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# A quasi-linear Schrödinger equation for large amplitude inertial oscillations in a rotating shallow fluid

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This paper derives a two-dimensional, quasi-linear Schrödinger equation in Lagrangian coordinates that describes the effects of weak pressure gradients on large amplitude inertial oscillations in a rotating shallow fluid. The coefficients of the equation are singular at values of the gradient of the wave amplitude that correspond to the vanishing of the Jacobian of the transformation from Lagrangian to Eulerian coordinates, but solutions do not appear to form singularities dynamically. Two regimes of high and moderate nonlinearity are identified, depending on whether or not phase differences in the components of the amplitude gradient are required to maintain a nonzero Jacobian. Numerical simulations show that moderately nonlinear solutions of the quasi-linear Schrödinger equation behave in a qualitatively similar way to solutions of a linear Schrödinger equation, whereas highly nonlinear solutions generate rapidly oscillating, small-scale waves.

*Keywords*: nonlinear waves, rotating fluids, inertial oscillations, quasi-linear Schrödinger equation

## 1. Introduction

We consider the non-dimensionalized rotation-dominated shallow water equations

$$\begin{aligned} &u_t + uu_x + vu_y - v + \varepsilon h_x = 0,\\ &v_t + uv_x + vv_y + u + \varepsilon h_y = 0,\\ &h_t + uh_x + vh_y + h(u_x + v_y) = 0, \end{aligned} \tag{1..1}$$

where $(x,y)$ are spatial, or Eulerian, coordinates, $(u,v)$ are the $(x,y)$ velocity components of the fluid and $h$ is the depth. The small dimensionless parameter $\varepsilon$ is given by

$$\varepsilon = \frac{gH}{f^2L^2}$$

where $g$ is the acceleration due to gravity, $f$ is the Coriolis parameter, $H$ is a typical depth of the fluid, and $L$ is a typical horizontal length scale of the fluid motion. Equivalently,

$$\varepsilon = \left(\frac{\mathrm{Ro}}{\mathrm{Fr}}\right)^2,$$

where the Rossby number Ro and the Froude number Fr are given in terms of a typical fluid velocity $U$ by

$$\mathrm{Ro} = \frac{U}{fL}, \qquad \mathrm{Fr} = \frac{U}{\sqrt{gH}}.$$

Thus, $\varepsilon \ll 1$ when Ro $\ll$ Fr and rotation dominates gravity.

The solutions we consider here are close to large-amplitude inertial oscillations, in which the dominant balance is between inertia $\vec{u}_t + \vec{u}\cdot\nabla\vec{u}$ and the Coriolis force $\vec{e}_z \times \vec{u}$. This differs from quasi-geostrophic solutions, in which the dominant balance is between the Coriolis force and the pressure gradient, and from the geostrophic adjustment of initially non-geostrophic solutions by means of inertia-gravity waves whose frequencies may differ substantially from the Coriolis frequency.

Liu and Tadmor (5) observed that (1..1) has large-amplitude, time-periodic solutions. When $\varepsilon = 0$, these solutions consist of spatially decoupled inertial oscillations at the Coriolis frequency, here non-dimensionalized to one. Cheng and Tadmor (3) proved that when the Coriolis force dominates pressure, smooth solutions have a longer lifespan than they would in the absence of rotation. In particular, they showed that smooth solutions of (1..1) close to the time-periodic solutions found in (5) have an enhanced life-span when $\varepsilon$ is small.

In this paper, we derive an asymptotic equation which describes the evolution of such solutions over a long time-scale of the order $\varepsilon^{-1}$. The main idea is to carry out the analysis in material, or Lagrangian, coordinates where, in the absence of pressure gradients, fluid particles undergo independent harmonic inertial oscillations.

Let $(\alpha,\beta)$ denote appropriate material coordinates and $\tau = \varepsilon t$ a 'slow' time. We show that (1..1) has asymptotic solutions of the form

$$u = A(\alpha,\beta,\tau)e^{-it} + \text{c.c.} + O(\varepsilon), \qquad v = -iA(\alpha,\beta,\tau)e^{-it} + \text{c.c.} + O(\varepsilon),$$

where the complex amplitude $A(\alpha,\beta,\tau)$ satisfies

$$\begin{aligned} 2iA_\tau + \frac{\partial}{\partial\alpha}&\left[\frac{A_\alpha - 2A_\beta\left(A_\alpha A_\beta^* - A_\alpha^* A_\beta\right)}{\left(1-4\left[|A_\alpha|^2 + |A_\beta|^2 + \left(A_\alpha A_\beta^* - A_\alpha^* A_\beta\right)^2\right]\right)^{3/2}}\right] \\ &+ \frac{\partial}{\partial\beta}\left[\frac{A_\beta + 2A_\alpha\left(A_\alpha A_\beta^* - A_\alpha^* A_\beta\right)}{\left(1-4\left[|A_\alpha|^2 + |A_\beta|^2 + \left(A_\alpha A_\beta^* - A_\alpha^* A_\beta\right)^2\right]\right)^{3/2}}\right] = 0. \end{aligned} \tag{1..2}$$

This equation is a two-dimensional, quasi-linear Schrödinger equation for $A$ with coefficients depending on $\nabla A$. For one-dimensional solutions independent of $\beta$, it reduces to

$$2iA_\tau + \left[\frac{A_\alpha}{\left(1-4|A_\alpha|^2\right)^{3/2}}\right]_\alpha = 0. \tag{1..3}$$

The local well-posedness of quasi-linear Schrödinger equations is analyzed in (4; 6), but we do not know of any previous derivation of a fully quasi-linear equation of this form in a physical problem. For example, quasi-linear Schrödinger equations for spin waves (1) have second-order terms roughly of the form $\Delta A + A\Delta|A|^2$, but in that case the coefficients of $\Delta A$ and $\Delta A^*$ depend on $A$ rather than $\nabla A$.

Equation (1..2) describes the slow evolution of the inertial oscillations of different fluid particles as a result of their interaction through weak pressure gradients. We refer to wave motions, such as this one, whose unperturbed or linearized behavior consists of spatially uncoupled oscillations with the same frequency as "constant-frequency waves." Some general features of constant-frequency waves are discussed in (2).

A significant feature of (1..2) is that its coefficients become singular if $\nabla A = (A_\alpha, A_\beta)$ is sufficiently large. Introducing the real quantity

$$\rho = |A_\alpha|^2 + |A_\beta|^2 + \left(A_\alpha A_\beta^* - A_\alpha^* A_\beta\right)^2, \tag{1..4}$$

we see that the coefficients in (1..2) become infinite as $\rho \to 1/4$. As we will show in Section 3., this limit corresponds to the vanishing of the Jacobian of the transformation from material to spatial coordinates, and the condition $0 \leqslant \rho < 1/4$ means that the Lagrangian to Eulerian map is smoothly invertible and positively oriented. We conjecture that this condition is preserved by the evolution of (1..2) for suitable smooth solutions, but do not attempt to prove it here. In all of our numerical solutions, we found that $\rho$ remained strictly less than $1/4$.

The condition $0 \leqslant \rho < 1/4$ motivates a distinction between two regimes for (1..2) which we refer to as "moderate" and "high" nonlinearity. It follows from (1..4) that $\rho \leqslant |\nabla A|^2$. Thus, if $|\nabla A|^2 < 1/4$, then $\rho < 1/4$ independently of the phases of $A_\alpha$ and $A_\beta$. We call this the moderately nonlinear regime. If $1/4 < |\nabla A|^2 < 1/2$ and $|A_\alpha|^2, |A_\beta|^2 < 1/4$, it is still possible to have $\rho < 1/4$ provided that $A_\alpha$ and $A_\beta$ are out of phase by an angle that is sufficiently close to $\pi/2$. We call this the highly nonlinear regime. Numerical simulations of (1..2) indicate that moderately nonlinear solutions behave like solutions of a linear Schrödinger equation, whereas highly nonlinear solutions develop rapid, small-scale oscillations.

