{1}{2k((c^-)^2 - (B^-)^2)}\widehat{\zeta_{1\tau\tau}}(k,\tau)e^{ky}
\end{aligned}
$$

$$
\begin{aligned}
&+ i\frac{(c^-)^2+(B^-)^2}{2c^-((c^-)^2-(B^-)^2)}\widehat{\zeta_{2\tau}}(k,\tau)e^{ky}\\
&+ \frac{1}{2}\int_0^{+\infty}\sigma(k-\sigma-|k-\sigma|)\widehat{\zeta_1}(k-\sigma,\tau)\widehat{\zeta_1}(\sigma,\tau)e^{(2\sigma-k)y}\,d\sigma,\\
b_3^+(k,y,\tau) = {}& \frac{((c_0^+)^2+(B_0^+)^2)U_1^+ + 2c_0^+B_0^+B_1^+}{2c_0^+((c_0^+)^2-(B_0^+)^2)}k\widehat{\zeta_1}(k,\tau)e^{-ky}\\
&+ \frac{1}{2k((c^+)^2-(B^+)^2)}\widehat{\zeta_{1\tau\tau}}(k,\tau)e^{-ky}\\
&- i\frac{(c^+)^2+(B^+)^2}{2c^+((c^+)^2-(B^+)^2)}\widehat{\zeta_{2\tau}}(k,\tau)e^{-ky}\\
&+ \frac{1}{2}\int_0^{+\infty}\sigma(k-\sigma-|k-\sigma|)\widehat{\zeta_1}(k-\sigma,\tau)\widehat{\zeta_1}(\sigma,\tau)e^{-(2\sigma-k)y}\,d\sigma.
\end{aligned}
\tag{5.51}
$$

- For $k < 0$:

$$
\begin{aligned}
a_3^+(k,y,\tau) = {}& -\frac{((c_0^+)^2+(B_0^+)^2)U_1^+ + 2c_0^+B_0^+B_1^+}{2c_0^+((c_0^+)^2-(B_0^+)^2)}k\widehat{\zeta_1}(k,\tau)e^{ky}\\
&- \frac{1}{2k((c^+)^2-(B^+)^2)}\widehat{\zeta_{1\tau\tau}}(k,\tau)e^{ky}\\
&+ i\frac{(c^+)^2+(B^+)^2}{2c^+((c^+)^2-(B^+)^2)}\widehat{\zeta_{2\tau}}(k,\tau)e^{ky}\\
&+ \frac{1}{2}\int_{-\infty}^{0}\sigma(k-\sigma+|k-\sigma|)\widehat{\zeta_1}(k-\sigma,\tau)\widehat{\zeta_1}(\sigma,\tau)e^{(2\sigma-k)y}\,d\sigma,\\
b_3^-(k,y,\tau) = {}& \frac{((c_0^-)^2+(B_0^-)^2)U_1^- + 2c_0^-B_0^-B_1^-}{2c_0^-((c_0^-)^2-(B_0^-)^2)}k\widehat{\zeta_1}(k,\tau)e^{-ky}\\
&+ \frac{1}{2k((c^-)^2-(B^-)^2)}\widehat{\zeta_{1\tau\tau}}(k,\tau)e^{-ky}\\
&- i\frac{(c^-)^2+(B^-)^2}{2c^-((c^-)^2-(B^-)^2)}\widehat{\zeta_{2\tau}}(k,\tau)e^{-ky}\\
&+ \frac{1}{2}\int_{-\infty}^{0}\sigma(k-\sigma+|k-\sigma|)\widehat{\zeta_1}(k-\sigma,\tau)\widehat{\zeta_1}(\sigma,\tau)e^{-(2\sigma-k)y}\,d\sigma.
\end{aligned}
\tag{5.52}
$$

Eliminating $\hat{\zeta}_3$ from the jump conditions (5.44) and (5.45), we get

$$
\left[\!\left[i\frac{\widehat{v_3}+\widehat{g_3}}{kc_0} \right]\!\right] = 0, \quad \left[\!\left[[p_3 + p_{1y}\zeta_1]^\wedge \right]\!\right] = 0,
$$

where

$$\widehat{g_3} = -\widehat{\zeta_{2\tau}} - \left[(U_1 + u_1)\zeta_{1\theta}\right]^{\wedge} + \widehat{v_{1y}\zeta_1},$$

and u_1 and v_1 are given by (5.28) and (5.29), respectively. (See Sec. A.3.)

