On Harish-Chandra bimodules for rational Cherednik algebras

by José Simental Rodríguez

B. S. Universidad Nacional Autónoma de México
M. S. Ohio University

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Ivan Loseu
Professor of Mathematics
Dedication

A mi mamá, mi papá, la Fer y el Kbto.
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Abstract of Dissertation

We study Harish-Chandra bimodules for rational Cherednik algebras $H_c(W)$ associated to a complex reflection group $W$ and parameter $c$. Our results allow to partially reduce the study of these to smaller algebras. We use this to classify those pairs of parameters $(c, c')$ for which there exist Harish-Chandra bimodules with full support, and we give a description of the category of all Harish-Chandra bimodules modulo those bimodules with proper support. When $W$ is the symmetric group, we produce an embedding from the category of Harish-Chandra $H_c$-bimodules to the category $O_c$, prove that its image is closed under subquotients, and find the irreducibles in its image. Finally, when $W$ is of cyclotomic type we produce a duality in the category of Harish-Chandra bimodules. We do this in the more general setting of quantized quiver varieties. Our methods are based on localization techniques, the study of partial KZ functors, the action of Namikawa-Weyl groups and restriction functors.
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Disclaimer

I hereby declare that the work in this thesis is that of the candidate alone, except where indicated in the text, and as described below.

Chapter 2 contains preliminary material and does not contain original results of the candidate.

Chapter 3 contains both original results of the candidate as well as results of other authors, as indicated in the text.

Chapters 4-6 are part of work of the author to be published as *Harish-Chandra bimodules over rational Cherednik algebras*. 27 pages. Submitted. Available at [https://arxiv.org/abs/1409.5465](https://arxiv.org/abs/1409.5465)

Chapter 7 contains both original results of the candidate, part of an ongoing work, and results of other authors, as indicated in the text.
Chapter 1

Introduction

In this work we study the representation theory of rational Cherednik algebras. Rational Cherednik algebras form an interesting class of algebras associated to the action of a complex reflection group $W$ and its reflection representation $R$, a precise definition is Definition 2.1.1. The rational Cherednik algebra depends on a parameter $c$, which is a conjugation invariant function $c : S \to \mathbb{C}$, where $S$ is the set of reflections of $W$. One of the reasons why this algebra is interesting is that it admits a filtration whose associated graded is the smash-product algebra $\mathbb{C}[R \oplus R^*]#W$, and so one can use the algebra $H_c$ to produce quantizations of the algebra of invariants $\mathbb{C}[R \oplus R^*]^W$, see [EG, CBH], among others.

A consequence of $H_c$ having a filtration with associated graded $\mathbb{C}[R \oplus R^*]#W$, together with the explicit presentation of $H_c$ given in Definition 2.1.1, is that we have three subalgebras in $H_c$, namely $\mathbb{C}[R], \mathbb{C}[R^*]$ and $CW$ and, moreover, as a vector space we have $H_c = \mathbb{C}[R] \otimes CW \otimes \mathbb{C}[R^*]$. So we can think of $CW$ as a “Cartan subalgebra” and of $\mathbb{C}[R], \mathbb{C}[R^*]$ as positive and negative “Levi subalgebras” of $H_c$, respectively. From this point of view, the representation theory of rational Cherednik algebras has many similarities to that of semisimple Lie algebras. For example, one has a category $O_c$, first defined in [GGOR] and that has been extensively studied in recent years, see, for example, [BEG2, BE, EGL, ES, Gi2, GL, L5, L6, L7, R, RSVV, Sh, SV, V, Wi], to name just a few.
In [BEG], a notion of Harish-Chandra bimodules for rational Cherednik algebras was introduced. The definition is similar to that of Harish-Chandra bimodules for universal enveloping algebras that arise in the study of projective functors, see [BG]. In particular, a Harish-Chandra \((H_c, H_{c'})\)-bimodule gives a functor \(B \otimes_{H_{c'}} \bullet : \mathcal{O}_{c'} \to \mathcal{O}_c\). These functors have been used in [L7] to produce derived equivalences between different categories \(\mathcal{O}_{c'}\). We remark, however, that as opposed to the Lie algebra setting, the functor of multiplying by a Harish-Chandra bimodule is not in general exact. Let us also remark that, while in the Lie algebra setting the categories of Harish-Chandra bimodules and category \(\mathcal{O}\) are very similar, cf. [BG, Section 5], this no longer needs to be the case in the Cherednik algebra setting, see Section 6.1.1.