An outline of the contents of the rest of this paper is as follows. In Section 2., we write out the Lagrangian formulation of the rotating shallow water equations. In Section 3., we derive the asymptotic equation (1..2). In Section 4., we show that (1..2) is Hamiltonian, and introduce a convenient generalization of the equation, given in (4..4) below. In Section 5., we derive an ellipticity condition for (4..4) that is is required for local well-posedness and verify that it is satisfied by equation (1..2) when $\rho < 1/4$. We also describe the regimes of moderate and high nonlinearity for (1..2). In Section 6., we consider harmonic, traveling wave solutions of (1..2) and show that they are linearly and modulationaly stable. In Section 7., we present some numerical solutions of (1..2), and in Section 8., we summarize our conclusions and open questions.

## 2. Lagrangian description

In this section we rewrite the rotating shallow water system of equations in material coordinates for use in the subsequent asymptotic analysis. Let $(\alpha, \beta)$ denote material coordinates and $x = x(\alpha, \beta, t)$, $y = y(\alpha, \beta, t)$ a deformation. In material coordinates, the system (1..1) becomes

$$\begin{aligned} &u_t - v + \varepsilon \frac{\partial(h,y)}{\partial(\alpha,\beta)} J^{-1} = 0, \qquad v_t + u + \varepsilon \frac{\partial(x,h)}{\partial(\alpha,\beta)} J^{-1} = 0, \\ &x_t = u, \qquad y_t = v, \qquad (hJ)_t = 0, \end{aligned} \tag{2..1}$$

where the $t$-derivative is a material time derivative taken holding $(\alpha, \beta)$ fixed, and

$$J = x_\alpha y_\beta - x_\beta y_\alpha, \qquad \frac{\partial(h,y)}{\partial(\alpha,\beta)} = h_\alpha y_\beta - h_\beta y_\alpha, \qquad \frac{\partial(x,h)}{\partial(\alpha,\beta)} = x_\alpha h_\beta - x_\beta h_\alpha. \tag{2..2}$$

By relabeling material particles if necessary, we may assume that $hJ = 1$. Then (2..1) becomes

$$
\begin{aligned}
&u_t - v + \varepsilon h \frac{\partial(h,y)}{\partial(\alpha,\beta)} = 0, \qquad v_t + u + \varepsilon h \frac{\partial(x,h)}{\partial(\alpha,\beta)} = 0, \\
&x_t = u, \qquad y_t = v, \qquad h = \frac{1}{J}.
\end{aligned}
\tag{2..3}
$$

We can eliminate $(u,v)$ from (2..3) to get a system for $(x,y)$ that is second order in time,

$$
x_{tt} - y_t + \varepsilon h \frac{\partial(h,y)}{\partial(\alpha,\beta)} = 0, \qquad y_{tt} + x_t + \varepsilon h \frac{\partial(x,h)}{\partial(\alpha,\beta)} = 0, \qquad h = \frac{1}{J},
$$

but it is more convenient for the asymptotic expansion to leave $(u,v)$ as dependent variables.

## 3. Asymptotic Expansion

Using the method of multiple scales, we introduce a "slow" time variable $\tau = \varepsilon t$ and look for an asymptotic solution of the rotation-dominated Lagrangian shallow water equations (2..3) of the form

$$
\begin{aligned}
u &= u_0(\alpha,\beta,t,\tau) + \varepsilon u_1(\alpha,\beta,t,\tau) + O(\varepsilon^2), \\
v &= v_0(\alpha,\beta,t,\tau) + \varepsilon v_1(\alpha,\beta,t,\tau) + O(\varepsilon^2), \\
x &= x_0(\alpha,\beta,t,\tau) + \varepsilon x_1(\alpha,\beta,t,\tau) + O(\varepsilon^2), \\
y &= y_0(\alpha,\beta,t,\tau) + \varepsilon y_1(\alpha,\beta,t,\tau) + O(\varepsilon^2), \\
h &= h_0(\alpha,\beta,t,\tau) + O(\varepsilon),
\end{aligned}
\tag{3..1}
$$

where $u_n$, $v_n$, $x_n$, $y_n$, $h_n$ are periodic functions of $t$. We use (3..1) in (2..3), expand time derivatives, and equate coefficients of powers of $\varepsilon$. At the order $\varepsilon^0$, we find that

$$
\begin{aligned}
&u_{0t} - v_0 = 0, \qquad v_{0t} + u_0 = 0, \\
&x_{0t} = u_0, \qquad\quad y_{0t} = v_0,
\end{aligned}
\tag{3..2}
$$

and at the order $\varepsilon$, we find that

$$
\begin{aligned}
u_{1t} - v_1 + u_{0\tau} + h_0 \left[ \frac{\partial(h,y)}{\partial(\alpha,\beta)} \right]_0 &= 0, \\
v_{1t} + u_1 + v_{0\tau} + h_0 \left[ \frac{\partial(x,h)}{\partial(\alpha,\beta)} \right]_0 &= 0.
\end{aligned}
\tag{3..3}
$$

We consider solutions for which the deformation reduces to the identity in the absence of a wave, in which case the solution of (3..2) is

$$
\begin{aligned}
u_0(\alpha,\beta,t,\tau) &= A(\alpha,\beta,\tau)e^{-it} + A^*(\alpha,\beta,\tau)e^{it}, \\
v_0(\alpha,\beta,t,\tau) &= -iA(\alpha,\beta,\tau)e^{-it} + iA^*(\alpha,\beta,\tau)e^{it}, \\
x_0(\alpha,\beta,t,\tau) &= \alpha + iA(\alpha,\beta,\tau)e^{-it} - iA^*(\alpha,\beta,\tau)e^{it}, \\
y_0(\alpha,\beta,t,\tau) &= \beta + A(\alpha,\beta,\tau)e^{-it} + A^*(\alpha,\beta,\tau)e^{it},
\end{aligned}
\tag{3..4}
$$

where $A$ is an arbitrary complex-valued function. The corresponding Jacobian

$$J_0 = x_{0\alpha} y_{0\beta} - x_{0\beta} y_{0\alpha}$$

and height $h_0$ are given by

$$\begin{aligned} &J_0 = Me^{-it} + N + M^* e^{it}, \qquad h_0 = \frac{1}{J_0}, \\ &M = iA_\alpha + A_\beta, \qquad N = 1 + 2i\left(A_\alpha A_\beta^* - A_\alpha^* A_\beta\right). \end{aligned} \tag{3..5}$$

The solvability condition for the system

$$u_{1t} - v_1 + F_0 = 0, \qquad v_{1t} + u_1 + G_0 = 0$$

to have a solution $(u_1, v_1)$ that is $2\pi$-periodic in $t$ is

$$\frac{1}{2\pi}\int_0^{2\pi} e^{it}(F_0 + iG_0)dt = 0.$$

Imposing this condition on (3..3) and using (3..4) in the result, we find that

$$2A_\tau + \frac{1}{2\pi}\int_0^{2\pi} h_0 e^{it}\left\{\left[\frac{\partial(h,y)}{\partial(\alpha,\beta)}\right]_0, +i\left[\frac{\partial(x,h)}{\partial(\alpha,\beta)}\right]_0\right\}dt = 0 \tag{3..6}$$

where the zero subscript denotes evaluation at the order zero terms.

