Problem 1. Consider the sequence

\[ a_1 = 1, \quad a_2 = \int_1^2 \frac{1}{x} \, dx, \quad a_3 = \frac{1}{2}, \quad \ldots, \quad a_{2k-1} = \frac{1}{k}, \quad a_{2k} = \int_{k-1}^{k} \frac{1}{x} \, dx \]

i. Prove that \( a_k \) is a monotonic decreasing sequence, converging to 0.
ii. Conclude that \( \sum (-1)^{k-1}a_k \) is a convergent series.
iii. Justify that

\[ \sum_{k=1}^{n} \frac{1}{k} \sim \ln n, \quad n \to \infty \]

Problem 2. Are the series

a. \( \sum_{k=1}^{+\infty} (-1)^{k+1} \frac{\ln k}{k - \ln k} \)

b. \( \sum_{k=1}^{+\infty} (-1)^k \frac{1}{\ln(k+1)} \)

absolutely convergent?

Problem 3. Find the interval of convergence of the following power series

a. \( \sum_{k=0}^{\infty} (2x)^k \)

b. \( \sum_{k=1}^{\infty} \frac{x^k}{k\sqrt{k}3^k} \)

Problem 4. Determine the radius of convergence \( R \) of the following power series

a. \( \sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!} x^k \)

b. \( \sum_{k=1}^{\infty} \left(1 + \frac{1}{k}\right)^k x^k \).

Answers.

2 a) Conditionally converges (alternating series + n-th test); b) Conditionally converges (alternating series test + direct comparison test with harmonic series

3 a) \( (-\frac{1}{2}, \frac{1}{2}) \); b) \([-3, 3])

4 a) \( R = 4 \); b) \( R = \frac{1}{e} \).