7/11 Notes

Midterm: Grades are up. Mean was 70.16.

Last Wednesday: computed the LU decomposition and used to solve a system of equations $Ax = b$.

We solved and in our example, we computed a unique solution: There is only one $x^*$ such that $Ax^* = b$.

What does this tell us about the matrix $A$?

$\Rightarrow$ A is invertible!

Is there another way to tell that $A$ is invertible directly? (Right now, we can use GE and see if it row reduces to the identity.)

$\Rightarrow$ YES! Recall that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

unless $ad-bc=0$ in which case there is no inverse.

What is $ad-bc$ called?

- The determinant!

FACT: All square matrices have a number associated with them called the determinant which tells us if the matrix is invertible.

The determinant of a square matrix, $A$, which we write $\det A$ and $|A|$. 

\[ \det A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \cdot 4 - 3 \cdot 2 = -2 \]
is defined as:

**Def.** The determinant of the $n \times n$ matrix $A$ is

\[
\det A = \sum_{\sigma} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}
\]

(1A1)

Here, the sum is over all permutations of the $n$ columns of $A$ (so then $a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$) is the product of the elements on the diagonal of the matrix whose columns have been permuted.

Here, $\operatorname{sgn}(\sigma) = 1$ if $\sigma$ is even, $-1$ if $\sigma$ is odd.

A permutation of columns is even/odd if it takes an even/odd number of column swaps to achieve the permutation.

**Example:**

\[
\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

We have to sum over all column permutations. There are only two: we can either not swap the columns or we can swap the columns. We can put columns in order.

\[
\begin{aligned}
12 & \text{ or } 21 \\
\begin{bmatrix} a & b \\ c & d \end{bmatrix} & \rightarrow \begin{bmatrix} b & a \\ d & c \end{bmatrix}
\end{aligned}
\]

permutated matrices

\[
\begin{aligned}
& \text{No swap} & \text{Swap} \\
\begin{bmatrix} a & b \\ c & d \end{bmatrix} & \rightarrow & \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\end{aligned}
\]

diagonal product

swap
\[ \begin{align*}
\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= 1 \cdot ad + (-1) \cdot bc \\
&= ad - bc
\end{align*} \]

**Example:**

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

**What are the possible perms?**

\begin{itemize}
  \item No swaps: 1 2 3 (columns in normal order)
  \item 1 swap: 1 3 2
  \item 1 swap: 3 2 1
  \item 1 swap: 2 1 3
  \item 2 swaps: 3 1 2
  \item 2 swaps: 2 3 1
\end{itemize}
\[ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \]

\[ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} 

\[ - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \]
This can be very cumbersome. Some people find the following easier:

**Cofactor formula:**

\[
\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in} \quad \text{where}
\]

\[
C_{ij} = (-1)^{i+j} \det M_{ij}
\]

Here, \( M_{ij} \) is called a minor of \( A \) and is just the submatrix of \( A \) when you remove the \( i \)th row of \( A \) and the \( j \)th column of \( A \).

To use this formula, we choose a row, \( i \), of our matrix and use the entries of this row as coefficients on the cofactors which are the determinants of the remaining submatrices.

**Example:** We'll visualize this process first on the matrix

\[
A = \begin{bmatrix}
2 & -1 & 0 \\
1 & 2 & -1 \\
0 & 1 & 2 \\
\end{bmatrix}
\]

\[
\begin{vmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & 1 & 2 \\
\end{vmatrix} = \begin{vmatrix}
2 & -1 \\
-1 & 2 \\
0 & 1 \\
\end{vmatrix} + \begin{vmatrix}
-1 & -1 \\
-1 & 2 \\
0 & 1 \\
\end{vmatrix} + \begin{vmatrix}
-1 & 2 \\
0 & 1 \\
0 & 2 \\
\end{vmatrix} + 0
\]

\[
\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}
\]

\[
= 2(-1) \begin{vmatrix}
2 & -1 \\
-1 & 2 \\
\end{vmatrix} + (-1)(-1) \begin{vmatrix}
-1 & -1 \\
-1 & 2 \\
\end{vmatrix} + 0(-1) \begin{vmatrix}
-1 & 2 \\
0 & 1 \\
0 & 2 \\
\end{vmatrix} + 0
\]

\[
= 2(4-1) + 1(-2-0) = 9
\]

Notice that 0 coefficients.
We could also have expanded around the second row:
\[
\begin{vmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 2
\end{vmatrix}
= \begin{vmatrix}
-1 & 0 \\
-1 & 2 \\
0 & 2
\end{vmatrix}
+ \begin{vmatrix}
2 & 0 \\
-1 & 2 \\
0 & 2
\end{vmatrix}
+ \begin{vmatrix}
2 & -1 \\
-1 & 2 \\
0 & -1
\end{vmatrix}
\]

\[
\det A = a_{21} C_{21} + a_{22} C_{22} + a_{23} C_{23}
\]

\[
= (-1)(-1)^{2+1} \begin{vmatrix}
-1 & 0 \\
-1 & 2
\end{vmatrix}
+ 2(-1)^{2+2} \begin{vmatrix}
2 & 0 \\
0 & 2
\end{vmatrix}
+ (-1)^{2+3} \begin{vmatrix}
2 & -1 \\
0 & -1
\end{vmatrix}
\]

\[
= (2 - 0) + 2(4 - 0) + (-2 - 0)
\]

\[
= -2 + 8 - 2 = 4
\]

