Problem 1.

a. Let $D$ be the solid region enclosed by the paraboloids $z = 4 - x^2 - y^2$ and $z = 3(x^2 + y^2)$. Find the volume of $D$

Sol The intersection of the paraboloids is given by $x^2 + y^2 = 1$. Using cylindrical coordinates, the region $D$ is described by

$$\begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 1 \\ 3r^2 \leq z \leq 4 - r^2 \end{cases}$$

The volume of $D$ is given by the integral

$$\text{Vol}(D) = \iiint_D dV = \int_0^{2\pi} \int_0^1 \int_{3r^2}^{4-r^2} r \, dz \, dr \, d\theta = 2\pi$$

b. Find the volume of the solid formed by the intersection of two cylinders of radius 1 that meet at right angles, $x^2 + z^2 = 1$, and $y^2 + z^2 = 1$.

Hint Fixed the limits of $z$ first.

Sol Considering the limits of $z$ are given by $-1 \leq z \leq 1$, the limits for each variable $x$ and $y$ are determined by the corresponding equation of each cylinder, i.e.

$$\begin{cases} -1 \leq z \leq 1 \\ -\sqrt{1 - z^2} \leq x \leq \sqrt{1 - z^2} \\ -\sqrt{1 - z^2} \leq y \leq \sqrt{1 - z^2} \end{cases}$$

Actually, for fixed $z$, the limits of $x$ and $y$ is a square. The volume is given by

$$\text{Vol}(D) = \iiint_D dV = \int_{-1}^{1} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} dy \, dx \, dz = \frac{16}{3}$$
Problem 2.

a. Evaluate the integral

$$\int \int_{R} x^2 y^4 \, dA,$$

where $R$ is the region bounded by $xy = 4$, $xy = 10$, $y = x$, and $y = 6x$, using the transformation $x = 2\sqrt{\frac{u}{v}}$, $y = 4\sqrt{uv}$.

Sol Using the transformation, the limits are set by

$$xy = 4 \quad \iff \quad u = \frac{1}{2}$$
$$xy = 10 \quad \iff \quad u = \frac{5}{4}$$
$$y = 6x \quad \iff \quad v = 3$$
$$y = x \quad \iff \quad v = \frac{1}{2}$$

On the other hand, the absolute value determinant of the Jacobian of the transformation is given by

$$|DT(u, v)| = \begin{vmatrix} u^{-1/2}v^{-1/2} & -u^{1/2}v^{-3/2} \\ 2u^{-1/2}v^{1/2} & 2u^{1/2}v^{-1/2} \end{vmatrix} = 4v^{-1}$$

Finally, the integral is given by

$$\int \int_{R} x^2 y^4 \, dA = \int_{\frac{1}{2}}^{\frac{5}{4}} \int_{\frac{1}{2}}^{3} 4u^4 v^3 4v^{-1} = 10(5^4 - 2^4)$$

b. Find the TNB frame for the curve

$$r(t) = (\cos t + t \sin t) \hat{i} + (\sin t - t \cos t) \hat{j} + \frac{1}{2} t^2 \hat{k},$$

at the point $t = 2\pi$.

Sol

$$\frac{dr}{dt} = t(\cos t \hat{i} + \sin t \hat{j} + 1\hat{k})$$

$$T = \frac{1}{\sqrt{2}}(\cos t \hat{i} + \sin t \hat{j} + 1\hat{k})$$

$$\frac{dT}{dt} = \frac{1}{\sqrt{2}}(-\sin t \hat{i} + \cos t \hat{j})$$

$$N = -\sin t \hat{i} + \cos t \hat{j}$$

Evaluating at $t = 2\pi$, $T(2\pi) = \frac{1}{\sqrt{2}}(1 \hat{i} + 1 \hat{k})$, and $N(2\pi) = 1 \hat{j}$. Finally,

$$B(2\pi) = T(2\pi) \times N(2\pi) = \frac{1}{\sqrt{2}}(-1 \hat{i} + 1 \hat{k}).$$
Problem 3.

a. Let \( \Gamma \) be the curve consisting on line segments connecting \((1, 0, 1) \rightarrow (1, 0, 0) \rightarrow (0, 0, 0) \rightarrow (1, \frac{\pi}{2}, 0)\). Compute the line integral
\[
\int_{\Gamma} \mathbf{F} \cdot d\mathbf{r}
\]
for the vector field
\[
\mathbf{F} = e^{yz} \mathbf{i} + (xze^{yz} + z \cos y) \mathbf{j} + (xye^{yz} + \sin y) \mathbf{k}.
\]
Sol For this problem, one can proceed in several ways. Using the component test is easy to see that \( \mathbf{F} \) is a conservative field, i.e., there exists a potential function \( f \) such that \( \mathbf{F} = \nabla f \). This potential function is given by
\[
f(x, y, z) = xe^{yz} + z \sin y + C
\]
And then, the line integral is given by the difference of potential in the endpoints,
\[
\int_{\Gamma} \mathbf{F} \cdot d\mathbf{r} = f(1, \frac{\pi}{2}, 0) - f(1, 0, 1) = 0
\]