Using (2..2) and (3..1), we may write the integrand in (3..6) as

$$\begin{aligned} &h_0 e^{it}\left\{\left[\frac{\partial(h,y)}{\partial(\alpha,\beta)}\right]_0 + i\left[\frac{\partial(x,h)}{\partial(\alpha,\beta)}\right]_0\right\} \\ &\qquad = \frac{1}{2}\frac{\partial}{\partial\alpha}\left\{h_0^2\left(e^{it} + 2A_\beta\right)\right\} + \frac{1}{2}\frac{\partial}{\partial\beta}\left\{h_0^2\left(ie^{it} - 2A_\alpha\right)\right\}. \end{aligned}$$

The use of this expression in (3..6) gives

$$2A_\tau + \frac{1}{2}\left[\frac{\partial}{\partial\alpha}\left(P + 2A_\beta Q\right) + \frac{\partial}{\partial\beta}\left(iP - 2A_\alpha Q\right)\right] = 0, \tag{3..7}$$

where

$$P = \frac{1}{2\pi}\int_0^{2\pi} h_0^2 e^{it}\,dt, \qquad Q = \frac{1}{2\pi}\int_0^{2\pi} h_0^2\,dt. \tag{3..8}$$

We evaluate these integrals by the method of residues. First, we consider $P$. Using (3..5) in (3..8), we get that

$$P = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}\,dt}{\left(Me^{-it} + N + M^* e^{it}\right)^2}. \tag{3..9}$$

We make the change of variables $z = e^{it}$ and write

$$\frac{M}{z} + N + M^* z = \frac{M^*}{z}(z - z_1)(z - z_2),$$

where $z_1 + z_2 = -N/M^*$ and $z_1 z_2 = M/M^*$. Then (3..9) becomes

$$P = \frac{1}{2\pi i (M^*)^2} \oint_{|z|=1} \frac{z^2}{\left[(z-z_1)(z-z_2)\right]^2}\, dz. \tag{3..10}$$

The roots $z_1$, $z_2$ are given explicitly by

$$z_1 = \frac{-1 - 2i\left(A_\alpha A_\beta^* - A_\alpha^* A_\beta\right) + \sqrt{1 - 4\left[|A_\alpha|^2 + |A_\beta|^2 + \left(A_\alpha A_\beta^* - A_\alpha^* A_\beta\right)^2\right]}}{2A_\beta^* - 2iA_\alpha^*}$$

$$z_2 = \frac{-1 - 2i\left(A_\alpha A_\beta^* - A_\alpha^* A_\beta\right) - \sqrt{1 - 4\left[|A_\alpha|^2 + |A_\beta|^2 + \left(A_\alpha A_\beta^* - A_\alpha^* A_\beta\right)^2\right]}}{2A_\beta^* - 2iA_\alpha^*}.$$

We assume that the discriminant beneath the square root is positive, meaning that

$$\rho = |A_\alpha|^2 + |A_\beta|^2 + \left(A_\alpha A_\beta^* - A_\alpha^* A_\beta\right)^2 < \frac{1}{4}. \tag{3..11}$$

It follows from (3..5) that

$$N - 2|M| \leqslant J_0 \leqslant N + 2|M|$$

over a period in $t$ and

$$N^2 - 4|M|^2 = 1 - 4\rho.$$

Thus, (3..11) corresponds to the condition that the Jacobian $J_0$ does not vanish over a period in $t$.

When (3..11) holds, we have $|z_1| < 1$ and $|z_2| > 1$. The integrand in (3..10) then has a pole of order two at $z_1$ inside the unit circle with residue

$$\operatorname{Res}\left\{\frac{z^2}{\left[(z-z_1)(z-z_2)\right]^2}; z = z_1\right\} = \frac{2z_1 z_2}{(z_2 - z_1)^3}.$$

Hence, by the residue theorem,

$$\begin{aligned} P &= \frac{2z_1 z_2}{(M^*)^2 (z_2 - z_1)^3} \\ &= \frac{-2\left(iA_\alpha + A_\beta\right)}{\left(1 - 4\left[|A_\alpha|^2 + |A_\beta|^2 + \left(A_\alpha A_\beta^* - A_\alpha^* A_\beta\right)^2\right]\right)^{3/2}}. \end{aligned} \tag{3..12}$$

Similarly, we find that

$$\begin{aligned} Q &= \frac{1}{2\pi i (M^*)^2} \oint_{|z|=1} \frac{z}{\left[(z-z_1)(z-z_2)\right]^2}\, dz \\ &= \frac{z_1+z_2}{(M^*)^2 (z_2-z_1)^3} \\ &= \frac{1+2i\left(A_\alpha A_\beta^* - A_\alpha^* A_\beta\right)}{\left(1-4\left[|A_\alpha|^2+|A_\beta|^2+\left(A_\alpha A_\beta^* - A_\alpha^* A_\beta\right)^2\right]\right)^{3/2}}. \end{aligned} \tag{3..13}$$

Using (3..12)–(3..13) in (3..7) and multiplying the result by $i$, we find that $A$ satisfies (1..2).

## 4. Hamiltonian structure

The quasi-linear Schrödinger equation (1..2) derived in Section 3. is Hamiltonian and has the complex canonical form

$$iA_\tau = \frac{\delta \mathscr{H}}{\delta A^*}, \tag{4..1}$$

where the Hamiltonian $\mathscr{H}$ is given by

$$\mathscr{H}(A,A^*) = \int \frac{1}{4\sqrt{1-4\left[A_\alpha A_\alpha^* + A_\beta A_\beta^* + \left(A_\alpha A_\beta^* - A_\alpha^* A_\beta\right)^2\right]}}\, d\alpha d\beta. \tag{4..2}$$

One can verify that this Hamiltonian corresponds to the Hamiltonian of the original rotating shallow water equations.

The energy density in (4..2) becomes infinite as $\rho \to 1/4$, where $\rho$ is defined in (1..4), as one would expect when an area element in material coordinates is compressed to zero in spatial coordinates. This singularity suggests that conservation of energy provides a mechanism for keeping $\rho$ strictly less than $1/4$. On its own, however, conservation of the integral quantity in (4..2) is not sufficient to imply that $\rho$ is bounded pointwise away from $1/4$, and it is an open question to prove that $\rho$ remains strictly less than $1/4$ under the evolution of (4..1).

More generally, we consider an equation of the form (4..1) with Hamiltonian

$$\mathscr{H}(A,A^*) = \int F\left(A_\alpha A_\alpha^* + A_\beta A_\beta^* - \left[i\left(A_\alpha A_\beta^* - A_\alpha^* A_\beta\right)\right]^2\right) d\alpha d\beta, \tag{4..3}$$

where $F : I \to \mathbb{R}$ is a smooth, real-valued function defined on an interval $I \subset \mathbb{R}$. Hamilton's equation is then

$$iA_\tau + \frac{\partial}{\partial \alpha}\left[f(\rho)\left(A_\alpha + 2iDA_\beta\right)\right] + \frac{\partial}{\partial \beta}\left[f(\rho)\left(A_\beta - 2iDA_\alpha\right)\right] = 0, \tag{4..4}$$

where $f = F'$, with a prime denoting the derivative with respect to $\rho$, and

$$D = i\left(A_\alpha A_\beta^* - A_\alpha^* A_\beta\right), \qquad \rho = |A_\alpha|^2 + \left|A_\beta\right|^2 - D^2. \tag{4..5}$$

For (1..2), we have

$$F(\rho) = \frac{1}{4\sqrt{1-4\rho}}, \qquad f(\rho) = \frac{1}{2(1-4\rho)^{3/2}}. \tag{4..6}$$

Equation (4..4) is invariant under rotations of $(\alpha,\beta)$ since both $|A_\alpha|^2 + |A_\beta|^2$ and $D^2$ are rotationally invariant scalars. For inertial oscillations, this rotational invariance is inherited from the original equations, while the reality of $\rho$, $D$ corresponds to the invariance of the equations under a phase shift of the oscillations.

In addition to conserving the Hamiltonian $\mathscr{H}$, (4..4) conserves the action $\mathscr{S}$, momentum $\vec{\mathscr{P}}$, and angular momentum $\mathscr{M}$, which are given by (8)

$$\begin{aligned}
\mathscr{S} &= \int |A|^2 \, d\alpha d\beta, \\
\vec{\mathscr{P}} &= i\int (A\nabla A^* - A^*\nabla A)\, d\alpha d\beta, \\
\mathscr{M} &= i\int \left\{\alpha\left(AA^*_\beta - A^*A_\beta\right) - \beta\left(AA^*_\alpha - A^*A_\alpha\right)\right\} d\alpha d\beta.
\end{aligned}$$

## 5. Ellipticity condition

In this section, we derive an ellipticity condition for (4..4), which is a necessary condition for the equation to be locally well-posed, and show that it is satisfied by (1..2). We do not prove a well-posedness result here, but we discuss the solution regimes of moderate and high nonlinearity for (1..2) that are suggested by this condition.

