NONLINEAR HYPERBOLIC SURFACE WAVES

JOHN K. HUNTER*

Abstract. We describe examples of hyperbolic surface waves and discuss their connection with initial boundary value and discontinuity problems for hyperbolic systems of PDEs that are weakly but not uniformly stable.

Key words. Nonlinear waves, conservation laws, initial-boundary value problems.

AMS(MOS) subject classifications. Primary 35L50, 35L65, 74J15, 74J30.

1. Introduction. Hyperbolic surface waves are described by initial boundary value problems (IBVPs) and discontinuity problems for hyperbolic systems of conservation laws that are weakly stable but not uniformly stable. Our aim here is to give an informal overview of such surface waves, with an emphasis on 'genuine' surface waves, or surface waves 'of finite energy,' that are localized near the boundary or discontinuity and decay exponentially into the interior. In particular, we are interested in understanding the effect of nonlinearity on such waves.

A well-known example of a genuine hyperbolic surface wave is the Rayleigh wave, or surface acoustic wave (SAW), that propagates on the stress-free boundary of an elastic half space. Rayleigh waves are important in seismology and electronics, where ultrasonic SAW devices are widely used as components in cell phones and other systems (although, at present, nearly always in a linear regime).

Another example of a genuine hyperbolic surface wave occurs in magnetohydrodynamics (MHD). We consider incompressible MHD for simplicity. A tangential discontinuity consists of a magnetic field and fluid velocity that are tangent to the discontinuity and jump across it. If the jump in the velocity is nonzero and the magnetic field is zero, then the tangential discontinuity is a vortex sheet, which is subject to the Kelvin-Helmholtz instability. If the magnetic field is large enough compared with the jump in the velocity, then it stabilizes the discontinuity. In that case, the discontinuity is weakly but not strongly stable and genuine surface waves propagate along it.

In a unidirectional surface wave, the displacement $y = \phi(x,t)$ of a weakly stable tangential discontinuity satisfies the following quadratically nonlinear, nonlocal asymptotic equation [1]

$$\phi_t + \mathbf{H}\left[\psi\psi_x\right]_x + \psi\phi_{xx} = 0 \qquad \psi = \mathbf{H}[\phi] \tag{1.1}$$

where $\mathbf{H}$ is the spatial Hilbert transform, defined by

$$\mathbf{H}[e^{ikx}] = -i(\operatorname{sgn} k)e^{ikx}.$$

*Department of Mathematics, University of California, Davis, California. Partially supported by NSF grant DMS-0604947.

Equation (1.1) provides a model equation for nonlinear hyperbolic genuine surface waves that plays an analogous role to the inviscid Burgers equation for bulk waves. It was first proposed by Hamilton, Il'insky, and Zabolotskaya [14] as a simplification of asymptotic equations for SAWs, so we refer to it as the HIZ equation for short.

Hyperbolic surface waves are nondispersive. As a result, their nonlinear behavior differs qualitatively from that of dispersive water waves that propagate on the free surface of a fluid. Even shallow water waves, which are nondispersive when their slope is sufficiently small, become dispersive once they steepen; by contrast, hyperbolic surface waves remain nondispersive however steep they become. Similarly, the Benjamin-Ono equation $\phi_t + \phi\phi_x + \mathbf{H}[\phi]_{xx} = 0$ has local nonlinearity and nonlocal linear dispersion, whereas the HIZ equation (1.1) has nonlocal nonlinearity and no dispersion.

2. Half-space hyperbolic IBVPs. The well-posedness of IBVPs for hyperbolic PDEs was studied by Kreiss [18] for systems and Sakamoto [26, 27] for wave equations. Majda [21, 22] extended this analysis to discontinuity problems for hyperbolic systems of conservation laws and used the results to study the stability of shock waves.

Consider a half-space IBVP in d-space dimensions that consists of a hyperbolic system of conservation laws for $u(x,t) \in \mathbb{R}^n$

$$u_t + \sum_{j=1}^{d} f^j(u)_{x_j} = 0 \qquad x_d > 0,\, t > 0 \tag{2.1}$$

with initial and boundary conditions

$$u = u_0 \quad \text{on } t = 0, \qquad h(u) = 0 \quad \text{on } x_d = 0. \tag{2.2}$$

Here, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $f^j : \mathbb{R}^n \to \mathbb{R}^n$ is the flux vector in the jth direction, and $h : \mathbb{R}^n \to \mathbb{R}^m$ defines the boundary conditions. Nonhyperbolic systems of conservation laws, such as the incompressible Euler or MHD equations, may be regarded as a limit of the corresponding compressible equations. We consider only inviscid equations here; for viscous problems, see *e.g.* [30].

