MAT 17A: CALCULUS FOR BIOLOGY AND MEDICINE
Midterm 1. February 1, 2019

Print name: [Key]  Print SID: [ ]

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<td>B01</td>
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<td>B08</td>
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Section: (circle one)

Read carefully the following instructions:

- PLEASE DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO.
- It is a violation of the University Honor Code to, in any way, assist another person in the completion of this exam.
- Put away all scrap paper, books, notebooks, cellphones and other electronic devices.
- Write your last name, first name and student ID on each page.
- Show all your work for full credit. If in doubt, write it out.
- Keep your work as neat as possible. If we can’t read it, we can’t grade it!
- You do NOT need to simplify your answers.
- Make sure you have 4 pages, including this cover page.
- You have until 6pm sharp to finish this exam.

Good luck!

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<th>Problem</th>
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Problem 1. Let \( Y = \log(y) \), \( X = \log(x) \). When graphing \( Y \) as a function of \( X \), we obtain a line passing through the points \((1, 4)\) and \((-1, 0)\).

(a) (5 points) Express \( Y \) as a function of \( X \).

\[
\begin{align*}
    m &= \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 0}{1 - (-1)} = \frac{4}{2} = 2 \\
    y &= mx + b. \\
    4 &= 2(1) + b \Rightarrow b = 2 \\
    y &= 2x + 2
\end{align*}
\]

(b) (5 points) Based on your answer from part (a), express \( y \) as a function of \( x \). Your final answer shouldn’t involve any logarithms.

\[
\begin{align*}
    y &= 2x + 2 \\
    10^y &= 10^{2x + 2} \\
    y &= 10^{2 \log(x) + 2} \\
    y &= (10^{\log(x)})^2 \cdot 10^2 \\
    y &= 100x^2
\end{align*}
\]
Problem 2. Consider the following recursion:

\[ a_{n+1} = \frac{1}{4} a_n + \frac{3}{4} \]

(a) (5 points) Find the fixed points of the recursion.

\[ \chi = \frac{1}{4} \chi + \frac{3}{4} \]

\[ \frac{3}{4} \chi = \frac{3}{4} \]

\[ \chi = 1 \]

(b) (5 points) Assume \( a_0 = 2 \). Find \( \lim_{n \to \infty} a_n \).

\[ a_0 = 2 \]
\[ a_1 = \frac{1}{4}(2) + \frac{3}{4} = \frac{5}{4} = 1 + \frac{1}{4} \]
\[ a_2 = \frac{1}{4} \left( \frac{5}{4} \right) + \frac{3}{4} = \frac{6}{12} + \frac{12}{12} = \frac{18}{12} = \frac{3}{2} = 1 + \frac{1}{2} \]
\[ a_3 = \frac{1}{4} \left( \frac{3}{2} \right) + \frac{3}{4} = \frac{9}{32} + \frac{24}{32} = \frac{33}{32} = 1 + \frac{1}{32} \]
\[ a_4 = \frac{1}{4} \left( \frac{33}{32} \right) + \frac{3}{4} = \frac{33}{128} + \frac{96}{128} = \frac{129}{128} = 1 + \frac{1}{128} \]

So we see that \( \lim_{n \to \infty} a_n = 1 \).
Problem 3. A certain isotope has a half-life of 84 years. Assume that, initially, we have 50 grams of the isotope.

(a) (5 points) Let \( N_t \) be the mass (in grams) of the isotope remaining after \( t \) periods of 84 years. Find a formula for \( N_t \).

\[
N_t = \left(\frac{1}{2}\right)^t \cdot 50
\]

In years, \( N_t = \left(\frac{1}{2}\right)^{t/84} \cdot 50 \) (but this is not what the question is asking).

(b) (5 points) How much time will have elapsed when 10 grams of the isotope remain?

\[
\left(\frac{1}{2}\right)^t \cdot 50 = 10
\]

\[
\left(\frac{1}{2}\right)^t = \frac{1}{5}
\]

\[
2^t = 5
\]

\[
t = \log_2(5) \text{ periods of 84 years, or}
\]

\[
84 \cdot \log_2(5) \text{ years.}
\]
Problem 4. Let \( y = 25(3^x) \).

(a) (3 points) If we want to graph this equation in a straight line, should we use a log-linear or a log-log plot?

This is a power function so a \underline{log-linear} plot.

(b) (7 points) Apply the transformation from part (a) and find the equation of the resulting straight line.

\[
\begin{align*}
y & = 25(3^x) \\
\log y & = \log (25 \cdot 3^x) \\
Y & = \log(25) + \log(3^x) \\
Y & = \log(25) + \log(3)^x
\end{align*}
\]
Problem 5. Consider the logistic recursion

\[ x_{t+1} = 2.8x_t(1 - x_t) \]

(a) (5 points) Find the fixed points of the recursion.

\[ \chi = 2.8\chi(1 - \chi) \]
\[ \chi = 2.8\chi - 2.8\chi^2 \]
\[ 0 = 1.8\chi - 2.8\chi^2 \]
\[ 0 = \chi(1.8 - 2.8\chi) \]
\[ \chi = 0 \]
\[ \chi = \frac{1.8}{2.8} \approx 0.64286 \]

(b) (2 points) If \( x_0 = 0.2 \), find \( \lim_{t \to \infty} x_t \).

\[ x_1 = 2.8(0.2)(1 - 0.2) = 0.448 \]
\[ x_2 = 2.8(0.448)(1 - 0.448) = 0.6924288 \]
\[ x_3 = 2.8(0.6924288)(1 - 0.6924288) = 0.5963923940556 \]
\[ x_4 = 2.8(0.5963923940556)(1 - 0.5963923940556) \approx 0.674 \]

(c) (3 points) As you have seen from part (a), if \( x_0 = 0 \), then \( x_1 = x_2 = \cdots = 0 \). Explain why this makes sense from a population point of view.

If the initial population is 0 at all times, the population will keep being 0.
Problem 6. (5 points) Let θ be an angle (in radians) such that \( \sin(\theta) = 1/5 \). Find \( \cos(\theta) \), assuming that \( 0 \leq \theta \leq \pi/2 \).

\[
\begin{align*}
\sin^2(\theta) + \cos^2(\theta) &= 1 \\
\frac{1}{25} + \cos^2(\theta) &= 1 \\
\cos^2(\theta) &= \frac{24}{25} \\
\cos(\theta) &= \sqrt{\frac{24}{25}} = \frac{2\sqrt{6}}{5}
\end{align*}
\]

Since \( 0 \leq \theta \leq \pi/2 \), \( \cos(\theta) \) is positive. So

\[
\cos(\theta) = \frac{2\sqrt{6}}{5}
\]

Problem 7. (5 points) Find \( \lim_{n \to \infty} \frac{n^5 + 2n + 1}{4n^5 - 7} \).

\[
\lim_{n \to \infty} \frac{n^5 + 2n + 1}{4n^5 - 7} = \frac{1}{4}
\]