Determinants:

Using determinants to find $A^{-1}$:

Start with matrix $A$.

Replace each entry with corresponding cofactor, i.e., with $\pm \det$ (smaller matrix obtained by removing corresponding row or column) from $A$.

Take transpose, divide by $\det A$.

E.g., $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$

$\det A$: use cofactor expansion.

$+ 0 \det \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} - 0 \det \begin{pmatrix} 1 & 3 \\ 0 & 5 \end{pmatrix} + 6 \det \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}$

Signs: $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

$\det A = 24$.

Find $A^{-1}$:
\[
A = \begin{pmatrix}
1 & 2 & 3 \\
0 & 4 & 5 \\
0 & 0 & 6
\end{pmatrix},
\]

\[
A^{-1} = \frac{1}{24} \begin{pmatrix}
\det(45) & -\det(05) & +\det(04) \\
-\det(26) & +\det(06) & -\det(00) \\
+\det(23) & -\det(03) & +\det(02)
\end{pmatrix}
\]

\[
= \frac{1}{24} \begin{pmatrix}
-24 & 0 & 0 \\
-12 & 6 & 0 \\
-2 & -5 & 4
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & -\frac{1}{2} & -\frac{1}{12} \\
0 & \frac{1}{4} & -\frac{5}{24} \\
0 & 0 & \frac{1}{6}
\end{pmatrix}
\]

Why this works:
row 1, cofactor expansion of \( \det \)

\[
A \cdot \frac{1}{\det} \left( \text{cofactor thing} \right)^T = \frac{1}{\det} \left( \begin{array}{ccc}
\times & \times & \times \\
\times & \times & \times \\
\text{row 2 cofactor expansion of } \det & \text{row 3 cofactor expansion}
\end{array} \right)
\]

row 1, column 2 of product is
(row 1 of A) \cdot (row 2 of cofactor matrix) - turns out to be cofactor expansion of \( \det(A \text{ with row 2 replaced by row 1}) \)
result is: each off-diagonal entry is 0.

so \[ A \cdot \frac{1}{\det A} \begin{pmatrix} \text{cofactor matrix} \end{pmatrix}^T = \frac{1}{\det A} \begin{pmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{pmatrix} = I. \]

Next chapter: Subspaces - vector spaces inside bigger vector spaces.

e.g., \( \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\} \) is a vector space, and it's contained in \( \mathbb{R}^3 \), so this is a subspace of \( \mathbb{R}^3 \).

e.g., \( C(\mathbb{R}) \) is the set of all continuous functions \( f: \mathbb{R} \to \mathbb{R} \). This is a vector space.

\( P \) is the set of all polynomials.

This is also a vector space.

\( P \) is a subspace of \( C(\mathbb{R}) \).

(Some people would write \( P \leq C(\mathbb{R}) \).)
Given a vector space, how to create subspaces? First, how to tell if something is a subspace?

Say, the set \( V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x + y + z = 0 \right\} \).

Check that this is a vector space:

- We know \( V \) is contained in \( \mathbb{R}^3 \).
- Additive closure: if \( v_1, v_2 \in V \), is \( v_1 + v_2 \in V \) also?
  (you could check this.)
- Multiplicative closure: if \( v \in V \) and \( c \in \mathbb{R} \), is \( cv \in V \) also?
  (you could check this.)

8 more properties: these are all either (true for \( V \) because they're true for \( \mathbb{R}^3 \)) or (true for \( V \) because they're true for \( \mathbb{R} \) and we've checked closure).

If \( V \) is a vector space and \( W \) is contained in \( V \), then to check whether \( W \) is a subspace, just need to check mult. and add. closure.
Fact: The set of polynomials $P$, is a vector space.

Say $V = \{ \text{polynomials } p(x) : p(2) = 2 \}$

$W = \{ \text{polynomials } p(x) : p'(5) = 0 \}$

Are either $V$ or $W$ vector spaces?

V, W are inside P so just check mul. + add. closure.

$V$: say $p(x), q(x) \in V$. Is $p + q$ guaranteed to be in $V$?

