Nonlinear variational surface waves

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Abstract

We derive nonlocal asymptotic equations for weakly nonlinear surface wave solutions of variational wave equations in a half-space. These equations are analogous to, but different from, equations that describe weakly nonlinear Rayleigh waves in elasticity and other hyperbolic conservation laws. We prove short time existence of smooth solutions of a simplified, but representative, asymptotic equation and present numerical solutions which show the formation of cusp-singularities. This singularity formation on the boundary is a different mechanism for the nonlinear breakdown of smooth solutions of hyperbolic IBVPs from the more familiar one of singularity formation in the interior.

1 Introduction

In this paper, we consider initial-boundary value problems (IBVPs) in a half-space $\mathbb{R}^d_+$ for systems of wave equations for $u : \mathbb{R}^d_+ \times \mathbb{R} \to \mathbb{R}^n$ that arise from variational principles of the form

$$\delta \int_{\mathbb{R}^d_+ \times \mathbb{R}} \left\{ \frac{1}{2}|u_t|^2 - W(u, \nabla u) \right\} dxdt = 0, \tag{1}$$

where the potential energy density $W(u, \nabla u)$ is a quadratic function of ∇u with coefficients depending on u.

The nonlinearity in the resulting variational, but non-conservative, wave equations differs qualitatively from the nonlinearity in conservative quasi-linear wave equations, such as nonlinear elasticity, where $W(\nabla u)$ is independent of u but not quadratic in ∇u. We are interested in comparing and contrasting the effects of nonlinearity on these types of waves.

We study IBVPs that are stable but not uniformly stable, which occurs when the IBVP has surface wave solutions. We focus on problems with "genuine" or "finite-energy" surface waves that propagate along the boundary and

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decay exponentially into the interior, rather than "radiative" surface waves that are coupled with bulk waves in the interior. The aim of this paper is to derive asymptotic equations for weakly nonlinear, variational genuine surface waves and, in particular, to show that singularities form in these waves on the boundary.

One example of such a variational system arises as a description of orientation waves in a massive director field [2, 14]. The orientation of the director is described by a unit vector field $\mathbf{n} : \mathbb{R}^3_+ \times \mathbb{R} \to \mathbb{S}^2$. The director field satisfies

$$\delta \int_{\mathbb{R}^3_+ \times \mathbb{R}} \left\{ \frac{1}{2}|\mathbf{n}_t|^2 - W(\mathbf{n}, \nabla\mathbf{n}) \right\} d\mathbf{x}dt = 0, \qquad \mathbf{n}\cdot\mathbf{n} = 1, \tag{2}$$

where $W(\mathbf{n}, \nabla\mathbf{n})$ is the Oseen-Frank energy function from the continuum theory of nematic liquid crystals [9, 23]

$$\begin{aligned} W(\mathbf{n}, \nabla\mathbf{n}) = \frac{1}{2}\alpha(\operatorname{div}\mathbf{n})^2 + \frac{1}{2}\beta(\mathbf{n}\cdot\operatorname{curl}\mathbf{n})^2 + \frac{1}{2}\gamma|\mathbf{n}\times\operatorname{curl}\mathbf{n}|^2 \\ + \frac{1}{2}\eta\left[\operatorname{tr}(\nabla\mathbf{n})^2 - (\operatorname{div}\mathbf{n})^2\right]. \end{aligned} \tag{3}$$

Here, α, β, γ, η are positive constants.

The potential energy density in (3) has a natural geometric origin: $W(\mathbf{n}, \nabla\mathbf{n})$ is the most general function that is quadratic in $\nabla\mathbf{n}$ and invariant under simultaneous orthogonal transformations $\mathbf{x} \mapsto Q\mathbf{x}$ and $\mathbf{n} \mapsto Q\mathbf{n}$ of the independent and dependent variables for all $Q \in SO(3)$. This symmetry is less restrictive than the harmonic map symmetry $\mathbf{x} \mapsto Q\mathbf{x}$ and $\mathbf{n} \mapsto Q'\mathbf{n}$ for all $Q, Q' \in SO(3)$, which implies that W is proportional to $|\nabla\mathbf{n}|^2$, and it allows the coefficients of the quadratic function of $\nabla\mathbf{n}$ to depend on $\mathbf{n}$. The term proportional to η in (3) is a null Lagrangian that corresponds to a surface energy term; it influences only the natural boundary conditions.

There has been extensive analysis of the initial value problem for (2)–(3) in one space dimension, including a proof of the existence of global weak solutions under suitable assumptions (see [25] and the references cited there). In addition, an almost global existence result for smooth, planar solutions in three space dimensions, without boundaries, is proved in [10]. A rigorous analysis of general weak solutions of (2)–(3) in several space dimensions is completely open (just as it is for hyperbolic conservation laws).

One motivation for the problems studied in this paper is an analogy between the director-field system (2)–(3) and nonlinear elasticity. The director-field system has two types of bulk waves, called splay and twist waves [2]. These waves are analogous to longitudinal p-waves and transverse s-waves, respectively, in elasticity.

The deformation gradients in weakly nonlinear elastic p-waves satisfy an inviscid Burgers equation,

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0,$$

and nonlinearity leads to the formation of shocks. After that, smooth solutions may be continued by unique weak entropy solutions [8]. On the other hand, weakly nonlinear splay waves in director fields satisfy the Hunter-Saxton (HS) equation [14],

$$\left[u_t + \left(\frac{1}{2}u^2\right)_x\right]_x = \frac{1}{2}u_x^2. \tag{4}$$

Nonlinearity causes the derivative of solutions to blow up, but after that they remain continuous. The HS equation is completely integrable [16] and has different classes of weak solutions, including ones that conserve energy as well as ones that dissipate energy [7]. Nonlinear effects on small-amplitude, transverse s waves and twist waves are weaker, and their amplitude functions satisfy cubically nonlinear asymptotic equations [2].

An elastic half-space supports Rayleigh waves, which are genuine surface waves whose energy is localized near the boundary. In a Rayleigh wave, the p and s bulk wave fields are coupled together through natural, stress-free boundary conditions. Weakly nonlinear Rayleigh waves are described by nonlocal asymptotic equations, and Hamilton, Il'insky, and Zabolotskaya [11] introduced a simplification of these equations

$$u_t + \mathrm{H}[hh_x]_x + hu_{xx} = 0, \qquad h = \mathrm{H}[u], \tag{5}$$

which we call the HIZ equation. In (5), and below, H denotes the spatial Hilbert transform, which is the linear singular-integral operator defined by

$$\mathrm{H}[e^{ikx}] = -i(\operatorname{sgn} k)e^{ikx}, \qquad \operatorname{sgn} k = \begin{cases} 1 & \text{if } k > 0, \\ 0 & \text{if } k = 0, \\ -1 & \text{if } k < 0. \end{cases}$$

The HIZ equation (5) also describes surface waves on a tangential discontinuity in MHD [1]. Thus, it serves as a model asymptotic equation for genuine surface wave solutions in hyperbolic conservation laws; it is analogous to the inviscid Burger's equation for bulk waves. Other asymptotic surface wave equations arise from hyperbolic conservation laws, but although they are more complicated than the HIZ equation, they have the same qualitative scaling and Hamiltonian properties [3].

The director-field system has surface wave solutions that are analogous to Rayleigh waves, in which the splay waves and twist waves are coupled through natural boundary conditions. In Section 8, we give an asymptotic equation for such weakly nonlinear, genuine surface waves.

The algebra involved in deriving this equation from the director-field system is extremely complicated. For most of this paper, we therefore study a model system of variational wave equations that exhibits the main features in a simpler setting. The only surface waves in IBVPs for scalar wave equations, such as the one considered by Majda [18], are radiative, and one needs at least two wave equations — as occurs in elasticity or director fields — to obtain genuine surface waves.

The model variational principle for two real-valued functions $u(x, y, t)$, $v(x, y, t)$ defined in the half-space $\mathbb{R}^2_+ = \{(x, y) : x > 0\}$ is

$$\delta \int_{\mathbb{R}^2_+ \times \mathbb{R}} \left\{ \frac{1}{2} \left[u_t^2 + v_t^2 \right] - W(u, v, \nabla u, \nabla v) \right\} dxdydt = 0, \tag{6}$$

with natural boundary conditions on $x = 0$, where the potential energy density W is given by

$$W(u, v, \nabla u, \nabla v) = \frac{1}{2}\alpha^2(u) \left[u_x^2 + u_y^2 \right] + \frac{1}{2}\beta^2(v) \left[v_x^2 + v_y^2 \right] - \eta \left[u_x v_y - u_y v_x \right]. \tag{7}$$

The bulk wave speeds $\alpha, \beta : \mathbb{R} \to \mathbb{R}_+$ in (7) are assumed to be smooth, positive functions, and η is a real, nonzero constant multiplying the null-Lagrangian $u_x v_y - u_y v_x$.

The corresponding IBVP consists of decoupled variational wave equations in $x > 0$,

$$u_{tt} = \alpha^2(u)\Delta u + \alpha(u)\alpha'(u)|\nabla u|^2, \qquad v_{tt} = \beta^2(v)\Delta v + \beta(v)\beta'(v)|\nabla v|^2, \tag{8}$$

with initial conditions for (u, u_t, v, v_t), and the boundary conditions on $x = 0$

$$\alpha^2(u)u_x - \eta v_y = 0, \qquad \beta^2(v)v_x + \eta u_y = 0. \tag{9}$$

In Section 2, we derive the Lopatinski condition for the linearization of (8)–(9), which is a necessary condition for the IBVP to be well-posed [6]. Serre [21] gives a full discussion of Lopatinski conditions for wave equations described by variational principles of the form (1). In Section 3, we describe the linearized surface wave solutions, and in Section 4, we use the method of multiple scales to derive an asymptotic equation for weakly nonlinear, genuine surface waves.

The result is a spectral asymptotic equation of the form

$$\hat{a}_t(k, t) + i \operatorname{sgn}(k) E_0 \int_{\mathbb{R}} \Lambda(-k, k - l, l)\hat{a}(k - l, t)\hat{a}(l, t)\, dl = 0, \tag{10}$$

where $\hat{a}(k, t)$ denotes the amplitude of the surface wave on the boundary as a function of a "slow" time t and the tangential wavenumber k, and the kernel Λ is given by

$$\begin{aligned} \Lambda(k_1, k_2, k_3) = {} & \left[A_0 - iB_0 \operatorname{sgn}(k_1k_2k_3) \right] \frac{|k_1k_2| + |k_2k_3| + |k_1k_3|}{|k_1| + |k_2| + |k_3|} \\ & - \left[C_0 - iD_0 \operatorname{sgn}(k_1k_2k_3) \right] \frac{k_1k_2 + k_2k_3 + k_1k_3}{|k_1| + |k_2| + |k_3|}. \end{aligned} \tag{11}$$

The constants A_0–E_0 are defined in (29)–(30). For definiteness, we consider solutions on the real line. The same equation applies to spatially periodic solutions after replacing integrals by sums.

The kernel $\Lambda(k_1, k_2, k_3)$ in (10) is an interaction coefficient that describes the strength of the quadratically nonlinear interactions between wavenumbers k_1, k_2, k_3 that satisfy the three-wave resonance condition

$$k_1 + k_2 + k_3 = 0.$$

Only wavenumbers that satisfy this condition appear in (10), and the value of Λ on other wavenumbers is irrelevant. It is, however, convenient to retain all three wavenumbers to exhibit the symmetry of $\Lambda(k_1, k_2, k_3)$. This "detailed balance" symmetry is a consequence of the fact that (10) is Hamiltonian [3]. Furthermore, the $\hat{a}(k,t)$ are complex canonical variables. In Section 5, we use the Hamiltonian formulation of (6) to derive the same asymptotic equation (10).

In Section 6, we show how to write (10) as a spatial equation for

$$a(x,t) = \int \hat{a}(k,t) e^{ikx}\, dk.$$

We also introduce a simplified asymptotic equation with the kernel

$$\Lambda(k_1, k_2, k_3) = \frac{1}{2}\left(|k_1| + |k_2| + |k_3|\right), \tag{12}$$

corresponding to $A_0 = C_0$ and $B_0 = D_0 = 0$ in (11). After rescaling a to remove an inessential constant, the associated spatial form of the equation is

$$a_t + \left(\frac{1}{2}a^2\right)_x = \mathrm{H}[a|\partial|a], \tag{13}$$

where $|\partial| = \mathrm{H}\partial$ has symbol $|k|$. Equation (13) plays an analogous role for surface wave solutions of variational wave equations to the HIZ equation (5) for hyperbolic conservation laws. It may also provide an example of a nonlocal surface wave equation that is easier to analyze than the HIZ equation. We summarize the various model asymptotic equations in Table 1.

