Nonlinear hyperbolic wave propagation in a one-dimensional random medium

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Abstract

We use an asymptotic expansion introduced by Benilov and Pelinovskiĭ to study the propagation of a weakly nonlinear hyperbolic wave pulse through a stationary random medium in one space dimension. We also study the scattering of such a wave by a background scattering wave. The leading-order solution is non-random with respect to a realization-dependent reference frame, as in the linear theory of O'Doherty and Anstey. The wave profile satisfies an inviscid Burgers equation with a nonlocal, lower-order dissipative and dispersive term that describes the effects of double scattering of waves on the pulse. We apply the asymptotic expansion to gas dynamics, nonlinear elasticity, and magnetohydrodynamics.

Key words: nonlinear waves, random media, O'Doherty-Anstey theory.

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1 Introduction

Nonlinear wave propagation in random media is a difficult and poorly understood subject. As in the case of deterministic waves (1), nondispersive, hyperbolic waves in random media have very different properties from dispersive waves, such as those modeled by a nonlinear Schrödinger equation with a random potential. In particular, waves modeled by nonlinear hyperbolic systems of conservation laws typically form shocks, and one would like to understand more about the propagation of shock waves through random media, such as turbulent flows. For surveys on linear wave scattering, see (2; 3); for a general discussion of nonlinear, nondispersive wave propagation in random media, see (4).

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The goal of this paper is to study a regime in which one can derive deterministic asymptotic equations for the propagation of a weakly nonlinear hyperbolic wave in a random media. These equations describe the propagation of a pulse through a stationary random medium, or wave, with a finite correlation lengthscale. In this respect, they differ from the equations derived by Majda and Rosales (5) for the resonant interaction of periodic, or almost periodic, weakly nonlinear hyperbolic waves.

The resulting theory may be regarded as a nonlinear generalization of the well-known O'Doherty-Anstey theory (6; 7; 8; 9; 10; 11) for linear waves. A key point is that there is a self-averaging phenomenon in the O'Doherty-Anstey theory, meaning that the leading-order solution for the wave is non-random in a suitable random reference frame. Therefore, provided one works in a properly chosen frame, closure problems do not arise and it is easy to account for the effects of weak nonlinearity.

The asymptotic expansion we use here was introduced by Benilov and Pelinovskiĭ (12) for a nonlinear wave equation, and generalized by Gurevich, Jeffrey, and Pelinovskiĭ (13) for first-order hyperbolic systems. We modify their expansion to account for spatially-dependent, instead of temporally-dependent, fluctuations in the random medium, and generalize it to the case of the scattering of one wave by another wave. We also apply the expansion to the propagation of waves in elasticity, gas dynamics, and magnetohydrodynamics.

A limitation of the resulting nonlinear O'Doherty-Anstey theory is that it only appears to apply in one space dimension, or at most in quasi-one-dimensional situations where waves propagate along rays and diffraction effects are negligable (see (11) for a linear theory). For example, the experimental observations of Hesselink and Sturtevant (14) on the propagation of weak shock waves through random media, where multiple ray-focusing occurs, lie outside the scope of this theory. One can derive equations of transonic small disturbance type with random coefficients to model these experiments (15; 16), but it is unclear how to derive a deterministic effective equation or make any further simplifications.

In this paper, we study the scattering of a pulse by a stationary random medium, and the scattering of a pulse by a stationary random background wave. In both cases, we suppose that the wave motion is governed by a hyperbolic system of conservation laws in one space dimension. Two physical examples that motivate the general theory are the scattering of an elastic wave by a random layered medium, and the scattering of a sound wave by a random entropy wave.

With respect to suitably nondimensionalized variables, we assume that the fluctuations in the medium or the background wave are small-amplitude and rapid, with amplitude of the order ε and wavelength of the order ε^2, where ε is a small parameter. We further assume that the fluctuations have zero spatial mean and are realizations of a second-order stationary random process with a sufficiently rapidly decaying correlation function. For simplicity, we suppose that the fluctuations are uniformly bounded in space (so we can Taylor expand the nonlinear terms), but it is reasonable to expect that, under suitable assumptions, the results remain valid even in the presence of large deviations, because the deviations would occur with very small probability over the space and time scales on which the asymptotic expansion is valid.

The pulse, which we call the primary wave, is assumed to have an amplitude and a width that are each of the order ε^2. With this scaling, the width of the pulse is of the same order as the length-scale of the fluctuations, and — in the case of a genuinely nonlinear primary wave — the effects of nonlinearity and scattering on the pulse are of the same order of magnitude, and both significantly affect the pulse on an order one time-scale.

The asymptotic solution for the pulse profile includes a random shift in the location of the pulse that typically converges in distribution to a Brownian motion as $\varepsilon \to 0$. The leading-order solution is non-random — as in the linear theory of O'Doherty and Anstey, to which the linearization of the equations we derive here reduces — in a realization-dependent reference frame moving with this Brownian motion. The self-averaging of the leading-order solution means that closure problems do not arise in the nonlinear theory.

The asymptotic equation for the pulse profile is an inviscid Burgers equation with a nonlocal, lower-order dissipative and dispersive term that describes the effects of double scattering on the pulse. In the simplest case, the equation for the pulse profile $u(x,t)$ has the normalized form

$$u_t + \left(\frac{1}{2}u^2\right)_x = \int_{-\infty}^{0} k(y) u_{xx}(x-y,t)\, dy, \tag{1}$$

where k is the covariance function of the random fluctuations. See (17) for a general discussion of such nonlocal equations.

The dissipative term on right-hand side of (1) may prevent the formation of shocks in the wave profiles. Indeed, for covariance functions k such that $k'(0^-) > 0$ (typical of rough media), there is a threshold value in the wave slope below which shocks do not form. We plan to investigate this phenomenon further in a separate paper (18).

The present paper is organized as follows.

In Section 2, we derive asymptotic equations for the profile of a linear pulse that is scattered by a random medium. We show that our results agree with those of the O'Doherty-Anstey theory derived in (11). In Section 3, we derive equations for the scattering of a nonlinear wave by a random medium. We treat quadratic nonlinearity in 3.1 and cubic nonlinearity in 3.2.

In Section 4, we generalize the solution derived in Section 3 for the scattering of a wave by a random medium to describe the scattering of one wave by another wave. We derive the equations starting from a strictly hyperbolic system of conservation laws in Subsection 4.1, and from a first-order hyperbolic system written in nonconservative form in Subsection 4.2.

In Section 5, we apply the results of Section 4 to one-dimensional gas dynamics to obtain an equation for the scattering of a nonlinear sound wave by stationary random entropy fluctuations. In Section 6, we apply the results of Section 3 to the scattering of longitudinal and transverse elastic waves by a random medium (see 6.1 and 6.2, respectively), and in 6.3 we apply the results of Section 4 to the scattering of longitudinal waves by random transverse waves. Finally, in Section 7, we apply our results to one-dimensional magnetohydrodynamics, where we consider magnetoacoustic-entropy wave interactions in 7.1 and magnetoacoustic-Alfvén wave interactions in 7.2.

2 Scattering of a linear wave by a nonuniform medium

We consider a strictly hyperbolic, first-order system of linear PDEs of the form

$$\mathbf{u}_t + \left[A\mathbf{u} + \varepsilon B(x/\varepsilon^2)\mathbf{u}\right]_x = \mathbf{0}. \tag{2}$$

Here, $A \in \mathbf{R}^{m\times m}$ is a constant $m \times m$ matrix, $\mathbf{u} = \mathbf{u}(x,t) \in \mathbf{R}^m$ is defined for $-\infty < x < \infty$ and $0 \le t$, ε is a small parameter, and the matrix $B(y) \in \mathbf{R}^{m\times m}$ is a realization of a zero-mean, second-order stationary random process in y that describes the small-amplitude, rapid fluctuations of the medium. We assume that the random process is continuous in quadratic mean and has a sufficiently rapidly decaying correlation function. For simplicity, we suppose that the realizations are uniformly bounded.

We assume that A has distinct, nonzero eigenvalues λ_i with corresponding left- and right-eigenvectors $\mathbf{l}_i$ and $\mathbf{r}_i$, respectively, normalized so that $\mathbf{l}_i^T\mathbf{r}_k = \delta_{ik}$ for $i, k = 1, 2, \ldots, m$.

We seek an asymptotic solution, $\mathbf{u}^\varepsilon$, of (2) of the form

$$\mathbf{u}^\varepsilon = \mathbf{u}_2(\xi_j, t) + \varepsilon\mathbf{u}_3(\xi_j, y, t) + \varepsilon^2\mathbf{u}_4(\xi_j, y, t) + O(\varepsilon^3), \quad \varepsilon \to 0, \tag{3}$$

where

$$y = \frac{x}{\varepsilon^2}, \qquad \xi_j = \frac{x - \lambda_j t - \varepsilon^3\phi(x/\varepsilon^2)}{\varepsilon^2}. \tag{4}$$

Here, we choose the subscripts in the perturbation terms in a way that is convenient for comparison with the nonlinear theory. It would be straightforward to include a dependence on the "slow" space variable x, as is required to study signaling problems, but since it is easy to modify the final equations to account for this, we omit the x-dependence to simplify the presentation.

The phase shift ϕ in (4) is a random process, which we will choose later. In this expansion, $\mathbf{u}_2$ describes a wave, which we call the primary wave, that propagates through the nonuniform medium, and $\mathbf{u}_3$ describes the secondary waves that are generated by the scattering of the primary wave by the fluctuations in the medium. We assume that $\mathbf{u}_2(\xi_j, t)$ decays to zero as $\xi_j \to \pm\infty$. Thus, this solution describes a wave pulse of width of the order ε^2 that propagates through a medium with fluctuations of amplitude of the order ε and length-scale of the order ε^2.

We use (3)–(4) in (2) and set the coefficients of ε^n equal to zero for $n = 0, 1, 2$ to obtain the hierarchy of equations

$$(A - \lambda_j I)\mathbf{u}_{2\xi_j} = \mathbf{0}, \tag{5}$$

$$(A - \lambda_j I)\mathbf{u}_{3\xi_j} + A\mathbf{u}_{3y} + (B - \phi_y A)\mathbf{u}_{2\xi_j} + B_y \mathbf{u}_2 = \mathbf{0}, \tag{6}$$

$$\begin{aligned}(A - \lambda_j I)\mathbf{u}_{4\xi_j} &+ A\mathbf{u}_{4y} \\ &+ (B - \phi_y A)\mathbf{u}_{3\xi_j} + (B\mathbf{u}_3)_y - \phi_y B\mathbf{u}_{2\xi_j} + \mathbf{u}_{2t} = \mathbf{0}.\end{aligned} \tag{7}$$

A solution of (5) is

$$\mathbf{u}_2 = a_j(\xi_j, t)\mathbf{r}_j, \tag{8}$$

where $a_j(\xi_j, t)$ is a scalar-valued function that decays to zero sufficiently rapidly as $\xi_j \to \pm\infty$.

We use (8) in (6) to get the equation

$$(A - \lambda_j I)\mathbf{u}_{3\xi_j} + A\mathbf{u}_{3y} + a_{j\xi_j}(B - \phi_y A)\mathbf{r}_j + a_j B_y \mathbf{r}_j = \mathbf{0}. \tag{9}$$

To solve (9) for $\mathbf{u}_3$, we expand $\mathbf{u}_3$ in a basis of right-eigenvectors of A,

$$\mathbf{u}_3 = \sum_{k=1}^{m} b_k(\xi_j, y, t)\mathbf{r}_k, \tag{10}$$

where each b_k is assumed to be uniformly bounded in y. We then multiply through (9) on the left by $\mathbf{l}_j^T$, and solve the resulting equation for b_j by the method of characteristics to get

$$\begin{aligned}b_j(\xi_j, y, t) = b_j^{(0)}(\xi_j, t) &- \frac{1}{\lambda_j}\beta_{jj}(y)a_j(\xi_j, t) + \frac{1}{\lambda_j}\beta_{jj}(\xi_j)a_j(\xi_j, t) \\ &- \frac{1}{\lambda_j}\left(\int_{\xi_j}^{y}\Big[\beta_{jj}(s) - \lambda_j\phi_s(s)\Big]\,ds\right)a_{j\xi_j}(\xi_j, t),\end{aligned} \tag{11}$$

where

$$\beta_{ik}(y) = \mathbf{l}_i^T B(y) \mathbf{r}_k. \tag{12}$$

The scattering coefficient β_{ik} describes the strength of an i-wave generated by the scattering of a k-wave by the fluctuations in the medium. The initial data, $b_j^{(0)}(\xi_j, t)$, is assumed to be integrable.

