1. Prove Helly’s Theorem: If $C$ is a finite collection of convex sets in $\mathbb{R}^d$ such that each $d + 1$ sets have nonempty intersection then the intersection of all sets in $C$ is nonempty.

2. If $A \subseteq \mathbb{R}^2$ has diameter less than or equal to 1, then $A$ can be covered by a circular disk of radius $\frac{1}{\sqrt{3}}$.

3. Given convex sets $K_1, \ldots, K_n \subseteq \mathbb{R}^d$, suppose that for each $d + 1$ of $K_1, \ldots, K_n$ there exists a ball of radius $r$ intersecting all $d + 1$. Prove then that there is a ball of radius $r$ intersecting all of $K_1, \ldots, K_n$.

4. Given convex sets $K_1, \ldots, K_n \subseteq \mathbb{R}^d$, suppose that the intersection of each $d + 1$ of the $K_i$ contains a ball of radius $r$. Prove then that there is a ball of radius $r$ contained in $K_1 \cap \ldots \cap K_n$.

5. Given 300 points in $\mathbb{R}^2$, prove there exists a point $p$ such that each closed halfplane determined by each line through $p$ contains at least 100 of the given points.

6. If $S \subseteq \mathbb{R}$, then $h(S) = 2$. That is, the $S$-Helly number of any $S$-subset of the real line is two.

7. In $\mathbb{R}^2$ we no longer have such a nice theorem. Some subsets of $\mathbb{R}^2$ have infinite Helly number. Show an explicit example.

8. For every $m \times d$ integer matrix $A$, bounds $l, u \in \mathbb{Z}^d$ and $b \in \mathbb{Z}^m$, the set of circuits of $A$ is the set of all edge-directions of the parametric polytopes $\{x : Ax = b, \ l \leq x \leq u\}$.

9. Prove the following lemma: Let $z_0$ and $z_1$ be feasible solutions to $A z = b, \ l \leq z \leq u$. Moreover, let

$$z_1 - z_0 = \sum_{i=1}^{r} \alpha_i g_i$$

be a nonnegative integer linear sign-compatible decomposition into Graver basis elements $g_i \in \mathcal{G}(A)$. Then for all choices of $\beta_1, \ldots, \beta_r \in \mathbb{Z}$ with $0 \leq \beta_j \leq \alpha_j, \ j = 1, \ldots, r$, the vector

$$z_0 + \sum_{i=1}^{r} \beta_i g_i$$

is also a feasible solution to $A z = b, \ l \leq z \leq u$. 


10. Suppose $X \subset \mathbb{Z}^d$ is a Hilbert bases for the pointed rational $cone(X)$ (remember this means $sg(X) = cone(X) \cap \mathbb{Z}^d$). Then any point in the semigroup $sg(X)$ can be represented as a non-negative linear combination of no more than $2d - 1$ (Cook, Fonlupt, Schrijver 1986).

11. Given a set $S$ of $n$ points in $\mathbb{R}^d$, a Tverberg $k$-coloring of $S$ is a partition of the points in $S$ into $k$ sets $S_1, ..., S_k$ such that $\bigcap_{i=1}^{k} \text{conv}(S_i)$ is nonempty.

Prove that any set of points in $\mathbb{R}^d$ with $n \geq (k - 1)(d + 1) + 1$ admits a Tverberg $k$-coloring.