Using (5.47), we may write the jump conditions as a system of equations for $(a_{3,0}^+, b_{3,0}^-)$ if $k > 0$, and a system for $(b_{3,0}^+, a_{3,0}^-)$ if $k < 0$. These systems are singular and nonhomogeneous. After some simplification, the solvability conditions for them imply that (see Sec. A.3)

- For $k > 0$:

$$\widehat{\zeta}_{1\tau\tau}(k,\tau) + \mu k^2 \widehat{\zeta}_1(k,\tau)$$
$$= \nu k \int_{-\infty}^{+\infty} (2k|\sigma| - k\sigma - \sigma|\sigma|)\widehat{\zeta}_1(k-\sigma,\tau)\widehat{\zeta}_1(\sigma,\tau) d\sigma.$$

- For $k < 0$:

$$\widehat{\zeta}_{1\tau\tau}(k,\tau) + \mu k^2 \widehat{\zeta}_1(k,\tau)$$
$$= \nu k \int_{-\infty}^{+\infty} (2k|\sigma| + k\sigma - \sigma|\sigma|)\widehat{\zeta}_1(k-\sigma,\tau)\widehat{\zeta}_1(\sigma,\tau)\, d\sigma.$$

Here, μ is given by (2.9) and ν is given by (2.12).

The transformation $k \mapsto -k$ and $\sigma \mapsto -\sigma$ in the equation for ζ_1 for $k > 0$ gives the equation for $k < 0$. The equation for $\zeta_1(k,\tau)$ with $-\infty < k < \infty$ is therefore

$$\widehat{\zeta}_{1\tau\tau}(k,\tau) + \mu k^2 \widehat{\zeta}_1(k,\tau) = \nu|k| \int_{-\infty}^{+\infty} \widetilde{\Lambda}(-k, k-\sigma, \sigma)\widehat{\zeta}_1(k-\sigma,\tau)\widehat{\zeta}_1(\sigma,\tau)\, d\sigma \quad (5.53)$$

where

$$\widetilde{\Lambda}(-k-\sigma, k, \sigma) = \operatorname{sgn}(k+\sigma)\big\{2(k+\sigma)|\sigma| - |k+\sigma|\sigma - \sigma|\sigma|\big\}. \quad (5.54)$$

We may write (5.53) and (5.54) as

$$\widehat{\zeta}_{1\tau\tau}(k,\tau) + \mu k^2 \widehat{\zeta}_1(k,\tau)$$
$$= i\nu k \int_{-\infty}^{+\infty} \{-2ik|\sigma| + i|k|\sigma + i\sigma|\sigma|\}\widehat{\zeta}_1(k-\sigma,\tau)\widehat{\zeta}_1(\sigma,\tau)\, d\sigma.$$

Using the convolution theorem and the fact that ∂_θ, $\mathbf{H}$ have symbols ik, $-i\operatorname{sgn} k$ to take the inverse Fourier transform of this equation, we find that $\zeta_1(\theta,\tau)$ satisfies

$$\zeta_{1\tau\tau} - \mu\zeta_{1\theta\theta} = \nu\{-2(\zeta_1\eta_{1\theta})_\theta + \mathbf{H}\left[\zeta_1\zeta_{1\theta}\right]_\theta + \zeta_1\eta_{1\theta\theta}\}_\theta, \quad \eta_1 = \mathbf{H}\left[\zeta_1\right].$$

Writing $\mathbf{H}\left[\zeta_1\zeta_{1\theta}\right] = \frac{1}{2}\mathbf{H}\left[\zeta_1^2\right]_\theta$ in this equation, using the identity

$$\mathbf{H}\left[\zeta_1^2\right] = \mathbf{H}\left[\eta_1^2\right] + 2\zeta_1\eta_1,$$

and simplifying the result, we get the spatial form of the asymptotic equation given in (2.11).

We may also write the spectral equation (5.53) as

$$\widehat{\zeta}_{1\tau\tau}(k,\tau) + \mu k^2 \widehat{\zeta}_1(k,\tau) = \nu|k| \int_{-\infty}^{+\infty} \Lambda(-k, k-\sigma, \sigma)\widehat{\zeta}_1(k-\sigma,\tau)\widehat{\zeta}_1(\sigma,\tau)\, d\sigma \qquad (5.55)$$

where the symmetrized kernel Λ is given by

$$\begin{aligned}\Lambda(-k-\sigma,k,\sigma) &= \frac{1}{2}[\widetilde{\Lambda}(-k-\sigma,k,\sigma) + \widetilde{\Lambda}(-k-\sigma,\sigma,k)] \\ &= \operatorname{sgn}(k+\sigma)\left\{ (k+\sigma)(|k|+|\sigma|) \right. \\ &\qquad \left. -\frac{1}{2}(k+\sigma)|k+\sigma| - \frac{1}{2}(k|k|+\sigma|\sigma|)\right\}. \end{aligned} \qquad (5.56)$$

We may write (5.56) as

$$\Lambda(-k-\sigma,k,\sigma) = \frac{2|k+\sigma||k||\sigma|}{|k+\sigma|+|k|+|\sigma|}, \qquad (5.57)$$

as can be seen by comparing the two expressions for Λ with the different possible choices for the signs of k, σ, and $k+\sigma$. The expression in (5.57) corresponds to the kernel in the spectral form of the normalized equation (1.1) given in (3.3)–(3.4).

