Write your solutions to the following problems neatly on a separate sheet of paper and turn in on Monday, July 11th at the beginning of class, 12:10 pm. Please write up your solutions in the ordering the problems are listed on this sheet.

1. Consider the system of equations below

\[ \begin{align*}
  x_1 - x_3 &= 2 \\
  2x_2 + x_4 &= 0 \\
  x_2 + x_3 &= 1 \\
  -2x_2 - 3x_4 &= -1
\end{align*} \]

(a) Write the system of equations above using a matrix, \( A \) and a vector, \( b \) as \( Ax = b \).
(b) List two ways of finding \( A^{-1} \) for the matrix you described above and use both to calculate \( A^{-1} \).
(c) Use \( A^{-1} \) to solve the system \( Ax = b \) you wrote in part (a).
(d) Check your answer by using Gaussian elimination on an augmented matrix representing \( Ax = b \).
(e) Check your answer by using LU factorization on \( A \) and forward and back solving with your matrices \( L \) and \( U \).

2. (a) Find an example of a 3 \( \times \) 3 invertible matrix (it must have at least 6 nonzero entries) and compute its inverse.
(b) Find an example of a 2 \( \times \) 2 noninvertible matrix (it must have 4 nonzero entries).

3. Consider the augmented matrix in RREF below:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & -1 & 1 \\
0 & 1 & -2 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

(a) Solutions to the system of equations this augmented matrix represents are vectors in which vector space?
(b) Write down the solution set.
(c) The solution set is a \( k \)-dimensional hyperplane. What is \( k \)?
(d) Write down a particular solution to this system of equations and show why it is a particular solution.
(e) Write down two different homogenous solutions to this system of equations and show why they are homogenous solutions.

4. Consider an augmented matrix with \( r \) many rows, \( c \) many LHS columns (before the bar), \( p \) many pivot variables and \( f \) many free variables.
(a) Write down an algebraic equation or inequality relating \( c, p \) and \( f \).
(b) Write down an algebraic equation or inequality relating \( r \) and \( p \).
(c) Is it possible to write down an equation relating \( r \) and \( p + f \)?
5. You are given an augmented matrix in RREF with \( r \) many rows and \( c \) many LHS columns (before the bar). Notice that the columns of the augmented matrix corresponding to the pivot variables, and the homogenous solutions you write in the solution set always form a basis for \( \mathbb{R}^c \).

(a) Write down an example augmented matrix in RREF with 3 rows, 4 columns and 2 pivot variables.

(b) Write down the solution set for your example.

(c) Check that the pivot columns and the homogenous solutions you wrote down in part (b) form a basis (for what vector space?).

(d) No need to be too rigorous or write a proof, but briefly explain why these vectors will always form a basis (and not just in your example).

6. (a) Write down an example augmented matrix in RREF whose solution set is a 4-dimensional hyperplane in \( \mathbb{R}^7 \).

(b) Can you write down (using ... symbols) an augmented matrix in RREF whose solution set is a \( k \)-dimensional hyperplane in \( \mathbb{R}^n \) where \( k < n \)? If so, write one down.

(c) Can you write down (using ... symbols) an augmented matrix in RREF whose solution set is a \( k \)-dimensional hyperplane in \( \mathbb{R}^n \) where \( k > n \)? If so, write one down.

7. Consider an augmented matrix in RREF with \( c \) many rows and \( c \) many LHS columns, all of which correspond to pivot variables.

(a) Solutions to this system of equations are in which vector space?

(b) The equation from any one row describes what geometric object?

(c) What type of geometric object is the solution set for such an augmented matrix?

(d) The solution set you described in part (c) is the intersection of \( c \) many geometric objects you named in part (b). If \( c = 2 \), draw an example of such a system of equations in \( \mathbb{R}^2 \).

8. The set of vectors in \( \mathbb{R}^4 \) whose first two components sum to 0 is a vector space. The operator which maps these vectors into \( \mathbb{R}^3 \) as

\[
\begin{align*}
y_1 &= x_2, \\
y_2 &= x_1, \\
y_3 &= x_3 + x_4
\end{align*}
\]

is a linear operator.

(a) Check that the set of vectors in \( \mathbb{R}^4 \) whose first two components sum to 0 satsifies additive closure.

(b) Check that the set of vectors in \( \mathbb{R}^4 \) whose first two components sum to 0 satsifies multiplicative closure.

(c) Prove that the operator described above is linear.

9. Consider the derivative operator \( \frac{d}{dx} : V \rightarrow W \) which acts on the vector space of cubic polynomials with real coefficients and one variable, \( V = \{ a_3x^3 + a_2x^2 + a_1x + a_0 : a_3, a_2, a_1, a_0 \in \mathbb{R} \} \).

(a) Show that this operator is linear.

(b) Choose the simplest vector space \( W \) that \( \frac{d}{dx} \) maps \( V \) into (i.e. the derivative of cubic polynomials are what type of polynomials?).

(c) Using the natural basis \( \{ x^3, x^2, x, 1 \} \), demonstrate how to represent an arbitrary element of \( V \) as an element of \( \mathbb{R}^4 \). Using the natural basis for your vector space described in part (b), represent an element of \( W \) as an element of \( \mathbb{R}^3 \).
(d) Find the matrix which represents \( \frac{d}{dx} \) and maps \( \mathbb{R}^4 \) to \( \mathbb{R}^3 \).
(e) Use this matrix and your answers from part (c) to solve the differential equation

\[
\frac{d}{dx} p(x) = x^2 + x - 1
\]

for a cubic polynomial, \( p(x) \).

10. Suppose that \( A \) is an \( n \times n \) invertible matrix. Use the fact that \( (AB)^T = B^T A^T \) to prove that \( (A^T)^{-1} = (A^{-1})^T \).