If $p(2) = 2$ and $q(2) = 2$,

does $(p + q)(2) = 2$?

No—e.g., if $p(x) = x$ and $q(x) = 2x - 2$,
then $(p + q)(2) = x + 2x - 2 \bigg|_{x=2} = 4$

So $V$ is not additively closed.

$W$: say $p(x), q(x)$ are in $W$. ($W$ is set of polynomials: $p'(5) = 0$).

So $p'(5) = 0$, $q'(5) = 0$.

Does $(p + q)'(5)$ have to be 0?
yes - because \((ptq)'(5) = p'(5) + q'(5) = 0 + 0 = 0\).

if \(c \in \mathbb{R}\), is \((cp)'(5) = 0\)? yes - differentiable,
so \(W\) is a subspace of \(P\).

To create a subspace of some vector space \(V\),
start with some vectors, include every other
vector necessary in order to be multi. + add. closed.

E.g., let's make a subspace of \(\mathbb{R}^3\).

Start with \(\left( \frac{1}{3} \right)\) and \(\left( \frac{4}{6} \right)\).

Make a subspace \(V\) of \(\mathbb{R}^3\) containing \(\left( \frac{1}{3} \right)\) and \(\left( \frac{4}{6} \right)\).

\(V\) must also contain \(\left( \frac{2}{6} \right), \left( \frac{10}{30} \right), \left( \frac{11}{33} \right)\), etc.

must contain \(\left\{ c \left( \frac{1}{3} \right): c \in \mathbb{R} \right\}\)
and \(\left\{ c \left( \frac{4}{6} \right): c \in \mathbb{R} \right\}\).

\(V\) must contain \(\left\{ a \left( \frac{1}{3} \right) + b \left( \frac{4}{6} \right): a, b \in \mathbb{R} \right\}\).

This is a subspace.

The set of all linear combinations of \(\left( \frac{1}{3} \right)\) and \(\left( \frac{4}{6} \right)\)
is a subspace of \(\mathbb{R}^3\).
Also called the span of \((\frac{1}{3})\) and \((\frac{4}{3})\).

The span of \(\{v_1, \ldots, v_n\}\) means
set of all linear combinations of \(v_1, \ldots, v_n\),
i.e., \(\text{span}(v_1, \ldots, v_n) = \{c_1v_1 + c_2v_2 + \ldots + c_nv_n : c_1, \ldots, c_n \in \mathbb{R}\}\)
and span of \((v_1, \ldots, v_n)\) is always a
subspace of \((\text{whatever space } v_1, \ldots, v_n \text{ live in})\).

\[\text{e.g., } \text{span}\left((\frac{1}{3})\right) \text{ is a subspace of } \mathbb{R}^3.\]
(it's the \(x\)-axis.)

\[\text{e.g., } \text{span}\left((\frac{1}{3}), (\frac{2}{3})\right) \text{ is a subspace of } \mathbb{R}^3.\]
(it's the \(xz\)-plane.)

\[\text{e.g., } \text{span}\left((\frac{1}{3}), (\frac{2}{3}), (\frac{0}{3})\right) \text{ is a subspace of } \mathbb{R}^3:\]
(it's \( \{x(\frac{1}{3}) + z(\frac{2}{3}) + y(\frac{0}{3}) : x, z, y \in \mathbb{R}\}\)
or \(\{x(\frac{1}{3}) \in \mathbb{R} : x \in \mathbb{R}\}\) which is \(\mathbb{R}^3\).

(Any vector space is a subspace of itself.)

\[\text{e.g., } \text{span}\left((\frac{2}{3})\right) \text{ is a subspace of } \mathbb{R}^2\]
(it's the line \(y = \frac{3}{2}x\).)
Two more ways of making subspaces.

Start with some linear function \( f: V \rightarrow W \).

The set \( \{ v \in V : f(v) = 0 \} \) is a subspace of \( V \), called the kernel of \( f \), written \( \ker f \).