We remark that the HIZ equation (5) can also be written in the spectral form (10) with a kernel proportional to

$$\Lambda(k_1, k_2, k_3) = \frac{2|k_1 k_2 k_3|}{|k_1| + |k_2| + |k_3|}.$$

This kernel is homogeneous of degree two, corresponding to the appearance of two spatial derivatives in (5), whereas the kernel (11) or (12) is homogeneous of degree one, corresponding to the appearance of one spatial derivative in (13). This difference in the scaling properties of the interaction coefficient Λ, when expressed with respect to complex canonical variables, is the fundamental qualitative difference between the weak nonlinearity in surface waves for conservation laws and variational wave equations.

In Section 7, we establish the short-time existence of smooth, spatially periodic solutions of (13) and show some numerical solutions; these indicate that

	Conservation Laws	Variational Equations
Bulk	$u_t + \left(\frac{1}{2}u^2\right)_x = 0$	$\left[u_t + \left(\frac{1}{2}u^2\right)_x\right]_x = \frac{1}{2}u_x^2$
Surface	$u_t + \mathrm{H}[hh_x]_x + hu_{xx} = 0, \quad h = \mathrm{H}[u]$	$u_t + \left(\frac{1}{2}u^2\right)_x = \mathrm{H}[u\vert\partial\vert u]$

Table 1: Model asymptotic equations for bulk and surface waves governed by hyperbolic conservation laws and variational principles of the form (1). The bulk-wave equations are the inviscid Burgers and HS equations; the surface wave equations are the HIZ equation and the equation derived in this paper.

smooth solutions break down in finite time and appear to form x^α-cusp singularities with $\alpha \approx 1/3$. The asymptotic solution in the interior of the half-space remains smooth, so this is a different mechanism for the breakdown of smooth solutions from the more familiar formation of singularities in the interior, which for variational waves is described by the HS equation.

Finally, in Section 8, we return to an analysis of surface wave solutions of the director-field equations. The result is an asymptotic equation (70) of the same form as (10), where the kernel $\Lambda(k_1, k_2, k_3)$ is given in (71). Like the model kernel, this kernel is a symmetric, homogeneous function of (k_1, k_2, k_3) of degree one, and it has an similar relationship to the model kernel as the full Rayleigh-wave kernel has to the HIZ kernel [3, 11].

There are many open questions about the nonlocal asymptotic equations for surface waves, such as (5) or (13), that we have described here. For example: a proof of singularity formation; the existence of global weak solutions, conservative or dissipative; and whether or not any of these equations are completely integrable. Although they are scalar equations, their intrinsic — and apparently non-removable — nonlocality seems to make their analysis significantly harder than the analysis of the inviscid Burgers or HS equations for bulk waves.

2 The Lopatinski condition

The linearization of the model IBVP (8)–(9) at $(u, v) = (0, 0)$ consists of pair of two-dimensional wave equations

$$u_{tt} = \alpha_0^2 \left(u_{xx} + u_{yy}\right), \qquad v_{tt} = \beta_0^2 \left(v_{xx} + v_{yy}\right), \tag{14}$$

in the half-space $x > 0$, where $\alpha_0 = \alpha(0)$ and $\beta_0 = \beta(0)$, with the boundary conditions on $x = 0$

$$\alpha_0^2 u_x - \eta v_y = 0, \qquad \beta_0^2 v_x + \eta u_y = 0. \tag{15}$$

In this section, we derive the Lopatinski condition for (14)–(15), which is a necessary condition for the well-posedness of the IBVP.

We look for Fourier-Laplace solutions of (14)–(15) of the form

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \hat{u}e^{-kx} \\ \hat{v}e^{-kx} \end{bmatrix} e^{\tau t + ily}, \tag{16}$$

where $l \in \mathbb{R}$, $\tau \in \mathbb{C}$ with $\Re\tau > 0$, and $k \in \mathbb{C}$ with $\Re k > 0$. Then (16) satisfies (14) if and only if:

$$\tau^2 = \alpha_0^2 \left(k^2 - l^2\right), \quad \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} = \begin{bmatrix} R \\ 0 \end{bmatrix}; \quad \text{or} \quad \tau^2 = \beta_0^2 \left(k^2 - l^2\right), \quad \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} = \begin{bmatrix} 0 \\ S \end{bmatrix}.$$

The corresponding solution of the PDE for a given tangential wavenumber l and complex frequency τ is

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} Re^{-k_\alpha x} \\ Se^{-k_\beta x} \end{bmatrix} e^{\tau t + ily},$$

where R and S are arbitrary constants and

$$k_\alpha = \sqrt{l^2 + \frac{\tau^2}{\alpha_0^2}}, \qquad k_\beta = \sqrt{l^2 + \frac{\tau^2}{\beta_0^2}}.$$

Here, we take the branch of the square root with positive real part, which defines k_α, k_β uniquely in $\Re\tau > 0$.

Using this solution in the boundary condition (15), we obtain the algebraic system

$$\begin{bmatrix} -\alpha_0^2 k_\alpha & -i\eta l \\ i\eta l & -\beta_0^2 k_\beta \end{bmatrix} \begin{bmatrix} R \\ S \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence, (14)–(15) has a nonzero solution of the form (16) if and only if $L(\tau, l) = 0$, where

$$L(\tau, l) = \alpha_0^2 \beta_0^2 k_\alpha k_\beta - \eta^2 l^2$$

is the Lopatinski determinant associated with the IBVP.

The IBVP is invariant under the rescaling $x \mapsto rx$, $y \mapsto ry$, $t \mapsto rt$, for all $r > 0$, and the reflection $y \mapsto -y$, $v \mapsto -v$, so we may set $l = 1$ without loss of generality. In that case, writing $k_\alpha = \gamma_\alpha$ and $k_\beta = \gamma_\beta$, we have

$$L(\tau, 1) = \alpha_0^2 \beta_0^2 \gamma_\alpha \gamma_\beta - \eta^2, \qquad \gamma_\alpha = \sqrt{1 + \frac{\tau^2}{\alpha_0^2}}, \quad \gamma_\beta = \sqrt{1 + \frac{\tau^2}{\beta_0^2}}. \tag{17}$$

A necessary condition for the well-posedness of the IBVP forward in time is the Lopatinski condition [6]

$$L(\tau, 1) \neq 0 \qquad \text{for all } \tau \in \mathbb{C} \text{ with } \Re\tau > 0. \tag{18}$$

In other words, there are no Fourier-Laplace modes that oscillate in the tangential spatial direction, decay in the normal spatial direction, and grow in time. If such modes do exist, then the IBVP is catastrophically unstable, since rescaling them gives finite-energy modes that grow arbitrarily quickly in time.

We define

$$a = \frac{\alpha_0^2}{|\eta|}, \qquad b = \frac{\beta_0^2}{|\eta|}, \qquad s^2 = \frac{\tau^2}{|\eta|}, \tag{19}$$

where $0 < a, b < \infty$. Then (18) is satisfied if and only if the quartic equation

$$ab\left(a + s^2\right)\left(b + s^2\right) = 1$$

has no roots for s with positive real part. Since

$$s^2 = \frac{1}{2}\left\{-(a+b) \pm \sqrt{\frac{4}{ab} + (a-b)^2}\right\}, \tag{20}$$

we get the following cases:

1. If $ab > 1$, then $s^2 < 0$, and there are four imaginary roots for τ, so (18) holds;

2. If $ab = 1$, then $s^2 < 0$ or $s^2 = 0$, and there are two imaginary and one double-zero root for τ, so (18) holds;

3. If $0 < ab < 1$, then $s^2 < 0$ or $s^2 > 0$, and there are two imaginary and two real roots for τ, one of which is positive, so (18) fails.

Thus, neglecting the marginal case $ab = 1$, we see that (14)–(15) satisfies the Lopatinski condition (18) if $ab > 1$, or $\alpha_0\beta_0 > |\eta|$. We assume from now on that this condition is satisfied.

3 Linearized surface waves

Next, we consider surface wave solutions of the linearized model IBVP, which correspond to Fourier-Laplace modes with a purely imaginary frequency τ that oscillate in time. The discussion in Section 2 shows that these modes exist for all parameter values.

We distinguish between two types of surface waves: (i) radiative, or leaky, waves that oscillate but do not decay in the normal spatial direction; (ii) genuine, or finite-energy, waves that decay exponentially in the normal spatial direction. Radiative surface waves are coupled with bulk waves in the interior of the half-space, whereas genuine surface waves are localized near the boundary.

To analyze the surface waves in more detail, it is convenient to write

$$\tau = -i\lambda, \qquad r^2 = \frac{\lambda^2}{|\eta|} = -s^2,$$

where $\lambda \in \mathbb{R}$. Since we have normalized the tangential wavenumber l to one, λ is the speed of the surface wave along the boundary. The normal spatial decay constants in (17) are then given by

$$\gamma_\alpha = \sqrt{1 - \frac{r^2}{a}}, \qquad \gamma_\beta = \sqrt{1 - \frac{r^2}{b}}.$$

For the negative square root in (20), we get

$$r^2 = \frac{1}{2}\left\{(a+b) + \sqrt{\frac{4}{ab} + (a-b)^2}\right\} > \frac{1}{2}\left\{(a+b) + |a-b|\right\} = \max(a,b).$$

Thus, γ_α and γ_β are both purely imaginary, and the surface waves are radiative. In this case, $|\lambda| > \max(\alpha_0, \beta_0)$, meaning that the surface waves are faster than the bulk waves.

On the other hand, for the positive square root in (20), we get

$$r^2 = \frac{1}{2}\left\{(a+b) - \sqrt{\frac{4}{ab} + (a-b)^2}\right\} < \frac{1}{2}\left\{(a+b) - |a-b|\right\} = \min(a,b).$$

Thus, γ_α and γ_β are both real and positive, and the surface waves are genuine, with both u and v decaying exponentially in the normal spatial direction. In this case, $|\lambda| < \min(\alpha_0, \beta_0)$, meaning that the surface waves are slower than the bulk waves.

We remark that it is these genuine surface-wave modes that lead to instability as ab decreases through 1; their wave-speeds coalesce at $\lambda = 0$ and become complex. Hunter and Thoo [15] carry out a bifurcation analysis of a similar problem in MHD.

The aim of this paper is to study the effect of weak nonlinearity on genuine variational surface waves. These problems typically lead to nonlocal asymptotic equations because the surface-wave speed is slower than the bulk-wave speeds, and what happens at one point of the boundary influences what happens elsewhere on the boundary through the half-space.

The qualitative behavior of radiative surface waves, which are faster than the bulk waves, is different from that of the genuine surface waves, and one typically obtains local asymptotic equations. Weakly nonlinear radiative surface waves in this problem could be analyzed in a similar way to the radiative surface waves on a compressible vortex sheet studied in [4].

4 Weakly nonlinear surface waves

Weakly nonlinear, genuine surface wave solutions of the model IBVP (14)–(15) may be derived by standard multiple-scale methods. The dominant nonlinear effects on surface waves are quadratic; for waves whose amplitude is of the order $\epsilon \ll 1$, they become significant on time-scales of the order ϵ^{-1} in a reference frame moving with the linearized wave speed.

In this section, we outline the multiple-scale expansion. Some details of the (lengthy) algebraic computations are given in the Appendix. In the next section, we outline an alternative derivation based on an expansion of the surface-wave Hamiltonian.

We introduce "fast" space variables $\theta = y - \lambda t$ and x, tangent and normal to the boundary, respectively, and a "slow" time variable $\tau = \epsilon t$. (We do not use τ to denote a complex frequency in this section, so this notation should cause no confusion.) We assume that the wave speeds $\alpha(u)$ and $\beta(v)$ in (14) have the expansions

$$\alpha^2(u) = \alpha_0^2 + 2\alpha_0\alpha_1 u + O(u^2), \qquad \beta^2(u) = \beta_0^2 + 2\beta_0\beta_1 v + O(v^2) \tag{21}$$

as $u, v \to 0$, where $\alpha_0 = \alpha(0)$, $\beta_0 = \beta(0)$, $\alpha_1 = \alpha'(0)$, $\beta_1 = \beta'(0)$, and $\alpha_0, \beta_0 > 0$. We then seek an asymptotic expansion as $\epsilon \to 0$ for a solution (u, v) of (14)–(15) of the form

$$\begin{aligned} u(x, y, t; \epsilon) &= \epsilon u_1(x, y - \lambda t, \epsilon t) + \epsilon^2 u_2(x, y - \lambda t, \epsilon t) + O(\epsilon^3), \\ v(x, y, t; \epsilon) &= \epsilon v_1(x, y - \lambda t, \epsilon t) + \epsilon^2 v_2(x, y - \lambda t, \epsilon t) + O(\epsilon^3), \end{aligned} \tag{22}$$

where λ is the linearized surface wave speed, and determine $u_1(x, \theta, \tau)$, $v_1(x, \theta, \tau)$ from the requirement that this expansion is formally valid for times $t = O(\epsilon^{-1})$.

