1. (Spring, 2012) For \( u \in L^1(0, \infty) \), consider the integral

\[
v(x) = \int_0^\infty \frac{u(y)}{x+y} \, dy
\]

defined for \( x > 0 \). Show that \( v(x) \) is infinitely differentiable away from the origin. Prove that \( v' \in L^1(\epsilon, \infty) \) for any \( \epsilon > 0 \). How does \( v(x) \) behave as \( x \to 0^+ \)?

2. Show that, for every \( \alpha \in (0; 1] \), there is a constant \( C_\alpha \) such that

\[
\int_0^\infty e^{-\lambda t} \sqrt{1+t^2} \, dt \leq C_\alpha \lambda^\alpha
\]

for all \( \lambda > 0 \), but the analogous inequality for \( \alpha = 0 \) is false.

3. Show, with justification, that

\[
\int_0^\infty \frac{x}{e^x - 1} \, dx = \sum_{n=1}^\infty \frac{1}{n^2}
\]

4. (Fall, 2009) For \( \epsilon > 0 \), let \( \eta_\epsilon \) denote the family of standard mollifiers on \( \mathbb{R}^2 \). Given \( u \in L^2(\mathbb{R}^2) \), define the functions

\[
u \epsilon = \eta_\epsilon * u.
\]

Prove that

\[
\epsilon \|Du\|_{L^2(\mathbb{R}^2)} \leq \|u\|_{L^2(\mathbb{R}^2)}
\]

where the constant \( C \) depends on the mollifying function, but not on \( u \).

5. Suppose that \( f_n, f \in L^p(\mathbb{R}) \), where \( 1 \leq p < \infty \), \( f_n \to f \) pointwise a.e., and \( \|f_n\|_{L^p} \to \|f\|_{L^p} \) as \( n \to \infty \). Show that \( \|f - f_n\|_{L^p} \to 0 \). Does this result remain true for \( p = \infty \)? HINT. Note that

\[
|a - b|^p \leq 2^{p-1} (|a|^p + |b|^p).
\]