For a stationary random process, β_{jj} with a finite correlation length, the integral term in (11) is typically of the order $y^{1/2}$ as $y \to \infty$, leading to a secular term in $\mathbf{u}_3$. We eliminate this term by choosing ϕ to satisfy

$$\phi_y(y) = \frac{1}{\lambda_j} \beta_{jj}(y). \tag{13}$$

It follows that the random correction to the location of the pulse is

$$\varepsilon\phi\left(\frac{x}{\varepsilon^2}\right) = \frac{1}{\lambda_j}\varepsilon \int_0^{x/\varepsilon^2} \beta_{jj}(y)\, dy, \tag{14}$$

where we assume that $\phi(0) = 0$ for definiteness.

If β_{jj} is a strictly stationary process that satisfies a suitable mixing condition, then $\varepsilon\phi(x/\varepsilon^2)$ converges in distribution to $\eta Z(x)$ as $\varepsilon \to 0$, where $Z(x)$ is a standard Brownian motion, and

$$\eta^2 = 2\int_0^\infty W_{jj}(y)\, dy,$$

with W_{jj} given below in (21) (see, for example, Theorem 4.32 in (19)).

Next, we multiply through (9) on the left by $\mathbf{l}_i^T$, $i \neq j$, and solve the resulting equation for b_i along characteristics to get

$$\begin{aligned} b_i(\xi_j, y, t) = {} & b_i^{(0)}\Big(\frac{\lambda_j - \lambda_i}{\lambda_j} y + \frac{\lambda_i}{\lambda_j}\xi_j, t\Big) - \frac{1}{\lambda_i}\beta_{ij}(y) a_j(\xi_j, t) \\ & + \frac{1}{\lambda_i}\beta_{ij}\Big(\frac{\lambda_j - \lambda_i}{\lambda_j} y + \frac{\lambda_i}{\lambda_j}\xi_j\Big) a_j\Big(\frac{\lambda_j - \lambda_i}{\lambda_j} y + \frac{\lambda_i}{\lambda_j}\xi_j, t\Big) \\ & - \frac{\lambda_j}{\lambda_i^2}\int_{\frac{\lambda_j - \lambda_i}{\lambda_j} y + \frac{\lambda_i}{\lambda_j}\xi_j}^{y} \beta_{ij}(s) a_{j\xi_j}\left(\frac{\lambda_j - \lambda_i}{\lambda_i}(y - s) + \xi_j, t\right) ds. \end{aligned} \tag{15}$$

Turning to (7), we derive a solvability condition for the solution $\mathbf{u}_4$ to be a bounded function of y. We substitute (8) and (10) into (7), and average the resulting equation with respect to y. We assume that $\mathbf{u}_3$ and $\mathbf{u}_4$ are uniformly bounded in y, and that averages over y are well defined. We then multiply through the averaged equation on the left by $\mathbf{l}_j^T$ and rearrange terms to get the equation

$$a_{jt} = \Big\langle \phi_y \beta_{jj} \Big\rangle a_{j\xi_j} - \sum_{k \neq j} \Big\langle \beta_{jk} b_{k\xi_j} \Big\rangle. \tag{16}$$

Here, the angular brackets denote the mean with respect to y, defined by

$$\big\langle f \big\rangle(\xi, t) = \lim_{L\to\infty} \frac{1}{L} \int_0^L f(\xi, y, t)\, dy. \tag{17}$$

For the second term on the right-hand side of (16), we substitute into it (15) to obtain, after replacing i by k,

$$\begin{aligned}\big\langle \beta_{jk} b_{k\xi_j} \big\rangle &= \frac{-1}{\lambda_k} \big\langle \beta_{jk}(y)\beta_{kj}(y) \big\rangle a_{j\xi_j}(\xi_j, t) \\ &\quad - \frac{\lambda_j}{\lambda_k^2} \Big\langle \int_0^{\frac{\lambda_k}{\lambda_j}(y-\xi_j)} \beta_{jk}(y)\beta_{kj}(y-\sigma) a_{j\xi_j\xi_j}\Big(\frac{\lambda_j - \lambda_k}{\lambda_k}\sigma + \xi_j, t\Big)\, d\sigma \Big\rangle.\end{aligned} \tag{18}$$

The average of the integral on the right-hand side of (18) is dominated by the limiting value of the integral for large y. Under appropriate hypotheses on the integrand, we may therefore replace the range of integration by $(0, \infty)$ if $\lambda_k/\lambda_j > 0$, or $(0, -\infty)$ if $\lambda_k/\lambda_j < 0$, and then bring the average with respect to y inside the integral. (See Lemma 1 in Appendix A for a proof.) This yields

$$\begin{aligned}\big\langle \beta_{jk} b_{k\xi_j} \big\rangle &= \frac{-1}{\lambda_k} \big\langle \beta_{jk}(y)\beta_{kj}(y) \big\rangle a_{j\xi_j}(\xi_j, t) \\ &\quad - \frac{\lambda_j}{\lambda_k^2} \int_0^{\operatorname{sgn}(\lambda_k/\lambda_j)\cdot\infty} \big\langle \beta_{jk}(y)\beta_{kj}(y-\sigma) \big\rangle a_{j\xi_j\xi_j}\Big(\frac{\lambda_j - \lambda_k}{\lambda_k}\sigma + \xi_j, t\Big)\, d\sigma,\end{aligned} \tag{19}$$

where

$$\operatorname{sgn}(\lambda_k/\lambda_j) = \begin{cases} 1 & \text{if } \lambda_k/\lambda_j > 0, \\ -1 & \text{if } \lambda_k/\lambda_j < 0. \end{cases}$$

Finally, using (13) and (19) in (16), we obtain the following equation for the wave profile, a_j, of the primary wave pulse:

$$\begin{aligned}a_{jt}(\xi_j, t) &- \sum_{k=1}^{m} \frac{1}{\lambda_k} W_{jk}(0) a_{j\xi_j}(\xi_j, t) \\ &= \sum_{k\neq j} \frac{\lambda_j}{\lambda_k^2} \int_0^{\operatorname{sgn}(\lambda_k/\lambda_j)\cdot\infty} W_{jk}(\sigma) a_{j\xi_j\xi_j}\Big(\frac{\lambda_j - \lambda_k}{\lambda_k}\sigma + \xi_j, t\Big)\, d\sigma,\end{aligned} \tag{20}$$

where

$$W_{jk}(\sigma) = \big\langle \beta_{jk}(y)\beta_{kj}(y-\sigma) \big\rangle \tag{21}$$

is a covariance of the scattering coefficients β_{jk} that are given by (12), and the angular brackets denote the mean with respect to y defined in (17). We assume that the spatial average in (21) exists. For example, if B in (2) is a strictly stationary random process with finite second-order moments, then the spatial average exists almost surely by the Birkhoff-Khintchine ergodic theorem (20).

According to (20), the wave profile $a_j(\xi_j, t)$ satisfies a linear, nonlocal evolution equation. The nonlocal term is a sum of convolutions of the second derivative of the profile with covariance functions of the random fluctuations in the medium.

To compare our result with the O'Doherty-Anstey theory, we consider the simplest case, such as the wave equation, in which there are two wave families. The normalized form of (20) for the wave profile $a(x, t)$ is

$$a_t = K * a_{xx}, \tag{22}$$

where $*$ denotes convolution, and

$$K(x) = \begin{cases} W(x) & \text{if } x \leq 0, \\ 0 & \text{if } x > 0. \end{cases}$$

The Fourier transform of (22) is

$$\hat{a}_t = -\xi^2 \hat{K}(\xi)\hat{a},$$

where $\hat{f}(\xi) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-i\xi x} f(x)\, dx$. Hence,

$$a(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{a}(0) e^{-\xi^2 \hat{K}(\xi) t} e^{i\xi x}\, d\xi,$$

which is the same as the O'Doherty-Anstey solution discussed in Sølna and Papanicolaou (11). The location, or travel time, of the pulse is shifted by a zero-mean random correction. The pulse shape is deterministic with respect to this random reference frame, and is modified from its initial shape by convolution with a kernel that describes the spreading of the pulse caused by the effects of doubly-scattered waves. It follows from Bochner's theorem that $\hat{K}(\xi)$ has nonnegative real part, which shows that this scattering has a dissipative effect on the pulse.

One difference between our derivation of O'Doherty-Anstey theory and the derivation in (11) is that we work in the space-time domain, whereas (11) works in the frequency domain. For linear problems, the two approaches are essentially equivalent, but for nonlinear problems, it is much easier to work exclusively in the space-time domain.

3 Scattering of a nonlinear wave by a nonuniform medium

In this section, we generalize the solution obtained in the previous section to describe the scattering of a nonlinear wave by a one-dimensional stationary random medium. A physical example is the scattering of a longitudinal or transverse elastic wave by random fluctuations in an elastic medium.

We consider a strictly hyperbolic system of conservation laws of the form

$$\mathbf{u}_t + \Big[\mathbf{f}(\mathbf{u}) + \varepsilon B(x/\varepsilon^2)\mathbf{u}\Big]_x = \mathbf{0}, \tag{23}$$

where $\mathbf{u}(x,t) \in \mathbf{R}^m$, $\mathbf{f}\colon \mathbf{R}^m \to \mathbf{R}^m$, the matrix B is stationary random process as in (2), and ε is a small parameter. We suppose that $\mathbf{f}$ has the Taylor expansion

$$\mathbf{f}(\mathbf{u}) = A\mathbf{u} + C(\mathbf{u},\mathbf{u}) + D(\mathbf{u},\mathbf{u},\mathbf{u}) + E(\mathbf{u},\mathbf{u},\mathbf{u},\mathbf{u}) + O(|\mathbf{u}|^5), \quad |\mathbf{u}| \to 0, \tag{24}$$

where $A = \nabla\mathbf{f}(\mathbf{0})$, $C = \frac{1}{2}\nabla^2\mathbf{f}(\mathbf{0})$, $D = \frac{1}{6}\nabla^3\mathbf{f}(\mathbf{0})$, and $E = \frac{1}{24}\nabla^4\mathbf{f}(\mathbf{0})$. We assume that A has distinct, nonzero eigenvalues λ_i with corresponding left- and right-eigenvectors $\mathbf{l}_i$ and $\mathbf{r}_i$, respectively, normalized so that $\mathbf{l}_i^T\mathbf{r}_k = \delta_{ik}$ for $i,k = 1,2,\ldots,m$.

3.1 Quadratic nonlinearity

We seek an asymptotic solution, $\mathbf{u}^\varepsilon$, of (23) of the form

$$\mathbf{u}^\varepsilon = \varepsilon^2\mathbf{u}_2(\xi_j,t) + \varepsilon^3\mathbf{u}_3(\xi_j,y,t) + \varepsilon^4\mathbf{u}_4(\xi_j,y,t) + O(\varepsilon^5), \quad \varepsilon \to 0, \tag{25}$$

where y and ξ_j are given as in (4):

$$y = \frac{x}{\varepsilon^2}, \qquad \xi_j = \frac{x - \lambda_j t - \varepsilon^3\phi(x/\varepsilon^2)}{\varepsilon^2}. \tag{26}$$

As before, the phase shift ϕ is a random process, which we will choose later, $\mathbf{u}_2$ describes the primary wave, and $\mathbf{u}_3$ describes the secondary waves that are generated by the scattering of the primary wave by the fluctuations in the medium. We assume that $\mathbf{u}_2(\xi_j,t)$ decays to zero as $\xi_j \to \pm\infty$. Thus, this solution describes a wave pulse of width of the order ε^2 that propagates through a medium with fluctuations of amplitude of the order ε and length scale of the order ε^2, as in the linear theory of Section 2. The amplitude of the pulse is of the order ε^2. With the scaling chosen here, the effects of quadratic nonlinearity and scattering of the wave by the medium are of the same order of magnitude, and both significantly influence the wave over propagation times of the order one.

