Preliminaries: Convexity

1. Prove that the image of a convex set under linear transformation is again a convex set.

2. For a set of points $A \subset \mathbb{R}^d$,

   \[ \text{conv}(A) = \left\{ \sum_{i=1}^{k} \gamma_i x_i : \sum_{i=1}^{k} \gamma_i = 1 \text{ and } \gamma_i \geq 0, x_i \in A \text{ for all } i \in \{1, ..., k\} \right\}. \]

3. Write down a detailed proof of the separation theorem: Let $C, D \subset \mathbb{R}^d$ be convex sets with $C \cap D = \emptyset$. Then there exists a hyperplane $h$ such that $C$ lies in one of the closed half-spaces determined by $h$, and $D$ lies in the opposite closed half-space. In other words, there exist a unit vector $a \in \mathbb{R}^d$ and a number $b \in \mathbb{R}$ such that for all $x \in C$ we have $\langle a, x \rangle \geq b$, and for all $x \in D$ we have $\langle a, x \rangle \leq b$. If $C$ and $D$ are closed and at least one of them is bounded, they can be separated strictly; in such a way that $C \cap h = D \cap h = \emptyset$.

4. A set $C \subset \mathbb{R}^d$ is a convex cone if it is convex and for each $x \in C$, the ray $\vec{0}x$ is fully contained in $C$. Let $C$ be a closed, convex cone in $\mathbb{R}^d$ and $b \not\in C$ a point. Prove that there exists a vector $a$ with $\langle a, x \rangle \geq 0$ for all $x \in C$ and $\langle a, b \rangle < 0$.

5. Let $X \subset \mathbb{R}^d$. Prove that $\text{diam}(\text{conv}(X)) = \text{diam}(X)$.

Caratheodory’s Theorem, Hilbert Bases, Circuits and Graver Bases

1. In the course of the proof of Caratheodory’s Theorem, given

   \[ x = \sum_{i=1}^{k} \lambda_i a_i \text{ with } \lambda_i > 0 \text{ and } \sum_{i=1}^{k} \lambda_i = 1, \]

   and also

   \[ \sum_{i=1}^{k} \mu_i a_i = 0, \text{ with } \sum_{i=1}^{k} \mu_i = 0 \text{ and not all } \mu_i = 0, \]

   we chose an index $j$ such that

   \[ \left| \frac{\mu_j}{\lambda_j} \right| \geq \left| \frac{\mu_i}{\lambda_i} \right| \]
for all $i \neq j$ and formed

$$(\lambda_1 - \frac{\lambda_j}{\mu_j})a_1 + ... + (\lambda_k - \frac{\lambda_j}{\mu_j})a_k.$$ 

(a) Show that

$$(\lambda_1 - \frac{\lambda_j}{\mu_j})a_1 + ... + (\lambda_k - \frac{\lambda_j}{\mu_j})a_k = x.$$ 

(b) Show that the coefficients of $a_1, ..., a_k$ are all non-negative and sum to 1, with the coefficient of $a_j$ equal to 0, so it represents $x$ as a convex combination of fewer than $k$ elements of $\{a_1, ..., a_k\}$.

2. Prove the Krein-Milman Theorem: If $S$ is a compact convex set then $S$ is the convex hull of its extreme points (those points that are not the midpoint of any segment of $S$).

3. Prove the Gordan-Dickson lemma: Let $\{p_1, p_2, ...\}$ be a sequence of points in $\mathbb{Z}_+^n$ such that $p_i \not\leq p_j$ whenever $i < j$. Then this sequence is finite.

4. Prove that the set $X = \text{cone}(A) \cap \mathbb{Z}^d$, where cone($A$) is a rational, polyhedral cone, is a finitely generated semigroup.

5. Suppose $X \subset \mathbb{Z}^d$ is a Hilbert bases for the pointed rational cone($X$) (remember this means $\text{sg}(X) = \text{cone}(X) \cap \mathbb{Z}^d$). Then any point in the semigroup $\text{sg}(X)$ can be represented as a non-negative linear combination of of no more than $2d - 1$ (Cook, Fonlupt, Schrijver 1986).

6. For every $m \times d$ integer matrix $A$, bounds $l, u \in \mathbb{Z}^d$ and $b \in \mathbb{Z}^m$, the set of circuits of $A$ is the set of all edge-directions of the parametric polytopes $\{x : Ax = b, l \leq x \leq u\}$.