Expanding derivatives, we may write (4..4) as a quasi-linear Schrödinger equation

$$iA_\tau + N[A] = 0, \tag{5..1}$$

where $N$ is a second-order, quasi-linear operator without lower-order terms of the form

$$N[A] = aA_{\alpha\alpha} + bA_{\alpha\beta} + cA_{\beta\beta} + \lambda A^*_{\alpha\alpha} + \mu A^*_{\alpha\beta} + \nu A^*_{\beta\beta}. \tag{5..2}$$

The real-valued coefficients $a$, $b$, $c$ and complex-valued coefficients $\lambda$, $\mu$, $\nu$ are functions of $\nabla A$, $\nabla A^*$; they are given explicitly by

$$\begin{aligned}
a &= \left[f(\rho) - 2D^2 f'(\rho)\right]\left(1 - 2\left|A_\beta\right|^2\right) + f'(\rho)\left|A_\alpha\right|^2, \\
b &= \left\{2\left[f(\rho) - 2D^2 f'(\rho)\right] + f'(\rho)\right\}\left(A_\alpha A^*_\beta + A^*_\alpha A_\beta\right), \\
c &= \left[f(\rho) - 2D^2 f'(\rho)\right]\left(1 - 2\left|A_\alpha\right|^2\right) + f'(\rho)\left|A_\beta\right|^2, \\
\lambda &= 2\left[f(\rho) - 2D^2 f'(\rho)\right]A_\beta^2 + f'(\rho)\left[A_\alpha^2 + 4iDA_\alpha A_\beta\right], \\
\mu &= -4\left[f(\rho) - 2D^2 f'(\rho)\right]A_\alpha A_\beta + 2f'(\rho)\left[A_\alpha A_\beta - 2iD\left(A_\alpha^2 - A_\beta^2\right)\right], \\
\nu &= 2\left[f(\rho) - 2D^2 f'(\rho)\right]A_\alpha^2 + f'(\rho)\left[A_\beta^2 - 4iDA_\alpha A_\beta\right],
\end{aligned} \tag{5..3}$$

where $\rho$ and $D$ are defined in (4..5).

The local linearized dispersion relation of (5..1)–(5..2) is obtained by 'freezing' coefficients in $N$ and looking for Fourier solutions of the resulting equation of the form

$$A(\alpha,\beta,\tau) = A_0 e^{i(\xi\alpha+\eta\beta-\gamma\tau)} + B_0 e^{-i(\xi\alpha+\eta\beta-\gamma\tau)}.$$

After some algebra, we find that the dispersion relation is

$$\gamma^2 = \sigma_N(\xi,\eta), \tag{5..4}$$

where

$$\begin{aligned}\sigma_N(\xi,\eta) = &\left[a^2 - |\lambda|^2\right]\xi^4 + \left[2ab - (\lambda\mu^* + \lambda^*\mu)\right]\xi^3\eta \\ &+ \left[2ac + b^2 - \left(\lambda\nu^* + \lambda^*\nu + |\mu|^2\right)\right]\xi^2\eta^2 \\ &+ \left[2bc - (\mu\nu^* + \mu^*\nu)\right]\xi\eta^3 + \left[c^2 - |\nu|^2\right]\eta^4.\end{aligned} \tag{5..5}$$

If (5..1)–(5..2) is to be locally well-posed in an appropriate Sobolev space, the 'frozen' equation cannot have growing Fourier modes, since then the homogeneity of $\sigma_N$ would imply that there are high-wavenumber modes with arbitrarily large growth rates. Neglecting degenerate cases in which $\sigma_N$ vanishes at a nonzero wavenumber, we see that for well-posedness $N$ must satisfy the following ellipticity condition

$$\sigma_N(\xi,\eta) \geqslant \sigma_0\left(\xi^2+\eta^2\right)^2 \text{ for some } \sigma_0 > 0 \text{ and all } (\xi,\eta)\in\mathbb{R}^2. \tag{5..6}$$

In other words, $\sigma_N$ must be positive-definite. A further 'non-trapping' condition for the bi-characteristics of $N$ is needed for a proof of local well-posedness (4; 7), but we do not investigate that condition here.

Before giving criteria for the ellipticity condition (5..6) to hold, we obtain some inequalities for the possible values of $\rho$ and $D$, and identify the physically relevant regime in the case of the quasi-linear Schrödinger equation for inertial oscillations.

From (4..5) we have $D^2 \leqslant 4|A_\alpha|^2|A_\beta|^2$ and

$$1-4\rho \leqslant \left(1-4|A_\alpha|^2\right)\left(1-4|A_\beta|^2\right). \tag{5..7}$$

For inertial oscillations, the inequality $1-4\rho > 0$ corresponds to the condition in (3..11) for the Jacobian $J_0$ of the deformation to be nonzero throughout a period, and if $\rho = 1/4$ then $J_0$ vanishes at some phase of the oscillation. If $1-4\rho > 0$ is to be feasible, then from (5..7) we must have either $|A_\alpha|^2, |A_\beta|^2 < 1/4$ or $|A_\alpha|^2, |A_\beta|^2 > 1/4$. We exclude the second case, since it would require that $A_\alpha$, $A_\beta$ are uniformly bounded away from zero if $\rho$ is not to pass through $1/4$, and therefore assume that $|A_\alpha|^2, |A_\beta|^2 < 1/4$. In that case, if $\rho < 1/4$ then $J_0 > 0$ is strictly positive throughout an inertial oscillation. Moreover, we have

$$\begin{aligned}\rho &\geqslant |A_\alpha|^2 + |A_\beta|^2 - 4|A_\alpha|^2|A_\beta|^2 = 4|A_\alpha|^2|A_\beta|^2 + |A_\alpha|^2\left(1-4|A_\beta|^2\right) + |A_\beta|^2\left(1-4|A_\alpha|^2\right) \\ &\geqslant 4|A_\alpha|^2|A_\beta|^2 \geqslant D^2.\end{aligned}$$

Thus, the physically relevant regime is

$$|A_\alpha|^2, |A_\beta|^2 < 1/4, \qquad D^2 \leqslant \rho < 1/4. \tag{5..8}$$

From (4..6), the asymptotic Hamiltonian density $F(\rho)$ with respect to material coordinates for inertial oscillations becomes infinite in the limit $\rho \to 1/4$ of a vanishing Jacobian. In fact, as the next result shows, even if the coefficient function $F(\rho)$ is smooth and well-defined for $\rho \geqslant 1/4$ the condition $\rho < 1/4$ is needed to ensure the well-posedness of (4..4).

PROPOSITION 5..1 Let $\sigma_N$ be given by (5..5) where the coefficients are defined in (5..3) and $\rho$, $D$ are defined in (4..5). Assume that

$$|A_\alpha|^2 < \frac{1}{4} \quad \text{and} \quad |A_\beta|^2 < \frac{1}{4}. \tag{5..9}$$

Then the ellipticity condition (5..6) holds if and only if

$$\rho < 1/4; \tag{5..10}$$

$$f^2 + \left(\rho - D^2\right) f f' > 0; \tag{5..11}$$

$$f^2 + 2\left(\rho - D^2\right) f f' + (1-4\rho) D^2 \left(f'\right)^2 > 0, \tag{5..12}$$

where $f$, $f'$ are evaluated at $\rho$. A sufficient condition for (5..6) to hold under the assumption (5..9) is that $\rho < 1/4$ and $f f' > 0$.

*Proof.* Using (5..3) in (5..5), we find after some algebra that the fourth-degree polynomial $\sigma_N$ may be factored into quadratic polynomials as

$$\sigma_N(\xi,\eta) = f\left(p\xi^2 + 2f'W\xi\eta + q\eta^2\right)\left(n\xi^2 + 4W\xi\eta + m\eta^2\right), \tag{5..13}$$

where

$$\begin{aligned} &p = f + 2\left(|A_\alpha|^2 - D^2\right) f', \qquad q = f + 2\left(|A_\beta|^2 - D^2\right) f', \\ &m = 1 - 4|A_\alpha|^2, \quad n = 1 - 4|A_\beta|^2, \quad W = A_\alpha A_\beta^* + A_\alpha^* A_\beta. \end{aligned} \tag{5..14}$$

The second factor on the right-hand side of (5..13) is independent of $f$ and arises from our assumption that the Hamiltonian density depends only $\nabla A$ only through $\rho$.