Acknowledgment

The first author was partially supported by the NSF under grant number DMS-1009538.

Appendix A. Algebraic Details

This appendix summarizes some of the algebraic details of the computations.

A.1. *The $O(\epsilon)$ equations*

We look for Fourier–Laplace solutions of (5.14) of the form

$$\widehat{\mathbf{U}}_1(k,y,\tau) = a(k,\tau)e^{-sy}\mathbf{R},$$

where a is an arbitrary complex-valued function. Substitution into (5.14) gives

$$[ik(A-\lambda_0 C) - sD]\mathbf{R} = 0. \qquad (A.1)$$

We compute that

$$\det[ik(A-\lambda_0 C) - sD] = ik^3(\lambda_0 - U_0)((\lambda_0-U_0)^2 - B_0^2)(k^2-s^2).$$

Hence, if $\lambda_0 - U_0 \neq 0$ and $(\lambda_0-U_0)^2 - B_0^2 \neq 0$, (A.1) has a nontrivial solution if and only if $s^2 = k^2$. If $s = k$, we find that $\mathbf{R}$ is given by (5.20). We get the solution for $s = -k$ by taking complex conjugates. This gives the solution (5.16).

Imposing the decay conditions (5.22) and (5.23), and evaluating of the jump conditions (5.15) and (5.16) at $y = 0$, we get the systems

- For $k > 0$:

$$\begin{bmatrix} 1 & 0 & k \\ 0 & 1 & -k \\ (c_0^+)^2 - (B_0^+)^2 & -[(c_0^-)^2 - (B_0^-)^2] & 0 \end{bmatrix} \begin{bmatrix} \alpha_1^+ \\ \beta_1^- \\ \widehat{\zeta}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{A.2}$$

- For $k < 0$:

$$\begin{bmatrix} 1 & 0 & -k \\ 0 & 1 & k \\ (c_0^+)^2 - (B_0^+)^2 & -[(c_0^-)^2 - (B_0^-)^2] & 0 \end{bmatrix} \begin{bmatrix} \beta_1^+ \\ \alpha_1^- \\ \widehat{\zeta}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{A.3}$$

Equations (A.2) and (A.3) have nontrivial solutions if and only if (5.24) holds, when λ_0 is given by (5.25).

A.2. *The $O(\epsilon^{3/2})$ equations*

Substituting (5.37) into (5.33) — where $\mathbf{R}$ and $\overline{\mathbf{R}}$ are given by (5.20), and $\mathbf{S}_0$ and $\mathbf{S}_1$ are given by (5.38) — and using (5.19) to simplify, we obtain the equation

$$(ka_2 + \alpha_{1\tau})e^{-ky}C\mathbf{R} + (kb_2 + \beta_{1\tau})e^{ky}C\overline{\mathbf{R}} = 0. \tag{A.4}$$

We determine a_2 and b_2. For this, we introduce the left null vectors $\mathbf{L}$ and $\overline{\mathbf{L}}$:

$$\mathbf{L} \cdot [i(A - \lambda_0 C) - D] = 0, \quad \overline{\mathbf{L}} \cdot [i(A - \lambda_0 C) + D] = 0,$$

and such that

$$\mathbf{L} \cdot C\overline{\mathbf{R}} = 1, \quad \mathbf{L} \cdot C\mathbf{R} = 0, \quad \overline{\mathbf{L}} \cdot C\mathbf{R} = 1, \quad \overline{\mathbf{L}} \cdot C\overline{\mathbf{R}} = 0.$$

We find that

$$\begin{aligned} \mathbf{L} &= \frac{1}{2(c_0^2 + B_0^2)}(c_0, ic_0, -B_0, -iB_0, c_0^2 - B_0^2), \\ \overline{\mathbf{L}} &= \frac{1}{2(c_0^2 + B_0^2)}(c_0, -ic_0, -B_0, iB_0, c_0^2 - B_0^2). \end{aligned} \tag{A.5}$$

We multiply (A.4) on the left by $\mathbf{L}$ and $\overline{\mathbf{L}}$ in turn, and apply the decay condition (5.36). This yields the following equations that we may solve for a_2 and b_2:

- For $k > 0$: $ka_2^+ + \alpha_{1\tau}^+ = 0, kb_2^- + \beta_{1\tau}^- = 0$.
- For $k < 0$: $ka_2^- + \alpha_{1\tau}^- = 0, kb_2^+ + \beta_{1\tau}^+ = 0$.

Therefore, using (5.26) and (5.27), we find that

- For $k > 0$:

$$a_2^+ = \widehat{\zeta_{1\tau}}, \quad b_2^- = -\widehat{\zeta_{1\tau}}.$$

- For $k < 0$:

$$a_2^- = \widehat{\zeta_{1\tau}}, \quad b_2^+ = -\widehat{\zeta_{1\tau}}.$$

Now we determine α_2 and β_2. First, we note that the decay condition (5.36) is satisfied if

- For $k > 0$: $\alpha_2^- = \beta_2^+ = 0$.
- For $k < 0$: $\alpha_2^+ = \beta_2^- = 0$.

