MAT 17A: CALCULUS FOR BIOLOGY AND MEDICINE
PRACTICE FOR MIDTERM I.

Problem 1 Assume that when we plot the relationship between \( x \) and \( y \) on a log-log plot, a straight line passing through the points \((0, 2)\) and \((-3, 4)\) appears. What is the relationship between \( x \) and \( y \)?

\[
\begin{align*}
X &= \log(x), \quad Y = \log(y) \\
Y &= mX + b.
\end{align*}
\]

Finding \( m \):
\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 4}{-3 - 0} = \frac{-2}{-3} = \frac{2}{3}
\]

Finding \( b \):
\[
Y = -\frac{2}{3}X + b. \quad \text{Take the point \((0, 2)\)}
\]
\[
2 = -\frac{2}{3} \cdot 0 + b \Rightarrow b = 2.
\]

\[
Y = \frac{2}{3}X + 2.
\]

\[
\begin{align*}
\log(Y) &= \log(10) \cdot 2^x \\
\log(Y) &= \log(10) + \log(2^x) \\
\log(Y) &= \log(10) + x \cdot \log(2)
\end{align*}
\]

\[
y = 100 \cdot (10^{\log(x)})^{-\frac{2}{3}}
\]

Problem 2 Let \( y = 10(2^x) \). Sketch the graph of this relationship on a log-linear plot. Be sure to state the slope and intercept of the resulting line.

\[
y = 10(2^x), \quad y = \log(y)
\]

\[
\begin{align*}
\log(y) &= \log(10 \cdot 2^x) \\
\log(y) &= \log(10) + \log(2^x) \\
\log(y) &= \log(10) + x \cdot \log(2)
\end{align*}
\]
Problem 3 Suppose we have 100 grams of an isotope with half-life of 12 hours. How much time will have elapsed when 10 grams remain?

Let $N(t) = $ amount of the isotope (in grams) at time $t$.

$t = $ time in 12-hour intervals!

So that

$N(t) = \frac{1}{2^t} 100$

We want to find $t$ so that $N(t) = 10$

$\frac{1}{2^t} 100 = 10$

So when 10 grams remain,

$\frac{1}{2^t} = \frac{10}{100}$

$\frac{1}{2^t} = \frac{1}{10}$

$2^t = 10$

$t = \log_2(10)$

We want to find $t$ so that $N(t) = 10$.

So when 10 grams remain, $\log_2(10)$ intervals of 12 hour will have elapsed. Another way of saying this is that $\frac{1}{2} \log_2(10)$ hour will have elapsed.

Problem 4 Let $\theta$ be such that $\sin(\theta) = \frac{1}{4}$. Assume also that $\pi/2 \leq \theta \leq \pi$. Find the values of $\cos(\theta)$ and $\tan(\theta)$.

We know $\sin^2(\theta) + \cos^2(\theta) = 1$.

$\left(\frac{1}{4}\right)^2 + \cos^2(\theta) = 1$

$\cos^2(\theta) = 1 - \left(\frac{1}{4}\right)^2 = \frac{15}{16}$

$\cos(\theta) = \pm \frac{\sqrt{15}}{4}$

Since $\frac{\pi}{2} \leq \theta \leq \pi$, $\cos(\theta)$ is negative.

So

$\cos(\theta) = -\frac{\sqrt{15}}{4}$

And $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\frac{1}{4}}{-\frac{\sqrt{15}}{4}} = \frac{-1}{\sqrt{15}}$

$\tan(\theta) = -\frac{1}{\sqrt{15}}$
Problem 5 Graph the following functions. Draw your own axes, but be sure to label any important information (axes, vertices etc.).

(a) \( f : (-2,2) \to \mathbb{R}, f(x) = (x-2)^2 \)

\[ y = x^2 \text{ is a parabola} \]
\[ y = (x-2)^2 \text{ is the same parabola shifted two units to the right} \]

Now we have to restrict the domain to \((-2,2)\)

(b) \( g : [-\pi,\pi] \to \mathbb{R}, g(t) = \cos(t) + 1 \)

\[ \cos(t) \text{ is} \]
\[ \cos(t) + 1 \text{ shifts one unit up, and we have to restrict the domain to } [-\pi, \pi] \]

(c) \( h : (-\infty,1) \cup (1,\infty) \to \mathbb{R}, h(x) = \frac{1}{x-1} + 3 \)

\[ \frac{1}{x} \text{ is the hyperbola} \]
\[ \frac{1}{x-1} \text{ shifts one unit to the right} \]
\[ \frac{1}{x-1} + 3 \text{ shifts three units up} \]
Problem 6 For each of the following recursions:

- Find the fixed points
- Decide whether the sequence has a limit when starting at the provided $a_0$. If it does, state what the limit is.