Neither the geometry nor the PDE define length or time parameters, and the IBVP is invariant under rescalings $x \mapsto cx$, $t \mapsto ct$ for any $c > 0$. Thus, the existence of a solution that is bounded in space and grows in time implies the existence of spatially-bounded solutions that grow arbitrarily quickly in time. As a result, there is a sharp division between weakly or strongly stable problems without growing modes and 'violently' unstable problems with growing modes. Unstable problems typically arise when the boundary conditions couple positive and negative energy bulk waves [19].

The linearization of (2.1)–(2.2) about a constant solution has the form

$$u_t + \sum_{j=1}^{d} A^j u_{x^j} = 0 \qquad \text{for } x_d,\, t > 0,$$

$$Cu = 0 \quad \text{on } x_d = 0, \qquad u = u_0 \quad \text{on } t = 0$$

where $A^j \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{m\times n}$ are constant matrices. For definiteness, We suppose that $x_d = 0$ is non-characteristic, meaning that A^d is non-singular.[1] Then a necessary condition for well-posedness is that the number of BCs m must equal the number of positive eigenvalues of A^d and the BCs must determine the associated wave-components that propagate into the half-space in terms of the wave-components that propagate out of the half-space.

A sufficient condition for the strong L^2-well-posedness of the IBVP in any number of space dimensions is provided by a uniform Kreiss-Sakamoto-Lopatinski condition, or uniform Lopatinski condition for short. To obtain this condition, we look for Fourier solutions of the IBVP that oscillate tangent to the boundary and decay away from it of the form

$$u(x,t) = e^{i(k_1x_1+\ldots+k_{d-1}x_{d-1})-i\omega t}U(x_d)$$

where $k = (k_1, \ldots, k_{d-1}) \in \mathbb{R}^{d-1}$, $\omega \in \mathbb{C}$, and $U(x_d) \to 0$ as $x_d \to +\infty$. Typically, U decays exponentially and given by

$$U(x_d) = \sum_{j=1}^{m} c_j e^{-\beta_j(\omega,k)x_d} r_j(\omega, k)$$

where $r_j(\omega, k) \in \mathbb{C}^n$ and $\beta_j(\omega, k) \in \mathbb{C}$ with $\operatorname{Re}\beta_j > 0$. In the case of repeated roots, we may obtain additional polynomial factors in x_d.

There are three alternatives that correspond in a rough sense (made precise by the Kreiss theory) to the growth, oscillation, or decay in time of appropriate Fourier solutions. (a) If there is a solution of the PDE and BC that grows in time and decays in space away from boundary, meaning that $\operatorname{Im}\omega > 0$ and $\operatorname{Re}\beta_j > 0$ for every $1 \le j \le m$ for some $k \in \mathbb{R}^{d-1}$, then the IBVP is subject to Hadamard instability and is ill-posed in any Sobolev space. (b) If there is a solution of the PDE and BC that oscillates in time and does not grow away from boundary, meaning that $\operatorname{Im}\omega = 0$ and $\operatorname{Re}\beta_j \ge 0$ for some $k \in \mathbb{R}^{d-1}$, *and* this solution is a limit of solutions of the PDE that grow in time and decay in space away from boundary, then the linearized problem is weakly stable in L^2-Sobolev spaces, but there is a 'loss' of derivatives in the energy estimates. (c) If there are no such solutions of the PDE and BC that grow or oscillate in time, then there are good L^2-energy estimates and the problem is strongly stable. This case occurs when the norm of a suitable $m \times m$ matrix $L(\omega, k)$ is uniformly bounded away from zero in $\operatorname{Im}\omega > 0$ and $k \in \mathbb{S}^{d-2}$. We then say that the IBVP satisfies a uniform Lopatinski condition. See [7] for a detailed description and derivation of these results.

[1] Characteristic IBVPs arise in many applications, for example in the study of discontinuities that move with a fluid such as the tangential discontinuities considered below, but it is more difficult to develop a general theory for them *c.f.* [7] for further discussion.

In the weakly stable case (b) there are two alternatives for the spatial behavior of the surface waves in the limit that they are approximated by spatially decaying solutions of the PDE with $\operatorname{Im}\omega \to 0^+$. (b1) If $\lim \operatorname{Re}\beta_j > 0$ for every j, then we get genuine surface waves that decay away from the boundary. (b2) If $\lim \operatorname{Re}\beta_j = 0$ for some j, then we get 'radiative' or 'leaky' surface waves that generate bulk waves which propagate away from the boundary into the interior.