It is known, thanks to [Gi2, L3], that the category of Harish-Chandra \((H_c, H_{c'})\)-bimodules is an abelian category with finitely many simples and such that every object has finite length. An interesting problem, then, is to describe the category \(\text{HC}(H_c, H_{c'})\) of Harish-Chandra \((H_c, H_{c'})\)-bimodules. In this work, we obtain several partial results towards this description. As a first step we show in Corollary 3.3.14 that the category \(\text{HC}(H_c, H_{c'})\) is equivalent to the category of representations of a finite-dimensional algebra. In the remainder of this introduction, we will give an overview of our results, as well as of previously known results on the structure of the category \(\text{HC}(H_c, H_{c'})\).

1.1 Harish-Chandra bimodules with full support

Let us say that a parameter \(c : S \to W\) is regular if the algebra \(H_c\) is simple, cf. Section 2.3.6. For example, it is known that if \(W\) is a Coxeter group every integral parameter \(c : S \to \mathbb{Z}\) is regular. In [BEG], Berest-Etingof-Ginzburg studied the category \(\text{HC}(H_c, H_{c'})\) in the case where \(W\) is a Coxeter group and \(c, c'\) are integral, in connection to the algebra of differential operators on \(c\)-quasi-invariants for the action of \(W\) on \(R\). In particular, they showed that when \(c\) is integral, then the algebra of differential operators on \(c\)-quasi-invariants is naturally
a bimodule over the spherical rational Cherednik algebra $A_c := eHce$, where $e \in CW$ is the trivial idempotent, and they used this to produce the following result.

**Theorem 1.1.1** ([BEG]). Assume $W$ is a Coxeter group and $c, c' : S \to \mathbb{Z}$ are parameters. Then, the category $\text{HC}(H_c, H_{c'})$ is equivalent to the category of representations of $W$. Moreover, if $c = c'$ then this is an equivalence of monoidal categories.

Theorem 1.1.1 was later extended by Spencer, [Sp], by considering the more general case in which $c, c'$ are regular, not necessarily integral.

**Theorem 1.1.2** ([Sp]). Let $W$ be a Coxeter group and $c, c' : S \to \mathbb{C}$ regular parameters. The following is true.

1. The category $\text{HC}(H_c, H_{c'})$ is nonzero if and only if there exists a character $\varepsilon : W \to \mathbb{C}^\times$ such that $c - \varepsilon c'$ is integral.

2. Assume that $c - \varepsilon c'$ is integral for a character $c' : W \to \mathbb{C}^\times$. Then, $\text{HC}(H_c, H_{c'})$ is equivalent to the category of representations of $W/W'$, where $W' := \langle s \in S : c(s) \notin \mathbb{Z} \rangle$.

If $c = c'$, this is an equivalence of monoidal categories.

Here, we extend Theorem 1.1.2 in two directions. Firstly, we remove the restriction that the parameters $c, c'$ are regular, at the cost of only looking at a certain quotient of the category $\text{HC}(H_c, H_{c'})$, which we are going to denote by $\overline{\text{HC}}(H_c, H_{c'})$, see Chapter 5 for a precise definition of $\overline{\text{HC}}(H_c, H_{c'})$. We remark that, if either $c$ or $c'$ is regular, then $\text{HC}(H_c, H_{c'}) = \overline{\text{HC}}(H_c, H_{c'})$.

Secondly, we remove the restriction that $W$ is a Coxeter group. In fact, in our result $W$ can be any complex reflection group. In this setting we still have a notion of integral parameters, although it is more complicated, see Section 2.3.5. However, the condition of having a character $\varepsilon : W \to \mathbb{C}^\times$ with $c - \varepsilon c'$ being integral turns out to be sufficient, but not necessary, to guarantee that the category $\overline{\text{HC}}(H_c, H_{c'})$ is nonzero.
To find necessary and sufficient conditions for \( \mathcal{H}C(H_c, H_{c'}) \) to be nonzero we need to introduce the action of a certain group on the space of parameters for the Cherednik algebra. This is the Namikawa-Weyl group, which is a product of symmetric groups, one for each orbit of reflection hyperplanes \( H \subseteq R \). This group coincides with the usual Namikawa-Weyl group for a symplectic resolution of \((R \oplus R^*)/W\) when the latter variety admits such a resolution, cf. [Nam]. In general, we can take the Namikawa-Weyl group of a \( \mathbb{Q} \)-factorial terminalization, which always exists, see [L10]. Here, we introduce the Namikawa-Weyl group, as well as its action on the space of parameters, by more elementary methods, see Section 5.3.