At the order ϵ, we find that (u_1, v_1) satisfies the linearized equations. The solution for (u_1, v_1) is a linear superposition of the Fourier-Laplace modes described in the previous section and is given by

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} (x, \theta, \tau) = \int_{\mathbb{R}} \hat{a}(k, \tau) \begin{bmatrix} R(k) e^{-\gamma_\alpha |k| x} \\ S(k) e^{-\gamma_\beta |k| x} \end{bmatrix} e^{ik\theta}\, dk, \tag{23}$$

where $\hat{a}(k, \tau)$ is an arbitrary complex-valued amplitude-function of the tangential wavenumber k and the slow time τ, with $\hat{a}(-k, \tau) = \hat{a}^*(k, \tau)$.

The decay constants

$$\gamma_\alpha = \sqrt{1 - \frac{\lambda^2}{\alpha_0^2}}, \quad \gamma_\beta = \sqrt{1 - \frac{\lambda^2}{\beta_0^2}} \tag{24}$$

in (23) satisfy equation (17),

$$\alpha_0^2 \beta_0^2 \gamma_\alpha \gamma_\beta = \eta^2, \tag{25}$$

and the wave speed λ is given by

$$\lambda^2 = \frac{1}{2} \left\{ \alpha_0^2 + \beta_0^2 - \sqrt{(\alpha_0^2 - \beta_0^2)^2 + \frac{4\eta^4}{\alpha_0^2 \beta_0^2}} \right\}. \tag{26}$$

Here, we assume the stability condition $\alpha_0 \beta_0 > |\eta|$, and we choose the sign of the square root that corresponds to genuine surface waves. The coefficients R

and S in (23) give the relative amplitudes of u and v, and are determined up to a scalar factor. A convenient choice is

$$R = \eta, \qquad S = i\alpha_0^2 \gamma_\alpha \operatorname{sgn}(k). \tag{27}$$

At the order ϵ^2, we find that (u_2, v_2) satisfies a nonhomogeneous linearized system, where the nonhomogeneous term depends on (u_1, v_1). A solution for (u_2, v_2) exists only if the nonhomogeneous term satisfies an appropriate solvability condition, which yields an evolution equation for the amplitude-function $\hat{a}(k, \tau)$. After some algebra, we find that

$$\hat{a}_\tau(k,\tau) + i \operatorname{sgn}(k) E_0 \int_{-\infty}^{\infty} \Lambda(-k, k-l, l) \hat{a}(k-l, \tau) \hat{a}(l, \tau)\, dl = 0, \tag{28}$$

where

$$E_0 = \lambda \left[\frac{1}{\gamma_\alpha^2} + \frac{1}{\gamma_\beta^2} - 2 \right]^{-1}, \tag{29}$$

and the kernel Λ is given by (11) with

$$A_0 = \frac{\eta \alpha_1}{\alpha_0}, \qquad B_0 = \frac{\alpha_0^2 \gamma_\alpha \beta_1}{\beta_0}, \qquad C_0 = \frac{A_0}{\gamma_\alpha^2}, \qquad D_0 = \frac{B_0}{\gamma_\beta^2}. \tag{30}$$

Rewriting the slow time τ as t, we get equation (10) stated in the introduction.

In summary, the weakly nonlinear, genuine surface wave solution of (14)–(15) has the asymptotic expansion (22)–(23), where the spectral wave-amplitude $\hat{a}(k, \tau)$ satisfies (28). Assuming that we have "prepared" initial data of the same form as (23), corresponding to a unidirectional surface wave, we supplement (28) with an initial condition $\hat{a}(k, 0) = \hat{a}_0(k)$.

5 Hamiltonian equations

In this section, we derive the equation for a weakly nonlinear surface wave by expanding its Hamiltonian up to cubic terms in the wave amplitude and evaluating the cubic terms on the linearized surface wave solution. This procedure is somewhat heuristic, but it involves less algebra than the multiple-scale approach and, as we will verify, it leads to the same result. It also provides an independent check on the multiple-scale analysis. See [24] for further explanation of the Hamiltonian formalism we use here.

5.1 Expansion of the Hamiltonian

The variational principle for the model IBVP (14)–(15) is (6)–(7). We denote the canonically conjugate momenta to u and v by $p = u_t$ and $q = v_t$, respectively. The corresponding Hamiltonian functional $\mathcal{H}$ is given by

$$\mathcal{H}(u, v, p, q) = \mathcal{A}(u, p) + \mathcal{B}(v, q) + \mathcal{E}(u, v),$$

where

$$\mathcal{A}(u,p) = \frac{1}{2}\int_{\mathbb{R}^2_+} \left[p^2 + \alpha^2(u)(u_x^2 + u_y^2)\right] \, dxdy,$$

$$\mathcal{B}(v,q) = \frac{1}{2}\int_{\mathbb{R}^2_+} \left[q^2 + \beta^2(v)(v_x^2 + v_y^2)\right] \, dxdy,$$

$$\mathcal{E}(u,v) = \int_{\mathbb{R}^2_+} \eta \left[u_y v_x - u_x v_y\right] \, dxdy.$$

Using the Taylor expansion (21) of the wave speeds in $\mathcal{A}$ and $\mathcal{B}$, we get that

$$\mathcal{A} = \mathcal{A}_2 + \mathcal{A}_3 + O(u^4), \qquad \mathcal{B} = \mathcal{B}_2 + \mathcal{B}_3 + O(v^4),$$

where

$$\mathcal{A}_2 = \frac{1}{2}\int_{\mathbb{R}^2_+} \left[p^2 + \alpha_0^2\left(u_x^2 + u_y^2\right)\right] \, dxdy, \quad \mathcal{A}_3 = \alpha_0\alpha_1 \int_{\mathbb{R}^2_+} u\left[u_x^2 + u_y^2\right] \, dxdy,$$

$$\mathcal{B}_2 = \frac{1}{2}\int_{\mathbb{R}^2_+} \left[q^2 + \beta_0^2\left(v_x^2 + v_y^2\right)\right] \, dxdy, \quad \mathcal{B}_3 = \beta_0\beta_1 \int_{\mathbb{R}^2_+} v\left[v_x^2 + v_y^2\right] \, dxdy.$$

Thus, the expansion of the Hamiltonian as $u, v \to 0$ is

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3 + O(u^4 + v^4), \tag{31}$$

where the quadratic and cubic terms $\mathcal{H}_2$ and $\mathcal{H}_3$, respectively, are given by

$$\mathcal{H}_2 = \mathcal{A}_2 + \mathcal{B}_2 + \mathcal{E}, \qquad \mathcal{H}_3 = \mathcal{A}_3 + \mathcal{B}_3.$$

To apply the Hamiltonian formalism, we separate the positive and negative frequency components of the wave amplitude. The linearized frequency of the surface wave is given by

$$\omega(k) = \lambda k$$

where the wave speed λ satisfies (26). For definiteness, we consider right-moving waves with $\lambda > 0$. In that case, the positive frequency components are the ones with positive wavenumbers, and the spatial Fourier coefficients

$$\{\tilde{a}(k,t), \tilde{a}^*(k,t) : k > 0\}$$

are complex-canonical conjugate variables for the wave, provided they are scaled appropriately.

The expression for (u, v, p, q) in a unidirectional, linear surface wave solution has the form

$$\begin{aligned}
u &= C_0 \int_0^\infty \left[R\tilde{a}(k,t)e^{iky-\gamma_\alpha kx} + R^*\tilde{a}^*(k,t)e^{-iky-\gamma_\alpha kx}\right] dk,\\
v &= C_0 \int_0^\infty \left[S\tilde{a}(k,t)e^{iky-\gamma_\beta kx} + S^*\tilde{a}^*(k,t)e^{-iky-\gamma_\beta kx}\right] dk,\\
p &= C_0 \int_0^\infty i\omega(k)\left[R\tilde{a}(k,t)e^{iky-\gamma_\alpha kx} - R^*\tilde{a}^*(k,t)e^{-iky-\gamma_\alpha kx}\right] dk,\\
q &= C_0 \int_0^\infty i\omega(k)\left[S\tilde{a}(k,t)e^{iky-\gamma_\beta kx} - S^*\tilde{a}^*(k,t)e^{-iky-\gamma_\beta kx}\right] dk,
\end{aligned} \tag{32}$$

where γ_α, γ_β are defined in (24), and from (27)

$$R = \eta, \qquad S = i\alpha_0^2\gamma_\alpha \tag{33}$$

for $k > 0$. We choose the positive scaling constant C_0 in (35) below, after we compute $\mathcal{H}_2$.

We use the linearized solution (32) in the Hamiltonian (31) and evaluate the resulting integrals with respect to (x, y).

A straightforward computation, using Parseval's theorem, shows that the quadratic term in the Hamiltonian is given by

$$\mathcal{H}_2(\tilde{a}, \tilde{a}^*) = \sigma_0 C_0^2 \int_0^\infty \lambda k\, \tilde{a}^*(k,t)\tilde{a}(k,t)dk,$$

where

$$\sigma_0 = \frac{\rho_0}{E_0}, \qquad \rho_0 = 2\pi\alpha_0^2\gamma_\alpha|R|^2 = 2\pi\beta_0^2\gamma_\beta|S|^2, \tag{34}$$

and E_0 is defined in (29). In particular, $\sigma_0 > 0$ when $\lambda > 0$. When expressed in terms of complex canonical variables, the quadratic part of the Hamiltonian has the form [24]

$$\mathcal{H}_2(\tilde{a}, \tilde{a}^*) = \int_0^\infty \omega(k)\tilde{a}^*(k,t)\tilde{a}(k,t)dk, \qquad \omega(k) = \lambda k.$$

This is the case if we choose

$$C_0 = \frac{1}{\sqrt{\sigma_0}}. \tag{35}$$

A longer computation [5] shows that the cubic term in the Hamiltonian (31) is given by

$$\mathcal{H}_3(\tilde{a}, \tilde{a}^*) = \rho_0 C_0^3 \int_0^\infty \int_0^\infty T(-l-m, l, m)\tilde{a}^*(l+m)\tilde{a}(l,t)\tilde{a}(m,t)\, dldm + \text{c.c.},$$

where ρ_0 is defined in (34) and the kernel T may be written as

$$\begin{aligned} T(k_1,k_2,k_3) = &\left[\frac{R\alpha_1}{\alpha_0} + \frac{S\beta_1}{\beta_0}\right] \frac{-k_1k_2 + k_2k_3 - k_1k_3}{-k_1 + k_2 + k_3} \\ &- \left[\frac{R\alpha_1}{\alpha_0\gamma_\alpha^2} + \frac{S\beta_1}{\beta_0\gamma_\beta^2}\right] \frac{k_1k_2 + k_2k_3 + k_1k_3}{-k_1 + k_2 + k_3} \end{aligned} \tag{36}$$

on $k_1 + k_2 + k_3 = 0$.