As an illustration, consider solutions of the scalar wave equation

$$u_{tt} = u_{xx} + u_{yy} \tag{2.3}$$

that propagate with speed λ along the boundary $y = 0$. If $|\lambda| < 1$, then the corresponding Fourier solution is given by

$$u(x, y, t) = e^{i(x-\lambda t) - \sqrt{1-\lambda^2}\, y}.$$

Thus, a surface wave that is 'subsonic' with respect to the speed of bulk waves is genuine. For example, the speed of a Rayleigh wave in isotropic elasticity is less than both the transverse and longitudinal bulk wave speeds. If $|\lambda| > 1$, then the Fourier solution is

$$u(x, y, t) = e^{i(x-\lambda t) + i\sqrt{\lambda^2-1}\, y},$$

and a 'supersonic' surface wave is radiative. The difference between 'subsonic' genuine waves and 'supersonic' radiative waves is a consequence of the geometrical effect that the phase speed along the boundary of a plane bulk wave propagating at an angle θ to a boundary is *greater* than its normal phase speed by a factor of $\sec\theta$.

Next, consider (2.3) in the half-space $y > 0$ with the initial and boundary conditions [23]

$$u = u_0, \quad u_t = v_0 \quad \text{on } t = 0, \qquad \gamma u_t + u_y = 0 \quad \text{on } y = 0 \tag{2.4}$$

where $\gamma \in \mathbb{R}$ is a parameter. The BC in (2.4) is invariant under a simultaneous spatial reflection and time reversal $y \mapsto -y$, $t \mapsto -t$, but not under $t \mapsto -t$ alone. Thus, unlike the free space problem, the IBVP is not reversible in time and IBVPs that are stable forward in time may be unstable backward in time.

The IBVP (2.3)–(2.4) is uniformly stable if $\gamma < 0$, weakly stable if $0 \le \gamma \le 1$, and unstable if $\gamma > 1$. For $\gamma > 1$, the mode that grows in t and decays in y is

$$u(x, y, t) = \exp\left[ix + \sigma(t - \gamma y)\right], \qquad \sigma = \frac{1}{\sqrt{\gamma^2 - 1}}.$$

For $\gamma < -1$, this mode grows in t only if it grows in y, consistent with the uniform stability. The weak stability for $0 < \gamma < 1$ is a consequence of the existence surface waves of the form

$$u(x,y,t) = \exp\left[ix + i\lambda(\gamma y - t)\right], \qquad \lambda = \pm\frac{1}{\sqrt{1-\gamma^2}}$$

that radiate bulk waves into the interior $y > 0$. These solutions also exist for $-1 < \gamma < 0$, but in that case they are not the limit of of solutions of the PDE that grow in time and decay in space, so their existence does not violate the uniform Lopatinski condition. Equivalently, the waves are coupled with bulk waves whose group velocity is directed toward the boundary, not away from it as in the weakly stable case [15].

This IBVP for a scalar wave equation leads to radiative surface waves in the weakly stable case, but at least two wave equations are required to obtain genuine surface waves. Thus, genuine surface waves do not arise in gas dynamics, only in larger systems such as elasticity or MHD.

3. Discontinuity problems. Discontinuity problems for hyperbolic system of conservation laws lead to analogous equations to the ones for IBVPs in which the location of the discontinuity appears as an additional variable. Consider a solution of (2.1) that contains a discontinuity — such as a shock wave, vortex sheet, or contact discontinuity — located at

$$x_d = \Phi\left(x_1, \ldots, x_{d-1}, t\right).$$

The PDE for $u = u^+$ in $x_d > \Phi$ and $u = u^-$ in $x_d < \Phi$ can be regarded as giving a single half-space PDE in twice the number of dependent variables. The jump conditions for (2.1) have the form

$$\Phi_t + W\left(u^+, u^-, \nabla\Phi\right) = 0, \quad h\left(u^+, u^-, \nabla\Phi\right) = 0 \qquad \text{on } x_d = \Phi,$$

which gives as an equation for the motion of the discontinuity as well as BCs on the discontinuity. Linearization of these equations about a planar discontinuity leads to similar equations to the linearized half-space equations.

If the unperturbed discontinuity is a Lax shock, then the boundary is non-characteristic. The Lax shock condition (that characteristics in one family enter the shock from both sides and characteristics in the other families cross the shock) implies that we have the correct number of boundary conditions: n jump conditions give $(n-1)$ BCs for the waves that propagate into the half-spaces on either side of shock and one condition for the shock location. We also require an 'evolutionary' condition, meaning that the jump conditions determine the wave components that are outgoing from the shock in terms of the incoming wave components.