Next, using (5.37), we evaluate the boundary and jump conditions (5.34) and (5.35) at $y = 0$ to obtain the system

- For $k > 0$:

$$\begin{bmatrix} 1 & 0 & k \\ 0 & 1 & -k \\ (c_0^+)^2 - (B_0^+)^2 & -((c_0^-)^2 - (B_0^-)^2) & 0 \end{bmatrix} \begin{bmatrix} \alpha_2^+ \\ \beta_2^- \\ \widehat{\zeta}_2 \end{bmatrix} = \begin{bmatrix} i(a_2^+ - \widehat{\zeta_{1\tau}})/(c_0^+ - B_0^+) \\ i(b_2^- + \widehat{\zeta_{1\tau}})/(c_0^- - B_0^-) \\ 2ic_0^+ a_2^+ - 2ic_0^- b_2^- \end{bmatrix}.$$

  Using (5.39) and (2.10), we find that the right-hand side vanishes so the system is homogeneous. Solving the system, we get

$$\alpha_2^+ = -k\widehat{\zeta}_2, \quad \beta_2^- = k\widehat{\zeta}_2.$$

- For $k < 0$:

$$\begin{bmatrix} 1 & 0 & -k \\ 0 & 1 & k \\ (c_0^+)^2 - (B_0^+)^2 & -((c_0^-)^2 - (B_0^-)^2) & 0 \end{bmatrix} \begin{bmatrix} \beta_2^+ \\ \alpha_2^- \\ \widehat{\zeta}_2 \end{bmatrix} = \begin{bmatrix} i(b_2^+ + \widehat{\zeta_{1\tau}})/(c_0^+ - B_0^+) \\ i(a_2^- - \widehat{\zeta_{1\tau}})/(c_0^- - B_0^-) \\ 2ic_0^+ b_2^+ - 2ic_0^- a_2^- \end{bmatrix}.$$

  Using (5.41) and (2.10), we find that the right-hand side vanishes so the system is homogeneous. Solving the system, we get

$$\alpha_2^- = -k\widehat{\zeta}_2, \quad \beta_2^+ = k\widehat{\zeta}_2.$$

In summary, we find that (5.37) is a solution of (5.33)–(5.35) if (5.39)–(5.42) holds.

A.3. *The $O(\epsilon^2)$ equations*

We calculate the components of $\widehat{\mathbf{H}}$, where $\mathbf{H}$ is given in (5.43). Toward this end, let

$$\chi_1^{\pm}(k, y, \tau) = k|k|\widehat{\zeta}_1(k, \tau)e^{\mp|k|y}, \quad \chi_2^{\pm}(k, y, \tau) = k^2\widehat{\zeta}_1(k, \tau)e^{\mp|k|y},$$

and let

$$\Upsilon_1^{\pm}(k,y,\tau) = \int_{-\infty}^{+\infty} \sigma|\sigma||k-\sigma|\widehat{\zeta}_1(k-\sigma,\tau)\widehat{\zeta}_1(\sigma)e^{\mp(|k-\sigma|+|\sigma|)y}\,d\sigma,$$

$$\Upsilon_2^{\pm}(k,y,\tau) = \int_{-\infty}^{+\infty} \sigma^2(k-\sigma)\widehat{\zeta}_1(k-\sigma,\tau)\widehat{\zeta}_1(\sigma)e^{\mp(|k-\sigma|+|\sigma|)y}\,d\sigma,$$

$$\Upsilon_3^{\pm}(k,y,\tau) = \int_{-\infty}^{+\infty} \sigma^2|k-\sigma|\widehat{\zeta}_1(k-\sigma,\tau)\widehat{\zeta}_1(\sigma)e^{\mp(|k-\sigma|+|\sigma|)y}\,d\sigma,$$

$$\Upsilon_4^{\pm}(k,y,\tau) = \int_{-\infty}^{+\infty} \sigma|\sigma|(k-\sigma)\widehat{\zeta}_1(k-\sigma,\tau)\widehat{\zeta}_1(\sigma)e^{\mp(|k-\sigma|+|\sigma|)y}\,d\sigma.$$