We use (24)–(26) in (23) and set the coefficients of ε^n equal to zero for $n = 2,3,4$ to obtain the hierarchy of equations

$$(A - \lambda_j I)\mathbf{u}_{2\xi_j} = \mathbf{0}, \tag{27}$$

$$(A - \lambda_j I)\mathbf{u}_{3\xi_j} + A\mathbf{u}_{3y} + (B - \phi_y A)\mathbf{u}_{2\xi_j} + B_y\mathbf{u}_2 = \mathbf{0}, \tag{28}$$

$$(A - \lambda_j I)\mathbf{u}_{4\xi_j} + A\mathbf{u}_{4y} + C(\mathbf{u}_2, \mathbf{u}_2)_{\xi_j} + (B - \phi_y A)\mathbf{u}_{3\xi_j} + (B\mathbf{u}_3)_y - \phi_y B\mathbf{u}_{2\xi_j} + \mathbf{u}_{2t} = \mathbf{0}. \tag{29}$$

These equations correspond to (5)–(7) in the linear perturbation equations, with the addition of a nonlinear term, $C(\mathbf{u}_2, \mathbf{u}_2)_{\xi_j}$, in (29).

Following the derivation in the linear theory, we obtain an equation for the profile, a_j, of the primary wave pulse:

$$a_{jt}(\xi_j, t) - \sum_{k=1}^{m} \frac{1}{\lambda_k} W_{jk}(0) a_{j\xi_j}(\xi_j, t) + \left(\frac{1}{2}\mathcal{G}_j a_j(\xi_j, t)^2\right)_{\xi_j} = \sum_{k \neq j} \frac{\lambda_j}{\lambda_k^2} \int_0^{\operatorname{sgn}(\lambda_k/\lambda_j)\cdot\infty} W_{jk}(\sigma) a_{j\xi_j\xi_j}\left(\frac{\lambda_j - \lambda_k}{\lambda_k}\sigma + \xi_j, t\right) d\sigma, \tag{30}$$

where β_{jk} and W_{jk} are given by (12) and (21), respectively, and

$$\mathcal{G}_j = 2\mathbf{l}_j^T C(\mathbf{r}_j, \mathbf{r}_j) \tag{31}$$

is a coefficient of quadratic nonlinearity.

We may identify $\mathcal{G}_j$ with Lax's coefficient of genuine nonlinearity (21; 22). To see this, we let $\lambda_j = \lambda(\mathbf{0})$, $\mathbf{l}_j = \mathbf{l}(\mathbf{0})$, and $\mathbf{r}_j = \mathbf{r}(\mathbf{0})$, and differentiate the equation

$$\left[\nabla\mathbf{f}(\mathbf{u}) - \lambda(\mathbf{u})I\right]\mathbf{r}(\mathbf{u}) = \mathbf{0}$$

with respect to $\mathbf{u}$ in the direction $\mathbf{r}(\mathbf{u})$. Multiplying on the left by $\mathbf{l}^T$, evaluating the resulting equation at $\mathbf{u} = \mathbf{0}$, and using (31), we get

$$\mathcal{G}_j = \nabla\lambda(\mathbf{u}) \cdot \mathbf{r}(\mathbf{u})\Big|_{\mathbf{u}=\mathbf{0}}. \tag{32}$$

Thus, $\mathcal{G}_j$ is the derivative of the j-wave velocity at $\mathbf{u} = \mathbf{0}$ in the direction of the corresponding right eigenvector.

In summary, we see that the wave profile, a_j, of the primary wave pulse satisfies an inviscid Burgers equation with a lower-order, nonlocal term on the right-hand side (30). The nonlocal term is dissipative and dispersive, and describes the effects of the double scattering of the primary wave by the random medium. The kernel $W_{jk}(\sigma)$ is a covariance function of the fluctuations in the medium. The coefficient, $\mathcal{G}_j$, of the quadratically nonlinear term describes the strength of the quadratically nonlinear self-interaction of the j-wave, which in general may lead to the formation of shocks. The coefficient $-\sum W_{jk}(0)/\lambda_k$ is a perturbation of the mean wave speed caused by the random fluctuations in the medium.

If the system of conservation laws is strictly hyperbolic and multiple primary pulses with different linearized velocities are present, then, to leading order in ε, the solutions superpose, because the pulses separate. Thus, one obtains a decoupled system of equations of the form (30).

In carrying out this expansion, we tacitly assume that the solutions remain smooth. Nevertheless, it is reasonable to expect that the asymptotic equation remains valid in the weak sense if shocks form (see (18) for numerical comparisons between solutions of the asymptotic equations and the full equations when shocks are present).

3.2 Cubic nonlinearity

In this section, we suppose that the primary wave is not genuinely nonlinear at $\mathbf{u} = \mathbf{0}$, meaning that $\mathcal{G}_j = 0$ in (32). A physical example is that of transverse elastic waves in an isotropic medium, which fail to be genuinely nonlinear at the undeformed state of the medium. The dominant nonlinear effect is then typically cubic instead of quadratic, and it is only significant if the primary wave amplitude is larger than in the case of genuinely nonlinear waves.

We therefore look for an asymptotic solution $\mathbf{u}^\varepsilon$ of (23) of the form

$$\mathbf{u}^\varepsilon = \varepsilon \mathbf{u}_1(\xi_j, t) + \varepsilon^2 \mathbf{u}_2(\xi_j, y, t) + \varepsilon^3 \mathbf{u}_3(\xi_j, y, t) + O(\varepsilon^4), \quad \varepsilon \to 0, \tag{33}$$

where ξ_j, y, and t are given by (23). In this expansion, $\mathbf{u}_1$ is the primary wave, and $\mathbf{u}_2$ is the scattered wave. We use (24), (26), and (33) in (23), and set the coefficients of ε^n equal to zero for $n = 1, 2, 3$ to obtain the hierarchy of equations

$$(A - \lambda_j I)\mathbf{u}_{1\xi_j} = \mathbf{0}, \tag{34}$$

$$(A - \lambda_j I)\mathbf{u}_{2\xi_j} + A\mathbf{u}_{2y} + C(\mathbf{u}_1, \mathbf{u}_1)_{\xi_j} + (B - \phi_y A)\mathbf{u}_{1\xi_j} + B_y \mathbf{u}_1 = \mathbf{0}, \tag{35}$$

$$\begin{aligned}(A - \lambda_j I)\mathbf{u}_{3\xi_j} + A\mathbf{u}_{3y} + 2C(\mathbf{u}_1, \mathbf{u}_2)_{\xi_j} + 2C(\mathbf{u}_1, \mathbf{u}_2)_y \\ + D(\mathbf{u}_1, \mathbf{u}_1, \mathbf{u}_1)_{\xi_j} + (B - \phi_y A)\mathbf{u}_{2\xi_j} + (B\mathbf{u}_2)_y \\ - \phi_y C(\mathbf{u}_1, \mathbf{u}_1)_{\xi_j} - \phi_y B\mathbf{u}_{1\xi_j} + \mathbf{u}_{1t} = \mathbf{0}.\end{aligned} \tag{36}$$

A solution of (34) is

$$\mathbf{u}_1 = a_j(\xi_j, t)\mathbf{r}_j, \tag{37}$$

where $a_j(\xi_j, t)$ is a scalar-valued function, which we assume decays sufficiently rapidly as $\xi_j \to \pm\infty$.

To solve (35) for $\mathbf{u}_2$, we look for a solution of the form

$$\mathbf{u}_2 = \mathbf{v}(\xi_j, y, t) + w(\xi_j, t)\mathbf{z}_j, \tag{38}$$

where $\mathbf{v} = \mathbf{v}(\xi_j, y, t)$, and $\mathbf{z}_j \in \mathbf{R}^m$. This assumption, together with (37), leads us to the following solvability conditions for (35):

$$w_{\xi_j}(A - \lambda_j I)\mathbf{z}_j + (a_j^2)_{\xi_j} C(\mathbf{r}_j, \mathbf{r}_j) = \mathbf{0}, \tag{39}$$

$$(A - \lambda I)\mathbf{v}_{\xi_j} + A\mathbf{v}_y + a_{j\xi_j}(B - \phi_y A)\mathbf{r}_j + a_j B_y \mathbf{r}_j = \mathbf{0}. \tag{40}$$

We choose $w(\xi_j, t) = a_j(\xi_j, t)^2$ in (39) so that $\mathbf{z}_j$ satisfies

$$(A - \lambda_j I)\mathbf{z}_j + C(\mathbf{r}_j, \mathbf{r}_j) = \mathbf{0}. \tag{41}$$

Note that we may solve (41) for $\mathbf{z}_j$ only if

$$\mathbf{l}_j^T C(\mathbf{r}_j, \mathbf{r}_j) = 0, \tag{42}$$

that is to say, precisely when $\mathcal{G}_j$ in (31) vanishes.

Solving the remaining equations in a similar way to the linear theory, and choosing ϕ as in (13), we find that the wave profile, a_j, of a primary wave pulse that is not genuinely nonlinear at $\mathbf{u} = \mathbf{0}$ satisfies the cubically nonlinear equation

$$\begin{aligned} a_{jt}(\xi_j, t) - \sum_{k=1}^{m} \frac{1}{\lambda_k} W_{jk}(0) a_{j\xi_j}(\xi_j, t) + \left(\frac{1}{3}\mathcal{H}_j a_j(\xi_j, t)^3\right)_{\xi_j} \\ = \sum_{k \neq j} \frac{\lambda_j}{\lambda_k^2} \int_0^{\operatorname{sgn}(\lambda_k/\lambda_j)\cdot\infty} W_{jk}(\sigma) a_{j\xi_j\xi_j}\left(\frac{\lambda_j - \lambda_k}{\lambda_k}\sigma + \xi_j, t\right) d\sigma, \end{aligned} \tag{43}$$

where

$$\mathcal{H}_j = 3\mathbf{l}_j^T \Big[2C(\mathbf{r}_j, \mathbf{z}_j) + D(\mathbf{r}_j, \mathbf{r}_j, \mathbf{r}_j)\Big], \tag{44}$$

and β_{jk} and W_{jk} are given by (12) and (21), respectively.

These equations are similar in form to the previous ones, except that the pulse profile satisfies a cubically nonlinear (or modified) inviscid Burgers equation with a nonlocal lower order dissipative term that describes the effects of wave scattering.

4 Wave-wave scattering

In this section, we generalize the solution derived in Section 3 for the scattering of a nonlinear wave by a nonuniform medium to describe the scattering of one wave (which we call the primary wave) by another wave (which we call the scattering wave). Physical examples include the scattering of a sound wave by an entropy wave in gas dynamics, the scattering of a longitudinal wave by a transverse wave in elasticity, and the scattering of a magnetoacoustic wave by an entropy or Alfvén wave in magnetohydrodynamics.

We find that the scattering of the primary wave by the scattering wave is significant only when the scattering wave fails to be genuinely nonlinear. The reason for this is that if the scattering wave is genuinely nonlinear, then it rapidly forms shocks and decays on an $O(\varepsilon)$ timescale before there is a significant accumulation of scattered waves. Thereafter, the scattering wave is negligable, and the primary wave propagates like a single wave. We therefore assume that the scattering wave is not genuinely nonlinear — which includes the case of linear degeneracy — as in all of the physical examples mentioned above.

4.1 The asymptotic solution

We consider a strictly hyperbolic system of conservation laws

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0, \tag{45}$$

where $\mathbf{u} = \mathbf{u}(x,t) \in \mathbf{R}^m$. We assume that the flux $\mathbf{f}\colon \mathbf{R}^m \to \mathbf{R}^m$ has the Taylor expansion in (24), where A has distinct eigenvalues λ_i with corresponding left- and right-eigenvectors $\mathbf{l}_i$ and $\mathbf{r}_i$, respectively, normalized so that $\mathbf{l}_i^T\mathbf{r}_k = \delta_{ik}$ for $i,k = 1,2,\ldots,m$.

We seek an asymptotic solution, $\mathbf{u}^\varepsilon$, of (45) of the form

$$\begin{aligned}\mathbf{u}^\varepsilon = {} & \varepsilon\mathbf{u}_1(\xi_p,t) + \varepsilon^2\mathbf{u}_2(\xi_j,\xi_p,t) + \varepsilon^3\mathbf{u}_3(\xi_j,\xi_p,t) + \varepsilon^4\mathbf{u}_4(\xi_j,\xi_p,t) \\ & + O(\varepsilon^5), \quad \varepsilon \to 0,\end{aligned} \tag{46}$$

where $j \neq p$,

$$\xi_j = \frac{x - \lambda_j t - \varepsilon^3\phi(\xi_p,t)}{\varepsilon^2}, \quad \xi_p = \frac{x - \lambda_p t}{\varepsilon^2}, \tag{47}$$

and ϕ is to be determined. In this expansion, $\mathbf{u}_1$ describes the background scattering wave, with linearized phase velocity λ_p, $\mathbf{u}_2$ describes the primary wave, with phase velocity λ_j, and $\mathbf{u}_3$ describes the scattered waves generated by the reflection of the j-wave off the p-wave.