Overcompressive shocks give linearized IBVPs with too many BCs, and such shocks typically split into multiple Lax shocks under arbitrary perturbations. Undercompressive shocks give linearized IBVPs with too few BCs, and extra conditions, or 'kinetic relations,' are required to determine their motion.

An evolutionary Lax shock is strongly stable in several space dimensions if it satisfies an analog of the uniform Lopatinski condition [21]. If the Lopatinski condition fails entirely, then there are modes that grow arbitrarily quickly, and the discontinuity is violently unstable as in the Kelvin-Helmholtz instability of a vortex sheet. If the discontinuity is weakly but not strongly stable, then surface waves propagate along the discontinuity.

Surface waves on a discontinuity may be radiative or genuine. Examples of discontinuities that support radiative surface waves include strong shocks in the compressible Euler equations for some equations of state [23], supersonic vortex sheets in a compressible fluid [3, 10], and detonation waves in combustion [24]. Examples of discontinuities that support genuine surface waves are tangential discontinuities in MHD [8, 29] and propagating phase boundaries in a van der Waals fluid [5, 6].

Short-time existence results have been obtained for various weakly stable problems, typically by the use of a Nash-Moser scheme to compensate for the loss of derivatives in the linearized energy estimates *e.g.* [8, 10, 25, 29]. The Nash-Moser scheme is a powerful method but because of its generality it makes relatively little use of the specific features of these problems, such as the distinction between genuine and radiative surface waves or the structure of the nonlinearity.

4. Magnetohydrodynamics. The non-conservative form of the system of MHD equations for a conducting, incompressible fluid with velocity $\mathbf{u}$, magnetic field $\mathbf{B}$, and total pressure p is

$$\begin{aligned} &\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{B}\cdot\nabla\mathbf{B} + \nabla p = 0, \qquad && \operatorname{div}\mathbf{u} = 0,\\ &\mathbf{B}_t + \mathbf{u}\cdot\nabla\mathbf{B} - \mathbf{B}\cdot\nabla\mathbf{u} = 0, && \operatorname{div}\mathbf{B} = 0. \end{aligned}$$

The jump conditions for a tangential discontinuity located at $\Phi(\mathbf{x},t) = 0$ are

$$\Phi_t + \mathbf{u}\cdot\nabla\Phi = 0, \quad \mathbf{B}\cdot\nabla\Phi = 0, \quad [p] = 0 \qquad \text{on } \Phi = 0$$

where $[p]$ denotes the jump in p across the discontinuity.

A planar tangential discontinuity located at $y = 0$, say, consists of arbitrary velocities and magnetic fields that are constant in each half-space $y > 0$ and $y < 0$, tangent to the discontinuity, and jump across it:

$$\mathbf{u} = \begin{cases} \mathbf{U}_+ & y > 0 \\ \mathbf{U}_- & y < 0 \end{cases}, \qquad \mathbf{B} = \begin{cases} \mathbf{B}_+ & y > 0 \\ \mathbf{B}_- & y < 0 \end{cases}.$$

If $\mathbf{B}_\pm = 0$ and $\mathbf{U}_+ \neq \mathbf{U}_-$, then the tangential discontinuity is an unstable vortex sheet. If the magnetic field components in the direction of the discontinuous velocity components are strong enough, specifically if [20]

$$\begin{aligned} &|\mathbf{B}_+|^2 + |\mathbf{B}_-|^2 > \frac{1}{2}\,|\mathbf{U}_+ - \mathbf{U}_-|^2\,,\\ &|\mathbf{B}_+ \times \mathbf{B}_-|^2 \geq \frac{1}{2}\left(|\mathbf{B}_+ \times \mathbf{U}_+|^2 + |\mathbf{B}_- \times \mathbf{U}_-|^2\right), \end{aligned}$$

then the tangential discontinuity is weakly stable, and genuine surface waves propagate along it.

For times that are not too long compared with a typical period, which we nondimensionalize to be of the order one, a unidirectional surface wave with small slope of the order ε consists of an arbitrary profile that propagates without change of shape at the linearized surface wave speed λ. Over longer times, of the order ε^{-1}, nonlinear effects distort the profile. Considering planar flows for simplicity, one finds that the weakly nonlinear solution for the location of the tangential discontinuity in a surface wave is given by [1]

$$y = \varepsilon\phi\left(x - \lambda t, \varepsilon t\right) + O(\varepsilon^2) \qquad \text{as } \varepsilon \to 0$$

where $\phi(x,t)$ satisfies (1.1) after the change of variables $x - \lambda t \mapsto x$, $\varepsilon t \mapsto t$ and a normalization.