Then, using (5.28)–(5.31), we find that

$$\begin{aligned}
&[U_1^{\pm}u_{1\theta}^{\pm}]^{\frown} = \mp ic_0^{\pm}U_1^{\pm}\chi_1^{\pm}, &&[B_1^{\pm}F_{1\theta}^{\pm}]^{\frown} = \pm B_0^{\pm}B_1^{\pm}\chi_1^{\pm},\\
&[U_1v_{1\theta}]^{\frown} = c_0U_1\chi_2, &&[B_1G_{1\theta}]^{\frown} = -B_0B_1\chi_2,\\
&[U_1^{\pm}F_{1\theta}^{\pm}]^{\frown} = \pm iB_0^{\pm}U_1^{\pm}\chi_1^{\pm}, &&[B_1^{\pm}u_{1\theta}^{\pm}]^{\frown} = \mp ic_0^{\pm}B_1^{\pm}\chi_1^{\pm},\\
&[U_1G_{1\theta}]^{\frown} = -B_0U_1\chi_2, &&[B_1v_{1\theta}]^{\frown} = c_0B_1\chi_2,
\end{aligned}$$

and

$$\begin{aligned}
&[u_1u_{1\theta}]^{\frown} = ic_0^2\Upsilon_1, &&[v_1u_{1y}]^{\frown} = -ic_0^2\Upsilon_2,\\
&[F_1F_{1\theta}]^{\frown} = iB_0^2\Upsilon_1, &&[G_1F_{1y}]^{\frown} = -iB_0^2\Upsilon_2,\\
&[u_1^{\pm}v_{1\theta}^{\pm}]^{\frown} = \mp(c_0^{\pm})^2\Upsilon_3^{\pm}, &&[v_1^{\pm}v_{1y}^{\pm}]^{\frown} = \pm(c_0^{\pm})^2\Upsilon_4^{\pm},\\
&[F_1^{\pm}G_{1\theta}^{\pm}]^{\frown} = \mp(B_0^{\pm})^2\Upsilon_3^{\pm}, &&[G_1^{\pm}G_{1y}^{\pm}]^{\frown} = \pm(B_0^{\pm})^2\Upsilon_4^{\pm},\\
&[u_1F_{1\theta}]^{\frown} = -ic_0B_0\Upsilon_1, &&[v_1F_{1y}]^{\frown} = ic_0B_0\Upsilon_2,\\
&[F_1u_{1\theta}]^{\frown} = -ic_0B_0\Upsilon_1, &&[G_1u_{1y}]^{\frown} = ic_0B_0\Upsilon_2,\\
&[u_1^{\pm}G_{1\theta}^{\pm}]^{\frown} = \pm c_0^{\pm}B_0^{\pm}\Upsilon_3^{\pm}, &&[v_1^{\pm}G_{1y}^{\pm}]^{\frown} = \mp c_0^{\pm}B_0^{\pm}\Upsilon_4^{\pm},\\
&[F_1^{\pm}v_{1\theta}^{\pm}]^{\frown} = \pm c_0^{\pm}B_0^{\pm}\Upsilon_3^{\pm}, &&[G_1^{\pm}v_{1y}^{\pm}]^{\frown} = \mp c_0^{\pm}B_0^{\pm}\Upsilon_4^{\pm}.
\end{aligned}$$

We calculate the Fourier transforms in (5.44) and (5.45). Using $\zeta_1(\theta,\tau) = \int_{-\infty}^{+\infty}\widehat{\zeta}_1(k,\tau)e^{ik\theta}\,dk$, (5.28) and (5.29), we find that

$$[u_1^{\pm}\zeta_{1\theta}]^{\frown} = \mp ic_0^{\pm}\int_{-\infty}^{+\infty}|\sigma|(k-\sigma)\widehat{\zeta}_1(k-\sigma,\tau)\widehat{\zeta}_1(\sigma,\tau)e^{\mp|\sigma|y}\,d\sigma,$$

$$[v_{1y}^{\pm}\zeta_1]^{\frown} = \pm ic_0^{\pm}\int_{-\infty}^{+\infty}\sigma|\sigma|\widehat{\zeta}_1(k-\sigma,\tau)\widehat{\zeta}_1(\sigma,\tau)e^{\mp|\sigma|y}\,d\sigma,$$

$$[p_{1y}\zeta_1]^{\frown} = (c_0^2-B_0^2)\int_{-\infty}^{+\infty}\sigma^2\widehat{\zeta}_1(k-\sigma,\tau)\widehat{\zeta}_1(\sigma,\tau)e^{\mp|\sigma|y}\,d\sigma.$$

We turn to the boundary and jump conditions (5.44) and (5.45) evaluated at $y = 0$:

$$\left[\!\!\left[i\frac{\widehat{v}_3 + \widehat{g}_3}{kc_0} \right]\!\!\right] = 0, \quad \left[\!\!\left[\left[p_3 + p_{1y}\zeta_1\right]^\frown \right]\!\!\right] = 0,$$

where

$$\widehat{g}_3 = -\widehat{\zeta}_{2\tau} - \left[(U_1 + u_1)\zeta_{1\theta}\right]^\frown + \widehat{v_{1y}\zeta_1},$$

and u_1 and v_1 are given by (5.28) and (5.29), respectively. Using (5.47), we write the boundary and jump conditions as the system of equations