From (34) and (35), we have $\rho_0 C_0^2 = E_0$. Thus, neglecting quartic and higher-degree terms, our final expression for the expanded surface-wave Hamiltonian is

$$\begin{aligned} \mathcal{H}(\tilde{a}, \tilde{a^*}) = &\int_0^\infty \lambda k\, \tilde{a}^*(k,t)\tilde{a}(k,t)dk \\ &+ C_0E_0 \int_0^\infty \int_0^\infty T(-l-m,l,m)\tilde{a}^*(l+m)\tilde{a}(l,t)\tilde{a}(m,t)\, dldm \\ &+ C_0E_0 \int_0^\infty \int_0^\infty T^*(-l-m,l,m)\tilde{a}(l+m,t)\tilde{a}^*(l,t)\tilde{a}^*(m,t)\, dldm. \end{aligned} \tag{37}$$

5.2 Hamilton's equation

The complex canonical form of Hamilton's equation for $\{\tilde{a}(k,t) : k \in \mathbb{R}_+\}$ is

$$i\tilde{a}_t(k,t) = \frac{\delta \mathcal{H}}{\delta \tilde{a}^*(k,t)}, \tag{38}$$

where $\delta\mathcal{H}/\delta\tilde{a}^*$ denotes the functional derivative of $\mathcal{H}(a,a^*)$ with respect to $\tilde{a}^*$. Hamilton's equation for the Hamiltonian (37) is

$$\begin{aligned} i\tilde{a}_t(k,t) = \lambda k\, \tilde{a}(k,t) + C_0 E_0 \int_0^k T(-k, k-l, l)\tilde{a}(k-l,t)\tilde{a}(l,t)\, dl \\ + 2C_0 E_0 \int_0^\infty T^*(-k-l, k, l)\tilde{a}(k+l,t)\tilde{a}^*(l,t)\, dl. \end{aligned} \tag{39}$$

Following [3], we rewrite equation (39) for the positive wavenumber components $\{\tilde{a}(k,t) : k \in \mathbb{R}_+\}$ as a convolution-type equation for all of the wavenumber components $\{\tilde{a}(k,t) : k \in \mathbb{R}\}$, where $\tilde{a}(-k,t) = \tilde{a}^*(k,t)$. The result is that

$$i\tilde{a}_t(k,t) = \lambda k\, \tilde{a}(k) + C_0 E_0 \operatorname{sgn} k \int_{-\infty}^{\infty} \Lambda(-k, k-l, l)\tilde{a}(k-l,t)\tilde{a}(l,t)\, dl, \tag{40}$$

where the kernel $\Lambda : \mathbb{R}^3 \to \mathbb{C}$ is given in terms of the kernel $T : \mathbb{R}_- \times \mathbb{R}_+^2 \to \mathbb{C}$ on $k_1 + k_2 + k_3$ as follows:

1. $\Lambda(k_1,k_2,k_3) = T(k_1,k_2,k_3)$ if $k_2, k_3 > 0$ and $k_1 < 0$;
2. $\Lambda(k_1,k_2,k_3) = T(k_2,k_1,k_3)$ if $k_1, k_3 > 0$ and $k_2 < 0$;
3. $\Lambda(k_1,k_2,k_3) = T(k_3,k_1,k_2)$ if $k_1, k_2 > 0$ and $k_3 < 0$;
4. $\Lambda(k_1,k_2,k_3) = T^*(-k_1,-k_2,-k_3)$ if $k_2, k_3 < 0$ and $k_1 > 0$;
5. $\Lambda(k_1,k_2,k_3) = T^*(-k_2,-k_1,-k_3)$ if $k_1, k_3 < 0$ and $k_2 > 0$;
6. $\Lambda(k_1,k_2,k_3) = T^*(-k_3,-k_1,-k_2)$ if $k_1, k_2 < 0$ and $k_3 > 0$.

By considering the different possible sign combinations of k_1, k_2, k_3, one can verify [5] that if $T(k_1,k_2,k_3)$ is given by (36), with R and S defined as in (33), then this expression for $\Lambda(k_1,k_2,k_3)$ agrees with (11) on $k_1 + k_2 + k_3 = 0$.

By comparing the multiple-scale solution (22)–(23) with the Hamiltonian solution (32), we see that the corresponding wave amplitudes are related by

$$\tilde{a}(k,t) = \frac{\epsilon}{C_0}\hat{a}(k,\epsilon t)e^{-i\lambda k t}.$$

The use of this expression for $\tilde{a}(k,t)$ in equation (40) gives equation (28) for $\hat{a}(k,\tau)$. The factor $e^{-i\lambda k t}$ in $\tilde{a}$ corresponds to a Galilean transformation which removes the linear term. This completes the verification that the Hamiltonian and the multiple-scale approaches lead to identical results.

Finally, we remark that a similar procedure can be applied to the Lagrangian instead of the Hamiltonian. One expands the Lagrangian up to cubic terms in the field variables and then evaluates these terms on the linearized surface wave solution. This gives a variational principle whose Euler-Lagrange equation is the asymptotic equation.

6 Spatial form of the asymptotic equation

In this section, we show how to rewrite the spectral form of the asymptotic equation (28) in spatial form, and we give a simplified, but representative, spatial equation.

We write (28) as

$$\hat{a}_\tau(k,\tau) + i\,\mathrm{sgn}(k) E_0 \hat{f}(k,\tau) = 0, \tag{41}$$

where

$$\hat{f}(k,\tau) = \int_{-\infty}^{\infty} \Lambda(-k, k-l, l)\hat{a}(k-l,\tau)\hat{a}(l,\tau)\, dl,$$

and let

$$a(\theta,\tau) = \int \hat{a}(k,\tau) e^{ik\theta}\, d\theta$$

denote a spatial amplitude, with corresponding notation for f. Then, taking the Fourier transform (41), we get that $a_\tau = E_0 \mathrm{H} f$ where H is the Hilbert transform.

To express f in terms of a, it is convenient to write the kernel Λ in (11), with coefficients (30), as

$$\begin{aligned}\Lambda(k_1,k_2,k_3) = {} & \frac{1}{2}\left[\left(1-\frac{1}{\gamma_\alpha^2}\right)A_0 - i\,\mathrm{sgn}(k_1k_2k_3)\left(1-\frac{1}{\gamma_\beta^2}\right)B_0\right]\Lambda_+(k_1,k_2,k_3) \\ & + \frac{1}{2}\left[\left(1+\frac{1}{\gamma_\alpha^2}\right)A_0 - i\,\mathrm{sgn}(k_1k_2k_3)\left(1+\frac{1}{\gamma_\beta^2}\right)B_0\right]\Lambda_-(k_1,k_2,k_3),\end{aligned} \tag{42}$$

where

$$\Lambda_+(k_1,k_2,k_3) = \frac{|k_1k_2|+|k_2k_3|+|k_1k_3|+k_1k_2+k_2k_3+k_1k_3}{|k_1|+|k_2|+|k_3|},$$
$$\Lambda_-(k_1,k_2,k_3) = \frac{|k_1k_2|+|k_2k_3|+|k_1k_3|-k_1k_2-k_2k_3-k_1k_3}{|k_1|+|k_2|+|k_3|}.$$

The kernels $\Lambda_\pm(k_1,k_2,k_3)$ may be written on $k_1+k_2+k_3=0$ in the equivalent form

$$\begin{aligned}\Lambda_+(k_1,k_2,k_3) &= \frac{1}{2}\left[\frac{|k_2k_3|+k_2k_3}{|k_1|} + \frac{|k_1k_3|+k_1k_3}{|k_2|} + \frac{|k_1k_2|+k_1k_2}{|k_3|}\right], \\ \Lambda_-(k_1,k_2,k_3) &= \frac{1}{2}\big[|k_1|+|k_2|+|k_3|\big],\end{aligned} \tag{43}$$

as one can verify by considering the different sign combinations of k_1, k_2, k_3.

Using the expressions in (43) and the convolution theorem, we can read off the spatial terms in the equation that correspond to the spectral terms. For example, if

$$\hat{f}_+(k) = \int \Lambda_+(-k, k-\xi, \xi)\hat{a}(k-\xi)\hat{a}(\xi)\, d\xi,$$
$$\hat{f}_-(k) = \int \Lambda_-(-k, k-\xi, \xi)\hat{a}(k-\xi)\hat{a}(\xi)\, d\xi,$$

then

$$f_+(x) = \frac{1}{2}|\partial|^{-1}\left\{(|\partial|a)^2 - (\partial a)^2\right\} + |\partial|\left\{|\partial|a \cdot |\partial|^{-1}a\right\} + \partial\left\{\partial a \cdot |\partial|^{-1}a\right\},$$
$$f_-(x) = \frac{1}{2}|\partial|\left(a^2\right) + a|\partial|a.$$

The terms in $\Lambda(k_1, k_2, k_3)$ with factors of $i\,\mathrm{sgn}(k_1k_2k_3)$ lead to an additional Hilbert transform on each function.

The spatial form of the full asymptotic equation follows from the previous discussion, but it is somewhat lengthy, and we will not write it out explicitly here. Instead, we consider a simplification of the asymptotic equation in which the only k-dependence of Λ comes from the terms proportional to $\Lambda_-(k_1, k_2, k_3)$. This corresponds to taking $\gamma_\alpha = \gamma_\beta = 1$ in (42), which arises in the limit $\lambda \to 0$ in the original problem, and $B_0 = 0$, meaning that the only nonlinearity comes from the u-equation.

The derivation of the asymptotic equation does not apply when $\lambda = 0$, since the linearized surface wave speeds coalesce at that point and the linear time-scale factor E_0 in (29) diverges as $\lambda \to 0$. Nevertheless, it is reasonable to approximate the coefficients of the kernel Λ, which describe the effects of nonlinearity, by their values at $\lambda = 0$.

We then get the spectral equation

$$\hat{a}_\tau(k,\tau) + \frac{1}{2}i\,\mathrm{sgn}(k)A_0E_0\int_{-\infty}^{\infty}\Big(|k| + |k-l| + |l|\Big)\hat{a}(k-l,\tau)\hat{a}(l,\tau)\, dl = 0.$$

The corresponding spatial equation for $a(\theta, \tau)$ is

$$a_\tau + A_0E_0\left\{\left(\frac{1}{2}a^2\right)_\theta - \mathrm{H}\left[a|\partial|a\right]\right\} = 0. \tag{44}$$

7 A simplified asymptotic equation

In this section, we consider the simplified asymptotic equation (44). Writing (x, t) instead of (θ, τ), so x is now a tangential spatial variable, and rescaling a to remove an inessential constant, we get the following equation for $a(x, t)$:

$$a_t + \left(\frac{1}{2}a^2\right)_x = \mathrm{H}\left[a|\partial|a\right]. \tag{45}$$

We can also write (45) as

$$a_t + \left(a^2\right)_x = [a, \mathrm{H}]\, |\partial| a, \tag{46}$$

where $[a, \mathrm{H}]$ denotes the commutator of multiplication by a and H. The right-hand side of (46) satisfies the estimate

$$\|\, [a, \mathrm{H}]\, |\partial| a \,\|_{L^2} \le C \|a\|_{\dot{H}^{1/2}}^2,$$

so it is a lower-order term for smooth solutions.

The Hamiltonian form of (45) is

$$a_t = \mathrm{H}\left[\frac{\delta \mathcal{H}}{\delta a}\right], \qquad \mathcal{H}(a) = \frac{1}{2}\int a^2 |\partial| a\, dx.$$

The Hilbert transform H is the spatial Hamiltonian operator corresponding to the spectral Hamiltonian operator of multiplication by $-i$ on positive wavenumbers. We can consider either free-space solutions of (45), with $x \in \mathbb{R}$, or spatially periodic solutions, with $x \in \mathbb{T}$, and interpret integrals over x as appropriate. Here, we will mostly consider spatially periodic solutions with zero mean.

The Hamiltonian $\mathcal{H}$ is conserved by smooth solutions of (45), but this is not particularly useful for analytical purposes since $\mathcal{H}$ is cubic and indefinite. An additional positive conserved quantity, associated with the invariance of (45) under spatial translations, is the momentum

$$\mathcal{P} = \frac{1}{2}\int a |\partial| a\, dx. \tag{47}$$

The mean $\int a\, dx$ is also conserved, but we do not know of any other conserved quantities for (45).

The momentum $\mathcal{P}$ plays the role of an entropy for dissipative weak solutions of (45). To show this, we consider the viscous equation

$$a_t + \left(\frac{1}{2}a^2\right)_x = \mathrm{H}\left[a|\partial| a\right] + \epsilon a_{xx}.$$

One finds that

$$\begin{aligned}
&(a|\partial|a)_t + \left(\frac{1}{2}a|\partial|(a^2) + a^2|\partial|a\right)_x \\
&\quad = \frac{1}{2}a_x|\partial|(a^2) + |\partial|a \cdot \mathrm{H}[a|\partial|a] + \epsilon\left(a|\partial|a_x + a_x|\partial|a\right)_x - 2\epsilon a_x|\partial|a_x.
\end{aligned}$$

Integrating this equation with respect to x, using the skew-adjointness of H, and assuming that the boundary terms vanish, we get that

$$\frac{1}{2}\frac{d}{dt}\int a|\partial|a\, dx = -\epsilon\int a_x|\partial|a_x\, dx \le 0.$$

7.1 Short-time existence of smooth solutions

We consider the following initial-value problem (IVP) for $a(x.t)$:

$$\begin{aligned} a_t + \left(\frac{1}{2}a^2\right)_x &= \mathrm{H}\left[a|\partial|a\right], \\ a(x,0) &= f(x). \end{aligned} \tag{48}$$

We look for spatially periodic solutions with zero mean,

$$a(x,t) = \sum_{k\in\mathbb{Z}} \hat{a}(k,t)e^{ikx},$$

where $\hat{a}(0,t) = 0$ and $\hat{a}(-k,t) = \hat{a}^*(k,t)$, and denote by $\dot{H}^s(\mathbb{T})$ the usual L^2-Sobolev space with norm

$$\|a\|_{\dot{H}^s} = \left(\sum_{k\in\mathbb{Z}} |k|^{2s}|\hat{a}(k)|^2\right)^{1/2}.$$

Theorem 7.1. *Suppose that $s > 3/2$ and $f \in \dot{H}^s(\mathbb{T})$. Then there is a unique local solution of (48) with*

$$a \in C(I; \dot{H}^s) \cap C^1(I; \dot{H}^{s-1})$$

defined on a time interval $I = (-T, T)$, where $T = C_s/\|f\|_{\dot{H}^s}$ for some constant $C_s > 0$.