7. (a) Determine the value of the Frobenius number of 5 and 12.
   (b) What is the largest number that is representable in exactly 2 different ways with 5 and 12?
   (c) Can you predict the value of the Frobenius number of $a$ and $b$?

8. Prove the following lemma: Let $z_0$ and $z_1$ be feasible solutions to $A z = b$, $l \leq z \leq u$. Moreover, let

$$z_1 - z_0 = \sum_{i=1}^{r} \alpha_i g_i$$

be a nonnegative integer linear sign-compatible decomposition into Graver basis elements $g_i \in \mathcal{G}(A)$. Then for all choices of $\beta_1, ..., \beta_r \in \mathbb{Z}$ with $0 \leq \beta_j \leq \alpha_j$,
\[ j = 1, ..., r, \] the vector
\[ z_0 + \sum_{i=1}^{r} \beta_i g_i \]
is also a feasible solution to \( Az = b, l \leq z \leq u. \)

9. Consider the optimization problem
\[
\min \left\{ (0 \ 1 \ 0 \ 2) \ z : \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} z = \begin{pmatrix} 10 \\ 21 \end{pmatrix}, z \in \mathbb{Z}_4^+ \right\}.
\]

(a) Use 4ti2 to compute the Graver basis \( \mathcal{G}(A) \) of \( A \).
(b) Find an initial integer point, \( z_0 \). Using the Graver basis \( \mathcal{G}(A) \), turn the integer point \( z_0 \) into a feasible solution \( z_1 \) of \( Az = \begin{pmatrix} 10 \\ 21 \end{pmatrix}, z \in \mathbb{Z}_4^+ \).
(c) Using the Graver basis \( \mathcal{G}(A) \), augment the feasible solution \( z_1 \) until an optimal solution \( z_{\min} \) is reached.

Helly, Radon and Tverberg

1. Give a proof of Radon’s lemma: If \( S \) is a set of \( n + 2 \) points in \( \mathbb{R}^n \) then \( S \) can be partitioned as \( S = A \cup B \) into two disjoint subsets \( A \) and \( B \) with nonempty intersection \( \text{conv}(A) \cap \text{conv}(B) \).
2. Prove Helly’s Theorem: If \( C \) is a finite collection of convex sets in \( \mathbb{R}^d \) such that each \( d + 1 \) sets have nonempty intersection then the intersection of all sets in \( C \) is nonempty.
3. If \( A \subset \mathbb{R}^2 \) has diameter less than or equal to 1, then \( A \) can be covered by a circular disk of radius \( \frac{1}{\sqrt{3}} \).
4. Given 300 points in \( \mathbb{R}^2 \), prove there exists a point \( p \) such that each closed halfplane determined by each line through \( p \) contains at least 100 of the given points.
5. Given convex sets \( K_1, ..., K_n \subset \mathbb{R}^d \), suppose that for each \( d + 1 \) of \( K_1, ..., K_n \) there exists a ball of radius \( r \) intersecting all \( d + 1 \). Prove then that there is a ball of radius \( r \) intersecting all of \( K_1, ..., K_n \).
6. Given convex sets \( K_1, ..., K_n \subset \mathbb{R}^d \), suppose that the intersection of each \( d + 1 \) of the \( K_i \) contains a ball of radius \( r \). Prove then that there is a ball of radius \( r \) contained in \( K_1 \cap ... \cap K_n \).
7. If \( S \subset \mathbb{R} \), then \( h(S) = 2 \). That is, the \( S \)-Helly number of any \( S \)-subset of the real line is two.
8. In $\mathbb{R}^2$ we no longer have such a nice theorem. Some subsets of $\mathbb{R}^2$ have infinite Helly number. Show an explicit example.

9. If $F$ is any subfield of $\mathbb{R}$ (e.g., $\mathbb{Q}(\sqrt{2})$), then what is the Helly number? $h(F^{d})$?

10. Given a set $S$ of $n$ points in $\mathbb{R}^d$, a Tverberg $k$-coloring of $S$ is a partition of the points in $S$ into $k$ sets $S_1, ..., S_k$ such that $\bigcap_{i=1}^{k} \text{conv}(S_i)$ is nonempty.

   Prove that any set of points in $\mathbb{R}^d$ with $n \geq (k - 1)(d + 1) + 1$ admits a Tverberg $k$-coloring.

11. Are the Tverberg partitions unique?