- For $k > 0$:

$$\begin{bmatrix} 1 & 1 \\ (c_0^+)^2 - (B_0^+)^2 & -((c_0^-)^2 - (B_0^-)^2) \end{bmatrix} \begin{bmatrix} a_{3,0}^+ \\ b_{3,0}^- \end{bmatrix}$$
$$= \begin{bmatrix} a_3^- + b_3^+ + i\left[\!\!\left[\dfrac{\widehat{g}_3}{c_0} \right]\!\!\right] \\ ((c_0^-)^2 - (B_0^-)^2)a_3^- - ((c_0^+)^2 - (B_0^+)^2)b_3^+ - \left[\!\!\left[\left[p_3 + p_{1y}\zeta_1\right]^\frown \right]\!\!\right] \end{bmatrix}.$$

- For $k < 0$:

$$\begin{bmatrix} 1 & 1 \\ (c_0^+)^2 - (B_0^+)^2 & -((c_0^-)^2 - (B_0^-)^2) \end{bmatrix} \begin{bmatrix} b_{3,0}^+ \\ a_{3,0}^- \end{bmatrix}$$
$$= \begin{bmatrix} a_3^+ + b_3^- - i\left[\!\!\left[\dfrac{\widehat{g}_3}{c_0} \right]\!\!\right] \\ ((c_0^-)^2 - (B_0^-)^2)b_3^- - ((c_0^+)^2 - (B_0^+)^2)a_3^+ - \left[\!\!\left[\left[p_3 + p_{1y}\zeta_1\right]^\frown \right]\!\!\right] \end{bmatrix}.$$

To obtain solvability conditions for the equations, above, we introduce the left null vector $\widetilde{\mathbf{l}}$:

$$\widetilde{\mathbf{l}} = ((c_0^+)^2 - (B_0^+)^2, -1) = (-((c_0^-)^2 - (B_0^-)^2), -1),$$

where we have used the relation (5.24). Then, multiplying the equations on the left by $\widetilde{\mathbf{l}}$, using the appropriate form of $\widetilde{\mathbf{l}}$ to match the $\pm$ superscripts, we find that

- For $k > 0$:

$$\begin{aligned} -2((c_0^-)^2 - (B_0^-)^2)a_3^- &+ 2((c_0^+)^2 - (B_0^+)^2)b_3^+ \\ &+ i\big((c_0^+)^2 - (B_0^+)^2\big)\frac{\widehat{g_3^+}}{c_0^+} + i\big((c_0^-)^2 - (B_0^-)^2\big)\frac{\widehat{g_3^-}}{c_0^-} \\ &+ \big((c_0^+)^2 - (B_0^+)^2\big)\widehat{p_{1y}^+\zeta_1} - \big((c_0^-)^2 - (B_0^-)^2\big)\widehat{p_{1y}^-\zeta_1} = 0. \end{aligned}$$

We substitute into this (5.51), using the fact that (5.21) and (2.10) imply that $c_0^+ + c_0^- = 0$ so that $[\![c_0^2]\!] = 0$. In so doing, among the terms we obtain are

$$-[\![B_0^2]\!] \int_0^{+\infty} \sigma(k-\sigma-|k-\sigma|)\widehat{\zeta}_1(k-\sigma,\tau)\widehat{\zeta}_1(\sigma,\tau)\, d\sigma$$
$$+ [\![B_0^2]\!] \int_{-\infty}^{+\infty} k|\sigma|\widehat{\zeta}_1(k-\sigma,\tau)\widehat{\zeta}_1(\sigma,\tau)\, d\sigma$$
$$- [\![B_0^2]\!] \int_{-\infty}^{+\infty} \sigma^2\widehat{\zeta}_1(k-\sigma,\tau)\widehat{\zeta}_1(\sigma,\tau)\, d\sigma.$$

To combine these terms, we make the change of variables $k-\sigma=\sigma'$ in the first integral:

$$\begin{aligned}
&\int_0^{+\infty} \sigma(k-\sigma-|k-\sigma|)\widehat{\zeta}_1(k-\sigma,\tau)\widehat{\zeta}_1(\sigma,\tau)\, d\sigma \\
&\quad= \int_k^{+\infty} \sigma(k-\sigma-|k-\sigma|)\widehat{\zeta}_1(k-\sigma,\tau)\widehat{\zeta}_1(\sigma,\tau)\, d\sigma \qquad (k-\sigma<0 \text{ and } k>0) \\
&\quad= \int_0^{-\infty} (k-\sigma')(\sigma'-|\sigma'|)\widehat{\zeta}_1(\sigma',\tau)\widehat{\zeta}_1(k-\sigma',\tau)\,(-d\sigma') \quad (k-\sigma=\sigma') \\
&\quad= \int_{-\infty}^{0} (k-\sigma)(\sigma-|\sigma|)\widehat{\zeta}_1(k-\sigma,\tau)\widehat{\zeta}_1(\sigma,\tau)\, d\sigma \qquad (\sigma'=\sigma) \\
&\quad= \int_{-\infty}^{+\infty} (k-\sigma)(\sigma-|\sigma|)\widehat{\zeta}_1(k-\sigma,\tau)\widehat{\zeta}_1(\sigma,\tau)\, d\sigma.
\end{aligned}$$