It is interesting to consider the dynamics of a tangential discontinuity in MHD near the onset of Kelvin-Helmholtz instability. This bifurcation problem differs from standard ones because all of the spatial modes of the solution become unstable simultaneously.

In a reference frame moving with the linearized surface wave speed at the bifurcation point where the left and right wave velocities coalesce and become complex, the location $y = \phi(x,t)$ of the discontinuity satisfies the normalized asymptotic equation [17]

$$\phi_{tt} + \left(\mathbf{H}\left[\psi\psi_x\right]_x + \psi\phi_{xx}\right)_x = \mu\phi_{xx} \qquad \psi = \mathbf{H}[\phi]. \tag{4.1}$$

Here, $\mu = \mathbf{B}_+^2 + \mathbf{B}_-^2 - \frac{1}{2}\left(\mathbf{U}_+ - \mathbf{U}_-\right)^2$ is the bifurcation parameter. If $\mu > 0$, then the tangential discontinuity is linearly stable and the asymptotic equation is a wave equation. For large μ, it reduces in a unidirectional approximation $\phi_t \sim \pm\sqrt{\mu}\phi_x$ to decoupled equations of the form (1.1). If $\mu < 0$, then the tangential discontinuity is linearly unstable, and the linearized asymptotic equation is a Laplace equation that is subject to Hadamard instability, corresponding to the Kelvin-Helmholtz instability of the discontinuity.

One might hope that nonlinearity has a regularizing effect on the Hadamard instability (as in the model equations studied in [4]). Preliminary numerical solutions of (4.1), however, suggest the reverse: finite amplitude effects appear to trigger a Kelvin-Helmholtz stability on the less stable side of the discontinuity where the magnetic field is weaker.

The apparent ill-posedness in any Sobolev space of vortex sheet solutions of the Euler and MHD equations raises questions about the validity of these equations as general physical models, especially for compressible flows where vortex sheets can be generated spontaneously from smooth initial data by the formation of shocks and triple points. Perhaps related to this instability is the apparent nonuniqueness of numerical solutions of the compressible Euler equations containing vortex sheets [13] and the nonuniqueness of low-regularity weak solutions [12].

5. The HIZ equation. Consider, for definiteness, solutions of the HIZ equation (1.1) that are 2π-periodic in space with the Fourier expansion

$$\phi(x,t) = \sum_{k\in\mathbb{Z}} \hat{\phi}(k,t)e^{ikx}. \tag{5.1}$$

The spectral form of (1.1) is

$$\hat{\phi}_t(k,t) + i(\operatorname{sgn} k)\sum_{n\in\mathbb{Z}} T(-k,k-n,n)\phi(k-n,t)\phi(n,t) = 0 \tag{5.2}$$

where the interaction coefficient $T : \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z} \to \mathbb{C}$ is given by

$$T(k,m,n) = \frac{2|kmn|}{|k|+|m|+|n|}. \tag{5.3}$$

The coefficient $T(k,m,n)$ describes the strength of the coupling between Fourier modes in the resonant three-wave interaction $m+n \to -k$. Only the values of $T(k,m,n)$ on $k+m+n = 0$ appear in (5.2), but it is convenient to show all three wave numbers explicitly.

The symmetry of $T(k,m,n)$ in (k,m,n) is equivalent to the existence of a canonical Hamiltonian structure for the HIZ equation; its spatial form is

$$\phi_t = \mathbf{H}\left[\frac{\delta\mathcal{H}}{\delta\phi}\right], \qquad \mathcal{H}(\phi) = \int \psi\phi_x\psi_x\, dx, \quad \psi = \mathbf{H}[\phi]. \tag{5.4}$$

The homogeneity of T of degree two is implied by a dimensional analysis of quadratically nonlinear nondispersive Hamiltonian waves, depending only on velocity parameters, that propagate on a boundary with codimension one [2].

It follows from (5.3) that

$$|T(k,m,n)| \le |kmn|^{1/2}\min\left\{|k|^{1/2}, |m|^{1/2}, |n|^{1/2}\right\} \tag{5.5}$$

on $k+m+n = 0$. This estimate implies a limit on the rate at which energy can be transferred from low to high wavenumbers by three-wave interactions, since the interaction coefficient is bounded by a factor that depends on the *lowest* wavenumber that participates in the interaction. Related to this spectral property is the following spatial form of (1.1)

$$\phi_t + [\mathbf{H},\psi]\,\psi_{xx} + \mathbf{H}\left[\psi_x^2\right] = 0 \tag{5.6}$$

where $[\mathbf{H},\psi] = \mathbf{H}\psi - \psi\mathbf{H}$ is the commutator of $\mathbf{H}$ with multiplication by ψ. As (5.6) shows, there is a cancelation of the second order spatial derivatives appearing in (1.1). Generalizing (5.5), we make the following definition.