We use (24) and (46)–(47) in (45) and set the coefficients of ε^n equal to zero for $n = 1,2,3,4$ to obtain the hierarchy of equations

$$(A - \lambda_p I)\mathbf{u}_{1\xi_p} = \mathbf{0}, \tag{48}$$

$$(A - \lambda_j I)\mathbf{u}_{2\xi_j} + (A - \lambda_p I)\mathbf{u}_{2\xi_p} + C(\mathbf{u}_1,\mathbf{u}_1)_{\xi_p} = \mathbf{0}, \tag{49}$$

$$(A-\lambda_j I)\mathbf{u}_{3\xi_j} + (A-\lambda_p I)\mathbf{u}_{3\xi_p} + 2C(\mathbf{u}_1,\mathbf{u}_2)_{\xi_j} + 2C(\mathbf{u}_1,\mathbf{u}_2)_{\xi_p} + D(\mathbf{u}_1,\mathbf{u}_1,\mathbf{u}_1)_{\xi_p} - \phi_{\xi_p}(A-\lambda_p I)\mathbf{u}_{2\xi_j} + \mathbf{u}_{1t} = \mathbf{0}, \quad (50)$$

$$\begin{aligned}(A-\lambda_j I)\mathbf{u}_{4\xi_j} &+ (A-\lambda_p I)\mathbf{u}_{4\xi_p} + 2C(\mathbf{u}_1,\mathbf{u}_3)_{\xi_j} \\ &+ C(\mathbf{u}_2,\mathbf{u}_2)_{\xi_j} + 2C(\mathbf{u}_1,\mathbf{u}_3)_{\xi_p} + C(\mathbf{u}_2,\mathbf{u}_2)_{\xi_p} \\ &+ 3D(\mathbf{u}_1,\mathbf{u}_1,\mathbf{u}_2)_{\xi_j} + 3D(\mathbf{u}_1,\mathbf{u}_1,\mathbf{u}_2)_{\xi_p} + E(\mathbf{u}_1,\mathbf{u}_1,\mathbf{u}_1,\mathbf{u}_1)_{\xi_p} \\ &- \phi_{\xi_p}(A-\lambda_p I)\mathbf{u}_{3\xi_j} - 2\phi_{\xi_p}C(\mathbf{u}_1,\mathbf{u}_2)_{\xi_j} + \mathbf{u}_{2t} = \mathbf{0}.\end{aligned} \quad (51)$$

A solution of (48) is

$$\mathbf{u}_1 = s_p(\xi_p, t)\mathbf{r}_p, \quad (52)$$

where s_p is a scalar-valued function. We assume that, for each t, the function $s_p(\xi_p, t)$ is a realization of a zero-mean, second-order stationary random process in ξ_p that is continuous in quadratic mean and has a sufficiently rapidly decaying correlation function. For simplicity, we also assume that $s_p(\xi_p, t)$ is uniformly bounded.

Using (52) in (49), we find that a solution of (49) is

$$\mathbf{u}_2 = a_j(\xi_j, t)\mathbf{r}_j + s_p(\xi_p, t)^2\mathbf{z}_p, \quad (53)$$

where a_j is a scalar-valued function, and $\mathbf{z}_p \in \mathbf{R}^m$ satisfies

$$(A-\lambda_p I)\mathbf{z}_p + C(\mathbf{r}_p, \mathbf{r}_p) = \mathbf{0}. \quad (54)$$

We assume that $a_j(\xi_j, t)$ decays sufficiently rapidly as $\xi_j \to \pm\infty$.

We may solve (54) for $\mathbf{z}_p$ only if

$$\mathbf{l}_p^T C(\mathbf{r}_p, \mathbf{r}_p) = 0. \quad (55)$$

This condition is satisfied if the scattering p-wave is not genuinely nonlinear at $\mathbf{u} = \mathbf{0}$.

To solve (50) for $\mathbf{u}_3$, we write $\mathbf{u}_3$ in the form

$$\mathbf{u}_3 = \mathbf{v}(\xi_j, \xi_p, t) + \mathbf{w}(\xi_p, t), \quad (56)$$

where $\mathbf{v} = \mathbf{v}(\xi_j, \xi_p, t), \mathbf{w} = \mathbf{w}(\xi_p, t) \in \mathbf{R}^m$. Then, using (52)–(53) and (56) in (50), and separating the terms that depend on ξ_p from those that depend on (ξ_j, ξ_p), we find that (50) is satisfied if

$$(A-\lambda_p I)\mathbf{w}_{\xi_p} + \left[2C(\mathbf{r}_p,\mathbf{z}_p) + D(\mathbf{r}_p,\mathbf{r}_p,\mathbf{r}_p)\right](s_p^3)_{\xi_p} + s_{pt}\mathbf{r}_p = \mathbf{0}, \quad (57)$$

$$(A-\lambda_j I)\mathbf{v}_{\xi_j} + (A-\lambda_p I)\mathbf{v}_{\xi_p} + 2C(\mathbf{r}_j,\mathbf{r}_p)[a_{j\xi_j}s_p + a_j s_{p\xi_p}] - (\lambda_j - \lambda_p)\phi_{\xi_p}a_{j\xi_j}\mathbf{r}_j = \mathbf{0}. \quad (58)$$

Multiplying through (57) on the left by $\mathbf{l}_p^T$, we find that the wave profile, s_p, of the scattering wave satisfies the following modified Burgers equation:

$$s_{pt}(\xi_p, t) + \left(\frac{1}{3}\mathcal{H}_p s_p(\xi_p, t)^3\right)_{\xi_p} = 0, \tag{59}$$

where

$$\mathcal{H}_p = 3\mathbf{l}_p^T\Big[2C(\mathbf{r}_p, \mathbf{z}_p) + D(\mathbf{r}_p, \mathbf{r}_p, \mathbf{r}_p)\Big]. \tag{60}$$

We may then solve (57) for $\mathbf{w}$; we omit the expression since we do not need it.

To solve for $\mathbf{v}$ in (58), we expand $\mathbf{v}$ in a basis of right-eigenvectors of A:

$$\mathbf{v} = \sum_{k=1}^{m} b_k(\xi_j, \xi_p, t)\mathbf{r}_k. \tag{61}$$

We then multiply through (58) on the left by $\mathbf{l}_j^T$, and solve the resulting equation for b_j by the method of characteristics to get

$$\begin{aligned} b_j(\xi_j, \xi_p, t) = {} & b_j^{(0)}(\xi_j, t) - \frac{1}{\lambda_j - \lambda_p}\Gamma^j_{jp} s_p(\xi_p, t) a_j(\xi_j, t) \\ & + \frac{1}{\lambda_j - \lambda_p}\Gamma^j_{jp} s_p(\xi_j, t) a_j(\xi_j, t) \\ & - \frac{1}{\lambda_j - \lambda_p}\left(\int_{\xi_j}^{\xi_p}\Big[\Gamma^j_{jp} s_p(s', t) - (\lambda_j - \lambda_p)\phi_s(s', t)\Big]\, ds'\right) a_{j\xi_j}(\xi_j, t), \end{aligned} \tag{62}$$

where

$$\Gamma^i_{kp} = 2\mathbf{l}_i^T C(\mathbf{r}_k, \mathbf{r}_p). \tag{63}$$

This interaction coefficient, Γ^i_{kp}, describes the strength of an i-wave produced by the quadratically nonlinear interaction of a k-wave with the p-wave. We assume that the initial data, $b_j^{(0)}(\xi_j, t)$, is integrable.

In a similar way to before, the integral term in (62) leads to secular terms unless we choose ϕ to satisfy

$$\phi_{\xi_p}(\xi_p, t) = \frac{1}{\lambda_j - \lambda_p}\Gamma^j_{jp} s_p(\xi_p, t).$$

Hence, if $\phi(0) = 0$, we have

$$\phi(\xi_p, t) = \frac{1}{\lambda_j - \lambda_p}\Gamma^j_{jp}\int_0^{\xi_p} s_p(y, t)\, dy. \tag{64}$$

Next, we multiply through (58) on the left by $\mathbf{l}_p^T$, and then solve the resulting equation for b_p along characteristics to get

$$
\begin{aligned}
b_p(\xi_j, \xi_p, t) = b_p^{(0)}(\xi_p, t) &+ \frac{1}{\lambda_j - \lambda_p}\Gamma^p_{jp} a_j(\xi_j, t) s_p(\xi_p, t) \\
&- \frac{1}{\lambda_j - \lambda_p}\Gamma^p_{jp} a_j(\xi_p, t) s_p(\xi_p, t) \\
&- \frac{1}{\lambda_j - \lambda_p}\Gamma^p_{jp}\Big(\int_{\xi_j}^{\xi_p} a_j(s', t)\, ds'\Big) s_{p\xi_p}(\xi_p, t).
\end{aligned} \tag{65}
$$

We multiply through (58) on the left by $\mathbf{l}_i^T$, $i \neq j, p$, and then solve for b_i along characteristics to get

$$
\begin{aligned}
&b_i(\xi_j, \xi_p, t) \\
&\quad = b_i^{(0)}\Big(\frac{\lambda_j - \lambda_i}{\lambda_j - \lambda_p}\xi_p + \frac{\lambda_i - \lambda_p}{\lambda_j - \lambda_p}\xi_j, t\Big) - \frac{1}{\lambda_i - \lambda_p}\Gamma^i_{jp} s_p(\xi_p, t) a_j(\xi_j, t) \\
&\quad + \frac{1}{\lambda_i - \lambda_p}\Gamma^i_{jp} s_p\Big(\frac{\lambda_j - \lambda_i}{\lambda_j - \lambda_p}\xi_p + \frac{\lambda_i - \lambda_p}{\lambda_j - \lambda_p}\xi_j, t\Big) a_j\Big(\frac{\lambda_j - \lambda_i}{\lambda_j - \lambda_p}\xi_p + \frac{\lambda_i - \lambda_p}{\lambda_j - \lambda_p}\xi_j, t\Big) \\
&\quad - \frac{\lambda_j - \lambda_p}{(\lambda_i - \lambda_p)^2}\Gamma^i_{jp}\int_{\frac{\lambda_j - \lambda_i}{\lambda_j - \lambda_p}\xi_p + \frac{\lambda_i - \lambda_p}{\lambda_j - \lambda_p}\xi_j}^{\xi_p} s_p(s', t) a_{j\xi_j}\Big(\frac{\lambda_j - \lambda_i}{\lambda_i - \lambda_p}(\xi_p - s') + \xi_j, t\Big)\, ds'.
\end{aligned} \tag{66}
$$

Turning to (51), we substitute into it (52)–(53), (56), and (61), and then average the resulting equation with respect to ξ_p. We assume that $\mathbf{u}_3$ and $\mathbf{u}_4$ are uniformly bounded in ξ_p, and that averages over ξ_p are well defined. We then multiply through the averaged equation on the left by $\mathbf{l}_j^T$ and rearrange terms to obtain the equation

$$
\begin{aligned}
\Big(a_j + \mathrm{Z}^j_p\big\langle s_p^2\big\rangle\Big)_t + \mathrm{H}^j_{jp}\big\langle s_p^2\big\rangle a_{j\xi_j} + \Big(\frac{1}{2}\mathcal{G}_j a_j^2\Big)_{\xi_j} \\
= \Gamma^j_{jp}\big\langle \phi_{\xi_p} s_p\big\rangle a_{j\xi_j} - \sum_{k\neq j}\Gamma^j_{kp}\big\langle s_p b_{k\xi_j}\big\rangle,
\end{aligned} \tag{67}
$$

where

$$
\mathrm{Z}^j_p = \mathbf{l}_j^T \mathbf{z}_p, \tag{68}
$$

$$
\mathrm{H}^j_{ip} = \mathbf{l}_j^T\Big[2C(\mathbf{r}_i, \mathbf{z}_p) + 3D(\mathbf{r}_i, \mathbf{r}_p, \mathbf{r}_p)\Big], \tag{69}
$$

$$
\mathcal{G}_j = 2\mathbf{l}_j^T C(\mathbf{r}_j, \mathbf{r}_j), \tag{70}
$$

and the brackets denote the mean with respect to ξ_p, defined by

$$
\big\langle f\big\rangle(\xi_j, t) = \lim_{L\to\infty}\frac{1}{L}\int_0^L f(\xi_j, \xi_p, t)\, d\xi_p. \tag{71}
$$

