Problem 1. (Spring, 2013) Consider the Hilbert space $L^2(0, 1)$ with inner product $\langle f, h \rangle = \int_0^1 f h \, dx$. Let

$$V = \left\{ f \in L^2(0, 1) : \int_0^1 x f(x) \, dx = 0 \right\}$$

and $g(x) = 1$. Find the closest element to $g$ in $V$.

Solution. By the projection theorem, the closest element to $g$ in $V$ is $h = Pg$ where $P : L^2(0, 1) \to L^2(0, 1)$ is the orthogonal projection onto $V$. Moreover, the error $g - h \in V^\perp$ is orthogonal to $V$.

Since $V = \langle x \rangle$, we have $V^\perp = \langle x \rangle^\perp = (\langle x \rangle)$. It follows that $g - h = kx$ for some constant $k$, so $h(x) = 1 - kx$. Then $h \in V$ if

$$\int_0^1 x h(x) \, dx = \int_0^1 (x - kx^2) \, dx = \frac{1}{2} - \frac{1}{3}k = 0,$$

so $k = 3/2$ and the closest element is

$$h(x) = 1 - \frac{3}{2}x.$$

Problem 1. (Fall, 2012) Show that the space of continuous functions on $[0, 1]$ with the sup-norm $\| f \| = \max |f(x)|$ is not a Hilbert space.

Solution. A norm $\| \cdot \|$ is derived from an inner product if and only if it satisfies the parallelogram law

$$\| f - g \|^2 + \| f + g \|^2 = 2 (\| f \|^2 + \| g \|^2),$$

but this fails for the sup-norm.

For example, consider $f(x) = 1$, $g(x) = x$. Then

$$\| f \| = 1, \quad \| g \| = 1, \quad \| f - g \| = 1, \quad \| f + g \| = 2,$$

and

$$\| f - g \|^2 + \| f + g \|^2 \neq 2 (\| f \|^2 + \| g \|^2).$$
Problem 4. (Fall, 2012) A bounded operator on a Hilbert space is normal if it commutes with its adjoint. Define $V : L^2(0, 1) \to L^2(0, 1)$ by

$$(V f)(x) = \int_0^x f(t) \, dt$$

and $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by $S(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots)$. Are either of $V$, $S$ normal?

Solution. Neither $V$ nor $S$ is normal.

To compute $V^*$, we consider for $f, g \in L^2(0, 1)$

$$\langle V f, g \rangle = \int_0^1 \left( \int_0^x f(t) \, dt \right) \overline{g(x)} \, dx = \int_0^1 f(t) \left( \int_t^1 g(x) \, dx \right) \, dt = \langle f, V^* g \rangle,$$

where

$$(V^* g)(x) = \int_x^1 g(t) \, dt.$$

The exchange in the order of integration is justified by Fubini’s theorem, since

$$\int_0^1 \left( \int_0^x |f(t)| \, dt \right) |g(x)| \, dx \leq \left( \int_0^1 |f(t)| \, dt \right) \left( \int_0^1 |g(x)| \, dx \right) \leq \|f\|_{L^2} \|g\|_{L^2}$$

is finite. If $f = 1$, then

$$(V f)(x) = \int_0^x 1 \, dt = x, \quad (V^* V f)(x) = \int_x^1 t \, dt = \frac{1}{2}(1 - x^2),$$

$$(V^* f)(x) = \int_x^1 1 \, dt = 1 - x, \quad (VV^* f)(x) = \int_0^x (1 - t) \, dt = x - \frac{1}{2} x^2,$$

so $V^* V \neq V V^*$.

To compute $S^*$, we consider for $x = (x_i)$ and $y = (y_i)$ in $\ell^2(\mathbb{N})$

$$\langle S x, y \rangle = \sum_{i=2}^{\infty} x_{i-1} \bar{y}_i = \sum_{i=1}^{\infty} x_i \bar{y}_{i+1} = \langle x, S^* y \rangle,$$

where $S^* y = (y_2, y_3, y_4, \ldots)$. If $x = (1, 0, 0, \ldots)$, then

$$S x = (0, 1, 0, 0, \ldots), \quad S^* S x = (1, 0, 0, \ldots),$$

$$S^* x = (0, 0, 0, \ldots), \quad SS^* x = (0, 0, 0, \ldots),$$

so $S^* S \neq SS^*$.