Definition 5.1. *An interaction coefficient* $T : \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ *has minimal spectral growth with exponent* $\mu \geq 0$ *if there is a constant* C *such that*

$$|T(k,m,n)| \leq C|kmn|^{1/2} \min\{|k|^{\mu}, |m|^{\mu}, |n|^{\mu}\}$$

for every $k, m, n \in \mathbb{Z}$ *such that* $k + m + n = 0$.

For example, the HIZ-coefficient (5.3) has minimal spectral growth with exponent $\mu = 1/2$. We then have the following local existence result [16].

Theorem 5.1. *Suppose that* $T : \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ *is a symmetric interaction coefficient that has minimal spectral growth with exponent* $\mu \geq 0$. *Let* $s > \mu + 2$ *and* $\phi_0 \in H^s(\mathbb{T})$. *Then there exists* $t_* > 0$, *depending only on* T *and* $\|\phi_0\|_{H^s}$, *and a unique local solution*

$$\phi \in C\left([-t_*, t_*]; H^s(\mathbb{T})\right) \cap C^1\left([-t_*, t_*]; H^{s-1}(\mathbb{T})\right)$$

of (5.1)–(5.2) with $\phi(0) = \phi_0$.

Thus, we get the short-time existence of smooth H^s-valued solutions of (1.1) for $s > 5/2$. The proof uses a standard energy method in Fourier space, and the main point of this result is the identification of the minimal spectral growth condition as a sufficient condition for well-posedness. This condition allows one to estimate a term such as $|k|^{\alpha}|m|^{\beta}T(k,m,n)$ by a term proportional to $|k|^{\alpha}|m|^{\beta}|n|^{\mu}$, thus avoiding a loss of derivatives. This corresponds to the use of integration by parts to eliminate higher derivatives in spatial estimates, although the nonlocality of the spatial form of (5.2) makes it more difficult to carry out the spatial estimates directly. If T does not satisfy the minimal spectral growth condition, then there appears to be a loss of derivatives in the energy estimates for solutions of (5.1)–(5.2), and it is not clear that local H^s-solutions exist for any $s \in \mathbb{R}$.

An interesting question is to characterize which boundary conditions in weakly stable IBVP problems with genuine surface waves for Hamiltonian or variational systems of PDEs lead to asymptotic equations whose interaction coefficients satisfy the minimal spectral growth condition. This would provide a possible criterion for the identification of weakly stable problems with a good nonlinear theory.

Numerical results obtained using a spectral viscosity method show the formation of singularities in smooth solutions of (1.1) in finite time (see Figure 1). The solution appears to remain continuous while its derivative blows up. Furthermore, solutions of a viscous regularization of (1.1) appear to converge to a weak solution of (1.1) in the zero-viscosity limit.

Using the distributional formula for the Hilbert transform,

$$\mathbf{H}[|x|^{\alpha}] = -c_{\alpha} \operatorname{sgn} x|x|^{\alpha} \quad c_{\alpha} = \tan\frac{\pi\alpha}{2} \qquad \text{for } \alpha \notin 2\mathbb{Z}+1,$$

we find that an exact steady distributional solution of (1.1) is

$$\phi(x) = A \operatorname{sgn} x\, |x|^{\alpha} \tag{5.7}$$

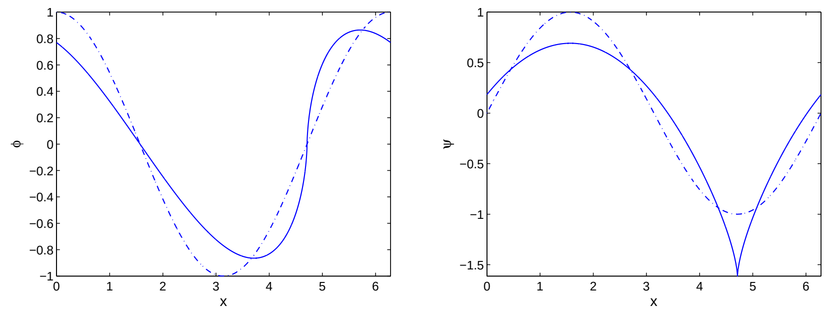

FIG. 1. *Formation of a singularity in the solution of (1.1) with initial data* $\phi(x,0) = \cos x$. *The dashed lines are the initial data and the solid lines are the solution at time* $t = 0.8$. *The plot on the left is* ϕ*; the plot on the right is* $\psi = \mathbf{H}\phi$.

