An Introduction to Voting Mechanisms

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May 18, 2016
Outline

1. Examples of Voting Methods
   1.1 Plurality Voting
   1.2 Instant-runoff Voting
   1.3 Borda Count

2. The Gibbard–Satterthwaite Theorem

3. More Voting Methods
   3.1 Random Ballot
   3.2 Approval Voting
   3.3 Range Voting

4. Conclusion
A single-winner voting method has the following ingredients:

- A society of individuals (or voters)
- A collection of available alternatives or candidates that affect the society; exactly one must be chosen
- A ballot: how a voter expresses preferences over the alternatives
- A social choice function that aggregates all the voters’ ballots and chooses a winning alternative, the social choice
Plurality Voting

Also called “first-past-the-post” (UK, Canada).

**The method**

- Each voter chooses one alternative
- The most-chosen alternative wins
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**Some current uses**

- Almost all US elections for public office
- Dozens of countries around the world
- Among all Western democracies, only the US, UK, and Canada use it to elect their legislatures
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Plurality is often combined with a possible runoff, creating a method called two-round majority (most common worldwide).
Plurality Voting

Advantages

▶ Simplicity

Disadvantages

▶ Vote splitting
▶ “Spoiler effect”
▶ “Lesser of two evils” dilemma
▶ Wasted votes
▶ “Favorite betrayal”

Uses the least possible amount of preference info from each voter (“just pick one”)

Duverger’s law: plurality favors a two-party system
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Instant-runoff Voting

Also called the “alternative vote” (UK and Canada), “preferential voting” (Australia), and “ranked choice voting” (US).
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The method

- Each voter linearly ranks the alternatives in order of preference (no ties allowed)
- Count all the first choices
- While no alternative has a majority of first choices
  - Eliminate the alternative with the fewest first choices
  - Update the rankings; recount the first choices
- Alternative with the majority of first choices wins
Example. 57 voters and 3 alternatives A, B, C.

<table>
<thead>
<tr>
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<tbody>
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</tr>
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(From rangevoting.org)
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Winner: A, after C is eliminated.
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Failure of **rational honest participation.**
Instant-runoff Voting

Advantages

- Fairly expressive for voters
- Mitigates the spoiler effect
- Complexity makes strategy difficult
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▶ Fairly expressive for voters
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Disadvantages

▶ Algorithm confuses voters; little trust in the results
▶ Cannot express indifference
▶ Non-monotonic
▶ Can be better to abstain than to vote honestly
▶ May favor a two-party system (see: Australia)
▶ “Reversal failure”: inverting preferences may preserve outcome
▶ Logically flawed? Why does fewest 1st-place votes ⇒ worst alternative?
Instant-runoff Voting

Some current uses

- Australian House of Representatives
- President of India
- In the US: San Francisco (mayor), Berkeley CA, Hugo Awards for science fiction, Oscar for Best Picture

Instant-runoff is advocated in the US by the nonprofit organization (fairvote.org). They are currently lobbying for instant-runoff in Maine (rcvmaine.com):
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Named after the 18th-century French mathematician Jean-Charles de Borda.

The method

- Assume there are $n \geq 1$ alternatives
- Each voter linearly ranks the alternatives in order of preference (no ties allowed)
- For each ballot, an alternative ranked in $k$-th place receives $n - k$ points
- Add up the points from all the ballots
- Most points wins

Some current uses

- National Assembly of Slovenia
- Heisman trophy
- MLB MVP
- Some universities and private organizations
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Example. Three friends (Alice, Bob, Carl) want to watch a movie. Choices: Casino Royale (CR) or The Terminator (TT).

They will vote with the Borda count to decide. True preferences:

Alice, Bob: CR > TT
Carl: TT > CR

Total points: CR = 6 and TT = 7. TT wins.

This is called strategic nomination or candidate cloning.
Criterion: independence of clones. Includes immunity to spoilers (plurality), teams (Borda), and crowds. 
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- Fairly expressive for voters
- Monotonic
- Not too complicated

Disadvantages

- Cannot express indifference
- Strategic nomination (teaming)
- "Burying" strategy: rank a not-least-preferred alternative last
- "Compromising" strategy: rank a not-most-preferred alternative first
- Compromising and burying are very tempting when there are 2 clear frontrunners; highly distorts the outcome

"My scheme is intended only for honest men." – de Borda
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Strategic manipulability is not a property of just a few “bad” voting methods.

In fact, all “rank-based” voting methods can be manipulated in at least one way...
The Gibbard–Satterthwaite Theorem

Proven independently by the philosopher Allan Gibbard (1973) and the economist Mark Satterthwaite (1975).

Let \( A \) be a nonempty finite set of \textit{alternatives} and \( V \) a nonempty finite set of \textit{voters}.
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Let $A$ be a nonempty finite set of alternatives and $V$ a nonempty finite set of voters.