The polynomial $\sigma_N$ is definite if and only if each of its factors is definite. The factor $n\xi^2 + 4W\xi\eta + m\eta^2$ is definite if and only if

$$mn > 4W^2, \tag{5..15}$$

and since $m, n > 0$ from (5..14) and (5..9), it is then positive-definite. Writing

$$W^2 = 4|A_\alpha|^2|A_\beta|^2 - D^2$$

and using (4..5), (5..14) we find that (5..15) is equivalent to (5..10).

We remark that if one of $|A_\alpha|^2$, $|A_\beta|^2$ is less than $1/4$ and the other is greater than $1/4$, then $m$, $n$ have opposite signs so $\sigma_N$ is not definite and the ellipticity condition always fails. In that case, however, we necessarily have $\rho > 1/4$.

The remaining factor $p\xi^2 + 2f'W\xi\eta + q\eta^2$ is definite if and only if

$$pq > \left(f'W\right)^2.$$

Using (4..5) and (5..14), we find that this condition is equivalent to (5..12). The factor is positive-definite if and only if $fp$, $fq$ are positive. Since they have the same sign when (5..12) holds, this is the case if and only if $fp + fq > 0$, which is equivalent to (5..11). This proves that (5..10)–(5..12) are necessary and sufficient conditions for (5..6) to hold under the assumption (5..9).

If $\rho < 1/4$ and (5..9) holds, then from (5..8) the coefficients $\rho - D^2$ and $1 - 4\rho$ in (5..11)–(5..12) are non-negative, and a sufficient condition for these inequalities to hold is that $f f' > 0$. □

If $f > 0$, as we may assume without loss of generality, then the sufficient condition $f' = F'' > 0$ is a convexity condition on the Hamiltonian (4..3). For the rotating shallow water Schrödinger equation, where $f$ is given by (4..6), we have $f, f' > 0$ so (5..6) is satisfied whenever $\rho < 1/4$.

This analysis suggests a distinction between two regimes, a moderately nonlinear regime in which nonlinear effects are not too strong, and a highly nonlinear regime in which nonlinear effects are very strong:

1. *Moderate nonlinearity*: $|A_\alpha|^2 + |A_\beta|^2 < 1/4$, in which case the ellipticity condition $\rho < 1/4$ holds independently of the relative phases of $A_\alpha$, $A_\beta$;

2. *High nonlinearity*: $|A_\alpha|^2 + |A_\beta|^2 > 1/4$ and $|A_\alpha|^2, |A_\beta|^2 < 1/4$, in which case the phase difference between $A_\alpha$, $A_\beta$ has to be sufficiently close to $\pi/2$ to ensure that $\rho < 1/4$.

In the one-dimensional case, all solutions are moderately nonlinear, but two-dimensional solutions may be highly nonlinear. For example, consider an amplitude function of the form

$$A(\alpha, \beta) = u(\alpha) + iv(\beta).$$

Then $A_\alpha A_\beta^*$ is purely imaginary, and we have $\rho = u_\alpha^2 + v_\beta^2 - 4u_\alpha^2 v_\beta^2$. Thus, the moderately nonlinear regime corresponds to

$$u_\alpha^2 + v_\beta^2 < \frac{1}{4}, \tag{5..16}$$

while the highly nonlinear regime corresponds to

$$\frac{1}{4} < u_\alpha^2 + v_\beta^2 < \frac{1}{2}, \qquad \max\{u_\alpha^2, v_\beta^2\} < \frac{1}{4}. \tag{5..17}$$

We show some numerical solutions of (1..2) with this type of initial data in Section 7. *c.f.* (7..4) below.

## 6. Stability of periodic waves

In this section, we consider periodic traveling wave solutions of (4..4) and show that they are stable. Equation (4..4) has the harmonic solutions

$$A = A_0 e^{i\vec{k}\cdot\vec{\alpha} - i\omega\tau}, \qquad \omega = f(\rho_0)\,|\vec{k}|^2, \qquad \rho_0 = |\vec{k}|^2 |A_0|^2, \tag{6..1}$$

where $\vec{\alpha} = (\alpha, \beta)$ is the space variable, $\vec{k} = (k, \ell)$ is a constant real wavenumber vector and $A_0$ is a constant complex amplitude. For the rotating shallow water equations, these solutions correspond to long-wavelength inertia-gravity waves. We will show that these solutions are both linearly and modulationaly stable. Thus, the the nonlinearity in (4..4) is defocusing.

### 6.1. *Linearized stability*

Since (4..4) is rotationally invariant, there is no loss of generality in assuming that $\vec{k} = (k, 0)$ in the unperturbed periodic wave (6..1). To determine the linearized stability of the solution (6..1), we use

$$A(\alpha, \beta, \tau) = A_0 e^{ik\alpha - i\omega\tau}\left[1 + B(\alpha, \beta, \tau)\right]$$

in (4..4) and linearize the resulting equation with respect to $B$. This gives

$$\begin{aligned} &iB_\tau + \omega B + \left(f_0 + \rho_0 f_0'\right)\left(B_{\alpha\alpha} + 2ikB_\alpha - k^2 B\right) \\ &\quad - \rho_0 f_0'\left(B^*_{\alpha\alpha} + k^2 B^*\right) + (1-2\rho_0)\, f_0 B_{\beta\beta} - 2\rho_0 f_0 B^*_{\beta\beta} = 0, \end{aligned} \tag{6..2}$$

where

$$\rho_0 = k^2|A_0|^2, \qquad f_0 = f\left(\rho_0\right), \qquad f_0' = f'\left(\rho_0\right).$$

Writing

$$B(\alpha,\beta,\tau) = u(\alpha,\beta,\tau) + iv(\alpha,\beta,\tau),$$

equating real and imaginary parts, and using the equation $\omega = f_0 k^2$, we get

$$\begin{aligned} &u_\tau + 2k\left(f_0 + \rho_0 f_0'\right) u_\alpha + \left(f_0 + 2\rho_0 f_0'\right) v_{\alpha\alpha} + f_0 v_{\beta\beta} = 0, \\ &v_\tau + 2k\left(f_0 + \rho_0 f_0'\right) v_\alpha - f_0 u_{\alpha\alpha} - (1-4\rho_0)\, f_0 u_{\beta\beta} + 2k^2\rho_0 f_0' u = 0. \end{aligned} \tag{6..3}$$

Looking for Fourier solutions

$$u = \hat{u}e^{i\xi\alpha + i\eta\beta - i\gamma\tau} + \text{c.c}, \qquad v = \hat{v}e^{i\xi\alpha + i\eta\beta - i\gamma\tau} + \text{c.c}$$

we find that the dispersion relation of (6..3) is

$$\left[\gamma - 2\left(f_0 + \rho_0 f_0'\right)\xi\right]^2 = \left[\left(f_0 + 2\rho_0 f_0'\right)\xi^2 + f_0\eta^2\right]\left[f_0\xi^2 + (1-4\rho_0)\, f_0\eta^2 + 2k^2\rho_0 f_0'\right]. \tag{6..4}$$

Thus, the periodic waves are linearly stable to both longitudinal and transverse perturbations if $0 < \rho_0 < 1/4$ and $f_0 f_0' > 0$, which is the case for the rotating shallow water waves.

In the high-frequency limit $\xi,\eta \to \infty$, when $\gamma = O(\xi^2)$ and $(k,\ell,\omega)$ is negligible compared with $(\xi,\eta,\gamma)$, the relation (6..4) becomes

$$\gamma^2 = \left[\left(f_0 + 2\rho_0 f_0'\right)\xi^2 + f_0\eta^2\right]\left[f_0\xi^2 + (1-4\rho_0)\, f_0\eta^2\right].$$

This result is consistent with the high-frequency, ‘frozen-coefficient’ dispersion relation (5..4) and (5..13), since for the unperturbed solution $A = A_0 e^{ik\alpha - i\omega\tau}$ we have from (5..14) that

$$p = f_0 + 2\rho_0 f_0', \qquad q = f_0, \qquad m = 1 - 4\rho_0, \qquad n = 1, \qquad W = 0.$$

### 6.2. *Modulational stability*

Next, we use Whitham’s averaged Lagrangian method (9) to derive modulation equations for locally periodic solutions of (4..4). The same results can be derived by the method of multiple scales.

