**Example:**

\[ \begin{align*}
\text{w} + \frac{1}{2}x + z &= 4 \\
2w + 3y &= 2 \\
-w + \frac{1}{2}x - 3y + z &= 2
\end{align*} \]

\[ \begin{bmatrix}
1 & \frac{1}{2} & 0 & 1 & 4 \\
2 & 0 & 3 & 0 & 2 \\
-1 & \frac{1}{2} & -3 & 1 & 2
\end{bmatrix} \]

\[
R_1 = R_1 \\
R_2 = R_2 - 2R_3 \\
R_3 = R_3 + R_1
\]

\[ \begin{bmatrix}
1 & \frac{1}{2} & 0 & 1 & 4 \\
0 & 1 & -3 & 2 & 0 \\
0 & 1 & -3 & 2 & 0
\end{bmatrix} \]

\[
R_1 = R_1 - \frac{1}{2}R_2 \\
R_2 = R_2 \\
R_3 = R_3 - R_2
\]

\[ \begin{bmatrix}
1 & 0 & \frac{3}{2} & 0 & 1 \\
0 & 1 & -3 & 2 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} \]

# Pivots?

# Free variables?

\[
\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -\frac{3}{2} \\ 3 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} : c,d \in \mathbb{R} \right\}
\]

**Notice that**

\[
\begin{bmatrix}
1 & \frac{1}{2} & 0 & 1 \\
2 & 0 & 3 & 0 \\
-1 & \frac{1}{2} & -3 & 1
\end{bmatrix} \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
4 \\
2 \\
2
\end{bmatrix} \quad \begin{bmatrix}
1 & \frac{1}{2} & 0 & 1 \\
2 & 0 & 3 & 0 \\
-1 & \frac{1}{2} & -3 & 1
\end{bmatrix} \begin{bmatrix}
-\frac{3}{2} \\
3 \\
1
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & \frac{1}{2} & 0 & 1 \\
2 & 0 & 3 & 0 \\
-1 & \frac{1}{2} & -3 & 1
\end{bmatrix} \begin{bmatrix}
0 \\
-2 \\
0
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

**Theorem:** Any solution to a linear system of equations is known as a particular solution. A homogeneous solution is a vector \( \mathbf{h} \) so that \( L\mathbf{h} = \mathbf{0} \).
Note then that \( p + ch \) is also a solution since by linearity of \( L \):
\[
L(p + ch) = Lp + cLh = v + c0 = v
\]

For our example,
\[
\begin{bmatrix}
1 & \frac{1}{2} & 0 & 1 \\
2 & 0 & 3 & 0 \\
-1 & \frac{1}{2} & -3 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
0 \\
1
\end{bmatrix}
+ \begin{bmatrix}
\frac{3}{2} \\
1 \\
0 \\
1
\end{bmatrix} + d \begin{bmatrix}
0 \\
-2 \\
0 \\
1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & \frac{1}{2} & 0 & 1 \\
2 & 0 & 3 & 0 \\
-1 & \frac{1}{2} & -3 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
0 \\
1
\end{bmatrix}
+ \begin{bmatrix}
\frac{3}{2} \\
1 \\
0 \\
1
\end{bmatrix} + d \begin{bmatrix}
0 \\
-2 \\
0 \\
1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
4 \\
2 \\
2
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} + d \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
= \begin{bmatrix}
4 \\
2 \\
2
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]
is a particular solution and \( \begin{bmatrix}
\frac{3}{2} \\
1 \\
0
\end{bmatrix} \) and \( \begin{bmatrix}
0 \\
-2 \\
1
\end{bmatrix} \) are homogenous solutions.

If \( L \mathbf{x} = \mathbf{v} \) is linear system of equations and \( p \) is a particular solution and \( h_1, h_2 \) are homogenous solutions, then
\[ p + ch_1 + tdh_2 \] is a particular solution (for choice of \( c, d \)).

Since
\[
L(p + ch_1 + dh_2) = Lp + cLh_1 + dLh_2 = v + c0 + d0 = v
\]
and \( ch_1 + dh_2 \) is a homogenous solution (for choice of \( c, d \)).