Proof. The proof is by a standard Galerkin method [22]. We omit the details and just give the required a priori energy estimate following [12]. The same estimate and proof applies to equations with the kernels (11) or (71).

The spectral form of (45) is

$$\hat{a}_t(k,t) + \frac{1}{2}i \operatorname{sgn} k \sum_{\xi\in\mathbb{Z}} \left(|k| + |k-\xi| + |\xi|\right) \hat{a}(k-\xi,t)\hat{a}(\xi,t) = 0.$$

It follows that

$$\begin{aligned} &\frac{d}{dt}\sum_{k\in\mathbb{Z}} |k|^{2s+1}|\hat{a}(k,t)|^2 \\ &\qquad + i \sum_{k,\xi\in\mathbb{Z}} k|k|^{2s}\left(|k| + |k-\xi| + |\xi|\right) \hat{a}(-k,t)\hat{a}(k-\xi,t)\hat{a}(\xi,t) = 0. \end{aligned}$$

Using the symmetry of the kernel, we can write this equation as

$$\frac{d}{dt}\sum_{k\in\mathbb{Z}} |k|^{2s+1}|\hat{a}(k,t)|^2 + \frac{1}{3}i \sum_{k,\xi\in\mathbb{Z}} \Lambda_s(-k,k-\xi,\xi)\hat{a}(-k,t)\hat{a}(k-\xi,t)\hat{a}(\xi,t) = 0,$$

where

$$\Lambda_s(k_1, k_2, k_3) = \left(k_1|k_1|^{2s} + k_2|k_2|^{2s} + k_3|k_3|^{2s}\right)(|k_1| + |k_2| + |k_3|)\,.$$

The following inequalities [12] hold on $k_1 + k_2 + k_3$:

$$|k_1| + |k_2| + |k_3| \leq \frac{2\sqrt{2}|k_1 k_2 k_3|^{1/2}}{\min(|k_1|^{1/2}, |k_2|^{1/2}, |k_3|^{1/2})},$$

$$\frac{\left|\, k_1|k_1|^{2s} + k_2|k_2|^{2s} + k_3|k_3|^{2s}\,\right|}{\min(|k_1|^{1/2}, |k_2|^{1/2}, |k_3|^{1/2})} \leq C_s\left(|k_1|^s|k_2|^s|k_3|^{1/2} + |k_2|^s|k_3|^s|k_1|^{1/2} + |k_3|^s|k_1|^s|k_2|^{1/2}\right),$$

where C_s denotes a generic constant depending on $s > 0$. From these inequalities, we get that

$$|\Lambda_s(k_1, k_2, k_3)| \leq C_s\left(|k_1|^{s+1/2}|k_2|^{s+1/2}|k_3| + |k_2|^{s+1/2}|k_3|^{s+1/2}|k_1| + |k_3|^{s+1/2}|k_1|^{s+1/2}|k_2|\right)$$

on $k_1 + k_2 + k_3 = 0$. By Young's inequality, we can then estimate

$$\left|\sum_{k,\xi\in\mathbb{Z}} \Lambda_s(-k, k-\xi, \xi)\hat{a}(-k,t)\hat{a}(k-\xi,t)\hat{a}(\xi,t)\right| \leq C_s\left\|\, |k|^{s+1/2}|\hat{a}| * |k|^{s+1/2}|\hat{a}| * |k||\hat{a}|\,\right\|_{\ell^1} \leq C_s\left\|\, |k|^{s+1/2}\hat{a}\,\right\|_{\ell^2}^2 \|\, |k|\hat{a}\,\|_{\ell^1}\,,$$

where, as usual,

$$\|\,\hat{a}\,\|_{\ell^p} = \left(\sum_{k\in\mathbb{Z}} |\hat{a}(k)|^p\right)^{1/p}.$$

Since $\|\,|k|\hat{a}\,\|_{\ell^1} \leq C_r \|\,|k|^r\hat{a}\,\|_{\ell^2}$ for $r > 3/2$, we conclude that

$$\frac{d}{dt}\sum_{k\in\mathbb{Z}} |k|^{2s+1}|\hat{a}(k,t)|^2 \leq C_s\left(\sum_{k\in\mathbb{Z}} |k|^{2s+1}|\hat{a}(k,t)|^2\right)^{3/2}$$

if $s + 1/2 > 3/2$, which gives an a priori $\dot{H}^r$-estimate for a if $r > 3/2$. □

The number of L^2-derivatives, $s > 3/2$, that are required for local existence for (45) is the same as the number required for local existence for the inviscid Burgers equation.

7.2 Numerical solutions

In this section, we show some numerical solutions of (48), which indicate that the spatial derivatives of smooth solutions blow up in finite time and that smooth solutions can be continued by weak dissipative solutions. The weak solutions appear to remain continuous after singularities form, with cusps rather than shocks, and they become continuous even if the initial data contains jump discontinuities.

We use a standard pseudo-spectral method with spectral viscosity and a fourth-order Runge-Kutta method in time. These numerical solutions are dissipative in nature; we do not address here the question whether or not (48) also has conservative weak solutions, as is the case for the HS equation that describes the corresponding bulk waves.

Figure 1 shows a surface plot of the solution with sinusoidal initial data for times $0 \le t \le 1$. The solution steepens in a similar way to solutions of the inviscid Burgers equation, and its derivative a_x blows up at $t \approx 0.55$. This singularity formation time is a little longer than the time $t = 0.5$ one would get by neglecting the lower-order term on the right-hand side of (46).

Figure 2 plots the momentum $\mathcal{P}$ of this solution as a function of time and shows the numerically computed spectrum at $t = 1$. The momentum is constant until the singularity forms; after that it decreases. In contrast to the inviscid Burgers equation, the solution appears to remain continuous even after its derivative blows up. The power-law numerical spectrum of the solution at $t = 1$, together with the following analytical solution, suggests that solutions have an x^α-singularity with $\alpha \approx 1/3$.

If α is not an odd integer, then

$$\mathrm{H}\left[\,|x|^\alpha\,\right] = -C_\alpha\,\mathrm{sgn}(x)|x|^\alpha, \quad C_\alpha \mathrm{H}\left[\,\mathrm{sgn}(x)|x|^\alpha\,\right] = |x|^\alpha, \quad C_\alpha = \tan\left(\frac{\pi\alpha}{2}\right)$$

in a distributional sense. Using this formula, we find that

$$a(x) = a_0\,\mathrm{sgn}(x)|x|^\alpha \tag{49}$$

is a distributional solution of the steady equation

$$\left(\frac{1}{2}a^2\right)_x = \mathrm{H}[a|\partial|a]$$

if α satisfies $C_{2\alpha-1} + C_\alpha = 0$. The solutions of this equation are

$$\alpha = 2n \pm \frac{1}{3}, \qquad n \in \mathbb{Z}.$$

The smallest positive solution is $\alpha = 1/3$. The corresponding spectral power law $|\hat{a}(k)| = \hat{a}_0|k|^{-4/3}$ is in approximate agreement with the numerical spectrum shown in Figure 2; there is, however, a slight discrepancy, and our numerically computed spectrum has an exponent α that is below 1.32.

This numerical solution of (45) is qualitatively similar to the numerical solution of the HIZ equation (5) with sinusoidal initial data, where a cusp singularity like (49) with $\alpha = 2/3$ appear to form [13].

In Figure 3, we show a numerical solution of (48) with discontinuous square-wave initial data

$$f(x) = \begin{cases} 1 & \text{for } \pi/4 < x < 5\pi/4, \\ 0 & \text{for } 0 < x < \pi/4 \text{ or } 5\pi/4 < x < 2\pi. \end{cases} \tag{50}$$

The global structure of the solution is similar to that of the solution for an inviscid Burgers equation, but it has a continuous cusp singularity instead of a shock and additional cusp singularities at each edge of the "expansion fan." In a periodic domain, these singularities hit each other and coalesce into one singularity.

8 Orientation waves in a director field

We conclude this paper with a description of a weakly nonlinear surface wave solution of the director-field equation. The derivation parallels the one for the model equation, but the algebra is more involved and we will only summarize the results. Additional details of the computations are given in [5].

We consider a half-space IBVP for the system of nonlinear hyperbolic partial differential equations for a massive director field that is obtained from the variational principle (2)–(3). The Euler-Lagrange equation in $x > 0$ is

$$\begin{aligned} \mathbf{n}_{tt} = \alpha \nabla(\operatorname{div} \mathbf{n}) - \beta \left[\operatorname{curl}(A\mathbf{n}) + A \operatorname{curl} \mathbf{n}\right] \\ + \gamma \left[\mathbf{B} \times \operatorname{curl} \mathbf{n} - \operatorname{curl}(\mathbf{B} \times \mathbf{n})\right] + \lambda \mathbf{n} \end{aligned} \tag{51}$$

where $A = \mathbf{n} \cdot \operatorname{curl} \mathbf{n}$ and $\mathbf{B} = \mathbf{n} \times \operatorname{curl} \mathbf{n}$; the natural boundary condition on $x = 0$ is

$$(\alpha - \eta)(\operatorname{div} \mathbf{n})\nu + \beta(A\mathbf{n} \times \nu) + \eta \left(\nu \cdot (\nabla \mathbf{n})^T\right) = \mu \mathbf{n}, \tag{52}$$

where $\nu = (-1, 0, 0)$ is the outward unit normal on the boundary $\partial\mathbb{R}^3_+$. The scalar fields λ and μ in (51) and (52) are Lagrange multipliers that enforce the constraint $\mathbf{n} \cdot \mathbf{n} = 1$ in the interior and on the boundary. They are given explicitly by

$$\lambda = -|\mathbf{n}_t|^2 + \alpha \left[|\nabla \mathbf{n}|^2 - |\operatorname{curl} \mathbf{n}|^2\right] + \left[\beta A^2 + \gamma |\mathbf{B}|^2\right] + (\alpha - \gamma) \operatorname{div} \mathbf{B},$$
$$\mu = (\alpha - \eta)(\operatorname{div} \mathbf{n})(\nu \cdot \mathbf{n}).$$

This system is more complicated than the model system. Due to the anisotropic nature of the equations, the behavior of solutions depends on the direction of wave propagation and on the direction of the normal to the boundary. Nevertheless, we get asymptotic equations for weakly nonlinear surface waves that are qualitatively similar to the ones arising from the model system.

For definiteness, we consider only the case of surface waves that are small perturbations of a constant director field $\mathbf{n}_0 = (0, 0, 1)$ that is tangent to the boundary.