(a) $a_{n+1} = \frac{a_n}{a_n + 2}; a_0 = 2.$

Fixed points: $x = \frac{x}{x+2}$, $x(x+2) = x$ $x^2 + 2x - x = 0$ $x^2 + x = 0$ $x(x+1) = 0$

Case 1: $x = 0$
Case 2: $x = -1$

$a_0 = 2$, $a_1 = \frac{2}{2+2} = \frac{1}{2}$ $a_2 = \frac{1}{2+2} = \frac{1}{4}$ $a_3 = \frac{1}{4+2} = \frac{1}{6}$ $a_4 = \frac{1}{6+2} = \frac{1}{8}$ $a_5 = \frac{1}{8+2} = \frac{1}{10}$

So $\lim_{n \to \infty} a_n = 0$

(b) $a_{n+1} = 2 \sqrt{a_n}; a_0 = 1.$

Fixed points: $x = 2 \sqrt{x}$, $x^2 - 4x = 0$, $x(x-4) = 0$

Case 1: $x = 0$
Case 2: $x = 4$

$a_0 = 1$, $a_1 = 2 \sqrt{1} = 2$, $a_2 = 2 \sqrt{2} \approx 2.8284$, $a_3 = 2 \sqrt{2.8284} \approx 3.36358$, $a_4 = 2 \sqrt{3.36358} \approx 3.668$, $a_5 = 2 \sqrt{3.668} \approx 3.8304$

So $\lim_{n \to \infty} a_n = 4$

(c) $a_{n+1} = 2 \cdot a_n(1 - a_n), a_0 = 0.5.$

Fixed points: $x = 2.1(1-x)$, $x = 2.1x - 2.1x^2 = 0$, $1.1x - 2.1x^2 = 0$ $x = (1.1 - 2.1)x$

Case 1: $x = 0$
Case 2: $x = 1 \frac{1}{2} = 0.523809529$

$a_0 = 0.5$, $a_1 = 2.1(0.5)(1-0.5) = 0.525$, $a_2 = 2.1(0.525)(1-0.525) = 0.52536975$, $a_3 = 2.1(0.5236975)(1-0.5236975) = 0.52382169492$

So $\lim_{n \to \infty} a_n = 1 \frac{1}{2.1}$
**Problem 7** Find the following limits.

(a) \[ \lim_{n \to \infty} \frac{n^2 + n - 2}{2n^2 + 1} \]

\[ \text{Same degree} \]

\[ \lim_{n \to \infty} \frac{n^3 + n - 2}{2n^2 + 1} = \frac{1}{2} \]

(b) \[ \lim_{n \to \infty} \frac{3^n}{n^{102} - n^3 + 1} \]

\[ \text{3^n exponential with base > 1} \]

\[ \lim_{n \to \infty} \frac{3^n}{n^{102} - n^3 + 1} \text{ due} \]

(c) \[ \lim_{n \to \infty} \frac{1}{n + 1} + \frac{1}{n^2 + 1} \]

\[ \lim_{n \to \infty} \frac{1}{n + 1} + \frac{1}{n^2 + 1} = 0 + 0 = 0 \]

\[ \lim \text{in } 0 \]

\[ \alpha, \quad \frac{1}{n + 1} + \frac{1}{n^2 + 1} = \frac{n^2 + 1 + n + 1}{(n+1)(n^2+1)} = \frac{n^2 + n + 2}{n^2 + n + 1} \]
Problem 8 Assume that when we plot the relationship between $x$ and $y$ on a log-linear plot, the line $2x + 3Y + 1 = 0$ appears. Find the relationship between $x$ and $y$.

$$2x + 3y = -1$$
$$3y = -1 - 2x$$
$$y = -\frac{1}{3} - \frac{2}{3}x$$

$$10^y = 10^{\frac{1}{3} - \frac{2}{3}x}$$
$$y = 10^{\frac{1}{3}} \cdot 10^{-\frac{2}{3}x}$$
$$y = 10^{\frac{1}{3}} \cdot (10^{-\frac{2}{3}})^x$$

Problem 9 Assume that, when plotting the relationship between $x$ and $y$ on a log-log plot, the line $X + 2Y = 1$ appears. Now let $X^* = \ln(x)$, $Y^* = \ln(y)$. Find the relationship between $X^*$ and $Y^*$.

$$X = \log x = \frac{\ln x}{\ln 10} = \frac{x^*}{\ln 10}$$

$$Y = \log y = \frac{\ln y}{\ln 10} = \frac{y^*}{\ln 10}$$

$$X + 2Y = 1 \quad \text{substituting}$$

$$\frac{x^*}{\ln 10} + 2 \frac{y^*}{\ln 10} = 1.$$

Multiply by $\ln 10$

$$x^* + 2y^* = \ln(10)$$
Problem 10  A strain of bacteria reproduces asexually every 20 minutes. Initially, there are 120 bacteria. Let $N_k$ be the number of bacteria after $k$ periods of 20 minutes.

(a) Write down a recursive relation with initial condition $N_0$ that describes how the population changes.

\[
N_0 = 120
\]

\[
N_{t+1} = 2N_t
\]

(b) Write down an explicit formula for $N_k$ as a function of $k$.

\[
N_k = 2^k \cdot 120
\]