Since
\[
L(ch_1 + dh_2) = cLh_1 + dLh_2 = c0 + d0 = 0.
\]
The matrix that describes the relationship between old and new matrices requires performing the correct EROs on the A matrix:

\[
\begin{bmatrix}
1 & 0 & 0 & 7 \\
0 & 1 & 2 & 2 \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & \frac{1}{2} & 0 & 1 & 4 \\
2 & 0 & 3 & 0 & 2 \\
1 & \frac{1}{2} & -3 & 1 & 2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} =
\begin{bmatrix}
1 & \frac{1}{2} & 0 & 1 & 4 \\
0 & 1 & -3 & 2 & 6 \\
0 & 1 & -3 & 2 & 6 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

**EXERCISE FOR HOME**

Check that

\[
\begin{bmatrix}
1 & -\frac{1}{2} & 0 \\
0 & 1 & 0 \\
0 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & \frac{1}{2} & 0 & 1 & 4 \\
0 & 1 & -3 & 2 & 6 \\
0 & 1 & -3 & 2 & 6
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & \frac{3}{2} & 0 & 1 \\
0 & 1 & -3 & 2 & 6 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

This lets us think about GE algebraically. We are multiplying both sides of equations by the same matrix:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 2 & 2 \\
0 & 1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & \frac{1}{2} & 0 & 1 \\
2 & 0 & 3 & 0 \\
1 & 1 & 2 & 1
\end{bmatrix} \begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
4 \\
2 \\
3
\end{bmatrix}
\]
In algebra, we have

\[ \begin{align*}
6x &= 12 \\
\implies 3^{-1}6x &= 3^{-1}12 \\
\implies 2x &= 4 \\
\implies 2^{-1}2x &= 2^{-1}4 \\
\implies 1x &= 2
\end{align*} \]

Now, dividing by a number solves equations here but multiplying by appropriate matrices solves our systems of linear equations. Just like we keep going until we have \( 1x = 2 \), we want to do GE until we have \( Iv = w \).

We can see:

To solve \( Ax = b \) for \( x \), we want to apply matrices to both sides of the matrix-vector equation until we have something as close to the identity as possible on the LHS:

\[ \begin{align*}
E_nE_{n-1}\ldots E_1Ax &= Ix = b' = E_nE_{n-1}\ldots E_1b
\end{align*} \]

Getting the identity on the LHS might not be possible. For example, our example has a \( 3 \times 2 \) matrix so it can't be the identity after GE (the identity is square), plus we get a row of all 0's in our \( Ax \), so it won't go to the identity.

(Handwritten note: Practice by working problems now this week.)

\( 2^{-1}3^{-1}e = 1 \) tells us that \( 6^{-1}6 = 1 \), or \( 6^{-1} \) “undoes” \( 6 \), \( E_nE_{n-1}\ldots E_1A = I \) tells us that the matrix that is the product of matrices \( E_nE_{n-1}\ldots E_1 \) “undoes” \( A \).
We figured out that:
\[
\begin{pmatrix}
1 & 0 & 0
0 & 1 & 0
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & -\frac{1}{2} & 0
0 & 1 & 0
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0
0 & 1 & 0
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0
0 & 0 & 1
0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0
0 & 1 & 0
0 & 0 & 1
\end{pmatrix}
\]

We call $E_6E_5E_4E_3E_2E_1$ the inverse of $A$ since it "undoes" the matrix $A$ and denote it $A^{-1} = E_6E_5E_4E_3E_2E_1$.

When we perform GE on $Ax=b$ where $A$ is square (and we don't end up with any rows of 0), we are finding the inverse matrix. If we knew the matrix $A^{-1}$, we could skip GE and just do a matrix multiplication

\[Ax = b\]
\[A^{-1}A x = A^{-1} b\]
\[I x = A^{-1} b\]
\[x = A^{-1} b\]

Knowing that $A^{-1}$ is what we need to solve $Ax=b$, we can find it by doing GE on $(A|I)$ until $(I|A^{-1})$.

