**Problem 5.** (Fall, 2011) Let \( u(x) = (1 + |\log x|)^{-1} \). Prove that \( u \in W^{1,1}(0,1) \) and \( u(0) = 0 \) but \( (u/x) \notin L^1(0,1) \).

**Solution.** Since \( u \in C^\infty(0,1) \) is smooth, its pointwise derivative \( v = u' \),
\[
v(x) = \frac{1}{x(1 + |\log x|)^2},
\]
is also its weak derivative (i.e., \( \int_0^1 u\phi' \, dx = -\int_0^1 v\phi \, dx \) for every \( \phi \in C^\infty_c(0,1) \)). The substitution \( t = 1 + |\log x| \) gives
\[
\int_0^1 \frac{1}{x(1 + |\log x|)^\alpha} \, dx = \int_1^\infty \frac{1}{t^\alpha} \, dt,
\]
which is finite if \( \alpha > 1 \) and infinite if \( \alpha \leq 1 \). It follows that \( v \in L^1(0,1) \) and \( u \in W^{1,1}(0,1) \). Moreover, \( u \) extends to an absolutely continuous function on \([0,1]\) with \( u(0) = \lim_{x \to 0^+} (1 + |\log x|)^{-1} = 0 \). The previous calculation (with \( \alpha = 1 \)) shows that \( (u/x) \notin L^1(0,1) \).

**Problem 6.** (Spring, 2011) Let \( C^{0,\alpha}([0,1]) \) be the Banach space of Hölder continuous functions on \([0,1]\) with exponent \( 0 < \alpha \leq 1 \) and norm
\[
\|u\|_{C^{0,\alpha}} = \sup_{x \in [0,1]} |u(x)| + \sup_{x \neq y \in [0,1]} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}}.
\]
Prove that the closed unit ball \( B = \{ u \in C^{0,\alpha}([0,1]) : \|u\|_{C^{0,\alpha}} \leq 1 \} \) in \( C^{0,\alpha}([0,1]) \) is a compact subset of \( C([0,1]) \) with the sup-norm topology.

**Solution.** By the Arzelà-Ascoli theorem, \( B \) is a compact subset of \( C([0,1]) \) if and only if it is closed, bounded, and equicontinuous. If \( u \in B \), then \( \|u\|_\infty \leq \|u\|_{C^{0,\alpha}} \leq 1 \), where \( \| \cdot \|_\infty \) denotes the sup-norm, so \( B \) is bounded, and \( |u(x) - u(y)| \leq |x - y|^\alpha < \epsilon \) if \( |x - y| < \epsilon^{1/\alpha} \), so \( B \) is equicontinuous. Finally, if \( u_n \in B \) and \( u_n \to u \) in \( C([0,1]) \), then \( u_n \to u \) pointwise and
\[
\frac{|u(x) - u(y)|}{|x - y|^\alpha} = \lim_{n \to \infty} \frac{|u_n(x) - u_n(y)|}{|x - y|^\alpha} \leq 1 \quad \text{for all } x \neq y \in [0,1]
\]
so \( u \in B \), and \( B \) is closed.
Problem 2. (Spring, 2012) Let \( X \subset L^2(0, 2\pi) \) be the set of functions \( u \) such that
\[
u(x) = \sum_{k \in \mathbb{Z}} a_k e^{ikx}, \quad |a_k| \leq \frac{1}{1 + |k|}.
\]
Prove that \( X \) is a compact subset of \( L^2(0, 2\pi) \).

Solution. The \( H^s \)-Sobolev norm of \( u \in X \) with Fourier coefficients \( a_k \) satisfies
\[
\|u\|_{H^s}^2 = \sum_{k \in \mathbb{Z}} (1 + |k|^2)^s |a_k|^2 \leq \sum_{k \in \mathbb{Z}} \frac{(1 + |k|^2)^s}{(1 + |k|)^2}.
\]
The series on the right converges if \( 2 - 2s > 1 \) or \( s < 1/2 \). It follows that \( X \) is a bounded subset of \( H^s(0, 2\pi) \) for \( 0 < s < 1/2 \), and the Rellich theorem implies that \( X \) is a precompact subset of \( L^2(0, 2\pi) \). Furthermore, if \( u_n \to u \) as \( n \to \infty \) in \( L^2(0, 2\pi) \) and \( u_n \in X \), then by the continuity of the inner product,
\[
|a_k| = \frac{1}{2\pi} \left| \int_0^{2\pi} u(x) e^{-ikx} \, dx \right| = \lim_{n \to \infty} \frac{1}{2\pi} \left| \int_0^{2\pi} u_n(x) e^{-ikx} \, dx \right| \leq \frac{1}{1 + |k|},
\]
so \( u \in X \), and \( X \) is closed, which proves that \( X \) is compact.

Remark. For completeness, we prove the version of Rellich’s theorem used here. (It wouldn’t be necessary to do this in an exam!)
If \( s > 1/2 \), then \( H^s \)-functions are Hölder continuous, and the result follows directly from Sobolev embedding and the Arzelà-Ascoli theorem: bounded sets in \( H^s \) are bounded in \( C^0, \alpha \) with \( \alpha = s - 1/2 > 0 \); so they are bounded and equicontinuous and therefore precompact in \( C([0, 2\pi]) \); which implies that they are precompact in \( L^2 \), since uniform convergence is stronger than \( L^2 \)-convergence.