**Theorem 1.1.3.** Let \( W \) be a complex reflection group, and \( c, c' : S \to \mathbb{C} \) parameters. Then, the following holds.

1. The category \( \mathcal{H}C(H_c, H_{c'}) \) is nonzero if and only if we can get from \( c \) to \( c' \) by a sequence of integral translations and actions of elements of the Namikawa-Weyl group.

2. If \( \mathcal{H}C(H_c, H_{c'}) \) is nonzero, then it is equivalent to the category of representations of \( W/W_c \), for an explicitly defined normal subgroup \( W_c \subseteq W \). If \( c = c' \), this is an equivalence of monoidal categories.

Let us remark that the condition of getting from \( c \) to \( c' \) by a sequence of integral translations and elements of the Namikawa-Weyl group is not complicated to check, and it actually expresses a relationship between the Hecke parameters \( q(c), q(c') \) associated to \( c \) and \( c' \), respectively. The parameters \( q(c), q(c') \) are a collection of nonzero complex numbers that are determined from \( c, c' \) via an exponential formula, see Section 2.3.4. We elaborate on the connection between the parameters \( q(c), q(c') \) and the condition from Theorem 1.1.3 in Section 5.3.5.

We also mention that the reason why the action of the Namikawa-Weyl group does not appear in Theorems 1.1.1, 1.1.2 is that in the case of a Coxeter group, the action of an element of the Namikawa-Weyl group can be obtained by multiplying by a character, followed by an
integral translation. Similarly, the reason why multiplication by a character does not appear in our Theorem 1.1.3 is that multiplication by a character is obtained by applying an element of the Namikawa-Weyl group followed by an integral translation. However, when $W$ is not a Coxeter group, the Namikawa-Weyl group is bigger than the group of characters. We elaborate on this in Section 5.3.5.

The definition of the subgroup $W_c \subseteq W$ is explicit, but technical, see Section 5.4. Here, we just mention that this subgroup satisfies the following properties.

1. It is a complex reflection group (but not, in general, a parabolic subgroup of $W$).

2. $W_c = W_{c'}$ if either $c - c'$ is integral, or $c$ and $c'$ belong to the same Namikawa-Weyl group orbit.

3. $W_c = \{1\}$ if and only if $c$ belongs to the Namikawa-Weyl group orbit of an integral parameter.


Note, in particular, that the structure (and even the number of irreducibles) of $\text{HC}(H_c, H_c)$ for regular $c$ really depends on the parameter $c$, which is in contrast with category $\mathcal{O}_c$, which is equivalent to the category of representations of $W$ whenever $c$ is regular.

### 1.2 Harish-Chandra bimodules for rational Cherednik algebras of type A

When $W = S_n$, acting on its reflection representation $R = \{(x_1, \ldots, x_n) \in \mathbb{C}^n : \sum_{i=1}^n x_i = 0\}$, we obtain information on the category $\text{HC}(H_c, H_c)$, not just its quotient $\overline{\text{HC}}$. First of all, recall that $\text{HC}(H_c, H_c') = \overline{\text{HC}}(H_c, H_c')$ if one of $c$, $c'$ is regular, so we will assume that both $c$ and $c'$ are singular. It is known that the set of singular parameters is given by those nonzero
rational numbers \( r/m \) where the fraction \( r/m \) is irreducible and \( 1 < m \leq n \). We will denote \( H_c(S_n) =: H_c(n) \).

**Theorem 1.2.1.** There is a functor \( \Phi_c : HC(H_c(n), H_c(n)) \to O_c(n) \), which is a fully faithful embedding whose image is closed under subquotients.

The functor \( \Phi_c \) is similar in spirit to the one used in [BG, Section 5], namely, it is taking tensor product with a projective Verma module. We remark, however, that Theorem 1.2.1 does not generalize beyond type \( A \). Indeed, in Section 6.1.1 we show via an example that if \( W \) is, say, a cyclic group, then there cannot exist a functor as the one in Theorem 1.2.1 for arbitrary \( c \).