b. Compute the area of that portion of the paraboloid \( x^2 + z^2 = y \), which is cut off by the plane \( y = 1 \).
Sol One solution for this problem is to define this surface implicitly using the function \( H(x, y, z) = x^2 - y + z^2 \) as \( H(x, y, z) = 0 \). Here, the projection of the surface on the \( xz \) plane is, \( R \), is given by the unit circle, and the normal vector is given by \( \mathbf{p} = \mathbf{j} \). On the other hand, the differential element of surface in this case is given by
\[
d\sigma = \frac{|\nabla H|}{|\nabla H \cdot \mathbf{p}|} = \sqrt{4x^4 + 1 + 4z^2} \, dx \, dz
\]
The area is given by
\[
A(S) = \iint_{R} \sqrt{4x^4 + 1 + 4z^2} \, dx \, dz = 2\pi \frac{1}{12} (5\sqrt{5} - 1)
\]
Problem 4. Let \( F = xz \hat{i} - y \hat{j} + x^2y \hat{k} \). The surface \( S \) consists of the three faces not in the \( xz \)-plane of the tetrahedron bounded by the three coordinate planes and the plane \( 3x + y + 3z = 6 \). Compute the surface integral
\[
\iint_S (\text{curl } F) \cdot \mathbf{n} \, d\sigma,
\]
where \( \mathbf{n} \) is the unit normal pointing out of the tetrahedron.

*Hint* Stokes' Theorem.

**Sol** Using Stokes' theorem, the integral is given by
\[
\iint_S (\text{curl } F) \cdot \mathbf{n} \, d\sigma = \int_{\Gamma} \mathbf{F} \cdot d\mathbf{r},
\]
where \( \Gamma \) is the boundary of the surface \( S \) (on the \( xz \)-plane). Parametrizing \( \Gamma \) in its three line segments connecting \((0, 0, 0) \rightarrow (0, 0, 2) \rightarrow (2, 0, 0) \rightarrow (0, 0, 0)\), \( \Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \), respectively. The line integrals are given by

\( \Gamma_1: r_1(t) = t \hat{k}, \quad 0 \leq t \leq 2 \). In this segment, \( \mathbf{F}(r_1) \cdot dr_1 = 0 \).

\( \Gamma_2: r_2(t) = 2t \hat{i} + (2 - 2t) \hat{k}, \quad 0 \leq t \leq 1 \). The line integral in this segment is given by
\[
\int_{\Gamma_2} \mathbf{F} \cdot dr_2 = \frac{4}{3}.
\]

\( \Gamma_3: r_3(t) = 2(1-t) \hat{i}, \quad 0 \leq t \leq 1 \). In this segment, \( \mathbf{F}(r_3) \cdot dr_3 = 0 \).

Finally,
\[
\iint_S (\text{curl } F) \cdot \mathbf{n} \, d\sigma = \int_{\Gamma_1} \mathbf{F} \cdot dr_1 + \int_{\Gamma_2} \mathbf{F} \cdot dr_2 + \int_{\Gamma_3} \mathbf{F} \cdot dr_3 = \frac{4}{3}.
\]
Problem 5. Let \( S \) be the surface of the unit cube, \( 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1 \), and let \( n \) be the unit outer normal to \( S \). If \( \mathbf{F} = x^2 \mathbf{i} + y^2 \mathbf{j} + z^2 \mathbf{k} \), use the divergence theorem to evaluate the surface integral \( \iint_S \mathbf{F} \cdot n \, d\sigma \). Verify the result by evaluating the surface integral directly.

Sol Using the divergence theorem, \( \nabla \cdot \mathbf{F} = 2x + 2y + 2z \), then,

\[
\iint_S \mathbf{F} \cdot n \, d\sigma = \iiint_D \nabla \cdot \mathbf{F} \, dV = \int_0^1 \int_0^1 \int_0^1 (2x + 2y + 2z) \, dx \, dy \, dz = 3
\]

On the other hand, for computing the flux integral directly, consider the 6 faces of the cube, denoted by \( S = S_{x=0} \cup S_{x=1} \cup S_{y=0} \cup S_{y=1} \cup S_{z=0} \cup S_{z=1} \).

\( S_{x=0} \): For this face, it is easy to see that the outward pointing normal vector is given by \(-\mathbf{i}\), but in this direction, the vector field has a 0 component. Therefore,

\[
\iint_{S_{x=0}} \mathbf{F} \cdot n \, d\sigma = 0.
\]

\( S_{y=0}, S_{z=0} \): Both integrals are 0, following the same argument developed for \( S_{x=0} \).

\( S_{x=1} \): This surface can be parametrized by \( r(u, v) = 1 \mathbf{i} + u \mathbf{j} + v \mathbf{k}, \) for \( 0 \leq u, v \leq 1 \). Then, \( n \, d\sigma = \mathbf{i} \), and over this surface \( \mathbf{F}(r) \cdot n \, d\sigma = 1 \). The flux is given by

\[
\iint_{S_{x=1}} \mathbf{F} \cdot n \, d\sigma = \int_0^1 \int_0^1 dy \, dz = 1
\]

\( S_{y=1}, S_{z=1} \): Both integrals are 1, following the same argument developed for \( S_{x=1} \).

Finally, the flux is given by

\[
\iint_S \mathbf{F} \cdot n \, d\sigma = \iint_{S_{x=0}} \mathbf{F} \cdot n \, d\sigma + \iint_{S_{x=1}} \mathbf{F} \cdot n \, d\sigma + \iint_{S_{y=0}} \mathbf{F} \cdot n \, d\sigma + \iint_{S_{y=1}} \mathbf{F} \cdot n \, d\sigma + \iint_{S_{z=0}} \mathbf{F} \cdot n \, d\sigma + \iint_{S_{z=1}} \mathbf{F} \cdot n \, d\sigma = 3
\]