Let $\mathcal{L}(A)$ be the set of all linear orders on $A$. These are binary relations on $A$ that are

(i) Transitive: $x \preceq y$ and $y \preceq z \implies x \preceq z \quad \forall x, y, z \in A$

(ii) Total: $x \preceq y$ or $y \preceq x \quad \forall x, y \in A$

(iii) Antisymmetric: $x \preceq y$ and $y \preceq x \implies x = y \quad \forall x, y \in A$
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We call $\preceq \in \mathcal{L}(A)$ a strict preference relation. Let

$$\mathcal{L}(A)^V = \prod_{i \in V} \mathcal{L}(A)$$

be the set of all profiles $(\preceq_i)_{i \in V}$ of strict preference relations.
The Gibbard–Satterthwaite Theorem

A function $F : \mathcal{L}(A)^V \to A$ is called a **social choice function**.

Examples: instant-runoff, Borda count, plurality voting.
The Gibbard–Satterthwaite Theorem

A function $F : \mathcal{L}(A)^V \rightarrow A$ is called a social choice function.

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Properties

- $F$ is **onto** if $F(\mathcal{L}(A)^V) = A$

  Every alternative has a possibility of winning
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**Properties**

- **\( F \) is onto** if \( F(\mathcal{L}(A)^V) = A \)
  
  Every alternative has a possibility of winning

- **\( F \) is strategy-proof** if for any \( i \in V \), any \((\preceq_j)_{j \in V} \in \mathcal{L}(A)^V\), and any \( \preceq'_i \in \mathcal{L}(A) \),

  \[
  \star \quad F(\preceq'_i, \preceq_{-i}) \preceq_i F((\preceq_j)_{j \in V}) \quad \star 
  \]

  where \( \preceq_{-i} = (\preceq_j)_{j \neq i} \)
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- $F$ is **dictatorial** if there is an $i \in V$ such that for any $(\preceq_j)_{j \in V} \in \mathcal{L}(A)^V$,
  
  $F((\preceq_j)_{j \in V}) \in \max(\preceq_i)$
The Gibbard–Satterthwaite Theorem

Theorem (Gibbard–Satterthwaite)

If $|A| \geq 3$ and $F : \mathcal{L}(A)^V \to A$ is onto and strategy-proof, then $F$ is dictatorial.

“For every reasonable*, deterministic, ordinal voting method over $\geq 3$ alternatives, there are situations in which lying pays.”

*onto, non-dictatorial
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Notes

- If $|A| = 2$, then plurality voting is a counterexample to G–S
- G–S still holds for non-strict preference relations (drop the antisymmetric requirement)
- Possible escape routes
  - Use a stochastic $F$
  - Use a non-ordinal voting system?
Random Ballot

The method

- Each voter chooses one alternative
- Select a voter uniformly at random
- Elect that voter’s choice

Current uses: none?
Random Ballot

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Current uses: none?

Advantages

- Onto, strategy-proof (avoids the G–S theorem!), independent of clones, rational participation

Disadvantages

- Democratic only in mathematical expectation
- Possibility for a terrible winner
Approval Voting

Introduced in 1977–1978, mainly by Steven Brams (political scientist) and Peter Fishburn (mathematician).

The method

- Each voter votes for any number of alternatives
- The alternative with the most votes wins
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Some current uses

- UN Secretary-General, MAA, AMS, ASA, INFORMS, Webby Awards, SFSU, Texas Green Party
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**Advocated** by a nonprofit, founded in 2011 (electology.org):
Approval Voting

Advantages

- Simplicity
- Always rational to vote for one’s favorite candidate
  - No wasted votes
- Tends to favor compromise candidates
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- Rational participation
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Disadvantages

▶ Not very expressive for voters; cannot express all preferences
▶ Wide range of sincere votes allows for manipulation
The method

- Let $N$ be a positive integer (a parameter)
- Each voter assigns a score from 0, 1, \ldots, $N$ to each alternative
- Highest summed score wins

Note: reduces to approval voting when $N = 1$. 
Range Voting

The method

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Some current uses

- German Pirate Party; no other political uses?
- Valedictorians, teaching evaluations
- Webby Awards, Mozilla
- Many TV competitions

Advocated by Warren Smith, Ph.D. (rangevoting.org) and The Center for Election Science.
Range Voting

Advantages

- Fairly simple
- Most expressive voting method
- Same advantages as approval voting

Disadvantages

- Temptation to exaggerate scores
- In most situations, “approval style” scoring is optimal (give only max/min scores)
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At the 2010 *Voting Power in Practice* workshop in Normandy, France, 22 professional voting theorists were asked:

“What is the best voting rule for your town to use to elect the mayor?”

They voted on 18 voting methods using approval voting.

Average number of approvals: 3.55.

Voting Theorists Vote About Voting Methods

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<th>Voting rule</th>
<th>Approvals</th>
<th>Approving percentage</th>
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<tbody>
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<td>Alternative vote</td>
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<td>Copeland</td>
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Conclusion

It’s tempting to believe that simply “holding a vote” will somehow locate the best alternative for society.

But the precise method used is crucially important.

The main concerns are:

1. What information do we ask voters to provide about their preferences?
2. From that information, how do we determine the best option for society as a whole?
3. How do we get voters to vote honestly and not try to cheat the system?

Satisfactory answers are necessary for a functioning democracy.