Entrainment of Coupled Oscillators
Balancing Order and Disorder in Control of a Complex System

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Motivation
Complex systems are ubiquitous, and often hard to control. Examples of complex systems include the brain, the world economy, gene regulatory networks, and many others.

Complex systems generally have many degrees of freedom, obey nonlinear dynamics, and exhibit self-organization. These features all make it difficult to robustly predict the result of applying a control signal.

⇒ Q: How can we understand generic limitations on control of self-organizing systems?

Main Objectives
• Study an analytically tractable model of control applied to a self-organizing system
• Draw conclusions about the trade-offs between order and disorder inherent in controlling a self-organizing system

Model Description
Our model consists of a globally-forced population of coupled, nonlinear oscillators. The reasons for choosing this model are twofold:
• Individual oscillators subject to periodic input display intrinsic and analytically tractable dynamics
• Coupled oscillators exhibit a synchronization transition, a canonical example of self-organization as a function of the balance between order and disorder

Entrainment
The basic unit of our system is a phase model of a forced, nonlinear oscillator. For sufficiently weak forcing ω(ϕ), we have [3]:

\[ \dot{\phi} = \omega + \Omega(\phi) \] (1)

where \( \omega \in S^1 \) is the phase of oscillation, \( \omega \) is the natural (unforced) frequency, and \( \Omega \) is the phase response curve (PRC). If \( \Omega \) is periodic with frequency \( \Omega \sim \omega \), i.e. \( \omega(\phi) = \omega(\phi+2\pi) \) with \( 2\pi \)-periodic, then \( \dot{\phi} = \omega - \Omega(\phi) \).

Synchronization
The other piece of our model is the theory of coupled oscillators, which goes back to Winfree [4] and Kuramoto [1]. The so-called Kuramoto model is the ODE

\[ \dot{\theta}_i = \omega_i + \sum_{j=1}^{N} S_{ij} \sin(\theta_j - \theta_i) \] (4)

The key features of the model are:
• Disorder: \( \omega \neq \omega_i \)
• Order: Coupling drives different phases together

At a critical coupling strength \( K_{c} \), order overcomes disorder and the oscillators begin to synchronize, despite their different frequencies. If \( \omega(\phi) \) are drawn from a probability distribution with density \( g(\omega) \), then

\[ K_{c} = \frac{\pi}{\langle |g(\omega)| \rangle} \] (5)

Entrainment & Synchronization
The general class of problem we propose to study is

\[ \dot{\phi}_i = \omega_i - \lambda(\phi_i) + \sum_{j=1}^{N} G(\phi_i - \phi_j) \] (6)

As a first case, we suppose approximate the case where \( G(\phi) = \sin(\phi) \). If \( \omega_i \) are distributed uniformly on \((-\pi, \pi)\), and \( \lambda(\phi) \) is a sawtooth function:

\[ \lambda(\phi) = \frac{\sin(\phi)}{\phi} \] (7)

We make these choices because in the absence of coupling, the phases settle down to \( (\phi_i - \pi) \), which is as far as possible from complete phase synchronization. Hence this is the most disordered possible frequency-locked state, and we quantify this disorder by finding the linear stability of \( \phi_i \) as a function of coupling strength \( K \).

Methods & Results
The tools we use are linear stability analysis in both the finite-dimensional case described above, and in an appropriately defined infinite-\( N \) limit.

In finite dimensions (i.e. for finite \( N \), linear stability of the fixed point \( \phi^* \) is determined by the eigenvalues of the Jacobian matrix

\[ J = (\partial_{\phi_j} \lambda(\phi_i))(\phi_i - \phi_j) \], whose elements are

\[ \lambda_1 = -\sum_{j=1}^{N} \frac{\sin(\phi)}{\phi} \] (8)

Bounding the eigenvalues of \( J \) using the Gershgorin circle theorem, and the fact \( |\sin(\phi)| \leq 1 \), we find that \( \phi^* \) is linearly stable provided \( K < \frac{\pi}{\pi} \). In other words, \( K_{c} \geq \frac{\pi}{\pi} \).

The infinite limit
Following much of the seminal work on the Kuramoto model, and in particular [2], we describe our system in the limit \( N \to \infty \) in terms of distributions over values of \( (\omega, \phi) \).

In particular, we take a distribution of natural frequencies with density \( g(\omega) \), and to every natural frequency \( \omega \) we associate a distribution over phases, \( \rho_\omega \in \mathcal{P}(S^1) \).

Conservation of oscillator number for every natural frequency gives a continuity equation:

\[ \partial_t \rho_\omega + \partial_\phi \omega \rho_\omega = 0 \] (9)

where \( D \omega \) is the derivative in the sense of distributions, and the phase velocity \( v_\omega \) gives \( \omega \) for an oscillator of natural frequency \( \omega \),

\[ v_\omega(\phi) = \omega - \lambda(\phi) + \sum_{j=1}^{N} G(\phi - \phi_j) \] (10)

The fixed point analogous to \( \phi^* \) is the family of distributions \( \rho_\infty \), \( \rho_\infty = \sum_j \rho_\omega \), in other words, the state in which an oscillator with frequency \( \omega \) has phase \( \phi = v_\omega(\phi) \).

To linearize the continuity equation (9) around \( \rho_\infty \), consider small perturbations, \( \rho = \rho_\infty + \rho_\omega \). Inserting this form into the continuity equation, we obtain at order \( \epsilon \):

\[ \partial_t \rho_\omega + \partial_\phi \omega \rho_\omega = 0 \] (10)

where \( L \) is linear. It turns out that permissible \( \eta \) have the form

\[ \eta = \frac{1}{\epsilon} (\omega - \lambda(\phi)) \] (11)

where \( \epsilon \in \mathcal{L}([\pi]), \). Thus we can consider \( L \) as an operator on the much simpler vector space \( \mathcal{L}([\pi]) \). In fact, we find that \( L \) is diagonal in the Fourier basis of \( \mathcal{L}([\pi]) \). The spectrum of \( L \) is

\[ \sigma(L) = \left\{ -\frac{1}{\epsilon} - \frac{1}{\epsilon^2} \right\} \] (12)

So we conclude that \( \phi^* \) is linearly unstable if \( K < \frac{\pi}{\pi} \) and linearly stable if \( K > \frac{\pi}{\pi} \). In other words, \( K_{c} = \frac{\pi}{\pi} \).

Conclusions
Forcing that entrains the oscillators has decreased the critical coupling strength for synchronization (see (5)):

\[ K_{c} = \frac{2}{\pi} - \frac{2}{\pi^2} \frac{1}{\pi} \]

In this sense, entrainment is an ordering influence on the system, despite phase diversity.

Forthcoming Research
• Realistic control signal and coupling functions : the sexton interaction function is not actually achievable given \( \sum \epsilon_i \leq \pi \), and coupling function \( G \) will generally not be sin. What conclusions still hold?
• Non-global coupling : Can non-global control help us analyze order-disorder trade-offs for more complicated states?
• Subharmonic Entrainment : How does the picture change if there is a diversity of response frequencies? Can structural features inform subharmonic selection?
• Adaptive coupling : Can the system learn in response to input?

References

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