We consider large-amplitude, slowly modulated asymptotic solutions of (4..4) of the form

$$A(\vec{\alpha},\tau) = a(\vec{\alpha},\tau)e^{iS(\vec{\alpha},\tau)}, \tag{6..5}$$

where $a$ is a real-valued amplitude function and $S$ is a real-valued phase. We let

$$\omega = -S_\tau, \qquad \vec{k} = \nabla S$$

denote the local frequency and wavenumber vector.

The Lagrangian for (4..4) is

$$\mathscr{L}(A,A^*) = \int \left\{ \frac{1}{2} i \left(A^* A_\tau - A A^*_\tau\right) - F\left( A_\alpha A^*_\alpha + A_\beta A^*_\beta - \left[ i\left( A_\alpha A^*_\beta - A^*_\alpha A_\beta \right)\right]^2 \right) \right\} d\vec{\alpha}\, d\tau,$$

where $F' = f$, and the variational principle is

$$\frac{\delta \mathscr{L}}{\delta A^*} = 0.$$

The averaged Lagrangian $\overline{\mathscr{L}}$ is obtained by setting

$$A = ae^{iS}, \quad A_\tau = -i\omega a e^{iS}, \quad \nabla A = ia\vec{k}e^{iS}$$

in the full Lagrangian. (The phase $S$ cancels, so it is not necessary to average the result over $S$.) This gives

$$\overline{\mathscr{L}}\left(\omega, \vec{k}, a\right) = \int \bar{L}\left(\omega, \vec{k}, a\right) d\vec{\alpha}\, d\tau, \qquad \bar{L}\left(\omega, \vec{k}, a\right) = \omega a^2 - F\left(|\vec{k}|^2 a^2\right). \tag{6..6}$$

The term $D = i(A_\alpha A^*_\beta - A^*_\alpha A_\beta)$ reduces to zero in this approximation and therefore does not affect the modulation equations.

The averaged Lagrangian equations are obtained from (6..6) by varying the amplitude $a$ and the phase $S$, which gives

$$\bar{L}_a = 0, \qquad \frac{\partial \bar{L}_\omega}{\partial \tau} - \nabla \cdot \bar{L}_{\vec{k}} = 0, \qquad \frac{\partial \vec{k}}{\partial \tau} + \nabla \omega = 0.$$

The equation $\bar{L}_a = 0$ is the nonlinear dispersion relation

$$\omega = |\vec{k}|^2 f\left(|\vec{k}|^2 a^2\right). \tag{6..7}$$

The remaining equations are

$$\frac{\partial}{\partial \tau}\left(a^2\right) + \nabla \cdot \left[ 2a^2 f\left(|\vec{k}|^2 a^2\right) \vec{k} \right] = 0, \qquad \frac{\partial \vec{k}}{\partial \tau} + \nabla \left[ |\vec{k}|^2 f\left(|\vec{k}|^2 a^2\right) \right] = 0.$$

Introducing

$$\rho = |\vec{k}|^2 a^2, \tag{6..8}$$

we may write these equations as

$$\rho_\tau + \nabla \cdot \left[ 2\rho f(\rho) \vec{k} \right] + 2\rho \vec{k} \cdot \nabla f(\rho) = 0, \qquad \vec{k}_\tau + \nabla \left[ |\vec{k}|^2 f(\rho) \right] = 0. \tag{6..9}$$

Freezing coefficients and looking for Fourier solutions proportional $e^{i\vec{\xi}\cdot\vec{\alpha} - i\gamma\tau}$, we find that the characteristic variety of (6..9) for gradient wavenumber vectors $\vec{k} = \nabla S$ is given by

$$\left[ \gamma - 2\left(f + \rho f'\right)(\vec{k}\cdot\vec{\xi}) \right]^2 = 2\rho f' \left\{ f \left[ |\vec{k}|^2 |\vec{\xi}|^2 - \left(\vec{k}\cdot\vec{\xi}\right)^2 \right] + \left(f + 2\rho f'\right)\left(\vec{k}\cdot\vec{\xi}\right)^2 \right\}.$$

This result agrees with the long-wave approximation $(\xi, \eta) \to 0$ of (6..4). It follows that if $f$, $f'$ are positive then $\gamma$ is real for all real $\vec{\xi}$, meaning that (6..9) is hyperbolic in $\rho > 0$. This is the case when $f$ is given by (4..6), so the locally periodic solutions (6..5) are modulationaly stable for the rotating shallow water equations. We remark that from (6..7) and (6..8)

$$\nabla_{\vec{k}} \omega(\vec{k}; a) = 2\left(f + \rho f'\right) \vec{k},$$

so the characteristic velocities of the modulation equations split into two branches centered around an advection at the 'linear' group velocity $\nabla_{\vec{k}} \omega$.

## 7. Numerical solutions

In this section, we show some numerical solutions of the Schrödinger equations (1..2) and (1..3). In particular, we illustrate the different behavior of moderately and highly nonlinear solutions in two space dimensions.

First, we consider the one-dimensional quasi-linear Schrödinger equation for $A(\alpha,\tau)$,

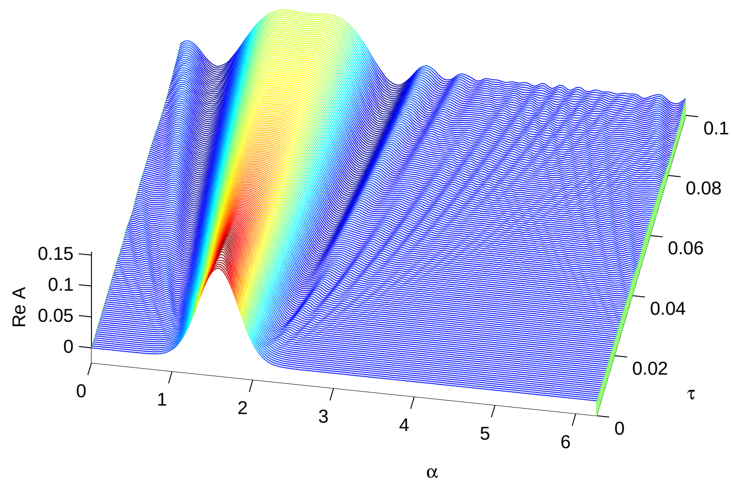


FIG. 1. Surface plot of $\mathrm{Re}A(\alpha,\tau)$ for the solution of (7..1) with initial data (7..2) for times $0 \leqslant \tau \leqslant 0.1$.

$$iA_\tau + \left[\frac{A_\alpha}{2\,(1-4|A_\alpha|^2)^{3/2}}\right]_\alpha = 0, \tag{7..1}$$

with $2\pi$-spatially periodic boundary conditions and initial data

$$A(\alpha,0) = 0.15\exp\left[-8(\alpha-\pi/2)^2\right] \qquad \text{for } 0 < \alpha < 2\pi. \tag{7..2}$$

This data corresponds to a periodic array of gaussian pulses with initially constant phase. In one space dimension, the solution is necessarily in the moderately nonlinear regime.

The coefficient of $A_\alpha$ in the spatial dispersive term in (7..1) changes by a factor of $(1-4\rho)^{-3/2}$ from its value at $A=0$, where $\rho = |A_\alpha|^2$, and this factor gives an indication of the strength of the nonlinearity. The maximum value of $\rho$ for the initial data (7..2) is approximately equal to 0.132, which gives a maximum value of the dispersion factor of $(1-4\rho)^{-3/2} \approx 3.10$.