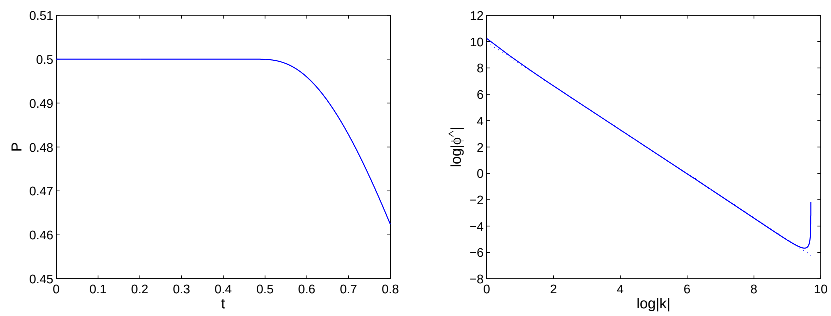

FIG. 2. *The plot on the left is the momentum* $P = (2\pi)^{-1}\int_0^{2\pi}\phi|\partial_x|\phi\,dx$ *as a function of time for the solution shown in Figure 1;* P *is conserved until the singularity develops at* $t \approx 0.5$. *The plot on the right is the spectrum for the numerical solution with* 32768 *modes at* $t = 0.8$. *The dotted line is a best linear fit to the power-law range of the spectrum, which gives* $|\hat{\phi}(k)| \sim C|k|^{-\alpha}$ *with* $\alpha \approx 1.66657$ *and* $\log C \approx 9.96648$.

where A is an arbitrary constant and α satisfies $(2\alpha - 1)c_{2\alpha} = (\alpha - 1)c_\alpha$. The unique solution of this equation with $\alpha, 2\alpha \notin 2\mathbb{Z}+1$ is $\alpha = 2/3$. As shown in Figure 2, numerical solutions have a power-law spectral decay with exponent close to 5/3, consistent with this analytical solution. Only singularities with $A > 0$ appear, presumably as the result of a viscous admissibility condition.

The validity of the asymptotic solution for MHD once the derivative of ϕ blows up is open to question. In particular, although the asymptotic solutions for the fluid velocity and magnetic field are smooth in the interior, they are unbounded near a singularity in the tangential discontinuity. Nevertheless, analogous weakly nonlinear asymptotic solutions for bulk waves correctly describe the formation and propagation of shocks and it is plausible that the asymptotic solution for surface waves remains valid in a weak sense. The asymptotic analysis therefore predicts the formation of singularities with Hölder-exponent 2/3 in the displacement of a weakly stable tangential discontinuity in incompressible MHD. This mechanism for singularity formation on the boundary in weakly stable IBVPs and discontinuities differs qualitatively from shock formation in the interior.

The numerical results suggest that (1.1) has global admissible weak solutions, but a proof of this conjecture is open. Apart from the energy $\int \psi\phi_x\psi_x \, dx$, which is not positive definite, the only globally conserved quantity for smooth solutions of (1.1) that we know of is the momentum $\int \phi|\partial_x|\phi \, dx$, which is the homogeneous $H^{1/2}$-norm of ϕ. This quantity is conserved for smooth solutions and decreases for weak solutions that satisfy a viscous admissibility condition, but it does not seem to be strong enough to imply the global existence of weak solutions.

6. Conclusion. Despite significant recent progress in the analysis of weakly stable IBVPs and discontinuity problems in hyperbolic conservation laws, much remains to be understood. It seems likely that one will need to identify and use crucial features of the nonlinear structure of these problems including, perhaps, their Hamiltonian or variational structure [28] and their Lagrangian formulation [9, 11]. There appears to be little hope of a reasonable nonlinear theory for 'violently' unstable IBVPs or discontinuities, such as vortex sheets. This raises serious questions about the consistency of hyperbolic conservation laws, which neglect all small-scale regularizing effects, as mathematical models for such problems.

REFERENCES

[1] G. Alì and J. K. Hunter, *Nonlinear surface waves on a tangential discontinuity in magnetohydrodynamics*, Quart. Appl. Math., **61** (2003), 451–474.

[2] G. Alì, J. K. Hunter and D. F. Parker, *Hamiltonian equations for nonlinear scale-invariant waves*, Stud. Appl. Math., **108** (2002) 305–321.

[3] M. Artola and A. J. Majda, *Nonlinear development of instabilities in supersonic vortex sheets I. The basic kink modes*, Physica D, **28** (1987), 253–281.

[4] J. T. Beale and D. Schaeffer, *Nonlinear behavior of model equations which are linearly ill-posed*, Comm. Partial Differential Equations, **13** (1988), 423–467.