**EXERCISE FOR HOME**: Check that

\[E_6E_5E_4E_3E_2E_1 = \begin{pmatrix}
-\frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & 1 & \frac{1}{2}
\end{pmatrix}\]

You can do the multiplications in any order, but can't reverse any multiplication.

$(AB)C = A(BC)$ (associative)

but

$AB \neq BA$ (not commutative)
To find $A^{-1}$ we do GE on

\[
\begin{bmatrix}
0 & 8 & 2 & 1 & 0 & 0 \\
1 & 2 & 0 & 0 & 1 & 0 \\
3 & 2 & 2 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 0 & 0 & 1 & 0 \\
0 & 8 & 2 & 1 & 0 & 0 \\
0 & -4 & 2 & 0 & 0 & -3 \\
\end{bmatrix}
\]

EXERCISE FOR HOME.

Continue GE on

\[
\begin{bmatrix}
1 & 2 & 0 & 0 & 1 & 0 \\
0 & 8 & 2 & 1 & 0 & 0 \\
0 & -4 & 2 & 0 & 0 & -3 \\
\end{bmatrix}
\]

until the LHS is the $3 \times 3$ identity matrix and check that the RHS is $A^{-1} = \begin{bmatrix}
\frac{1}{6} & \frac{1}{2} & \frac{1}{6} \\
\frac{1}{12} & \frac{1}{4} & \frac{-1}{12} \\
\frac{1}{16} & \frac{-1}{4} & \frac{1}{3} \\
\end{bmatrix}$.

Like I said before, getting the identity matrix on the LHS of the RREF of an AM is not always possible. We have a special name for matrices when this is possible.

Def: A matrix $A$ is invertible if its RREF has the identity matrix as the LHS.

What does this tell us about the geometry of the solution set?

Take $Ax = b$ to RREF

\[
\begin{bmatrix}
1 & 0 & 0 & | & a \\
0 & 1 & 0 & | & b \\
0 & 0 & 1 & | & c \\
\end{bmatrix}
\]

tells us there is just one solution!

Note: For $A$ to be invertible, it must be square!
Like we talked about on Friday, if $A$ is a linear operator (matrix) and $b$ is known then $Ax = b$ has either

1. One solution
2. No solutions
3. Infinitely many solutions

Recall our examples from Friday:

$$3 \begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$$\begin{array}{l}
\hline \\
1 \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5/4 \\ 4/3 \end{bmatrix} \\
\end{array}$$

$$\begin{array}{l}
\hline \\
2 \begin{bmatrix} 1 & -2 \\ -4 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 20 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \end{bmatrix} \\
\end{array}$$

$$\begin{array}{l}
\hline \\
4 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\end{array}$$
In case 1, the matrix was invertible and there was 1 solution.
In cases 2, 3, 4 the 2x2 matrix was not invertible and there was no solution and infinitely many solutions (in the form of a line and a plane).

If there are 3 variables, so solutions are in 3D, then each equation defines a plane. There are 5 possibilities:

1. The planes have a unique point of intersection (there can be more than 3 planes, i.e. equations).
2. No solutions since some of the equations are contradictory so all planes don't intersect.
3. Line since the planes intersect in a common line.
4. Plane since all of the planes coincide geometrically (or there was only one plane, i.e. equation to begin with).
5. All of 3D space (R^3) since there are no restrictions on solution at all \((0,0,0)(\frac{x}{2}) = (0,0,0)\).

In general, when there are k unknowns, there are k+2 possibilities, where there are k+1 no solution, one solution, line, plane, and various dimension hyperplanes.

When we look at the solution set, we have

- Solutions \( \mathbb{F} \) = 8 particular solution + linear combinations of homogeneous solutions
- The number of homogeneous solutions here tells us the size (or dimension) of the solution set.
  - No homogeneous solutions \( \Rightarrow \) one solution
  - 1 homogeneous solution \( \Rightarrow \) solution set is a line
  - k homogeneous solutions \( \Rightarrow \) solution set is a