Figure 1 shows a numerical solution of (7..1)–(7..2), computed by a pseudo-spectral method with $2^{13}$ Fourier modes and a fourth-order Runge-Kutta method in time with a fixed time step. For comparison, we show the solution of a linear Schrödinger equation with the same initial data in Figure 2 for longer times. Despite the fact that the initial data is not small, the solutions have a similar structure; the main difference is the faster dispersion of the pulse for the nonlinear equation as a result of

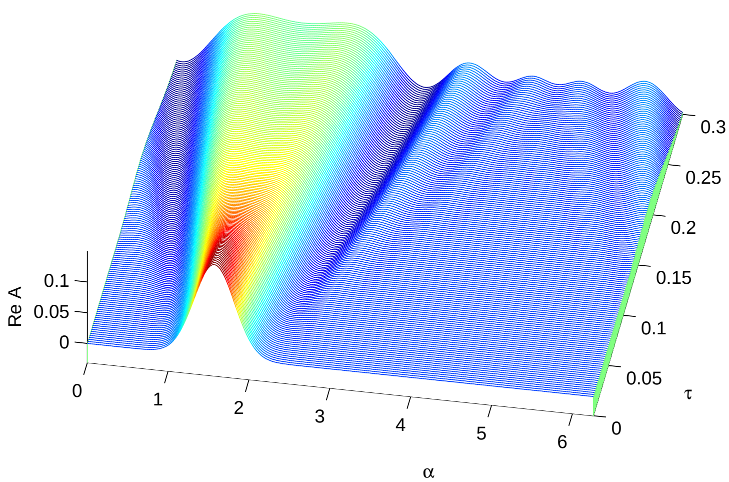


FIG. 2. Surface plot of $\mathrm{Re}\,A(\alpha,\tau)$ for the solution of the linear Schrödinger equation $iA_\tau + (1/2)A_{\alpha\alpha} = 0$ with the same initial data (7..2) as in Figure 1 for times $0 \leqslant \tau \leqslant 0.3$.

its larger dispersion coefficient at higher amplitudes. In addition, we see the generation of some small, fast, high-wavenumber modes in the nonlinear solution. Numerical simulations of the one-dimensional Schrödinger equation with other initial data gave similar results.

Next, we consider a localized two-dimensional solution with initial data

$$A(\alpha,\beta,0) = 20\exp(-8[(\alpha-\pi)^2+(\beta-\pi)^2])\,[(\alpha-\pi)+i(\beta-\pi)]^7 \tag{7..3}$$

for $0<\alpha<2\pi$, $0<\beta<2\pi$. This initial data is a pulse centered at $(\pi,\pi)$ with amplitude proportional to $r^7e^{-8r^2}$ and phase equal to $7\theta$. For this data, we have

$$\max\left(|A_\alpha|^2+|A_\beta|^2\right) = 0.1387, \quad \max\left[i\left(A_\alpha A_\beta^* - A_\alpha^* A_\beta\right)\right]^2 = 0.0060, \quad \max\rho = 0.1363,$$

where $\rho$ is defined in (1..4). Figure 3 shows a numerical solution of (1..2) for $A(\alpha,\beta,\tau)$ at $\tau = 0.05$ with initial data (7..3) and $2\pi$-periodic boundary conditions in $\alpha$, $\beta$. The solution is computed by the use of a pseudo-spectral method with $512\times512$ Fourier modes and a fourth order, fixed-step Runge-Kutta method in time.

For comparison, we show the solution of a linear Schrödinger equation with the same initial data at a later time in Figure 4. As in the previous one-dimensional example, the solutions have a very similar structure, with the generation of small, high wavenumber modes ahead of the pulse in the nonlinear equation. Other numerical experiments with different localized initial data also show a surprising small influence of nonlinearity on solutions of (1..2) beyond an enhanced rate of dispersion. One reason appears to be that spatial dispersion causes the derivatives of the solution to decrease rapidly in time, after which the behavior of the solution is essential linear.

Finally, we show two spatially periodic solutions of (1..2) with initial data of the form

$$A(\alpha,\beta,0) = a_0\left(\cos\alpha + i\cos\beta\right). \tag{7..4}$$

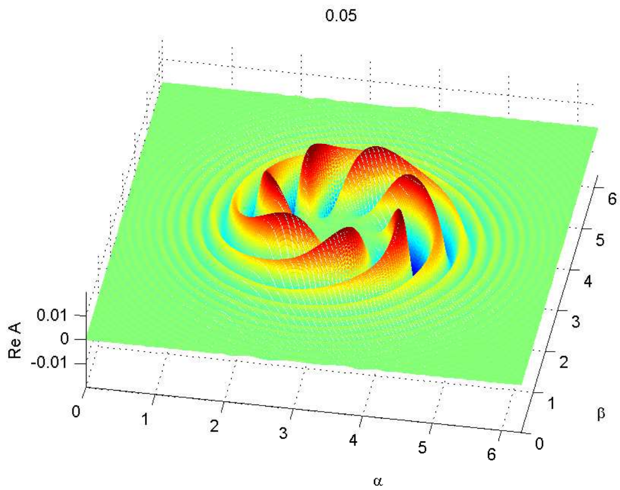


FIG. 3. A surface plot of $\mathrm{Re}A(\alpha,\beta,\tau)$ for the solution of (1..2) at time $\tau = 0.05$ with initial data (7..3).

If $a_0^2 < 1/8$ this data is in the moderately nonlinear regime (5..16) for all $(\alpha,\beta)$, while if $1/8 < a_0^2 < 1/4$ this data is in the highly nonlinear regime (5..17) when $|\nabla A|^2$ is near its maximum value.

Figures 5–6 show contour plots of the derivatives of the solution of (1..2) at $\tau = 0.1$ with $2\pi$-periodic boundary conditions in $\alpha$ and $\beta$ and initial data (7..4) for a moderately nonlinear case $a_0^2 = 1/9$. In Figure 5, we plot $|\nabla A|^2$, and in Figure 6, we plot $D = i(A_\alpha A_\beta^* - A_\alpha^* A_\beta)$. The solution is computed by a pseudo-spectral method with $1024 \times 1024$ Fourier modes and a fourth-order, fixed-step Runge-Kutta method in time. The maximum value of $\rho$ for this initial data is $\rho \approx 0.173$, leading to a maximum value of $(1-4\rho)^{-3/2} \approx 5.83$ for the dispersion factor. The solution remains relatively smooth, although it develops sharp transitions in $D$ at $\alpha = \pi$ and $\beta = \pi$, which become sharper with further evolution in time.

Next, in Figures 7–8, we show corresponding contour plots of the solution of (1..2) at $\tau = 0.005$ with initial data (7..4) for a highly nonlinear case $a_0^2 = 1/5$. This problem is stiff, with a dispersion factor $(1-4\rho)^{-3/2}$ that varies initially from 1 to 125. As shown in Figure 9, the condition $\rho < 1/4$ continues to hold everywhere in our numerical solution, but the behavior of the highly nonlinear solution is qualitatively different from that of the previous moderately nonlinear solution. We see the generation of small-scale waves at the 'corners' of the square-shaped level curves of $|\nabla A|$. The amplitude of these waves does not grow. At later times, they propagate into the low dispersion regions where $|\nabla A|$ is close to zero; they remain trapped in these regions, and exhibit a kind of "phase turbulence" which requires further study.

## 8. Conclusions

We have derived a fully quasi-linear, two-dimensional Schrödinger equation that gives an asymptotic description in Lagrangian coordinates of large amplitude inertial oscillations and long inertia-gravity waves in a rotation-dominated shallow fluid. The coefficients of the equation become singular at values of the velocity gradient that correspond to a loss of smooth invertibility in the Lagrangian to Eulerian map.

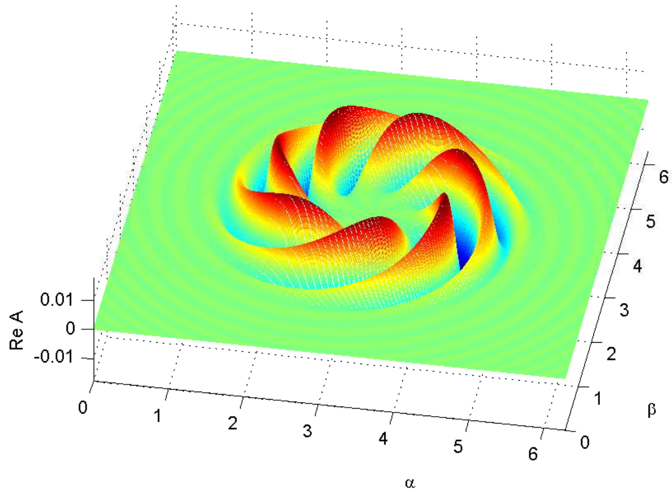


FIG. 4. A surface plot of $\mathrm{Re}\,A(\alpha,\beta,\tau)$ for the solution of the linear Schrödinger equation $iA_\tau + (1/2)\left(A_{\alpha\alpha} + A_{\beta\beta}\right) = 0$ at $\tau = 0.10$ with the same initial data (7..3) as in Figure 4.