We split the second term on the right-hand side of (67) into two cases: $k = p$

and $k \neq p$. For $k = p$, we substitute (65) into the second term to obtain

$$\left\langle s_p b_{p\xi_j} \right\rangle = \frac{1}{\lambda_j - \lambda_p} \Gamma^p_{jp} \left\langle s_p(\xi_p, t)^2 \right\rangle a_{j\xi_j}(\xi_j, t); \tag{72}$$

and for $k \neq p$, we substitute (66) into the second term to obtain, after replacing i by k,

$$\begin{aligned} \left\langle s_p b_{k\xi_j} \right\rangle = &-\frac{1}{\lambda_k - \lambda_p} \Gamma^k_{jp} \left\langle s_p(\xi_p, t)^2 \right\rangle a_{j\xi_j}(\xi_j, t) \\ &- \frac{\lambda_j - \lambda_p}{(\lambda_k - \lambda_p)^2} \Gamma^k_{jp} \left\langle \int_0^{\frac{\lambda_k - \lambda_p}{\lambda_j - \lambda_p}(\xi_p - \xi_j)} s_p(\xi_p, t) s_p(\xi_p - \sigma, t) \right. \\ &\left. \qquad a_{j\xi_j\xi_j}\left(\frac{\lambda_j - \lambda_k}{\lambda_k - \lambda_p} \sigma + \xi_j, t \right) d\sigma \right\rangle. \end{aligned} \tag{73}$$

Finally, using (64), (72)–(73), and Lemma 1 in (67), we obtain the following equation for the wave profile, a_j, of the primary wave pulse:

$$\begin{aligned} &\left(a_j(\xi_j, t) + \mathrm{Z}^j_p \left\langle s_p(\xi_p, t)^2 \right\rangle \right)_t \\ &\qquad + \mathcal{C}_{jp} W_p(0, t) a_{j\xi_j}(\xi_j, t) + \left(\frac{1}{2} \mathcal{G}_j a_j(\xi_j, t)^2 \right)_{\xi_j} \\ &\qquad = \sum_{k \neq j, p} \Lambda_{jpk} \int_0^{\operatorname{sgn}(\mu_{kpj}) \cdot \infty} W_p(\sigma, t) a_{j\xi_j\xi_j}(\xi_j - \mu_{jkp}\sigma, t)\, d\sigma, \end{aligned} \tag{74}$$

where

$$\begin{aligned} \mathcal{C}_{jp} &= \mathrm{H}^j_{jp} + \frac{1}{\lambda_j - \lambda_p} \Gamma^p_{jp} \Gamma^j_{pp} - \sum_{k \neq p} \frac{1}{\lambda_k - \lambda_p} \Gamma^k_{jp} \Gamma^j_{kp}, \\ \Lambda_{jpk} &= \frac{\lambda_j - \lambda_p}{(\lambda_k - \lambda_p)^2} \Gamma^k_{jp} \Gamma^j_{kp}, \\ \mu_{kjp} &= \frac{\lambda_k - \lambda_j}{\lambda_p - \lambda_j}, \\ W_p(\sigma, t) &= \left\langle s_p(\xi_p, t) s_p(\xi_p - \sigma, t) \right\rangle, \end{aligned}$$

Γ^i_{kp}, Z^j_p, H^j_{jp}, and $\mathcal{G}_j$ are given by (63) and (68)–(70), respectively, and s_p satisfies (59).

In summary, we have derived an asymptotic solution that describes the scattering of a pulse in one (primary) wave-field of the hyperbolic system of conservation laws in (45) by stationary random fluctuations in another (scattering) wave-field, under the assumption that the scattering wave is not genuinely nonlinear. The solution is given by (46)–(47), where $\mathbf{u}_1$ is given by (52), and $\mathbf{u}_2$ is given by (53).

The profile s_p of the scattering wave satisfies a cubically nonlinear, modified Burgers equation (59). If the scattering wave is linearly degenerate, then the the coefficient of the cubically nonlinear vanishes, and the scattering wave profile does not change at leading order on the $O(1)$ time-scale of validity of the asymptotic solution. The random phase shift ϕ in the primary wave is given in terms of the scattering wave profile by (64).

The profile a_j of the primary wave satisfies an inviscid Burgers equation with a nonlocal right-hand side (74) that describes the effects of double scattering on the primary wave. The kernel W_p of the nonlocal term is a covariance function of the profile of the scattering wave. The coefficients Λ_{jpk} describe the strength of the j-wave produced by double scattering: a quadratically nonlinear interaction of the j-wave with the p-wave to produce a k-wave, followed by a quadratically nonlinear interaction of the k-wave with the p-wave to produce the j-wave. The coefficient, $\mathcal{G}_j$ describes the strength of the quadratically nonlinear self-interaction of the j-wave, and $\mathcal{C}_{jp}W_p(0,t)$ is a correction to the mean j-wave speed caused by the presence of the other waves.

4.2 Nonconservative form

In carrying out the asymptotic expansion for specific systems, it is often convenient to write (45) in the nonconservative form

$$\mathbf{u}_t + A(\mathbf{u})\mathbf{u}_x = \mathbf{0}, \tag{75}$$

where $A : \mathbf{R}^m \to \mathbf{R}^{m\times m}$. For use in the following applications, we summarize the expressions for the coefficients of the resulting asymptotic equations in terms of A.

We assume that the matrix $A(\mathbf{u})$ has the Taylor expansion

$$\begin{aligned} A(\mathbf{u}) = A_0 + \nabla A_0(\mathbf{u}) + \frac{1}{2}\nabla^2 A_0(\mathbf{u},\mathbf{u}) + \frac{1}{6}\nabla^3 A_0(\mathbf{u},\mathbf{u},\mathbf{u}) \\ + O(|\mathbf{u}|^4), \quad |\mathbf{u}| \to 0, \end{aligned} \tag{76}$$

where $A_0 = A(\mathbf{0})$, and that A_0 has distinct eigenvalues λ_i with corresponding left- and right-eigenvectors $\mathbf{l}_i$ and $\mathbf{r}_i$, respectively, such that $\mathbf{l}_i^T\mathbf{r}_k = \delta_{ik}$ for $i,k = 1,2,\ldots,m$. We look for an asymptotic solution, $\mathbf{u}^\varepsilon$, of (75) of the form (46)–(47), and follow the derivation in Section 4.1. The result is that the wave profile, s_p, of the background scattering wave satisfies (59), and the wave profile, a_j, of the primary wave pulse satisfies (74), where the coefficients are now given by

$$
\begin{aligned}
\mathcal{H}_p &= \mathbf{l}_p^T\Big[\nabla A_0(\mathbf{r}_p)\mathbf{z}_p + \nabla A_0(\mathbf{z}_p)\mathbf{r}_p + \frac{1}{2}\nabla^2 A_0(\mathbf{r}_p,\mathbf{r}_p)\mathbf{r}_p\Big],\\
\mathrm{Z}_p^j &= \mathbf{l}_j^T\mathbf{z}_p,\\
\mathcal{C}_{jp} &= \mathrm{H}_{pj}^j + \frac{1}{\lambda_j-\lambda_p}(\Gamma_{pp}^j\Gamma_{pj}^p - (\Gamma_{pj}^j)^2) - \sum_{k\neq j,p}\frac{1}{\lambda_k-\lambda_p}\Gamma_{pk}^j\Gamma_{jp}^k,\\
\mathcal{G}_j &= \mathbf{l}_j^T\nabla A_0(\mathbf{r}_j)\mathbf{r}_j,\\
\Lambda_{jpk} &= \frac{\lambda_j-\lambda_p}{(\lambda_k-\lambda_p)^2}\Gamma_{pk}^j\tilde{\Gamma}_{kjp}\\
\mathrm{H}_{pj}^j &= \mathbf{l}_j^T\Big[\nabla A_0(\mathbf{z}_p)\mathbf{r}_j + \frac{1}{2}\nabla^2 A_0(\mathbf{r}_p,\mathbf{r}_p)\mathbf{r}_j\Big],\\
\Gamma_{pk}^j &= \mathbf{l}_j^T\nabla A_0(\mathbf{r}_p)\mathbf{r}_k,\\
\tilde{\Gamma}_{kjp} &= \mu_{kjp}\Gamma_{jp}^k + \mu_{kpj}\Gamma_{pj}^k,\\
\mu_{kjp} &= \frac{\lambda_k-\lambda_j}{\lambda_p-\lambda_j},\\
W_p(\sigma,t) &= \Big\langle s_p(\xi_p,t)s_p(\xi_p-\sigma,t)\Big\rangle,
\end{aligned}
\tag{77}
$$

and $\mathbf{z}_p$ satisfies

$$
(A_0-\lambda_p I)\mathbf{z}_p + \frac{1}{2}\nabla A_0(\mathbf{r}_p)\mathbf{r}_p = \mathbf{0},
$$

the angular brackets denote the mean with respect to ξ_p defined by (71), and the random shift ϕ is given by (64).

5 Gas dynamics

In this section, we apply the results of Section 4 to obtain an equation for the scattering of a nonlinear sound wave pulse by stationary random entropy fluctuations. Equations for the resonant interaction of periodic sound and entropy waves are studied in (23).

The one-dimensional compressible Euler equations are (1)

$$
\begin{aligned}
\rho_t + (\rho v)_x &= 0,\\
(\rho v)_t + (\rho v^2 + p)_x &= 0,\\
\left(\rho e + \frac{1}{2}\rho v^2\right)_t + \left(\rho e v + \frac{1}{2}\rho v^3 + p v\right)_x &= 0,
\end{aligned}
\tag{78}
$$

where ρ is the density, v is the velocity, p is the pressure, and e is the internal energy per unit mass. For simplicity, we consider an ideal gas with constant specific heats, c_p and c_v. In that case,

$$e = \frac{1}{\gamma - 1}\frac{p}{\rho}, \quad p = \kappa\rho^\gamma \exp(S/c_v), \quad c^2 = \frac{\gamma p}{\rho}, \tag{79}$$

where $\gamma = c_p/c_v > 1$ is the ratio of specific heats, S is the entropy, and c is the sound speed. We consider perturbations of a uniform background state with density ρ_0, velocity $v_0 = 0$, entropy S_0, and sound speed $c_0 = \sqrt{\gamma p_0/\rho_0}$.

We may write (78) in the nonconservative form (75)–(76), where

$$\mathbf{u} = \begin{bmatrix} \rho - \rho_0 \\ v \\ S - S_0 \end{bmatrix}, \qquad A_0 = \begin{bmatrix} 0 & \rho_0 & 0 \\ (p_\rho)_0/\rho_0 & 0 & (p_S)_0/\rho_0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues and corresponding left- and right-eigenvectors of A_0 are

$$\lambda_1 = -c_0, \quad \lambda_2 = 0, \quad \lambda_3 = c_0,$$

$$\mathbf{r}_1 = \begin{bmatrix} \rho_0 \\ -c_0 \\ 0 \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} (p_S)_0 \\ 0 \\ -c_0^2 \end{bmatrix}, \quad \mathbf{r}_3 = \begin{bmatrix} \rho_0 \\ c_0 \\ 0 \end{bmatrix}.$$

The 1- and 3-waves are left- and right-moving sound waves, respectively, and the 2-wave is an entropy wave.

We take the entropy wave to be the background scattering wave, which is consistent with the expansion in Section 4 because the entropy wave is linearly degenerate. Then (59) for the wave profile fluctuations of the entropy wave is

$$s_{2t} = 0, \tag{80}$$

so that $s_2 = s_2(\xi_2)$. The asymptotic equation (74), (77) for the wave profile of the right-moving sound wave is

$$a_{3t} - 2\mathcal{M}\left\langle s_2^2 \right\rangle a_{3\xi_3} + \left(\frac{1}{2}\mathcal{G}a_3^2\right)_{\xi_3} = 5\mathcal{M}\int_{-\infty}^{0} W(\sigma) a_{3\xi_3\xi_3}(\xi_3 - 2\sigma, t)\, d\sigma, \tag{81}$$

where

$$W(\sigma) = \left\langle s_2(\xi_2) s_2(\xi_2 - \sigma) \right\rangle,$$

and the angular brackets denote an average with respect to ξ_2. The coefficients (81) are given by

$$\mathcal{M} = \frac{c_0^5}{4\gamma^2 c_v^2}, \quad \mathcal{G} = \frac{(\gamma+1)c_0}{2}.$$

The equation for the left-moving sound wave can be obtained by replacing the wave speed c_0 by $-c_0$.

