MAT 17A: CALCULUS FOR BIOLOGY AND MEDICINE
Midterm 2. March 1, 2019

Print name: key

Section   TA   Time
B01       Nate  4:10pm
B02       Graham  7:10pm
B03       Graham  6:10pm

Section: (circle one) B04   Ye   5:10pm
B05       Carter  5:10pm
B06       Qianhui  8:10pm
B07       Ye   6:10pm
B08       Qianhui  7:10pm

Read carefully the following instructions:

• PLEASE DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO.

• It is a violation of the University Honor Code to, in any way, assist another person in the completion of this exam.

• Put away all scrap paper, books, notebooks, cellphones and other electronic devices. Only a scientific calculator is allowed.

• Write your last name, first name and student ID on each page.

• Show all your work for full credit. If in doubt, write it out.

• Keep your work as neat as possible. If we can’t read it, we can’t grade it!

• Unless otherwise indicated, you do NOT need to simplify your answers.

• Make sure you have 3 pages, including this cover page.

• You have until 6pm sharp to finish this exam.

Good luck!

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Problem 1. Find the derivatives of the following functions. [4 points each]

(a) \( f(t) = \frac{4t^7 + 2t^2 + e^t}{\pi} \).

\[ f'(t) = \frac{28t^6 + 4t}{\pi} \]

(b) \( f(x) = \frac{x^3 + 3\tan(x) + 1}{\cos(x)} \).

\[ f'(x) = \frac{(3x^2 + 3\sec^2(x))(\cos(x)) + \sin(x)(x^3 + 3\tan(x) + 1)}{\cos^2(x)} \]

\[ = \frac{3x^2\cos(x) + 3\sec(x) + x^3\sin(x) + 3\tan(x)\sin(x) + \sin(x)}{\cos^2(x)} \]

(c) \( j(x) = 3\sin(x)e^x - 2x\ln(x) \).

\[ j'(x) = 3\cos(x)e^x + 3\sin(x)e^x - 2\ln(x) - 2 \]
Problem 2. Find the derivatives of the following functions. [6 points each]

(a) \( k(x) = \cos(\sqrt{x^2 + e^x}) \).
\[
 k'(x) = -\sin \left( \sqrt{x^2 + e^x} \right) \left( \frac{2x}{2} + e^x \right) \cdot \frac{1}{\sqrt{x^2 + e^x}} - \frac{e^x}{2} 
\]

(b) \( m(x) = \frac{e^{x^2} \cos^2(x)(x + 2)}{(x^2 + 2x)^x} \).
\[
 m'(x) = \frac{1 - 2 \sin(x)x}{\cos(x)} + \frac{1}{x^2 + 2x} - 7x \ln(\frac{x^2 + 2x}{x^2}) - \frac{3x(2x + 4)}{x^2 + 2x} 
\]
\[
 m'(x) = \left( \frac{1 - 2 \tan(x) + \frac{1}{x^2 + 2x} - 7x \ln(\frac{x^2 + 2x}{x^2}) - \frac{3x(2x + 4)}{x^2 + 2x}}{x^2 + 2x} \right) e^{x^2} \cos(x)(x + 2) 
\]
Problem 3. The curve described by the equation \( x^{2/3} + y^{2/3} = 1 \) is known as an astroid. Here is its graph

(a) Use implicit differentiation to find the equation of the tangent line to the astroid \( x^{2/3} + y^{2/3} = 1 \) at the point \( \left( \frac{1}{8}, \frac{-\sqrt[3]{27}}{8} \right) \). [9 points]

\[
\frac{2}{3} x^{-1/3} + \frac{2}{3} y^{-1/3} y' = 0
\]

\[
y' = -\frac{1}{3} \frac{y^{-2/3} x^{-1/3}}{y^{-1/3}} = \left( \frac{y}{x} \right)^{\frac{1}{3}}
\]

\[
\text{Slope} \left( \frac{-\sqrt[3]{27}}{8} \right) = \left( -\sqrt[3]{27} \right)^{1/3} = -\sqrt[3]{3}
\]

\[
y - \frac{-\sqrt[3]{27}}{8} = -\sqrt[3]{3} \left( x - \frac{1}{8} \right)
\]

\[
y = -\sqrt[3]{3} x + \frac{3\sqrt[3]{3}}{8} + \frac{\sqrt[3]{27}}{8}
\]

