Research summary for experts

1 Heegaard splittings, structure and genera

The first line of investigation I would like to highlight concerns Heegaard splittings, both their structure and their genera. Some time ago, I proved structural theorems for Heegaard splittings of Seifert fibered spaces (joint with Yoav Moriah) and, more generally, graph manifolds. See [2] and [6]. These structural theorems described how to build Heegaard splittings for members of the given family of manifolds. For a Seifert fibered space M, the structural theorem immediately gives the Heegaard genus of M. For a graph manifold, the Heegaard genus can be determined by a computation whose complexity depends on the complexity of the graph underlying the graph manifold. Interestingly, this allowed Richard Weidmann and me to prove that the discrepancy between the Heegaard genus and the rank of the fundamental group of a graph manifold can be arbitrarily large. See [10].

I am interested in a phenomenon known as "degeneration" of Heegaard genus. This phenomenon concerns the behavior of Heegaard genus under gluings. For instance, in the case of Dehn surgery, Heegaard genus can go down. Work of Moriah-Rubinstein, Rieck-Sedgwick, and Futer-Purcell points to the specialized settings in which this degeneration occurs. I hope to construct more examples exhibiting this degeneration in an effort to close the gap between what is known about when this phenomenon can't occur and when it can and does occur. Dehn surgery constitutes a special case of gluing two manifolds together along boundary components. In the more general context of gluing manifolds I accomplished several things: In [8], I established a lower bound on the genus of a manifold obtained by gluing two manifolds along boundary components. This work generalized joint work with Martin Scharlemann, see [3], [4], [5] and [7]. In addition, in joint work with Richard Weidmann, I provided examples of degeneration of genus when two manifolds are glued along a torus. Here too, I hope to construct more examples, in an effort to fully describe the phenomenon of degeneration of Heegaard genus.

2 Thin manifold decompositions

The second line of investigation I would like to highlight concerns manifold decompositions. Heegaard splittings correspond to Morse functions, specifically self-indexing Morse functions. It turns out that more general handle decompositions, those corresponding to Morse functions that are not necessarily self-indexing, are also of interest. Scharlemann-Thompson defined a notion of width for such handle decompositions.

Thin manifold decompositions, those minimizing the width, possess many fortuitous properties. Cerf theory provides enough structure to relate all Morse functions on a manifold and therefore also all possible handle decompositions of the manifold. In [11], I explore this relation by defining the width complex. This investigation is ongoing.

My second book, joint with Saito and Scharlemann, describes handle decompositions of manifolds more generally. See [15]. They key contribution in the book lies in the development of fork complexes. Fork complexes provide a description of certain graphs underlying handle decompositions of manifolds. In particular, they provide a descriptive tool that can be used to assess issues pertaining to complexity. Complexity drives much of the current discussion of algorithms. In joint work with Kristof Huszar, I plan to obtain thin manifold decompositions for particular families of manifolds. These thin manifold decompositions will enable certain algorithms to run more efficiently on these families of manifolds, most notably fixed parameter tractable algorithms.

3 Surface complexes

The third line of investigation I would like to highlight concerns surface complexes. Codimension 1 submanifolds of a manifold M provide insight into the geometry, topology, and self-automorphisms of M. This is readily seen in the plentiful and beautiful work accomplished concerning the curve complex of a surface. One dimension up, Kakimizu defined an analogous complex on knot complements. Specifically, the Kakimizu complex is the flag complex in which vertices correspond to isotopy classes of Seifert surfaces and edges correspond to pairs of vertices with disjoint representatives. This definition mimics the definition of the curve complex, but requires adjustment in the context of more general 3-manifolds.

In [13], Piotr Przytycki and I proved that the Kakimizu complex of a knot is contractible. Building on this foundational result, I found two, mutually informative, ways to generalize this construction to all 3-manifolds. One such generalization, called the Kakimizu complex, depends not only on a given manifold M, but also on a second relative homology class $\alpha \in H_2(M, \partial M)$. The other proceeds from the definition of the curve complex and/or Kakimizu complex of a knot complement and builds a tower of complexes to ensure that the resulting complex is connected. The advantage of the former lies in the tools available for its study. The advantage of the latter is a cleaner formulation. With the help of the former, the latter can be computed in certain instances. See [16] and [17].

My research group, consisting, in addition to myself, of postdoctoral researcher Josh Howie and graduate students Andrew Alameda and Neetal Neel, is currently investigating classes of knots with a view towards computing their Kakimizu complexes. Sutured manifold theory proves extremely useful in studying Kakimizu complexes of knots. Indeed, some of the results concerning Kakimizu complexes of knots could benefit from being described in terms of sutured manifold theory, as accomplished so elegantly in [1].

References

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