Therefore, after combining all the terms in the solvability condition for $k>0$, we obtain the following wave equation for ζ_1:

$$\begin{aligned}
&\widehat{\zeta}_{1\tau\tau}(k,\tau) + \mu k^2\widehat{\zeta}_1(k,\tau) \\
&\qquad = -\frac{1}{2}[\![B_0^2]\!]k \int_{-\infty}^{+\infty} (2k|\sigma| - k\sigma - \sigma|\sigma|)\widehat{\zeta}_1(k-\sigma,\tau)\widehat{\zeta}_1(\sigma,\tau)\, d\sigma.
\end{aligned}$$

- For $k<0$:

$$\begin{aligned}
&2\big((c_0^+)^2 - (B_0^+)^2\big)a_3^+ - 2\big((c_0^-)^2 - (B_0^-)^2\big)b_3^- \\
&\quad -i\big((c_0^+)^2 - (B_0^+)^2\big)\frac{\widehat{g_3^+}}{c_0^+} - i\big((c_0^-)^2 - (B_0^-)^2\big)\frac{\widehat{g_3^-}}{c_0^-} \\
&\quad +\big((c_0^+)^2 - (B_0^+)^2\big)\widehat{p_{1y}^+\zeta_1} - \big((c_0^-)^2 - (B_0^-)^2\big)\widehat{p_{1y}^-\zeta_1} = 0.
\end{aligned}$$

We substitute into this (5.52), and make the change of variables $k-\sigma=\sigma'$ in the term

$$\int_{-\infty}^{0} \sigma(k-\sigma+|k-\sigma|)\widehat{\zeta}_1(k-\sigma,\tau)\widehat{\zeta}_1(\sigma,\tau)\, d\sigma$$

to rewrite the integral $\int_{-\infty}^{0} \to \int_{-\infty}^{+\infty}$. Then, after combining and simplifying all the terms in the solvability condition for $k < 0$, we obtain the following wave equation for ζ_1

$$\widehat{\zeta}_{1\tau\tau}(k,\tau) + \mu k^2 \widehat{\zeta}_1(k,\tau) = -\tfrac{1}{2}[\![B_0^2]\!]k \int_{-\infty}^{+\infty} (2k|\sigma| + k\sigma - \sigma|\sigma|)\widehat{\zeta}_1(k-\sigma,\tau)\widehat{\zeta}_1(\sigma,\tau)\, d\sigma.$$

These equations are equivalent to (5.53) and (5.54).

Appendix B. Comparison with the Unidirectional Equation

In this section, we compare the two-way wave equation obtained above for surface waves on a tangential discontinuity near the onset of instability with the one-way wave equation obtained in [1] for unidirectional surface waves on a linearly stable discontinuity.

Suppose, as in Sec. 2, that the tangential velocities and magnetic fields of the unperturbed discontinuity are $U^\pm$, $B^\pm$ where $\Delta(U,B) > 0$, and the location of the discontinuity is given by $y = \zeta(x,t;\epsilon)$. The asymptotic solution for a unidirectional wave is given by

$$\zeta(x,t;\epsilon) \sim \epsilon\zeta_1(x - \lambda t, \epsilon t) + O(\epsilon^2)$$

as $\epsilon \to 0^+$, where λ is one of the wave speeds λ_1, λ_2 in (2.5). The equation for the displacement $\zeta_1(\tilde{\theta},\tilde{\tau})$ derived in [1] may be written as

$$\delta\zeta_{1\tilde{\tau}} + f(\zeta_1) = 0, \tag{B.1}$$

where the constant δ is given by

$$\delta = \left[\!\!\left[\frac{c}{c^2 - B^2}\right]\!\!\right] = \frac{c^+}{(c^+)^2 - (B^+)^2} - \frac{c^-}{(c^-)^2 - (B^-)^2} \tag{B.2}$$

with $c^\pm = \lambda - U^\pm$, and

$$f(\zeta) = \frac{1}{2}\mathbf{H}\left[\eta^2\right]_{\tilde{\theta}\tilde{\theta}} + \eta\zeta_{\tilde{\theta}\tilde{\theta}}, \quad \eta = \mathbf{H}\left[\zeta\right]. \tag{B.3}$$

Here, $\mathbf{H}$ denotes the Hilbert transform with respect to $\tilde{\theta}$.

