Sample Prelim Problems
Hilbert Spaces
Fall 2012

1. (Fall, 2007) Let $H$ be the weighted $L^2$ space

$$H = \left\{ f : \mathbb{R} \to \mathbb{C} \mid \int_{\mathbb{R}} e^{-|x|} |f(x)|^2 \, dx < \infty \right\}$$

with inner product

$$(f, g) = \int_{\mathbb{R}} e^{-|x|} \overline{f(x)} g(x) \, dx$$

Let $T : H \to H$ be the translation operator

$$(Tf)(x) = f(x + 1).$$

Compute the adjoint $T^*$ and operator norm $\|T\|$. 

2. (Fall, 2011) Let $H$ be a complex Hilbert space and denote by $\mathcal{B}(H)$ the Banach space of all bounded linear transformations (operators) of $H$ considered with the operator norm.

(a) What does it mean for $A \in \mathcal{B}(H)$ to be compact? Give a definition of compactness of an operator $A$ in terms of properties of the image of bounded sets, e.g., the set \{$Ax \mid x \in H, \|x\| \leq 1$}. 

(b) Suppose that $H$ is separable and let \{\$e_n\}$\_{n \geq 0}$ be an orthonormal basis of $H$. For $n \geq 0$, let $P_n$ denote the orthogonal projection onto the subspace spanned by $e_0, \ldots, e_n$. Prove that $A \in \mathcal{B}(H)$ is compact iff the sequence $(P_nA)_{n \geq 0}$ converges to $A$ in norm.

3. (Spring, 2010) Suppose that $h : [0, 1]^2 \to [0, 1]^2$ is a continuously differentiable function from the square to the square with continuously differentiable inverse $h^{-1}$. Define an operator $T$ on the Hilbert space $L^2([0, 1]^2)$ by the formula $T(f) = f \circ h$. Prove that $T$ is a well-defined bounded operator on this Hilbert space.

4. An operator $A \in \mathcal{B}(H)$ is normal if it commutes with its adjoint. Define $V : L^2(0, 1) \to L^2(0, 1)$ by

$$(Vf)(x) = \int_0^x f(t) \, dt$$
and $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by

$$S(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots)$$

Are either of $V$, $S$ normal?

5. Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_n : n \in \mathbb{N}\}$.

(a) If $(x_k)_{k=1}^{\infty}$ is a sequence in $\mathcal{H}$, show that $x_k \rightharpoonup 0$ as $k \to \infty$ if and only if $(e_n, x_k) \to 0$ as $k \to \infty$ for every $n \in \mathbb{N}$

and $\{\|x_k\| : k \in \mathbb{N}\}$ is bounded. (You can assume the Theorem that every weakly convergent sequence is bounded.)

(b) Give an example of a sequence $(x_k)$ in $\mathcal{H}$ such that $(e_n, x_k) \to 0$ as $k \to \infty$ for every $n \in \mathbb{N}$ but $(x_k)$ does not converge weakly to 0.

6. If $T \in \mathcal{B}(\mathcal{H})$ is a bounded linear operator on a Hilbert space $\mathcal{H}$, prove that $T$ is compact if and only if it maps weakly convergent sequences to strongly convergent sequences.

7. (a) If $A, B \in \mathcal{B}(\mathcal{H})$ are bounded linear operators on a Hilbert space $\mathcal{H}$, prove that

$$\|AB\| \leq \|A\| \|B\|.$$  

(b) If $A \in \mathcal{B}(\mathcal{H})$, prove that $\|A^*\| = \|A\|$ and

$$\|A\|^2 = \|A^*A\|.$$  

8. Define $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$(Tf)(x) = \int_{-\infty}^{x} e^{-(x-y)} f(y) \, dy.$$  

Show that $T$ is well-defined and bounded.

(b) Show that

$$\lambda = \frac{1}{1 + i\omega}$$

is in the continuous spectrum of $T$ for every $\omega \in \mathbb{R}$. 

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