8.1 The Lopatinski condition

We expand the solution of (51)–(52) as $\epsilon \to 0$ as

$$\mathbf{n}(\mathbf{x}, t) = \mathbf{n}_0 + \epsilon \mathbf{n}'(\mathbf{x}, t) + O(\epsilon^2), \qquad \mathbf{n}_0 = (0, 0, 1).$$

Since $\mathbf{n}$ is a unit vector, $\mathbf{n}'$ is orthogonal to $\mathbf{n}_0$, and we write it as

$$\mathbf{n}'(\mathbf{x}, t) = (u(\mathbf{x}, t), v(\mathbf{x}, t), 0)\,, \tag{53}$$

where $\mathbf{x} = (x, y, z)$. The linearization of the Euler-Lagrange equation (51) in $x > 0$ is

$$\begin{aligned} u_{tt} &= \alpha u_{xx} + \beta u_{yy} + \gamma u_{zz} + (\alpha - \beta) v_{xy}, \\ v_{tt} &= \beta v_{xx} + \alpha v_{yy} + \gamma v_{zz} + (\alpha - \beta) u_{xy}, \end{aligned} \tag{54}$$

in $x > 0$, and the linearization of the boundary condition (52) on $x = 0$ is

$$\begin{aligned} -\alpha u_x + (\eta - \alpha) v_y &= 0, \\ -\beta v_x + (\beta - \eta) u_y &= 0. \end{aligned} \tag{55}$$

We look for Fourier-Laplace solutions of the form

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \hat{u} e^{-kx} \\ \hat{v} e^{-kx} \end{bmatrix} e^{\tau t + ily + imz}, \tag{56}$$

where $l, m \in \mathbb{R}$, $\tau \in \mathbb{C}$ with $\Re(\tau) > 0$, and $k \in \mathbb{C}$ with $\Re(k) > 0$. Using (56) in (54), we find that (56) is a solution of the PDE if

$$\begin{bmatrix} \tau^2 - \alpha k^2 + \beta l^2 + \gamma m^2 & (\alpha - \beta) ikl \\ (\alpha - \beta) ikl & \tau^2 - \beta k^2 + \alpha l^2 + \gamma m^2 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It follows that (k, l, m, τ) satisfies the linearized dispersion relation

$$\left[\tau^2 - \alpha k^2 + \beta l^2 + \gamma m^2\right] \left[\tau^2 - \beta k^2 + \alpha l^2 + \gamma m^2\right] + (\alpha - \beta)^2 k^2 l^2 = 0,$$

and

$$\begin{bmatrix} u \\ v \end{bmatrix} = R \begin{bmatrix} ik_\alpha \\ l \end{bmatrix} e^{\tau t + ily + imz - \gamma_\alpha x} + S \begin{bmatrix} -l \\ ik_\beta \end{bmatrix} e^{\tau t + ily + imz - \gamma_\beta x}, \tag{57}$$

where R, S are arbitrary constants. The decay constants γ_α, γ_β in (57) are given by

$$\gamma_\alpha = \sqrt{l^2 + \frac{\tau^2 + \gamma m^2}{\alpha}}, \qquad \gamma_\beta = \sqrt{l^2 + \frac{\tau^2 + \gamma m^2}{\beta}},$$

where we take the branch of the square root with positive real part.

Using (57) in the boundary condition (55), we obtain the algebraic equation

$$\begin{bmatrix} \eta l^2 + \gamma m^2 + \tau^2 & i\eta l \gamma_\beta \\ -i\eta l \gamma_\alpha & \eta l^2 + \gamma m^2 + \tau^2 \end{bmatrix} \begin{bmatrix} R \\ S \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{58}$$

This equation has a nontrivial solution if and only if $L(\tau, l, m) = 0$, where L is the Lopatinski determinant

$$L(\tau, l, m) = \left(\eta l^2 + \gamma m^2 + \tau^2\right)^2 - \eta^2 l^2 \gamma_\alpha \gamma_\beta. \tag{59}$$

The Lopatinski condition for this problem is then

$$L(\tau, l, m) \neq 0 \qquad \text{for all } l, m \in \mathbb{R} \text{ and } \tau \in \mathbb{C} \text{ with } \Re(\tau) > 0. \tag{60}$$

If the Lopatinski condition for a second-order, variational IBVP fails, then it fails for a real value of τ (see Theorem 3.4 in [21]). Thus, writing

$$a = \frac{\alpha}{\eta}, \qquad b = \frac{\beta}{\eta}, \qquad c = \frac{\gamma}{\eta}, \qquad x = \frac{\tau^2}{\eta}, \tag{61}$$

we see that the Lopatinski condition (60) is satisfied if and only if there are no $l, m \in \mathbb{R}$ with $l^2 + m^2 = 1$ such that the polynomial equation

$$ab\left(l^2 + cm^2 + x\right)^4 - l^4\left(al^2 + cm^2 + x\right)\left(bl^2 + cm^2 + x\right) = 0 \tag{62}$$

has a real, strictly positive root for x. (Recall that we assume $\alpha, \beta, \gamma, \eta > 0$.)

If $l = 1$ and $m = 0$, corresponding to a tangential wavenumber vector that is orthogonal to the unperturbed director field $\mathbf{n}_0$, then one can show that (62) has no real, positive root x if

$$\frac{1}{a} + \frac{1}{b} \leq 4. \tag{63}$$

We numerically computed the roots of (62) as l and m vary over $l^2 + m^2 = 1$, and found that if the parameters satisfy (63), then either $x < 0$ or $x \in \mathbb{C}$ with $\Re x < 0$. Although we do not have a proof, the numerical results suggest that the Lopatinski condition holds when (63) is satisfied (and $a, b, c > 0$).

8.2 Linearized surface waves

Surface wave solutions correspond to Fourier-Laplace solutions with purely imaginary frequency $\tau = -i\omega_0$ with $\omega_0 \in \mathbb{R}$ such that $L(-i\omega_0, l, m) = 0$ for some $l, m \in \mathbb{R}$. It follows from (59) that the linearized surface-wave dispersion relation of (54)–(55) for $\omega_0(l, m)$ is

$$\left(\eta l^2 + \gamma m^2 - \omega_0^2\right)^2 = \eta^2 l^2 \gamma_\alpha \gamma_\beta, \tag{64}$$

where

$$\gamma_\alpha = \sqrt{l^2 + \frac{\gamma m^2 - \omega_0^2}{\alpha}}, \qquad \gamma_\beta = \sqrt{l^2 + \frac{\gamma m^2 - \omega_0^2}{\beta}}. \tag{65}$$

It is convenient to introduce the scaled parameters (61). To find genuine surface wave solutions, we first look for real, negative roots $x = -\omega_0^2/\eta$ of (62).

If such a root exists, then we check that γ_α, γ_β are real numbers. Finally, we verify that

$$\left(l^2 + cm^2 + x\right)^2 - l^2\sqrt{l^2 + \frac{cm^2 + x}{a}}\sqrt{l^2 + \frac{cm^2 + x}{b}} = 0.$$

This last step is necessary to rule out extraneous solution due to squaring the Lopatinski condition to obtain (62).

There is a simple explicit solution of these equations with

$$\omega_0^2 = \gamma m^2, \qquad \gamma_\alpha = \gamma_\beta = |l|, \tag{66}$$

and

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ i \operatorname{sgn} l \end{bmatrix} e^{i(ly + mz \pm \sqrt{\gamma} mt) - |l|x}.$$

This is a genuine surface wave solution if $l \neq 0$. It turns out, however, that the quadratically nonlinear effects on this transverse surface wave vanish, so it is not of interest for the present analysis.

This is not the only solution, however. For example, in Table 2 we show the roots of (62) with $a = 2$, $b = 0.5$, $c = 3$ for several values of l with $l^2 + m^2 = 1$. The root x_4 corresponds to the explicit solution (66). The root x_3 is a second real, negative root. As Table 3 shows, it corresponds to a genuine surface wave.

In summary, surface wave solutions to the IBVP (54)–(55) can be written in the form

$$\begin{bmatrix} u \\ v \end{bmatrix} = \left\{ R \begin{bmatrix} i\gamma_\alpha \\ l \end{bmatrix} e^{-\gamma_\alpha x} + S \begin{bmatrix} -l \\ i\gamma_\beta \end{bmatrix} e^{-\gamma_\beta x} \right\} e^{i(ly + mz - \omega_0 t)}.$$

Here, γ_α and γ_β are given by (65), the surface-wave frequency ω_0 satisfies

$$\left(\eta l^2 + \gamma m^2 - \omega_0^2\right)^2 - \eta^2 l^2 \sqrt{l^2 + \frac{\gamma m^2 - \omega_0^2}{\alpha}}\sqrt{l^2 + \frac{\gamma m^2 - \omega_0^2}{\beta}} = 0,$$

and R, S satisfy (58). This solutions represents a genuine surface wave whose phase velocity along the boundary is at an angle $\phi = \tan^{-1}(m/l)$ to $\mathbf{n}_0 = (0, 0, 1)$. General solutions may be obtained by Fourier-superposition of these solutions.

8.3 Weakly nonlinear surface waves

Asymptotic equations for weakly nonlinear surface wave solutions may derived following the procedure for the model problem. The algebra involved in a direct multiple-scale expansion is prohibitively complicated, but one can derive the equation by expanding the Hamiltonian or Lagrangian. Austria [5] gives details for the expansion of the Lagrangian. In this paper, we simply report the result.

Let $\epsilon > 0$ be a small dimensionless parameter. We introduce a "slow" time variable $\tau = \epsilon t$ and a "fast" phase variable

$$\theta = ly + mz - \omega_0 t,$$

where $(l, m) \in \mathbb{R}^2$ is a fixed nonzero vector, ω_0 satisfies (64), and $\gamma_\alpha, \gamma_\beta > 0$ are given in (65). The asymptotic solution has an expansion as $\epsilon \to 0$ of the form

$$\mathbf{n}(x, \theta, \tau; \epsilon) = \mathbf{n}_0 + \epsilon \mathbf{n}_1(x, \theta, \tau) + O(\epsilon^2) \tag{67}$$

where $\mathbf{n}_0 = (0, 0, 1)$, and

$$\mathbf{n}_1(x, \theta, \tau) = (u(x, \theta, \tau), v(x, \theta, \tau), 0) \,. \tag{68}$$

The first-order term (u, v) is the linearized surface wave solution

$$\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix} = \int_{\mathbb{R}} \hat{a}(\rho, \tau) \Bigg\{ & R(\rho) \begin{bmatrix} i\,\mathrm{sgn}(\rho)\gamma_\alpha \\ l \end{bmatrix} e^{-\gamma_\alpha |\rho| x} \\ & + S(\rho) \begin{bmatrix} l \\ -i\,\mathrm{sgn}(\rho)\gamma_\beta \end{bmatrix} e^{-\gamma_\beta |\rho| x} \Bigg\} e^{i\rho\theta}\, d\rho, \end{aligned} \tag{69}$$

where we represent the solution as a Fourier integral with respect to ρ, and

$$R(\rho) = \eta l^2 + \gamma m^2 - \omega_0^2, \qquad S(\rho) = -i\,\mathrm{sgn}(\rho) l \eta \gamma_\alpha.$$

Expanding the surface-wave Lagrangian up to cubic terms in $\hat{a}$, one finds that the corresponding Euler-Lagrange equation for $\hat{a}(\rho, \tau)$ has the form

$$\hat{a}_\tau(\rho, \tau) + i\,\mathrm{sgn}(\rho) \int_{\mathbb{R}} \Lambda\left(-\rho, \rho - \xi, \xi\right) \hat{a}(\rho - \xi) \hat{a}(\xi)\, d\xi = 0. \tag{70}$$

The kernel $\Lambda(k_1, k_2, k_3)$ in (70) is given on $k_1 + k_2 + k_3 = 0$ by

$$\begin{aligned} \Lambda(k_1, k_2, k_3) = & \frac{\Gamma_0(k_1, k_2, k_3)}{|k_1| + |k_2| + |k_3|} \\ & + \frac{\mathrm{sgn}(k_2 k_3)\Gamma_1(k_1, k_2, k_3)}{\gamma_\alpha |k_1| + \gamma_\beta(|k_2| + |k_3|)} + \frac{\mathrm{sgn}(k_1 k_3)\Gamma_1(k_2, k_1, k_3)}{\gamma_\alpha |k_2| + \gamma_\beta(|k_1| + |k_3|)} \\ & + \frac{\mathrm{sgn}(k_1 k_2)\Gamma_1(k_3, k_2, k_1)}{\gamma_\alpha |k_3| + \gamma_\beta(|k_1| + |k_2|)} \\ & + \frac{\mathrm{sgn}(k_1)\Gamma_2(k_1, k_2, k_3)}{\gamma_\alpha(|k_2| + |k_3|) + \gamma_\beta |k_1|} + \frac{\mathrm{sgn}(k_2)\Gamma_2(k_2, k_1, k_3)}{\gamma_\alpha(|k_1| + |k_3|) + \gamma_\beta |k_2|} \\ & + \frac{\mathrm{sgn}(k_3)\Gamma_2(k_3, k_2, k_1)}{\gamma_\alpha(|k_1| + |k_2|) + \gamma_\beta |k_3|}, \end{aligned} \tag{71}$$