The set \( \{ w \in W : w = f(\text{something}) \} \) is a subspace of \( W \), called the image of \( f \), im \( f \) (or range of \( f \)).

E.g., take \( f: \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) given by

\[
f \left( \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) = \begin{pmatrix} x + y \\ 0 \end{pmatrix}.
\]

Is it linear? \( f(v_1 + v_2) \) must always equal \( f(v_1) + f(v_2) \).
\( f(cv_1) \) must always equal \( cf(v_1) \).

(You could check that this is linear.)

\[
\ker f = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : f \left( \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.
\]

\[
= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : (x + y) = 0 \right\} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x = -y \right\}
\]

This must be a subspace of \( \mathbb{R}^3 \).

Why? if \( v_1, v_2 \in \ker f \), check that \( v_1 + v_2 \in \ker f \).
if \( v_1, v_2 \in \ker f \), then \( f(v_1) = 0 \), \( f(v_2) = 0 \).
so \( f(v_1 + tv_2) = f(v_1) + f(tv_2) = 0 + 0 = 0 \)

so \( v_1 + tv_2 \in \ker f \).

if \( v_1 \in \ker f \) and \( c \in \mathbb{R} \), then \( f(v_1) = 0 \)

so \( f(cv_1) = c f(v_1) = c \cdot 0 = 0 \) so \( cv_1 \in \ker f \).

\[
\text{im } f = \{ \mathbf{v} \in \mathbb{R}^2 : \mathbf{v} = f(\text{something}) \}.
\]

\[
= \{ \mathbf{v} \in \mathbb{R}^2 : \mathbf{v} = (x, y) = (x + y) \mathbf{e}_2 = \{ (c) \in \mathbb{R}^2 : c \in \mathbb{R} \} \}.
\]

the \( x \)-axis.

"opposite idea" of span. "linear dependence + independence."

Say \[
A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}
\]

\[
A \mathbf{x} = \mathbf{0} \quad (\text{homogeneous linear system})
\]

\[
\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]

\( (0, 0) \) must be a solution (trivial solution).

Are there any other solutions?

Rephrase in terms of vectors:
\[ \begin{align*}
  x_1 + 2x_2 + 3x_3 &= 0 \\
  4x_1 + 5x_2 + 6x_3 &= 0 \\
  7x_1 + 8x_2 + 9x_3 &= 0
\end{align*} \]
\[ \text{or } X_1 \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + X_2 \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + X_3 \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \]

Solve for \( x_1, x_2, x_3; \) \( x_1 = x_2 = x_3 = 0 \) is the trivial solution. Is there a nontrivial solution?

In general, start with some vectors \( v_1, v_2, \ldots, v_n \in V, \) some vector space.

The equation \( c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = 0, \)

where \( c_1, \ldots, c_n \) are unknown real numbers,

must have a trivial solution, \( c_1 = c_2 = \ldots = c_n = 0. \)

Is there a nontrivial solution?

If yes, then \( \{v_1, \ldots, v_n\} \) is linearly dependent.

If no, then \( \{v_1, \ldots, v_n\} \) is linearly independent.

E.g., say \( v_1 = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}. \)

Are these linearly dependent or linearly independent?

Does \( c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 \) have a nontrivial solution?
\[ c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 \]

\[ c_1\left(\frac{1}{2}\right) + c_2\left(\frac{2}{3}\right) + c_3\left(\frac{3}{9}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

\[ \begin{cases} c_1 + 2c_2 + 3c_3 = 0 \\ 4c_1 + 5c_2 + 6c_3 = 0 \\ 7c_1 + 8c_2 + 9c_3 = 0 \end{cases} \]

\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}

either: matrix is invertible \(\iff\) exactly one solution

\[\iff\] no non-trivial solutions.

or: matrix is singular \(\iff\) infinitely many solutions

\[\iff\] there are non-trivial solutions.

in fact, \(\text{det}(\text{this matrix}) = 0\) so

system has non-trivial solutions so

\(\{v_1, v_2, v_3\}\) are linearly dependent.

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