Since $\mathcal{G} \neq 0$, we may normalize (81) by introducing the variables

$$\bar{u}(\bar{x},\bar{t}) = \frac{2\mathcal{G}}{5\mathcal{M}} a_3(\xi_3, t), \quad \partial_{\bar{t}} = \frac{2}{5\mathcal{M}}\partial_t - \frac{4}{5}\left\langle s_2^2 \right\rangle \partial_{\xi_3}, \quad \bar{x} = \xi_3.$$

This transforms (81) into

$$\bar{u}_{\bar{t}} + \left(\frac{1}{2}\bar{u}^2\right)_{\bar{x}} = \int_{-\infty}^{0} W\left(\frac{\sigma}{2}\right) \bar{u}_{\bar{x}\bar{x}}(\bar{x}-\sigma,\bar{t})\, d\sigma. \tag{82}$$

In summary, we have derived an asymptotic solution of (78) of the following form:

$$\begin{bmatrix} \rho \\ v \\ S \end{bmatrix} \sim \begin{bmatrix} \rho_0 \\ 0 \\ S_0 \end{bmatrix} + \varepsilon s_2(\xi_2) \begin{bmatrix} p_0/c_v \\ 0 \\ -c_0^2 \end{bmatrix} + \varepsilon^2 a_3(\xi_3, t) \begin{bmatrix} \rho_0 \\ c_0 \\ 0 \end{bmatrix} + O(\varepsilon^3),$$

where the multiple-scale variables, ξ_2 and ξ_3, are evaluated at

$$\xi_2 = \frac{x}{\varepsilon^2}, \quad \xi_3 = \frac{x - c_0 t - \varepsilon^3 \phi\left(x/\varepsilon^2\right)}{\varepsilon^2}, \quad \phi\left(\frac{x}{\varepsilon^2}\right) = \frac{-c_0^2}{2\gamma c_v} \int_0^{x/\varepsilon^2} s_2(y)\, dy.$$

The entropy wave profile s_2 is a stationary random function of ξ_2, while the sound wave profile a_3 decays to zero as $\xi_3 \to \pm\infty$. From (80), we see that s_2 is given by its initial value. The random correction ϕ to the location of the sound wave pulse is determined from s_2, and the profile, a_3, of the sound wave satisfies an inviscid Burgers equation with a nonlocal right-hand side (81), whose kernel W is a covariance of the entropy wave profile.

6 Elasticity

In this section, we apply the general theory derived above to elastic waves in an isotropic medium. We will consider three different cases: the scattering of longitudinal waves by a random medium, leading to a quadratically nonlinear equation; the scattering of transverse waves by a random medium, leading to a cubically nonlinear equation; and the scattering of a longitudinal wave by a transverse wave. Equations for the resonant interaction of periodic elastic waves are derived and analyzed in (24).

We consider an elastic wave propagating in one space-dimension, say in the x_1-direction, through an isotropic hyperelastic medium; for simplicity, we consider plane polarized waves. The deformation is given by

$$\mathsf{x}_1 = x_1 + u_1(x_1, t),$$
$$\mathsf{x}_2 = x_2 + u_2(x_1, t),$$

where $(\mathsf{x}_1, \mathsf{x}_2)$ are spatial coordinates: (x_1, x_2) is a point in the reference medium and (u_1, u_2) is the displacement defined for $-\infty < x_1 < \infty$ and $0 \le t$. To simplify the notation, we let $x = x_1$ in the following.

In the absence of body forces, the equations of motion are (25)

$$\begin{aligned} \rho u_{1tt} &= (g(u_{1x}, u_{2x}^2))_x, \\ \rho u_{2tt} &= (u_{2x} h(u_{1x}, u_{2x}^2))_x, \end{aligned} \tag{83}$$

where $\rho\colon \mathbf{R} \to \mathbf{R}$ is the density, and $g, h\colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ are constitutive functions. For a hyperelastic body,

$$g(p,q) = \frac{\partial \sigma}{\partial p}(p,q), \quad h(p,q) = 2\frac{\partial \sigma}{\partial q}(p,q), \tag{84}$$

where $\sigma\colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is the strain-energy density.

We assume that g and h have the Taylor expansions

$$g(w_1, w_2^2) = a w_1 + \frac{1}{2} c_1 w_1^2 + \frac{1}{2} c_2 w_2^2 + \frac{1}{6} d_1 w_1^3 + \frac{1}{6} d_2 w_1 w_2^2 + \text{h.o.t.}, \tag{85}$$
$$h(w_1, w_2^2) = \alpha + \frac{1}{2}\gamma w_1 + \frac{1}{6}\delta_1 w_1^2 + \frac{1}{6}\delta_2 w_2^2 + \text{h.o.t.}, \tag{86}$$

as $w_k \to 0$, where "h.o.t." denotes higher-order terms. Here $a, c_1, c_2, d_1, d_2, \alpha, \gamma, \delta_1, \delta_2$ are material constants that are related to the Lamé moduli λ and μ, the density ρ, and the longitudinal and transverse wave speeds, c_l and c_t, respectively, by

$$a = 2\mu + \lambda = \rho c_l^2, \quad \alpha = \mu = \rho c_t^2; \tag{87}$$

the other constants are higher-order elastic moduli, where (84) implies that $\gamma = 2c_2$ and $\delta_1 = d_2$.

6.1 Longitudinal waves in a nonuniform medium

To write (83) as a first-order hyperbolic system of conservation laws of the form (23), we let m_k and w_k denote the momentum and the displacement gradient, respectively:

$$\begin{aligned} m_1 &= \rho u_{1t}, \qquad & m_2 &= \rho u_{2t}, \\ w_1 &= u_{1x}, \qquad & w_2 &= u_{2x}. \end{aligned} \tag{88}$$

We assume that the Lamé moduli and density in (87) have the expansions

$$\lambda(x) = \lambda_0 + \varepsilon\lambda_1(x/\varepsilon^2) + O(\varepsilon^2), \quad \mu(x) = \mu_0 + \varepsilon\mu_1(x/\varepsilon^2) + O(\varepsilon^2),$$
$$\rho(x) = \rho_0 + \varepsilon\rho_1(x/\varepsilon^2) + O(\varepsilon^2), \quad \varepsilon \to 0.$$

Then, using (85)–(86) and (88), and neglecting higher-order terms, the equations of motion (83) may be put in the form (23) with

$$\mathbf{u} = \begin{bmatrix} m_1 \\ m_2 \\ w_1 \\ w_2 \end{bmatrix}, \qquad \mathbf{f}(\mathbf{u}) = \begin{bmatrix} -aw_1 - \frac{1}{2}c_1 w_1^2 - \frac{1}{2}c_2 w_2^2 \\ -\alpha w_2 - c_2 w_1 w_2 \\ -m_1/\rho_0 \\ -m_2/\rho_0 \end{bmatrix},$$

$$B(x/\varepsilon^2) = \begin{bmatrix} 0 & 0 & -(2\mu_1(x/\varepsilon^2) + \lambda_1(x/\varepsilon^2)) & 0 \\ 0 & 0 & 0 & -\mu_1(x/\varepsilon^2) \\ \rho_0^{-2}\rho_1(x/\varepsilon^2) & 0 & 0 & 0 \\ 0 & \rho_0^{-2}\rho_1(x/\varepsilon^2) & 0 & 0 \end{bmatrix}. \tag{89}$$

From this, it follows that

$$A = \nabla\mathbf{f}(\mathbf{0}) = \begin{bmatrix} 0 & 0 & -\rho_0 c_l^2 & 0 \\ 0 & 0 & 0 & -\rho_0 c_t^2 \\ -1/\rho_0 & 0 & 0 & 0 \\ 0 & -1/\rho_0 & 0 & 0 \end{bmatrix}. \tag{90}$$

The eigenvalues and corresponding right-eigenvectors of A are

$$\kappa_1 = -c_t, \quad \kappa_2 = -c_l, \quad \kappa_3 = c_l, \quad \kappa_4 = c_t, \tag{91}$$

$$\mathbf{r}_1 = \begin{bmatrix} 1 \\ 0 \\ 1/(\rho_0 c_l) \\ 0 \end{bmatrix}, \qquad \mathbf{r}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1/(\rho_0 c_t) \end{bmatrix},$$

$$\mathbf{r}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1/(\rho_0 c_t) \end{bmatrix}, \qquad \mathbf{r}_4 = \begin{bmatrix} 1 \\ 0 \\ -1/(\rho_0 c_l) \\ 0 \end{bmatrix}. \tag{92}$$

The eigenvalues satisfy

$$-c_l < -c_t < c_t < c_l,$$

and, respectively, the 1- and 4-waves are left- and right-moving longitudinal waves, and the 2- and 3-waves are left- and right-moving transverse waves.

The asymptotic solution of (83) for a right-moving longitudinal wave is

$$\begin{bmatrix} m_1 \\ m_2 \\ w_1 \\ w_2 \end{bmatrix} \sim \varepsilon^2 a_4(\xi_4, t) \begin{bmatrix} 1 \\ 0 \\ -1/(\rho_0 c_l) \\ 0 \end{bmatrix} + O(\varepsilon^3),$$

where the multiple-scale variable, ξ_4, is evaluated at

$$\xi_4 = \frac{x - c_t t - \varepsilon^3 \phi(x/\varepsilon^2)}{\varepsilon^2},$$

$$\phi\left(\frac{x}{\varepsilon^2}\right) = \frac{1}{2\rho_0 c_l c_t} \int_0^{x/\varepsilon^2} \left[2\mu_1(y) + \lambda_1(y) - c_l^2 \rho_1(y)\right] dy,$$

where ϕ is the random correction to the location of the pulse, and a_4 satisfies

$$a_{4t} - \frac{1}{c_l}\mathcal{M} a_{4\xi_4} + \left(\frac{1}{2}\mathcal{G} a_4^2\right)_{\xi_4} = \frac{1}{c_l}\int_{-\infty}^{0} W(\sigma) a_{4\xi_4\xi_4}(\xi_4 - 2\sigma)\, d\sigma. \tag{93}$$

Here, the coefficients are

$$\mathcal{G} = \frac{-c_1}{2\rho_0^2 c_l^2},$$

$$\mathcal{M} = \left\langle \beta_{11}(y)^2 \right\rangle + \left\langle \beta_{14}(y)^2 \right\rangle, \quad W(\sigma) = \left\langle \beta_{14}(y)\beta_{14}(y-\sigma) \right\rangle,$$

$$\beta_{11}(y) = \frac{-1}{2\rho_0 c_l}\left[2\mu_1(y) + \lambda_1(y) - c_l^2 \rho_1(y)\right],$$

$$\beta_{14}(y) = \frac{-1}{2\rho_0 c_l}\left[2\mu_1(y) + \lambda_1(y) + c_l^2 \rho_1(y)\right].$$

The equation for the left-moving wave can be obtained by replacing the wave speeds c_t by $-c_t$ and c_l by $-c_l$.

Since $\mathcal{G} \neq 0$, the change of variables

$$\bar{u}(\bar{x}, \bar{t}) = 2\mathcal{G} c_l a_4(\xi_4, t), \quad \partial_{\bar{t}} = 2c_l \partial_t - 2\mathcal{M}\partial_{\xi_4}, \quad \bar{x} = \xi_4$$

puts (93) in the normalized form (82).

Thus, we see that the wave profile, a_4, of the right-moving longitudinal wave satisfies an inviscid Burgers equation with a nonlocal right-hand side (93). The nonlocal term on the right-hand side originates from the scattering of a right-moving longitudinal wave into a left-moving longitudinal wave, and the scattering of the left-moving longitudinal wave back into the right-moving longitudinal wave. In an isotropic medium, the scattering of longitudinal waves does not produce any transverse waves, and vice versa, which explains why only one scattering term is present on the right-hand side of (93).

6.2 Transverse waves in a nonuniform medium

Transverse waves are not genuinely nonlinear at an undeformed state due to the reflectional symmetry of the waves (26; 27; 28). Hence, to include nonlinear effects in the scattering of transverse waves, we use the expansion (43).