Of course, the natural question now is to describe the image of the functor \( \Phi_c \). We cannot do this in general, but we can describe the irreducible modules in the image of \( \Phi_c \). We remark, however, that the image of \( \Phi_c \) is not closed under extensions, so knowing which irreducibles it contains is not enough to describe it.

Let us describe the irreducible modules in the image of \( \Phi_c \). The irreducible modules in category \( O_c \) are parametrized by partitions of \( n \). If \( \lambda \) is such a partition, it is easy to see that there exists a unique way to express \( \lambda \) as \( \lambda = m\mu + \nu \), where the partition \( \nu \) is \( m \)-regular, that is, no two consecutive parts of \( \nu \) differ by more than \( m - 1 \). Let us say that an \( m \)-regular partition \( \nu \) is \( m \)-trivial if \( \nu = ((m - 1), \ldots , (m - 1), d, 0, \ldots ) \) for some \( 0 \leq d < m - 1 \).

**Theorem 1.2.2.** Assume \( c = r/m > 0 \). Let \( \lambda \) be a partition of \( n \), and decompose it as \( \lambda = m\mu + \nu \), where \( \nu \) is \( m \)-regular. Then, the irreducible module \( L_c(\lambda) \) corresponding to the partition \( \lambda \) belongs to the image of \( \Phi_c \) if and only if \( \nu \) is \( m \)-trivial.

Now assume \( c, c' \) are different, singular parameters. So \( c = r/m, c' = r'/m' \).

**Lemma 1.2.3.** The category \( HC(H_c, H_{c'}) \) is zero unless \( m = m' \).

Thanks to the previous lemma, we can assume that \( c = r/m, c' = r'/m \). Let us set \( \ell := \lfloor n/m \rfloor \) We construct a filtration \( \{0\} = HC(H_c, H_{c'}(n))^{\ell+1} \subseteq HC(H_c(n), H_{c'}(n))^{\ell} \subseteq \cdots \subseteq \)
HC(H_c(n), H_{c'}(n))^1 \subseteq HC(H_c(n), H_{c'}(n))^0 by Serre subcategories. Let us set HC(H_c(n), H_{c'}(n))_i := HC(H_c(n), H_{c'}(n))^i / HC(H_c(n), H_{c'}(n))^{i+1} for i = 0, \ldots, \ell. Then, we show the following result.

**Theorem 1.2.4.** Let i \in \{0, 1, \ldots, \ell\}.

1. Assume n - im \neq 0. Then, HC(H_c(n), H_{c'}(n))_i \neq 0 if and only if either c + c' or c - c' is an integer. If this is the case, then HC(H_c(n), H_{c'}(n))_i is equivalent to the category of representations of S_i, and this is an equivalence of monoidal categories when c = c'.

2. Assume n = im. Then, HC(H_c(n), H_{c'}(n))_i is equivalent to the category of representations of S_i. If c = c', this is an equivalence of monoidal categories.

Let us remark that the filtration mentioned above is the filtration by supports that has already been constructed by Losev, [L3], in a more general setting that we will review in Section 3.2.2. What is new here is the description of the subquotients.

### 1.3 Duality

Our last main result is a description of duality for certain types of rational Cherednik algebras, namely, those associated to the groups G(\ell, 1, n) := S_n \ltimes (\mathbb{Z}/\ell\mathbb{Z})^n, viewed as the group of monomial (n \times n)-matrices whose nonzero entries are \ell-roots of 1. In fact, we construct this duality in a more general setting: quantizations of quiver varieties.

Quantized quiver varieties fall into the more general setting of quantizations of symplectic resolutions, see [BPW, BLPW]. Let us assume that X is a smooth, symplectic variety with a \mathbb{C}^\times-action rescaling the symplectic form and such that the natural map X \to X_0 is a resolution of singularities, where X_0 is the affine variety Spec(\mathbb{C}[X]). By a quantization of X, we mean a filtered sheaf of algebras \mathcal{A} on the conical topology (where open = Zariski-
open + \( \mathbb{C}^* \)-stable) of \( X \), together with a Poisson isomorphism \( \iota : \text{gr} \ A \to \mathcal{G}_X \).\(^1\) The sheaf \( A \) is supposed to satisfy some technical assumptions, see Section 7.2.2. A quantization of \( X_0 \) is a filtered algebra \( A \) together with an isomorphism of Poisson algebras \( \text{gr} \ A \to \mathbb{C}[X] \).