where Γ_0, Γ_1, Γ_2 are given by

$$\begin{aligned} & \Gamma_0(k_1, k_2, k_3) \\ & = c_1(\gamma - \beta)\left(l^2 - \gamma_\beta^2\right)\left[\left(k_1^2 + k_2^2 + k_3^2\right) - \left(|k_1| + |k_2| + |k_3|\right)^2\right] \\ & - c_1(\gamma - \beta)\left(l^2 - \gamma_\beta^2\right)\left(k_1|k_1| + k_2|k_2| + k_3|k_3|\right)\mathrm{sgn}(k_1 k_2 k_3) \\ & - c_2(\eta - \alpha)\left(l^2 - \gamma_\alpha^2\right)\gamma_\alpha^2\left(k_1|k_1| + k_2|k_2| + k_3|k_3|\right)\mathrm{sgn}(k_1 k_2 k_3) \\ & + c_2(\eta - \alpha)\left(l^2 - \gamma_\alpha^2\right) l^2 \left(k_1^2 + k_2^2 + k_3^2\right) \\ & - c_1(\gamma - \eta)\gamma_\beta^2 \left(|k_1| + |k_2| + |k_3|\right)^2 \\ & + c_2(\gamma - \eta)\gamma_\alpha^2 l^2 \left(|k_1| + |k_2| + |k_3|\right)^2, \end{aligned} \tag{72}$$

$$
\begin{aligned}
&\Gamma_1(k_1,k_2,k_3)\\
&= c_3(\gamma-\beta)\left(l^2-\gamma_\beta^2\right)\left(l^2-\mathrm{sgn}(k_1k_2)\gamma_\alpha\gamma_\beta\right)\left(k_1^2-k_2^2\right)\\
&+ c_3(\gamma-\beta)\left(l^2-\gamma_\beta^2\right)\left(l^2-\mathrm{sgn}(k_1k_3)\gamma_\alpha\gamma_\beta\right)\left(k_1^2-k_3^2\right)\\
&- c_3(\eta-\alpha)\left(l^2-\gamma_\alpha^2\right)\left(l^2-\mathrm{sgn}(k_2k_3)\gamma_\beta^2\right)k_1^2\\
&- c_3(\gamma-\eta)l^2\left(\mathrm{sgn}(k_1)\gamma_\alpha-\mathrm{sgn}(k_3)\gamma_\beta\right)\left(\gamma_\alpha|k_1|+\gamma_\beta(|k_2|+|k_3|)\right)k_2\\
&- c_3(\gamma-\eta)l^2\left(\mathrm{sgn}(k_1)\gamma_\alpha-\mathrm{sgn}(k_2)\gamma_\beta\right)\left(\gamma_\alpha|k_1|+\gamma_\beta(|k_2|+|k_3|)\right)k_3\\
&- c_3(\gamma-\eta)\left(l^2-\mathrm{sgn}(k_2k_3)\gamma_\beta^2\right)\left(\gamma_\alpha\gamma_\beta(|k_1k_2|+|k_1k_3|)+l^2k_1^2\right),
\end{aligned}
\tag{73}
$$

$$
\begin{aligned}
&\Gamma_2(k_1,k_2,k_3)\\
&= c_4(\gamma-\beta)\left(l^2-\gamma_\beta^2\right)\gamma_\alpha\left(\mathrm{sgn}(k_3)-\mathrm{sgn}(k_2)\right)\left(k_3^2-k_2^2\right)\\
&- c_4(\eta-\alpha)\left(l^2-\gamma_\alpha^2\right)\left(\mathrm{sgn}(k_1)\gamma_\beta-\mathrm{sgn}(k_3)\gamma_\alpha\right)k_2^2\\
&- c_4(\eta-\alpha)\left(l^2-\gamma_\alpha^2\right)\left(\mathrm{sgn}(k_1)\gamma_\beta-\mathrm{sgn}(k_2)\gamma_\alpha\right)k_3^2\\
&- c_4(\gamma-\eta)\left(\mathrm{sgn}(k_1)\gamma_\beta-\mathrm{sgn}(k_3)\gamma_\alpha\right)\left(\gamma_\alpha^2|k_2k_3|+\gamma_\alpha\gamma_\beta|k_1k_2|+l^2k_2^2\right)\\
&- c_4(\gamma-\eta)\left(\mathrm{sgn}(k_1)\gamma_\beta-\mathrm{sgn}(k_2)\gamma_\alpha\right)\left(\gamma_\alpha^2|k_2k_3|+\gamma_\alpha\gamma_\beta|k_1k_3|+l^2k_3^2\right)\\
&- c_4(\gamma-\eta)\left(l^2-\mathrm{sgn}(k_2k_3)\gamma_\alpha^2\right)\left(\gamma_\alpha(|k_2|+|k_3|)+\gamma_\beta|k_1|\right)k_1.
\end{aligned}
\tag{74}
$$

The constants are given by

$$
\begin{aligned}
c_1 &= \gamma_\alpha\left[\frac{m\left(\eta l\gamma_\alpha\right)^2\eta l^2}{4E_0}\right]\\
c_2 &= \frac{\gamma_\beta}{\gamma_\alpha^2}\left[\frac{m\left(\eta l\gamma_\alpha\right)^2}{4E_0}\right]\left(\eta l^2+\gamma m^2-\omega_0^2\right)\\
c_3 &= \left[\frac{m\left(\eta l\gamma_\alpha\right)^2}{4E_0}\right]\left(\eta l^2+\gamma m^2-\omega_0^2\right)\\
c_4 &= \gamma_\beta\left[\frac{m\left(\eta l\gamma_\alpha\right)^2\eta l^2}{4E_0}\right],
\end{aligned}
$$

where

$$
E_0=\left(\frac{l^2+\gamma_\alpha^2}{2\gamma_\alpha}\right)|R|^2+\left(\frac{l^2+\gamma_\beta^2}{2\gamma_\beta}\right)|S|^2-il\left[RS^*-R^*S\right].
$$

We note that for the surface wave solution (66), we get

$$
\begin{aligned}
\Gamma_0(k_1,k_2,k_3) &= (\gamma-\eta)l^2(c_2l^2-c_1)\left(|k_1|+|k_2|+|k_3|\right)^2,\\
\mathrm{sgn}(k_2k_3)\Gamma_1(k_1,k_2,k_3) &= (\gamma-\eta)l^2(c_3l)\left(|k_1|+|k_2|+|k_3|\right)^2,\\
\mathrm{sgn}(k_1)\Gamma_2(k_1,k_2,k_3) &= (\gamma-\eta)l^2(c_4)\left(|k_1|+|k_2|+|k_3|\right)^2,
\end{aligned}
$$

and $3(c_3l-c_4)+c_2l^2-c_1=0$. Thus, $\Lambda(k_1,k_2,k_3)=0$ for all k_1,k_2, k_3 with $k_1+k_2+k_3=0$, so the quadratically nonlinear effects drop out, and one would have to include higher-order terms in the expansion (67) to determine the nonlinear behavior of these surface waves.

If $\omega_0^2 \neq \gamma m^2$, then the asymptotic equation is typically nontrivial, and we get a qualitatively similar kernel to the one we obtained for the model equation.

9 Appendix

In this Appendix, we briefly describe the derivation of the multiple-scale solution (22) of (14)–(15) for a weakly nonlinear surface wave. Further details of the computations are given in [5].

We introduce multiple-scale variables $\theta = y - \lambda t$, x, $\tau = \epsilon t$ and expand partial derivatives with respect to x, y, t as

$$\partial_x = \partial_x, \qquad \partial_y = \partial_\theta, \qquad \partial_t = -\lambda\partial_\theta + \epsilon\partial_\tau.$$

We use these expansions and (21)–(22) in (14)–(15) and equate coefficients of ϵ and ϵ^2 to zero in the resulting equations.

At the order ϵ, we find that (u_1, v_1) satisfies

$$(\alpha_0^2 - \lambda^2)u_{1\theta\theta} + \alpha_0^2 u_{1xx} = 0, \qquad (\beta_0^2 - \lambda^2)v_{1\theta\theta} + \beta_0^2 v_{1xx} = 0 \tag{75}$$

in $x > 0$, with the boundary condition

$$\alpha_0^2 u_{1x} - \eta v_{1\theta} = 0, \qquad \beta_0^2 v_{1x} + \eta u_{1\theta} = 0 \tag{76}$$

on $x = 0$. The slow time τ occurs as a parameter in these equations.

At the order ϵ^2, we find that (u_2, v_2) satisfies

$$\begin{aligned}(\alpha_0^2 - \lambda^2)u_{2\theta\theta} + \alpha_0^2 u_{2xx} &= -2\lambda u_{1\theta\tau} \\ &\quad + \alpha_0\alpha_1\left[u_{1x}^2 - (u_1^2)_{xx} + u_{1\theta}^2 - (u_1^2)_{\theta\theta}\right] \\ (\beta_0^2 - \lambda^2)v_{2\theta\theta} + \beta_0^2 v_{2xx} &= -2\lambda v_{1\theta\tau} \\ &\quad + \beta_0\beta_1\left[v_{1x}^2 - (v_1^2)_{xx} + v_{1\theta}^2 - (v_1^2)_{\theta\theta}\right]\end{aligned} \tag{77}$$

in $x > 0$, with the boundary condition

$$\begin{aligned}\alpha_0^2 u_{2x} - \eta v_{2\theta} &= -\alpha_0\alpha_1 (u_1^2)_x, \\ \beta_0^2 v_{2x} + \eta u_{2\theta} &= -\beta_0\beta_1 (v_1^2)_x\end{aligned} \tag{78}$$

on $x = 0$.

The Fourier-Laplace solution of the leading-order equations (75)–(76) has the form

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}(x, \theta, \tau) = \int_{\mathbb{R}} \hat{a}(k, \tau) \begin{bmatrix} R(k)e^{-\gamma_\alpha |k| x} \\ S(k)e^{-\gamma_\beta |k| x} \end{bmatrix} e^{ik\theta}\, dk, \tag{79}$$

where $\hat{a}(-k, \tau) = \hat{a}^*(-k, \tau)$. This expression satisfies (75) if

$$\gamma_\alpha = \sqrt{1 - \frac{\lambda^2}{\alpha_0^2}}, \qquad \gamma_\beta = \sqrt{1 - \frac{\lambda^2}{\beta_0^2}},$$

and it satisfies (76) if

$$\begin{bmatrix} -\alpha_0^2\gamma_\alpha|k| & -i\eta k \\ i\eta k & -\beta_0^2\gamma_\beta|k| \end{bmatrix} \begin{bmatrix} R(k) \\ S(k) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{80}$$

Equation (80) has a nontrivial solution for (R, S) if $\alpha_0^2\beta_0^2\gamma_\alpha\gamma_\beta = \eta^2$, which implies that

$$\lambda^2 = \frac{1}{2}\left\{\alpha_0^2 + \beta_0^2 \pm \sqrt{(\alpha_0^2 - \beta_0^2)^2 + \frac{4\eta^4}{\alpha_0^2\beta_0^2}}\right\}.$$

We choose the negative sign, as in (26), in which case γ_α and γ_β are real and positive for $\alpha_0\beta_0 > |\eta|$. We also take $R = \eta$ and $S = i\,\mathrm{sgn}(k)\alpha_0^2\gamma_\alpha$.

Next, using the Fourier-Laplace transform, we solve the non-homogeneous system of PDEs (77) for (u_2, v_2), where (u_1, v_1) is given by (79). After some algebra, we find that

$$\begin{aligned} u_2 = &\int_{\mathbb{R}} \left[R_0(k) + i\,\mathrm{sgn}(k)\frac{\eta}{\alpha_0^2\gamma_\alpha}\lambda\hat{a}_\tau(k,\tau)x\right] e^{ik\theta - \gamma_\alpha|k|x}\,dk \\ &+ \iint_{U_1} R_1(k,l)\hat{a}(k,\tau)\hat{a}(l,\tau)x e^{i(k+l)\theta - \gamma_\alpha(|k|+|l|)x}\,dkdl \\ &+ \iint_{U_2} R_2(k,l)\hat{a}(k,\tau)\hat{a}(l,\tau)e^{i(k+l)\theta - \gamma_\alpha(|k|+|l|)x}\,dkdl, \\ v_2 = &\int_{\mathbb{R}} \left[S_0(k) - \frac{\alpha_0^2\gamma_\alpha}{\beta_0^2\gamma_\beta}\lambda\hat{a}_\tau(k,\tau)x\right] e^{ik\theta - \gamma_\beta|k|x}\,dk \\ &+ \iint_{U_1} S_1(k,l)\hat{a}(k,\tau)\hat{a}(l,\tau)x e^{i(k+l)\theta - \gamma_\beta(|k|+|l|)x}\,dkdl \\ &+ \iint_{U_2} S_2(k,l)\hat{a}(k,\tau)\hat{a}(l,\tau)e^{i(k+l)\theta - \gamma_\beta(|k|+|l|)x}\,dkdl, \end{aligned} \tag{81}$$

where $U_1 = \{(l,k) \in \mathbb{R}^2 : \mathrm{sgn}\,k = \mathrm{sgn}\,l\}$, $U_2 = \{(k,l) \in \mathbb{R}^2 : \mathrm{sgn}\,k \neq \mathrm{sgn}\,l\}$,

$$R_1(k,l) = \frac{\alpha_0\alpha_1\eta^2\left[\gamma_\alpha^2\left((|k|+|l|)^2 - |k||l|\right) - \left((k+l)^2 - kl\right)\right]}{2\alpha_0^2\gamma_\alpha(|k|+|l|)}$$

$$R_2(k,l) = -\frac{\alpha_0\alpha_1\eta^2\left[\gamma_\alpha^2\left((|k|+|l|)^2 - |k||l|\right) - \left((k+l)^2 - kl\right)\right]}{\alpha_0^2\gamma_\alpha^2\left[(|k|+|1|)^2 - (k+l)^2\right]}$$

$$S_1(k,l) = -\frac{\beta_0\beta_1(\alpha_0^2\gamma_\alpha)^2\,\mathrm{sgn}(k)\,\mathrm{sgn}(l)\left[\gamma_\beta^2\left((|k|+|l|)^2 - |k||l|\right) - \left((k+l)^2 - kl\right)\right]}{2\beta_0^2\gamma_\beta(|k|+|l|)}$$

$$S_2(k,l) = \frac{\beta_0\beta_1(\alpha_0^2\gamma_\alpha)^2\,\mathrm{sgn}(k)\,\mathrm{sgn}(l)\left[\gamma_\beta^2\left((|k|+|l|)^2 - |k||l|\right) - \left((k+l)^2 - kl\right)\right]}{\beta_0^2\gamma_\beta^2\left[(|k|+|l|)^2 - (k+l)^2\right]},$$

and $R_0(k)$, $S_0(k)$ are arbitrary functions of integration.