This argument doesn’t work directly if \( 0 < s \leq 1/2 \), when \( H^s \)-functions needn’t even be continuous, but we can fix it up. The idea is to approximate a bounded sequence of \( H^s \)-functions uniformly in \( L^2 \) by sequences of smooth functions (we simply truncate their Fourier series), apply the Arzelà-Ascoli theorem and a diagonal argument to show that there is a subsequence of the original sequence all of whose approximate subsequences converge uniformly, and conclude that the subsequence converges in \( L^2 \).
Theorem 1. If $s > 0$, then $H^s(0, 2\pi)$ is compactly embedded in $L^2(0, 2\pi)$.

Proof. We need to show that a bounded sequence in $H^s$ has a subsequence that converges strongly in $L^2$. If

$$u(x) = \sum_{k \in \mathbb{Z}} a_k e^{ikx}, \quad a_k = \frac{1}{2\pi} \int_0^{2\pi} u(x) e^{-ikx},$$

we use as norms

$$\|u\|_{L^2} = \left( \frac{1}{2\pi} \int_0^{2\pi} |u|^2 \, dx \right)^{1/2} = \left( \sum_{k \in \mathbb{Z}} |a_k|^2 \right)^{1/2},$$

$$\|u\|_{H^s} = \left( \sum_{k \in \mathbb{Z}} (1 + k^2)^s |a_k|^2 \right)^{1/2}.$$  \hfill (1)

For $N \in \mathbb{N}$, we denote the orthogonal projection $u^N \in C^\infty([0, 2\pi])$ of $u \in L^2(0, 2\pi)$ onto the space of trigonometric polynomials of degree less than or equal to $N$ by

$$u^N(x) = \sum_{|k| \leq N} a_k e^{ikx}.$$  \hfill (1)

If $u \in H^s$, then

$$\|u - u^N\|_{L^2} = \left( \sum_{|k| > N} |a_k|^2 \right)^{1/2} \leq \frac{1}{(1 + N^2)^{s/2}} \left( \sum_{|k| > N} (1 + k^2)^s |a_k|^2 \right)^{1/2} \leq \|u\|_{H^s} (1 + N^2)^{s/2}.$$  \hfill (1)

Now suppose that $(u_n)$ is a bounded sequence in $H^s$ with $\|u_n\|_{H^s} \leq R$ for all $n \in \mathbb{N}$. Denoting the Fourier coefficients of $u_n$ by $a_{n,k}$, we have

$$|u^N_n(x)| \leq \sum_{|k| \leq N} |a_{n,k}| \leq (1 + 2N)^{1/2} \left( \sum_{|k| \leq N} |a_{n,k}|^2 \right)^{1/2} \leq C_N R,$$
where \( C_N \) is a generic constant depending on \( N \), and
\[
\left| u_n^N(x) - u_n^N(y) \right| \leq \sum_{|k| \leq N} |a_{n,k}| \cdot |e^{ikx} - e^{iky}|
\leq \sum_{|k| \leq N} |a_{n,k}| \cdot \sqrt{2} |kx - ky|
\leq \sqrt{2} \left( \sum_{|k| \leq N} k^2 \right)^{1/2} \left( \sum_{|k| \leq N} |a_{n,k}|^2 \right)^{1/2} |x - y|
\leq C_N R|x - y|.
\]

It follows that \( \{ u_n^N : n \in \mathbb{N} \} \) is a bounded, equicontinuous subset of \( C([0, 2\pi]) \) for every \( N \in \mathbb{N} \), so it is precompact by the Arzelà-Ascoli theorem.

Using a diagonal argument, we can extract a subsequence \( (u_{n_j}) \) of the original sequence \( (u_n) \) such that \( (u_{n_j}^N) \) converges uniformly as \( j \to \infty \) for every \( N \in \mathbb{N} \). To do this, choose a subsequence \( (u_{n_{j1}}) \) of \( (u_n) \) so that \( (u_{n_{j1}}^N) \) converges uniformly, then choose a subsequence \( (u_{n_{j2}}) \) of \( (u_{n_{j1}}) \) so that \( (u_{n_{j2}}^N) \) converges uniformly, and so on to get successive subsequences \( (u_{n_{jM}}) \) such that \( (u_{n_{jM}}^N) \) converges uniformly as \( j \to \infty \) for every \( 1 \leq M \leq N \), and define \( u_{n_j} = u_{n_{jM}}^N \).

Using (1) and the inequality \( \|u\|_{L^2} \leq \sqrt{2\pi} \|u\|_{L^\infty} \), we get that
\[
\|u_n - u_{n_j}\|_{L^2} \leq \|u_n - u_{n_i}^N\|_{L^2} + \|u_{n_i}^N - u_{n_j}^N\|_{L^2} \leq \frac{2R}{(1 + N^2)^{s/2}} + \sqrt{2\pi} \|u_{n_i}^N - u_{n_j}^N\|_{L^\infty}.
\]

Given \( \epsilon > 0 \), choose \( N \) sufficiently large that
\[
\frac{2R}{(1 + N^2)^{s/2}} < \frac{\epsilon}{2}.
\]

Since \( (u_{n_i}^N) \) converges uniformly, it is uniformly Cauchy, and there exists \( J \in \mathbb{N} \) such that
\[
\sqrt{2\pi} \|u_{n_i}^N - u_{n_j}^N\|_{L^\infty} < \frac{\epsilon}{2} \quad \text{for all } i, j > J.
\]

It follows that \( \|u_n - u_{n_j}\|_{L^2} < \epsilon \) for all \( i, j > J \), so the subsequence \( (u_{n_j}) \) is Cauchy in \( L^2 \), and therefore it converges in \( L^2 \).