(b) Note that \( y' \) is not well-defined at the points \((1, 0), (0, 1), (-1, 0)\) and \((0, -1)\). Without referring to your equation for \( y' \) and referring only to the graph pictured above, explain why this must be the case. [3 points]

The graph has sharp points at \((1, 0), (0, 1), (-1, 0)\) \& \((0, -1)\). So the tangent line is not defined there, and neither is its slope.
Problem 4. Find the following limits. Show all your work for full credit. [3 points each]

(a) \( \lim_{t \to \infty} \frac{e^{2t} - 4}{e^{3t} + 1} \).

\[
\lim_{t \to \infty} \frac{e^{2t} - 4}{e^{3t} + 1} = \lim_{t \to \infty} \frac{e^{2t} - e^{3t}}{1 + e^{3t}} = 0
\]

(b) \( \lim_{t \to -\infty} \frac{e^{2t} - 4}{e^{3t} + 1} \).

Set \( z = -t \), so \( z \to 0 \) as \( t \to -\infty \).

\[
\lim_{t \to -\infty} \frac{e^{2t} - 4}{e^{3t} + 1} = \lim_{z \to 0} \frac{e^{2z} - 4}{e^{3z} + 1} = -4
\]

(c) \( \lim_{z \to 0} \frac{\sin(4z)}{z} \).

Set \( t = 4z \), so \( z = t/4 \).

\[
\lim_{z \to 0} \frac{\sin(4z)}{z} = \lim_{t \to 0} \frac{\sin(t)}{t/4} = 4 \lim_{t \to 0} \frac{\sin(t)}{t} = 4
\]

(d) \( \lim_{x \to \infty} \cos(x^2 + 3x + \pi) e^{-x} \). Note that \(-1 \leq \cos(x^2 + 3x + \pi) \leq 1\).

So \(-e^{-x} \leq \cos(x^2 + 3x + \pi) e^{-x} \leq e^{-x}\).

By the Sandwich Theorem, \( \lim_{x \to \infty} \cos(x^2 + 3x + \pi) e^{-x} = 0 \).
Problem 5. Water is being poured into an inverted right circular conical tank at the rate of 10 cm³/sec. The tank has a height of 30 cm and the radius on top is 10 cm.

(a) At what rate is the water level rising when the water is 4 cm deep? [9 points]

\[
\frac{dV}{dt} = 10 \text{ cm}^3/\text{sec} \quad V = \frac{\pi}{3} r^2 h. \quad \text{Wrote: } \frac{dh}{dt}.
\]

\[
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\[
V = \frac{\pi}{3} \ \frac{h^3}{9} = \frac{\pi h^3}{27}
\]

\[
\frac{dV}{dt} = \frac{\pi h^2}{9} \frac{dh}{dt}.
\]

\[
10 \text{ cm}^3/\text{sec} = \pi \left(\frac{16}{9}\right) \text{ cm}^2 \frac{dh}{dt}.
\]

So \( \frac{dh}{dt} = \frac{90}{16\pi} \text{ cm/sec}, \) the water level is rising at a rate of \( \frac{90}{16\pi} \text{ cm per second} \).

(b) Without doing any calculations, do you expect the water level to be rising at a faster or at a slower rate than part (a) when the water is 10 cm deep? Explain your answer for full credit. [3 points]

The tank is wider at 10 cm than it is at 4 cm, so more water is needed to increase the water level. Thus, we expect the water level to be increasing at a slower rate.