We have verified that the equation satisfies an ellipticity condition required for local well-posedness, and distinguished two regimes of high and moderate nonlinearity depending on whether or not phase differences in the components of the velocity gradient are required to avoid a singularity. We do not observe the spontaneous formation of singularities in numerical solutions, but rigorous proofs of the non-occurrence of singularities and the local, or global, well-posed of the Schrödinger equation are open questions.

Periodic traveling wave solutions of the Schrödinger equation are linearly and modulationaly stable, so nonlinearity does not appear to focus waves. In numerical simulations, the qualitative behavior of moderately nonlinear solutions is remarkably similar to that of solutions of a linear Schrödinger equation, but for highly nonlinear solutions, we observe the generation of small-scale waves from low-wavenumber initial data. The evolution of the resulting small-scale wave patterns over longer times is complicated and requires further study.

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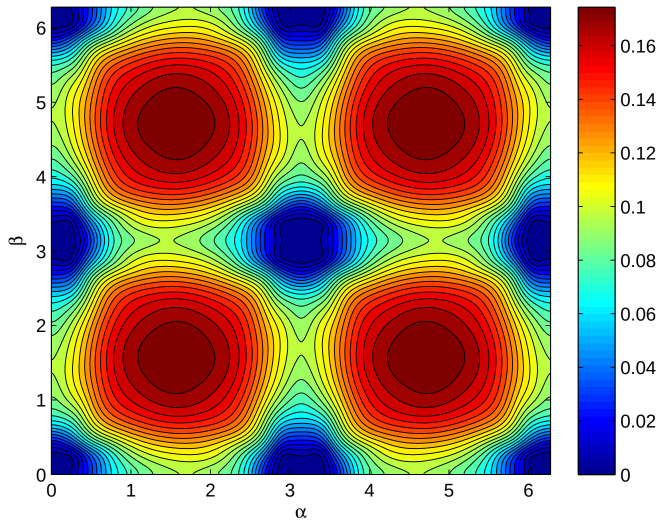


FIG. 5. A contour plot of $|A_\alpha|^2 + |A_\beta|^2$ for the solution of (1..2) at time $\tau = 0.1$ with initial data (7..4) where $a_0^2 = 1/9$.


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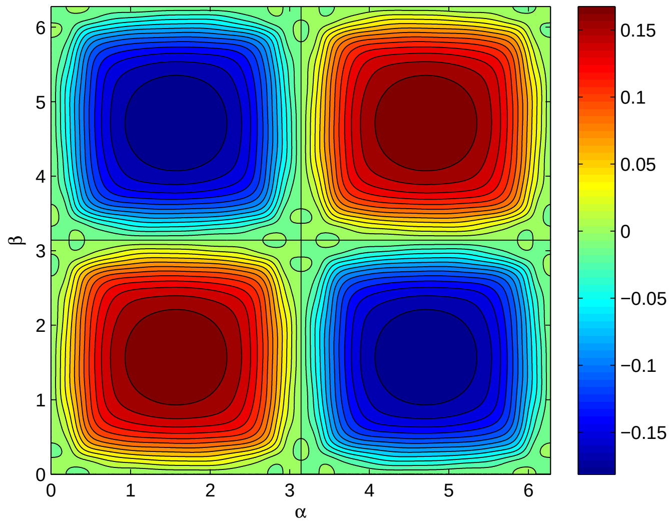


FIG. 6. A contour plot of $i(A_\alpha A_\beta^* - A_\alpha^* A_\beta)$ for the solution of (1..2) at time $\tau = 0.1$ with initial data (7..4) where $a_0^2 = 1/9$.

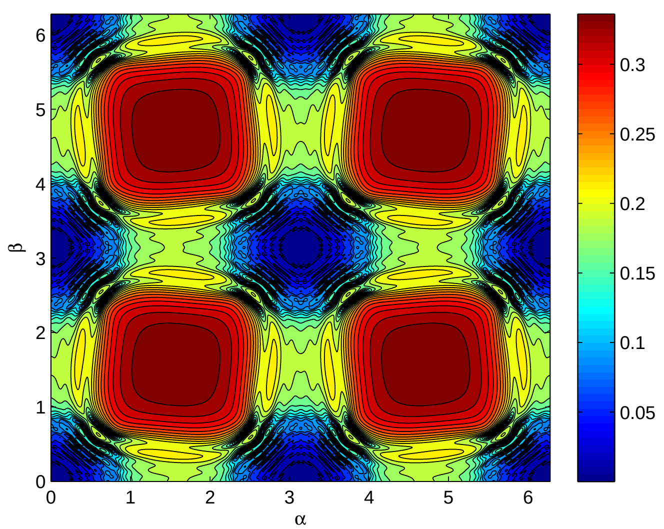


FIG. 7. A contour plot of $|A_\alpha|^2 + |A_\beta|^2$ for the solution of (1..2) at time $\tau = 0.005$ with initial data (7..4) where $a_0^2 = 1/5$.

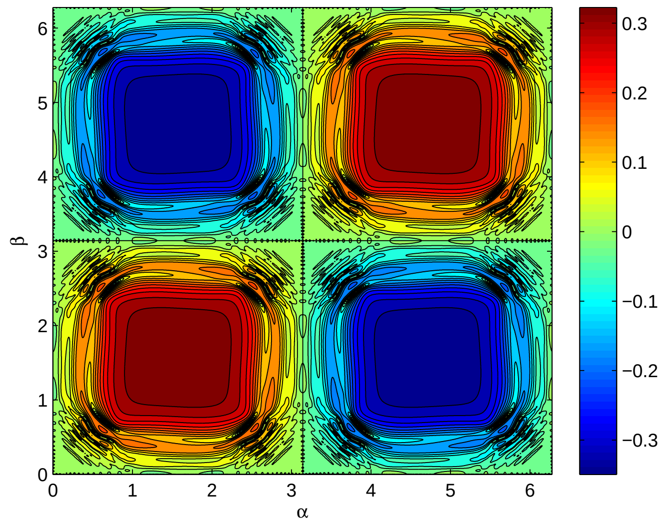


FIG. 8. A contour plot of $i(A_\alpha A_\beta^* - A_\alpha^* A_\beta)$ for the solution of (1..2) at time $\tau = 0.005$ with initial data (7..4) where $a_0^2 = 1/5$.

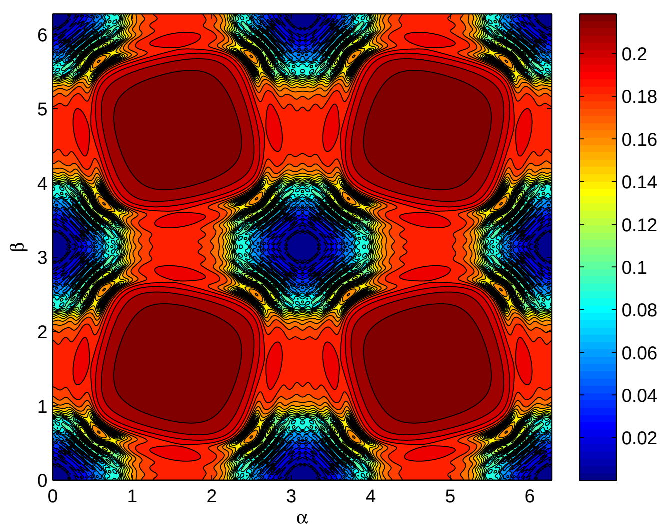


FIG. 9. A contour plot of $\rho$, defined in (1..4), for the solution of (1..2) at time $\tau = 0.005$ with initial data (7..4) where $a_0^2 = 1/5$.