[5] S. Benzoni-Gavage, *Stability of subsonic planar phase boundaries in a van der Waals fluid*, Arch. Rat. Mech. Anal. **150** (1999), 23–55.

[6] S. Benzoni-Gavage and H. Freistühler, *Effects of surface tension on the stability of dynamical liquid-vapor interfaces*, Arch. Ration. Mech. Anal., **174** (2004), 111–150.

[7] S. Benzoni-Gavage and D. Serre, *Multi-dimensional Hyperbolic Partial Differential Equations*, Oxford University Press, Oxford, 2007.

[8] G.-Q. Chen and Y.-G. Wang, *Existence and stability of compressible current-vortex sheets in three-dimensional magnetohydrodynamics*, Arch. Rational Mech. Anal., **187** (2008), 369–408.

[9] A. Cheng, D. Coutand and S. Shkoller, *On the Motion of Vortex Sheets with Surface Tension in the 3D Euler Equations with Vorticity*, Comm. Pure Appl. Math., **61** (2008), 1715–1752.

[10] J. F. Coulombel and P. Secchi, *Stability of compressible vortex sheet in two space dimensions*, Indiana Univ. Math. J. **53** (2004), 941-1012.

[11] D. Coutand and S. Shkoller, *Well-posedness of the free-surface incompressible Euler equations with or without surface tension*, J. Amer. Math. Soc., **20** (2007), 829–930.

[12] C. de Lellis, and L. Székelyhidi, *The Euler equations from the point of view of differential inclusions*, Mat. Ital. **9** (2008), 873–879.

[13] V. Elling, *A possible counter example to well posedness of entropy solutions and to Godunov scheme convergence*, Math. Comp., **75** (2006), 1721–1733.

[14] M. F. Hamilton, Yu. A. Il'insky and E. A. Zabolotskaya, *Evolution equations for nonlinear Rayleigh waves*, J. Acoust. Soc. Am., **97** (1995), 891–897.

[15] R. L. Higdon, *Initial-boundary value problems for linear hyperbolic systems*, SIAM Review, **28** (1986), 177–217.

[16] J. K. Hunter, *Short-time existence for scale-invariant Hamiltonian waves*, J. Hyperbolic Differential Equations, **3** (2006), 247–267.

[17] J. K. Hunter and J. B. Thoo, *A weakly nonlinear Kelvin-Helmholtz instability in MHD*, in preparation.

[18] H.-O. Kreiss, *Initial boundary value problems for hyperbolic systems*, Comm. Pure Appl. Math., **23** (1970), 277–298.

[19] S. C. Lashmore-Davies, *Negative energy waves*, J. Plasma Physics, **71** (2005), 101–109.

[20] L. D. Landau, L. P. Pitaevskii and E. M. Lifschitz, *The Electrodynamics of Continuous Media*, Pergamon Press, New York, 2nd ed., 1984.

[21] A. J. Majda, *The stability of multidimensional shock fronts*, Mem. Amer. Math. Soc., **41** (1983), no. 275.

[22] A. J. Majda, *The existence of multi-dimensional shock fronts*, Mem. Amer. Math. Soc., **43** (1983), no. 281, AMS, Providence, 1983.

[23] A. J. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Dimensions*, Springer-Verlag, 1984.

[24] A. J. Majda, and R. R. Rosales, *A theory for spontaneous Mach stem formation in reacting shock fronts. I. The basic perturbation analysis*, SIAM J. Appl. Math., **43** (1983), 1310–1334.

[25] M. Sable-Tougeron, *Existence pour un problème de l'élastodynamique Neumann non linéaire en dimension* 2, Arch. Rational Mech. Anal., **101** (1988), 261–292.

[26] R. Sakamoto, *Mixed problems for hyperbolic equations. I. Energy inequalities*, J. Math. Kyoto Univ., **10** (1970), 349–373.

[27] R. Sakamoto, *Mixed problems for hyperbolic equations. II. Existence theorems with zero initial datas and energy inequalities with initial datas, J. Math. Kyoto Univ.*, **10** (1970), 403–417.

[28] D. Serre, *Second order initial boundary-value problems of variational type*, J. Functional Anal., **236** (2006), 409–446.

[29] Y. Trakhinin, *Existence of compressible current-vortex sheets: variable coefficients linear analysis*, Arch. Ration. Mech. Anal. **177** (2005), 331-366.

[30] K. Zumbrun, *Multidimensional stability of planar viscous shock waves*, Advances in the theory of shock waves, 307–516, Progr. Nonlinear Differential Equations Appl., **47**, Birkhäuser Boston, Boston, MA, 2001.