Using (79) and (81) in the second-order boundary condition (78) and simplifying the result, we get a singular linear system of equations for (R_0, S_0) of the form

$$\begin{bmatrix} -\alpha_0^2\gamma_\alpha|k| & -i\eta k \\ i\eta k & -\beta_0^2\gamma_\beta|k| \end{bmatrix} \begin{bmatrix} R_0(k) \\ S_0(k) \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \tag{82}$$

where

$$\begin{aligned} F_1 &= -i\,\mathrm{sgn}(k)\eta\lambda\hat{a}_\tau(k,\tau) + \int_{-\infty}^{\infty} \Gamma_1(k-l,l)\hat{a}(k-l,\tau)\hat{a}(l,\tau)\,dl \\ F_2 &= \frac{\alpha_0^2\gamma_\alpha}{\gamma_\beta}\lambda\hat{a}_\tau(k,\tau) + \int_{-\infty}^{\infty} \Gamma_2(k-l,l)\hat{a}(k-l,\tau)\hat{a}(l,\tau)\,dl. \end{aligned} \tag{83}$$

The kernels $\Gamma_i(k-l,l)$ in F_i are given by

$$\Gamma_1(k-l,l) = \begin{cases} P_2(k-l,l), & \text{if } k>0 \text{ and } -\infty<l<0 \\ P_1(k-l,l), & \text{if } k>0 \text{ and } 0<l<k \\ P_2(k-l,l), & \text{if } k>0 \text{ and } k<l<+\infty \\ P_2(k-l,l), & \text{if } k<0 \text{ and } -\infty<l<k \\ P_1(k-l,l), & \text{if } k<0 \text{ and } k<l<0 \\ P_2(k-l,l), & \text{if } k<0 \text{ and } 0<l<+\infty \end{cases}$$

and

$$\Gamma_2(k-l,l) = \begin{cases} Q_2(k-l,l), & \text{if } k>0 \text{ and } -\infty<l<0 \\ Q_1(k-l,l), & \text{if } k>0 \text{ and } 0<l<k \\ Q_2(k-l,l), & \text{if } k>0 \text{ and } k<l<+\infty \\ Q_2(k-l,l), & \text{if } k<0 \text{ and } -\infty<l<k \\ Q_1(k-l,l), & \text{if } k<0 \text{ and } k<l<0 \\ Q_2(k-l,l), & \text{if } k<0 \text{ and } 0<l<+\infty \end{cases}$$

where

$$\begin{aligned} P_1(k,l) &= \frac{\alpha_0\alpha_1\gamma_\alpha\eta^2}{2}\left[(|k|+|l|)^2 + |k||l| + \frac{(k+l)^2-kl}{\gamma_\alpha^2}\right]\frac{1}{|k|+|l|} \\ P_2(k,l) &= \frac{\alpha_0\alpha_1\gamma_\alpha\eta^2}{2}\left[|k||l| - (k+l)^2 + \frac{(k+l)^2-kl}{\gamma_\alpha^2}\right]\frac{|k|+|l|}{|k||l|-kl} \\ &\quad + i\frac{\beta_1(\alpha_0^2\gamma_\alpha)^2\eta}{2\beta_0}\left[(|k|+|l|)^2 - |k||l| - \frac{(k+l)^2-kl}{\gamma_\beta^2}\right]\frac{(k+l)\,\mathrm{sgn}(k)\,\mathrm{sgn}(k)}{|k||l|-kl} \end{aligned}$$

and

$$\begin{aligned} Q_1(k,l) &= -\frac{\beta_0\beta_1\gamma_\beta(\alpha_0^2\gamma_\alpha)^2}{2}\left[(|k|+|l|)^2 + |k||l| + \frac{(k+l)^2-kl}{\gamma_\beta^2}\right]\frac{\mathrm{sgn}(k)\,\mathrm{sgn}(l)}{|k|+|l|} \\ Q_2(k,l) &= -\frac{\beta_0\beta_1\gamma_\beta(\alpha_0^2\gamma_\alpha)^2}{2}\left[|k||l| - (k+l)^2 + \frac{(k+l)^2-kl}{\gamma_\beta^2}\right]\frac{(|k|+|l|)\,\mathrm{sgn}(k)\,\mathrm{sgn}(l)}{|k||l|-kl} \\ &\quad + i\frac{\alpha_1\eta^3}{2\alpha_0}\left[(|k|+|l|)^2 - |k||l| - \frac{(k+l)^2-kl}{\gamma_\alpha^2}\right]\frac{(k+l)}{|k||l|-kl}. \end{aligned}$$

The solvability condition for (82) is

$$\eta F_1 - i\,\mathrm{sgn}(k)\alpha_0^2\gamma_\alpha F_2 = 0.$$

Using (83) in this condition and simplifying the result, we get an equation for $\hat{a}(k,\tau)$, which may be written as

$$\hat{a}_\tau(k,\tau) + i\,\mathrm{sgn}(k)\int_{-\infty}^{+\infty}\Lambda(-k,k-l,l)\hat{a}(k-l,\tau)\hat{a}(l,\tau)\,dl = 0,$$

where

$$\Lambda(-k,k-l,l) = \frac{\gamma_\alpha\gamma_\beta\eta^2}{\lambda(\alpha_0^2\gamma_\alpha^2+\beta_0^2\gamma_\beta^2)}\left[\frac{1}{\alpha_0^2\gamma_\alpha\eta}\Gamma_1(k-l,l) - \frac{i\,\mathrm{sgn}(k)}{\eta^2}\Gamma_2(k-l,l)\right].$$

Finally, by considering the possible signs of $-k$, $k-l$, l one can verify that this expression for $\Lambda(-k,k-l,l)$ agrees with the expression given in (11) for $\Lambda(k_1,k_2,k_3)$ on $k_1+k_2+k_3=0$.

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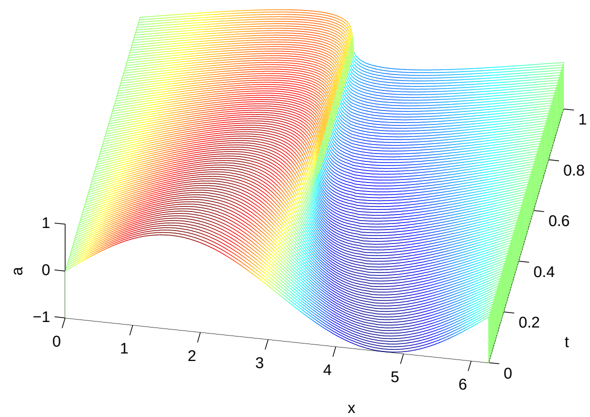

Figure 1: Numerical solution of (48) for $0 \leq t \leq 1$ with initial data $f(x) = \sin x$.

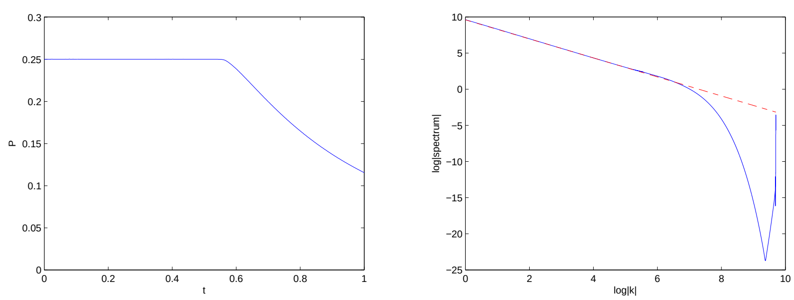

Figure 2: Left: The momentum $\mathcal{P}$ defined in (47) as a function of time for the solution shown in Figure 1. Right: The numerical spectrum $\log |\hat{a}(k,t)|$ at $t = 1.0$ as a function of $\log |k|$. The dashed line is a best linear fit to the spectrum for $8 \leq |k| \leq 128$, and is $\log |\hat{a}(k,t)| \approx -1.315 \cdot \log |k| + 9.591$. The numerical solution is computed for wavenumbers $|k| \leq 2^{14}$, after dealiasing. Spectral viscosity is switched on starting at $|k| = 256$.

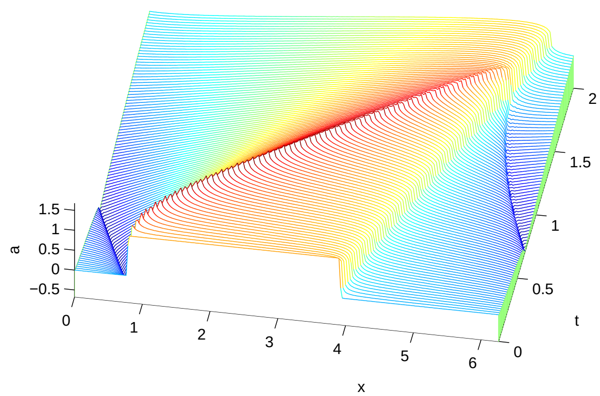

Figure 3: Numerical solution of (48) with square-wave initial data (50).

l	x_1	x_2	x_3	x_4
0.000	-3.0000	-3.0000	-3.0000	-3.0000
0.125	-2.9810 + 0.0082i	-2.9810 - 0.0082i	-2.9599	-2.9531
0.250	-2.9239 + 0.0326i	-2.9239 - 0.0326i	-2.8397	-2.8125
0.375	-2.8288 + 0.0734i	-2.8288 - 0.0734i	-2.6393	-2.5781
0.500	-2.6956 + 0.1304i	-2.6956 - 0.1304i	-2.3587	-2.2500
0.625	-2.5245 + 0.2038i	-2.5245 - 0.2038i	-1.9980	-1.8281
0.750	-2.3152 + 0.2935i	-2.3152 - 0.2935i	-1.5571	-1.3125
0.875	-2.0679 + 0.3994i	-2.0679 - 0.3994i	-1.0360	-0.7031
1.000	-1.7826 + 0.5217i	-1.7826 - 0.5217i	-0.4348	0

Table 2: Roots of the quartic polynomial (62) when $a = 2$, $b = 0.5$, and $c = 3$ and $l^2 + m^2 = 1$. Solutions with $x < 0$ correspond to surface waves with speed $\lambda = (-\eta x)^{1/2}$.

l	x_3	$(\alpha/\eta)^{1/2}\gamma_\alpha$	$(\beta/\eta)^{1/2}\gamma_\beta$
0.000	-3.0000	0.0000	0.0000
0.125	-2.9599	0.0245	0.0010
0.250	-2.8397	0.0978	0.0041
0.375	-2.6393	0.2201	0.0092
0.500	-2.3587	0.3913	0.0163
0.625	-1.9980	0.6114	0.0255
0.750	-1.5571	0.8804	0.0367
0.875	-1.0360	1.1984	0.0499
1.000	-0.4348	1.5652	0.0652

Table 3: Decay constants for the negative root x_3 in Table 2, showing that it corresponds to a genuine surface wave.