We consider the same set of equations as in the previous subsection. The asymptotic solution of (83) for a right-moving transverse wave is given by

$$\begin{bmatrix} m_1 \\ m_2 \\ w_1 \\ w_2 \end{bmatrix} \sim \varepsilon a_3(\xi_3, t) \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1/(\rho_0 c_t) \end{bmatrix} + O(\varepsilon^3),$$

where the multiple-scale variable, ξ_3, is evaluated at

$$\xi_3 = \frac{x - c_l t - \varepsilon^3 \phi(x/\varepsilon^2)}{\varepsilon^2},$$
$$\phi\left(\frac{x}{\varepsilon^2}\right) = \frac{1}{2\rho_0 c_l c_t} \int_0^{x/\varepsilon^2} \left[\mu_1(y) - c_t^2 \rho_1(y)\right] dy,$$

where ϕ is the random correction to the location of the pulse. The profile, a_3, of the transverse wave satisfies

$$a_{3t} - \frac{1}{c_t} \mathcal{N} a_{3\xi_3} + \left(\frac{1}{3} \mathcal{H} a_3^3\right)_{\xi_3} = \frac{1}{c_t} \int_{-\infty}^{0} V(\sigma) a_{3\xi_3\xi_3}(\xi_3 - 2\sigma)\, d\sigma. \tag{94}$$

Here, the coefficients are

$$\mathcal{H} = \frac{-3c_2^2}{4\rho_0^4 c_t^3 (c_l^2 - c_t^2)} + \frac{\delta_2}{4\rho_0^3 c_t^3}, \tag{95}$$

$$\mathcal{N} = \left\langle \beta_{22}(y)^2 \right\rangle + \left\langle \beta_{23}(y)^2 \right\rangle, \qquad V(\sigma) = \left\langle \beta_{23}(y) \beta_{23}(y - \sigma) \right\rangle,$$
$$\beta_{22}(y) = \frac{-1}{2\rho_0 c_t} \left[\mu_1(y) - c_t^2 \rho_1(y)\right], \qquad \beta_{23}(y) = \frac{-1}{2\rho_0 c_t} \left[\mu_1(y) + c_t^2 \rho_1(y)\right].$$

The equation for the left-moving wave can be obtained by replacing the wave speed c_t by $-c_t$.

Since $\mathcal{H} \neq 0$, the normalized form of (94) is

$$\bar{u}_{\bar{t}} + \left(\frac{1}{3} \bar{u}^3\right)_{\bar{x}} = \int_{-\infty}^{0} V\left(\frac{\sigma}{2}\right) \bar{u}_{\bar{x}\bar{x}}(\bar{x} - \sigma)\, d\sigma,$$

where

$$\bar{u}(\bar{x}, \bar{t}) = 2\mathcal{H}^{1/2} c_t a_3(\xi_3, t), \quad \partial_{\bar{t}} = 2c_t \partial_t - 2\mathcal{N} \partial_{\xi_3}, \quad \bar{x} = \xi_3.$$

Thus, for transverse elastic waves in an isotropic medium, we obtain a cubically nonlinear modified Burgers equation with a nonlocal right-hand side (94), instead of the quadratically nonlinear equation we obtained for longitudinal waves.

6.3 Scattering of longitudinal waves by transverse waves

In Subsections 6.1 and 6.2, we considered elastic waves propagating in a nonuniform isotropic hyperelastic medium. In this subsection, we consider waves propagating in a uniform medium with constant density, $\rho = \rho_0$, and constant Lamé moduli, $\lambda = \lambda_0$ and $\mu = \mu_0$, in (87). Transverse waves are not genuinely nonlinear at $\mathbf{u} = \mathbf{0}$, so according to the general theory developed in Section 4 we may consider the scattering of a longitudinal wave by a transverse wave.

The corresponding asymptotic solution of (83) for right-moving transverse and longitudinal waves is

$$\begin{bmatrix} m_1 \\ m_2 \\ w_1 \\ w_2 \end{bmatrix} \sim \varepsilon s_3(\xi_3, t) \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1/(\rho_0 c_t) \end{bmatrix} + \varepsilon^2 a_4(\xi_4, t) \begin{bmatrix} 1 \\ 0 \\ -1/(\rho_0 c_l) \\ 0 \end{bmatrix} + O(\varepsilon^3),$$

where the multiple-scale variables, ξ_3 and ξ_4, are evaluated at

$$\xi_3 = \frac{x - c_t t}{\varepsilon^2}, \quad \xi_4 = \frac{x - c_l t}{\varepsilon^2}.$$

In this case, the presence of a transverse wave does not affect the propagation speed of the longitudinal wave, so there is no random phase shift. The wave profiles s_3 and a_4 satisfy

$$s_{3t} + \left(\frac{1}{3}\mathcal{H} s_3^3\right)_{\xi_3} = 0, \tag{96}$$

$$\begin{aligned} &\left(a_4 + 4\mathcal{M}\rho_0^2 c_l c_t^3 \left\langle s_3^2 \right\rangle\right)_t + \mathcal{C}\left\langle s_3^2 \right\rangle a_{4\xi_4} + \left(\frac{1}{2}\mathcal{G} a_4^2\right)_{\xi_4} \\ &\quad = \mathcal{M} c_2 (c_l - c_t)^2 \int_{-\infty}^{0} W(\sigma, t) a_{4\xi_4\xi_4}\left(\xi_4 - \frac{c_l + c_t}{2c_t}\sigma, t\right) d\sigma. \end{aligned} \tag{97}$$

Here, $\mathcal{H}$ is given by by (95), and

$$\mathcal{M} = \frac{c_2}{16\rho_0^4 c_l c_t^5 (c_l - c_t)}, \quad \mathcal{G} = \frac{-c_1}{2\rho_0^2 c_l^2}, \quad W(\sigma, t) = \left\langle s_3(\xi_3, t) s_3(\xi_3 - \sigma, t) \right\rangle,$$

$$\mathcal{C} = \frac{-c_1 c_2}{4\rho_0^4 c_l c_t^2 (c_l^2 - c_t^2)} + \frac{d_2}{12\rho_0^3 c_l c_t^2} + \frac{c_2^2}{4\rho_0^4 c_l c_t^3 (c_l - c_t)} - \frac{c_2^2}{8\rho_0^4 c_l c_t^4}.$$

The equations for the scattering of the right-moving longitudinal wave by the left-moving transverse wave can be obtained by replacing c_t by $-c_t$. The asymptotic equations for the scattering of the left-moving longitudinal wave by the transverse waves can be obtained by replacing the wave speed c_l by $-c_l$ in the above equations.

If s_3 is a smooth solution of (96), then $\left\langle s_3^2 \right\rangle$ is constant. In that case, we may write (97) in the normalized form

$$\bar{u}_{\bar{t}} + \left(\frac{1}{2}\bar{u}^2\right)_{\bar{x}} = \int_{-\infty}^{0} W\left(\frac{\sigma}{c}, \bar{t}\right) \bar{u}_{\bar{x}\bar{x}}(\bar{x} - \sigma, \bar{t})\, d\sigma,$$

where

$$\bar{u}(\bar{x}, \bar{t}) = \frac{\mathcal{G}c}{\Lambda} a_4(\xi_4, t), \quad \partial_{\bar{t}} = \frac{c}{\Lambda}\partial_t + \frac{\mathcal{C}c}{\Lambda}\left\langle s_3^2 \right\rangle \partial_{\xi_4}, \quad \bar{x} = \xi_4,$$
$$c = \frac{c_l + c_t}{2c_t}, \quad \Lambda = \mathcal{M}c_2(c_l - c_t)^2.$$

Thus, we see that in the scattering of a longitudinal wave by a transverse wave, the wave profile, s_3, of the transverse wave satisfies a modified Burgers equation with a cubically nonlinear term (96). On the other hand, the wave profile, a_4, of the longitudinal wave satisfies an inviscid Burgers equation with a nonlocal right-hand side (97), whose kernel is a covariance function of the transverse wave profile, s_3.

7 Magnetohydrodynamics

In this section, as a final application of the general theory, we study the scattering of magnetoacoustic waves by random entropy and Alfvén waves. Equations for the resonant interaction of periodic waves in magnetohydrodynamics are derived in (29; 30).

In the absence of body forces, the one-dimensional, compressible magnetohydrodynamics (MHD) equations may be written as (31)

$$\begin{aligned}
&B_{1t} = B_{1x} = 0, \\
&B_{2t} + (B_2 v_1 - B_1 v_2)_x = 0, \\
&B_{3t} + (B_3 v_1 - B_1 v_3)_x = 0, \\
&\rho_t + (\rho v_1)_x = 0, \\
&(\rho v_1)_t + \left(\rho v_1^2 + p + \frac{1}{2}|\mathbf{B}|^2\right)_x = 0, \\
&(\rho v_2)_t + (\rho v_1 v_2 - B_1 B_2)_x = 0, \\
&(\rho v_3)_t + (\rho v_1 v_3 - B_1 B_3)_x = 0, \\
&\left(\frac{1}{2}\rho|\mathbf{v}|^2 + \rho e + \frac{1}{2}|\mathbf{B}|^2\right)_t \\
&\quad + \left[\left(\frac{1}{2}\rho|\mathbf{v}|^2 + \rho e + p + |\mathbf{B}|^2\right) v_1 - B_1 \mathbf{B}\cdot\mathbf{v}\right]_x = 0,
\end{aligned} \tag{98}$$

where $\mathbf{B} = [B_1 \ \ B_2 \ \ B_3]^T$ is the magnetic field, $\mathbf{v} = [v_1 \ \ v_2 \ \ v_3]^T$ is the velocity field, ρ is the density, p is the pressure, and e is the internal energy per unit mass. We note that (98) implies that B_1 is constant, say $B_1 = B_{1,0}$.

We assume the ideal gas equation of state in (79), and consider perturbations of a uniform background state with magnetic field $\mathbf{B}_0 = [B_{1,0} \ \ B_{2,0} \ \ 0]^T$, velocity field $\mathbf{v}_0 = [0 \ \ 0 \ \ 0]^T$, density ρ_0, entropy S_0, and sound speed c_0.

We may write (98) in the nonconservative form (75)–(76), where

$$\mathbf{u} = [B_2 - B_{2,0} \quad B_3 \quad v_1 \quad v_2 \quad v_3 \quad \rho - \rho_0 \quad S - S_0]^T,$$

$$A_0 = \begin{bmatrix} 0 & 0 & B_{2,0} & -B_{1,0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -B_{1,0} & 0 & 0 \\ B_{2,0}/\rho_0 & 0 & 0 & 0 & 0 & (p_\rho/\rho)_0 & (p_S/\rho)_0 \\ -B_{1,0}/\rho_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -B_{1,0}/\rho_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of A_0 are

$$\lambda_1 = -c_f, \quad \lambda_2 = -c_a, \quad \lambda_3 = -c_s, \quad \lambda_4 = 0, \quad \lambda_5 = c_s, \quad \lambda_6 = c_a, \quad \lambda_7 = c_f,$$

where

$$c_s^2 = \frac{1}{2}\left(c_*^2 - \sqrt{c_*^4 - 4c_0^2 c_a^2}\right), \quad c_f^2 = \frac{1}{2}\left(c_*^2 + \sqrt{c_*^4 - 4c_0^2 c_a^2}\right),$$
$$c_*^2 = c_0^2 + c_a^2 + \hat{c}_a^2, \quad c_0^2 = \frac{\gamma p_0}{\rho_0}, \quad c_a^2 = \frac{B_{1,0}^2}{\rho_0}, \quad \hat{c}_a^2 = \frac{B_{2,0}^2}{\rho_0}.$$

Here, c_0 denotes the sound speed at $\mathbf{u} = \mathbf{0}$, c_a denotes the Alfvén speed, and c_s and c_f, respectively, denote the slow and fast magnetoacoustic speeds. The wave speeds satisfy

$$c_s \leq c_0, \quad c_a \leq c_f.$$

The corresponding right-eigenvectors of A_0 are

$$\mathbf{r}_{1,7} = \begin{bmatrix} B_{2,0}c_f^2/(c_f^2 - c_a^2) \\ 0 \\ \mp c_f \\ \pm B_{1,0}B_{2,0}c_f/(\rho_0(c_f^2 - c_a^2)) \\ 0 \\ \rho_0 \\ 0 \end{bmatrix}, \qquad \mathbf{r}_{2,6} = \begin{bmatrix} 0 \\ -B_{1,0} \\ 0 \\ 0 \\ \mp c_a \\ 0 \\ 0 \end{bmatrix},$$

$$\mathbf{r}_{3,5} = \begin{bmatrix} B_{2,0}c_s^2/(c_s^2 - c_a^2) \\ 0 \\ \mp c_s \\ \pm B_{1,0}B_{2,0}c_s/(\rho_0(c_s^2 - c_a^2)) \\ 0 \\ \rho_0 \\ 0 \end{bmatrix}, \qquad \mathbf{r}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \rho_0 \\ -\rho_0 c_0^2/(p_S)_0 \end{bmatrix}.$$

Here, the 1- and the 7-waves are left- and right-moving fast magnetoacoustic waves, the 2- and 6-waves are left- and right-moving Alfvén waves, the 3- and 5-waves are left- and right-moving slow magnetoacoustic waves, and the 4-wave is an entropy wave.