The quadratically nonlinear operator (B.3) in (B.1) is the same as the one that appears in (2.11), which may be written as

$$\zeta_{1\tau\tau} - \mu\zeta_{1\theta\theta} = \nu f(\zeta_1)_\theta. \tag{B.4}$$

In order to compare (B.1) with the two-way wave equation (B.4), we suppose that U, B are given by (2.6) with $\mu > 0$, and consider the limit as $\epsilon \to 0^+$ when

$(U, B) \to (U_0, B_0)$. We consider, for definiteness, a surface wave with speed $\lambda = \lambda_2$. Then from (2.5), we have

$$\lambda = \lambda_0 + \epsilon^{1/2}\sqrt{\mu} + O(\epsilon). \tag{B.5}$$

Using (2.6), (2.7) and (B.5) in (B.2), we find after some algebra that

$$\delta = \frac{2\sqrt{\mu}}{\nu}\epsilon^{1/2} + O(\epsilon) \quad \text{as} \quad \epsilon \to 0^+, \tag{B.6}$$

where μ and ν are defined in (2.9) and (2.12), respectively. Moreover, writing the multiple scale variables $\tilde{\theta} = x - \lambda t$, $\tilde{\tau} = \epsilon t$ used in (B.1) in terms of the multiple scale variables $\theta = x - \lambda_0 t$, $\tau = \epsilon^{1/2} t$ used in (B.4), we get

$$\tilde{\theta} = \theta - (\lambda - \lambda_0)t, \quad \tilde{\tau} = \epsilon^{1/2}\tau.$$

From (B.5), we therefore have to leading order in ϵ that

$$\tilde{\theta} = \theta - \sqrt{\mu}\tau, \quad \tilde{\tau} = \epsilon^{1/2}\tau,$$

which implies that $\partial_\tau + \sqrt{\mu}\partial_\theta = \epsilon^{1/2}\partial_{\tilde{\tau}}$.

Using this approximation together with (B.6) in (B.1), we find that in the limit $\epsilon \to 0^+$, the equation becomes

$$\zeta_{1\tau} + \sqrt{\mu}\zeta_{1\theta} + \frac{\nu}{2\sqrt{\mu}} f(\zeta_1) = 0. \tag{B.7}$$

The operator f is independent of $\tilde{\tau}$ and translation-invariant in $\tilde{\theta}$, so it is still given by (B.3) with $\tilde{\theta}$ replaced by θ.

On the other hand, as $\mu \to \infty$ unidirectional solutions of (B.4) with linearized speed $\sqrt{\mu}$ may be obtained by use of a similar approximation to (3.5), which gives after an integration with respect to θ that

$$-2\sqrt{\mu}\left(\zeta_{1\tau} + \sqrt{\mu}\zeta_{1\theta}\right) = \nu f\left(\zeta_1\right).$$

Comparing this result with (B.7), we see that the two equations agree in their common domain of validity.

Adjusting signs as appropriate, we find similarly that the equations for left-moving waves with speed $\lambda_1 \sim \lambda_0 - \epsilon^{1/2}\sqrt{\mu}$ also agree. Thus, (B.4) is the simplest generalization of the equations for left and right moving unidirectional waves near a point where their wave speeds coalesce.

References

[1] G. Alì and J. K. Hunter, Nonlinear surface waves on a tangential discontinuity in magnetohydrodynamics, *Quart. Appl. Math.* **61** (2003) 451–474.

[2] D. M. Ambrose, Singularity formation in a model for the vortex sheet with surface tension, *Math. Comput. Simulation* **80** (2009) 102–111.

[3] J. T. Beale and D. Schaeffer, Nonlinear behavior of model equations which are linearly ill-posed, *Commun. Partial Differential Equations* **13** (1988) 423–467.

[4] S. Benzoni-Gavage and M. D. Rosini, Weakly nonlinear surface waves and subsonic phase boundaries, *Comput. Math. Appl.* **57** (2009) 1463–1484.

[5] M. Brio and J. K. Hunter, Asymptotic equations for conservations laws of mixed type, *Wave Motion* **16** (1992) 57–64.
[6] G.-Q. Chen and Y.-G. Wang, Existence and stability of compressible current-vortex sheets in three-dimensional magnetohydrodynamics, *Arch. Ration. Mech. Anal.* **187** (2008) 369–408.
[7] J. K. Hunter, Nonlinear surface waves, in *Nonlinear Conservation Laws and Applications*, eds. A. Bressan, G.-Q. Chen, M. Lewicka and D. Wang, IMA Volumes in Mathematics and its Applications, Vol. 153 (Springer, 2011), pp. 303–314.
[8] L. D. Landau, L. P. Pitaevskii and E. M. Lifschitz, *The Electrodynamics of Continuous Media* 2nd edn. (Pergamon Press, New York, 1984).
[9] D. F. Parker, Nonlinear surface waves and nonlocality, in *Trends in Applications of Mathematics to Mechanics*, Berichte aus der Mathematik (Shaker-Verlag, Aachen, 2005), pp. 303–314.
[10] Y. Trakhinin, Existence of compressible current-vortex sheets: Variable coefficients linear analysis, *Arch. Ration. Mech. Anal.* **177** (2005) 331–366.