This cofactor formula makes the following fact clear. Make sure you understand why this must be true. **THM:** If \( A \) has a row consisting of entirely zeros then
\[
\det A = 0
\]

Similarly, ensure you can make sense of the following fact:

**FACT:** The determinant of a diagonal matrix is the product of its diagonal entries.

**Example:**

\[
\det \begin{bmatrix}
-1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 3
\end{bmatrix}
= (-1)(-1)^{1+1} (2 \cdot 3 - 0 \cdot 0) + 0(-1)^{1+2} (0 \cdot 3 - 0 \cdot 0) +
0(-1)^{1+3} (0 \cdot 0 - 2 \cdot 0)
= -18 + 0 + 0
= -18
\]
Notice that we didn't say anything about the dimensions of the matrix. These formulas for computing the determinant hold regardless of their size (provided they are square).

**Example:**

\[ \text{det}(I) = ? \]

**Ans:**

\[ \text{det}(I) = 1 \]

\[ \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \cdot 1 - 0 \cdot 0 = 1 \]

\[ \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(1 \cdot 1 - 0 \cdot 0) + 0 \cdot \ldots + 0 \cdot \ldots = 1 \]

*TIP:* If \( A \) and \( A' \) differ by a row swap, then \( \text{det} A' = -\text{det} M \)

**Example:**

\[ \text{det} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = 1 \cdot 3 - 1 \cdot 2 = 1 \]

\[ \text{det} \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} = 2 \cdot 1 - 3 \cdot 1 = -1 \]

This tells us something about elementary row operation matrices! Consider the 3x3 ERO matrix which swaps rows 1+2 in a matrix that has 3 rows:

\[ \text{det} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0 \cdot \ldots + 1 \cdot (-1)^{1+2} (1 \cdot 1 - 0 \cdot 0) + 0 \cdot \ldots \]
Now, consider a matrix $A$ and $A'$ where $A'$ is $A$ with the $i^{th}$ and $j^{th}$ rows swapped. Let $E_{ij}$ denote the ERO matrix which does this swap. We know

$$E_{ij} A = A'$$

and we know $\det A' = -\det A$ and $\det E_{ij} = -1$

so then

$$\det E_{ij} A = \det A' = -\det A = \det E_{ij} \det A$$

$$\Rightarrow \det E_{ij} A = \det E_{ij} \det A$$

What about an ERO matrix that multiplies a row by $c$?

**Example:**

$$\det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 2$$

THM: If $A'$ is $A$ with the $i^{th}$ row multiplied by $c$, then

$$\det A' = c \det A = \det E_i^c \det A$$

We have almost all of the ERO matrices we regularly use.

What are we missing?

- adding multiples of one row to another!
Suppose $E_{ij}^c$ is the ERO matrix which adds $c$·row $j$ to row $i$.

Example:

$$E_{12}^2 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1(1 \cdot 1 - 0 \cdot 0) + 2(-1)^{1+2}(0 \cdot 1 - 0 \cdot 0) + 0 = 1$$

**THM:** If $A'$ is $A$ with a multiple ($c$) of row $j$ added to row $i$, so $A' = E_{ij}^c A$, then

$$\det A' = \det A = \det E_{ij}^c \det A$$

**Additional THM:** For any square matrix $A$, $\det A \neq 0$ if and only if $A$ is invertible.

(This says that if $A$ is invertible then $\det A \neq 0$ and if $\det A \neq 0$ then $A$ is invertible.)

Some hints:

1) You can also expand (use the cofactor formula) along a column (rather than a row).

2) Zeros are your friend! If there is a row or column with a lot of zeros, expanding along it saves a lot of computation.
**FACT:** If $A$ is a square matrix, then $\det A^r = \det A$.

**THM:** If $A$ is invertible then

$$\det A^{-1} = \frac{1}{\det A}$$

**Example:**

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad - bc} & -\frac{b}{ad - bc} \\ -\frac{c}{ad - bc} & \frac{a}{ad - bc} \end{bmatrix}$$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{d}{ad - bc} \cdot \frac{a}{ad - bc} - \frac{-b}{ad - bc} \cdot -\frac{c}{ad - bc}$$

$$= \frac{ad}{(ad - bc)^2} - \frac{bc}{(ad - bc)^2} = \frac{ad - bc}{(ad - bc)^2} \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \frac{1}{ad - bc} = \frac{1}{\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}}$$