The entropy wave and the Alfvén waves are linearly degenerate. Thus, we take these waves, in turn, to be the background scattering wave.

The asymptotic solution of (98) for the scattering of a right-moving fast magnetoacoustic wave by an entropy wave is

$$\begin{bmatrix} B_2 \\ B_3 \\ v_1 \\ v_2 \\ v_3 \\ \rho \\ S \end{bmatrix} = \begin{bmatrix} B_{2,0} \\ 0 \\ 0 \\ 0 \\ 0 \\ \rho_0 \\ S_0 \end{bmatrix} + \varepsilon s_4(\xi_4) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \rho_0 \\ -\rho_0 c_0^2 c_v / p_0 \end{bmatrix}$$

$$+ \varepsilon^2 a_7(\xi_7, t) \begin{bmatrix} B_{2,0} c_f^2/(c_f^2 - c_a^2) \\ 0 \\ c_f \\ -B_{1,0} B_{2,0} c_f/(\rho_0(c_f^2 - c_a^2)) \\ 0 \\ \rho_0 \\ 0 \end{bmatrix} + O(\varepsilon^3). \quad (99)$$

The asymptotic equation (59) for the entropy wave profile, s_4, is

$$s_{4t} = 0, \tag{100}$$

so that $s_4 = s_4(\xi_4)$. The asymptotic equation (74) for the wave profile a_7 of the right-moving fast magnetoacoustic wave is

$$\begin{aligned} a_{7t} - \mathcal{M}_f(3-\delta_f)\left\langle s_4^2 \right\rangle a_{7\xi_7} + \left(\frac{1}{2}\mathcal{G}_f a_7^2\right)_{\xi_7} \\ = \mathcal{M}_f(5 - 3\delta_f) \int_{-\infty}^0 W_4(\sigma) a_{7\xi_7\xi_7}(\xi_7 - 2\sigma, t)\, d\sigma \\ + \Lambda_f^s \int_{-\infty}^0 W_4(\sigma) a_{7\xi_7\xi_7}\left(\xi_7 - \frac{c_f + c_s}{c_s}\sigma, t\right) d\sigma \\ + \Lambda_f^{-s} \int_0^\infty W_4(\sigma) a_{7\xi_7\xi_7}\left(\xi_7 + \frac{c_f - c_s}{c_s}\sigma, t\right) d\sigma. \end{aligned} \tag{101}$$

Here, the multiple-scale variables, ξ_4 and ξ_7, are evaluated at

$$\xi_4 = \frac{x}{\varepsilon^2}, \quad \xi_7 = \frac{x - c_f t - \varepsilon^3 \phi(x/\varepsilon^2)}{\varepsilon^2}, \quad \phi\left(\frac{x}{\varepsilon^2}\right) = \frac{-\delta_s}{2} \int_0^{x/\varepsilon^2} s_4(y)\, dy,$$

where ϕ is the random correction to the location of the pulse, and the coefficients are

$$\begin{aligned}
\mathcal{M}_{f,s} &= \frac{1}{4}\delta_{s,f} c_{f,s}, \\
\mathcal{G}_{f,s} &= \left(\frac{\gamma+1}{2}\delta_{s,f} + \frac{3}{2}\delta_{f,s}\right) c_{f,s}, \\
\Lambda^s_{f,s} &= \mathcal{M}_{f,s}\left[\left(\frac{c_{f,s} + c_{s,f}}{c_{s,f}}\right)^2 \delta_{f,s} - \delta_{s,f}\right], \\
\delta_{f,s} &= \frac{c_{f,s}^2 - c_0^2}{c_{f,s}^2 - c_{s,f}^2}, \\
W_4(\sigma) &= \left\langle s_4(\xi_4) s_4(\sigma - \xi_4) \right\rangle,
\end{aligned}$$

and $\Lambda^{-s}_{f,s}$ is obtained from $\Lambda^s_{f,s}$ by replacing the wave speed c_s by $-c_s$.

The equations for a left-moving fast magnetoacoustic wave can be obtained by replacing the wave speed c_f by $-c_f$, and the equations for a slow magnetoacoustic wave can be obtained by exchanging the subscripts f and s.

We see from (100) that the wave profile, s_4, of the linearly degenerate entropy wave is given by its initial value. The profile a_7 of the magnetoacoustic wave satisfies (101). The nonlocal terms on the right-hand side of (101) originate from the scattering of the right-moving fast magnetoacoustic wave by the entropy wave into the left-moving fast magnetoacoustic wave, the left-moving slow magnetoacoustic wave, and the right-moving slow magnetoacoustic wave, respectively, followed by the scattering of these waves by the entropy wave back into the right-moving fast magnetoacoustic wave.

7.2 Magnetoacoustic-Alfvén wave interaction

The asymptotic solution for the scattering of a right-moving fast magnetoacoustic wave by a right-moving Alfvén wave is given by (99) with $s_4(\xi_4)$ replaced by $s_6(\xi_6)$ and the associated eigenvector $\mathbf{r}_4$ replaced by $\mathbf{r}_6$. The asymptotic equation (59) for the wave profile fluctuations of the right-moving Alfvén wave is

$$s_{6t} = 0, \tag{102}$$

so that $s_6 = s_6(\xi_6)$. The asymptotic equation (74) for the wave profile of the right-moving fast magnetoacoustic wave is

$$a_{7t} + 2\mathcal{N}_f c_a c_f \left\langle s_6^2 \right\rangle a_{7\xi_7} + \left(\frac{1}{2}\mathcal{G}_f a_7^2\right)_{\xi_7} \\ = \mathcal{N}_f (c_a - c_f)(2c_a + c_f) \int_{-\infty}^{0} W_6(\sigma) a_{7\xi_7\xi_7}\left(\xi_7 - \frac{c_f + c_a}{2c_a}\sigma, t\right) d\sigma. \quad (103)$$

Here, $\mathcal{N}_f = \mathcal{M}_f/(4c_0^2)$,

$$W_6(\sigma) = \left\langle s_6(\xi_6) s_6(\sigma - \xi_6) \right\rangle$$

and the multiple-scale variables, ξ_6 and ξ_7, are evaluated at

$$\xi_6 = \frac{x}{\varepsilon^2}, \quad \xi_7 = \frac{x - c_f t}{\varepsilon^2}.$$

The random phase shift ϕ is zero in this case. The equation for the left-moving fast magnetoacoustic wave can be obtained by replacing the wave speed c_f by $-c_f$. The nonlocal term on the right-hand side of (103) originates from the scattering of the right-moving fast magnetoacoustic wave by the Alfvén wave into a left-moving fast magnetoacoustic wave, and the scattering of this wave by the Alfvén wave back into the rigth-moving fast magnetoacoustic wave.

The asymptotic equations for the slow magnetoacoustic-Alfvén wave interaction can be obtained by exchanging the subscripts f and s. The asymptotic equations for the scattering of the magnetoacoustic waves by the left-moving Alfvén wave can be obtained by replacing the wave speed c_a by $-c_a$ in the above equations.

Since $\mathcal{G}_f \neq 0$, the normalized form of (103) is

$$\bar{u}_{\bar{t}} + \left(\frac{1}{2}\bar{u}^2\right)_{\bar{x}} = \int_{-\infty}^{0} W_6\left(\frac{\sigma}{c}\right) \bar{u}_{\bar{x}\bar{x}}(\bar{x} - \sigma, \bar{t})\, d\sigma,$$

where

$$\bar{u}(\bar{x}, \bar{t}) = \frac{\mathcal{G}_f c}{\Lambda} a_7(\xi_7, t), \quad \partial_{\bar{t}} = \frac{c}{\Lambda}\partial_t + \frac{2\mathcal{N}_f c_a c_f c}{\Lambda}\left\langle s_6^2 \right\rangle \partial_{\xi_7}, \quad \bar{x} = \xi_7,$$
$$c = \frac{c_f + c_a}{2c_a}, \quad \Lambda = \mathcal{N}_f (c_a - c_f)(2c_a + c_f).$$

As in the case of magnetoacoustic-entropy wave interaction, we see from (102) that the wave profile, s_6, of the linearly degenerate scattering Alfvén wave is given by its initial value. On the other hand, the wave profile, a_7, of the scattered right-moving fast magnetoacoustic wave again satisfies an inviscid Burgers equation with a nonlocal right-hand side (103), whose kernel is a covariance function of the Alfvén wave profile.

A Averaging lemma

Lemma 1 *Suppose that* $F\colon \mathbf{R}^2 \to \mathbf{R}$ *is a Lebesgue measurable function such that:*

(a) $\dfrac{1}{L}\displaystyle\int_0^L F(y,\sigma)\,dy$ *converges pointwise a.e. in* σ *as* $L \to \infty$*;*

(b) there is a function $h \in L^1(\mathbf{R})$ *such that*

$$\frac{1}{L}\int_0^L |F(y,\sigma)|\,dy \leq h(\sigma)$$

for all $L > 0$*.*

Let

$$\left\langle F \right\rangle(\sigma) = \lim_{L\to\infty} \frac{1}{L}\int_0^\infty F(y,\sigma)\,dy,$$

and for $\lambda > 0$ *and* $y_0 > 0$*, let*

$$G(y) = \int_0^{\lambda(y-y_0)} F(y,\sigma)\,d\sigma.$$

Then

$$\left\langle G \right\rangle = \lim_{L\to\infty} \frac{1}{L}\int_0^L G(y)\,dy$$

exists and is given by

$$\left\langle G \right\rangle = \int_0^\infty \left\langle F \right\rangle(\sigma)\,d\sigma.$$

Proof By Fubini's theorem,

$$\begin{aligned}\int_0^L G(y)\,dy &= \int_0^L \left(\int_0^{\lambda(y-y_0)} F(y,\sigma)\,d\sigma\right) dy \\ &= -\int_{-\lambda y_0}^0 \int_0^{\sigma/\lambda+y_0} F(y,\sigma)\,dy\,d\sigma + \int_0^{\lambda(L-y_0)} \int_{\sigma/\lambda+y_0}^L F(y,\sigma)\,dy\,d\sigma.\end{aligned}$$

Dividing by L, taking the limit as $L \to \infty$, and applying the Lebesgue dominated convergence theorem, we find that

$$\begin{aligned}\left\langle G \right\rangle &= \int_0^\infty \lim_{L\to\infty} \left(\chi_L(\sigma)\frac{1}{L}\int_{\sigma/\lambda+y_0}^L F(y,\sigma)\,dy\right) d\sigma \\ &= \int_0^\infty \left\langle F \right\rangle(\sigma)\,d\sigma,\end{aligned}$$

where χ_L is the characteristic function of the interval $[0, \lambda(L-y_0)]$. □

In the notation used in the main part of the paper, we may write this result as

$$\left\langle \int_0^{\lambda(y-y_0)} F(y,\sigma)\, d\sigma \right\rangle = \int_0^{\infty} \left\langle F \right\rangle(\sigma)\, d\sigma,$$

where the angular brackets denote an average with respect to y. More generally, for $\lambda \neq 0$ and $y_0 \in \mathbf{R}$, we have

$$\left\langle \int_0^{\lambda(y-y_0)} F(y,\sigma)\, d\sigma \right\rangle = \int_0^{\mathrm{sgn}(\lambda)\cdot\infty} \left\langle F \right\rangle(\sigma)\, d\sigma,$$

where

$$\mathrm{sgn}(\lambda) = \begin{cases} 1 & \text{if } \lambda > 0, \\ -1 & \text{if } \lambda < 0. \end{cases}$$

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