Let us remark that, if \( A \) is a quantization of \( X \), then \( A := \Gamma(X, A) \) is a quantization of \( X_0 \). We have functors of localization and global sections \( \Gamma(X, \cdot) : A \text{-mod} \to A \text{-mod}, \text{Loc} : A \text{-mod} \to A \text{-mod} \). These functors, however, are not always an equivalence. For example, for \( X_0 \) we can take the nilpotent cone in a semisimple Lie algebra \( g \), and \( X \) is the Steinberg variety. The quantizations of \( X \) are naturally parametrized by weights. The Beilinson-Bernstein localization theorem tells us that the functors of localization and global sections are inverse equivalences precisely when the weight is regular and dominant. If the global sections and localization functors are an equivalence, we say that localization holds at \( A \).

For quantizations \( A, A' \) of \( X \), with \( A := \Gamma(X, A), A' := \Gamma(X, A') \), we can define categories of Harish-Chandra \((A, A')\)-bimodules, as well as of Harish-Chandra \((A, A')\)-bimodules, see Section 7.3. Let us denote by \( \mathcal{HC}(A, A') \) the category of Harish-Chandra \((A, A')\)-bimodules, and by \( \mathcal{HC}(A, A') \)-the category of Harish-Chandra \((A, A')\)-bimodules. We have functors \( \Gamma : \mathcal{HC}(A, A') \leftrightarrow \mathcal{HC}(A, A') : \text{Loc} \) which are quasi-inverse equivalences when localization holds both at \( A \) and \( (A')^{\text{opp}} \).

The category \( \mathcal{HC}(A, A) \) has a natural duality: the homological duality. By standard homological algebra results, for a Harish-Chandra \( A \)-bimodule \( B \), the complex \( R\text{Hom}_{A\text{-bimod}}(B, A) \) has nonzero homology only at degree \(-\dim X\). So there is a natural duality \( \mathcal{HC}(A, A) \to \mathcal{HC}(A, A)^{\text{opp}} \). Of course, by taking global sections we would like to have a duality \( \mathcal{HC}(A, A) \to \mathcal{HC}(A, A)^{\text{opp}} \). But localization does not need to hold at \( A \) and \( A^{\text{opp}} \) simultaneously.

Luckily, when \( X \) is a smooth Nakajima quiver variety there is a way to fix this. In fact, there is an isomorphism \( A \to A^{\text{opp}} \) for any quantization of \( X_0 \). A small technical issue is that

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\(^1\)following [BPW, BLPW], to avoid confusions with category \( \mathcal{O} \), we denote the structure sheaf of a variety \( X \) by \( \mathcal{G}_X \).
this isomorphism does \textit{not} induce the identity at the associated graded level. So we need to slightly generalize the definition of a Harish-Chandra bimodule to take this into account. Let us state the duality theorem, adapted for type A rational Cherednik algebras.

\textbf{Theorem 1.3.1.} Let $c \in \mathbb{C}$ be not a negative rational number. Let $N \in \mathbb{Z}$ be such that $-c + N \not\in \mathbb{Q}_{<0}$. Then, there is an equivalence of categories

$$\mathcal{D} : \text{HC}(H_c(n), H_c(n)) \to \text{HC}(H_{-c+N}(n), H_{-c+N}(n))^\text{opp}$$

The restriction $c \not\in \mathbb{Q}_{<0}$ is not very important and, in fact, it can be removed, see Corollary 7.4.2.

\section*{1.4 Structure of the dissertation}

Let us now briefly mention how is this work structured. In Chapter 2, we recall basic definitions and properties of rational Cherednik algebras and their categories $\mathcal{O}_c$. In particular, we recall the connection of the category $\mathcal{O}_c$ to Hecke algebras via the KZ functor, and the Bezrukavnikov-Etingof parabolic restriction functors. This chapter contains no new results.

In Chapter 3, we recall some basic known facts about the category of Harish-Chandra bimodules, as well as proving some new results. We recall the original definition due to Berest-Etingof-Ginzburg, [BEG2], as well as a very useful equivalent definition due to Losev, [L3]. We also recall Losev's restriction functors, first in their naive (but easier) version, and then in their equivariant version, which is the version that will be most useful for us. In this chapter, we prove some new results concerning the structure of the category of Harish-Chandra bimodules. For example, we show that the tensor product of two irreducible bimodules vanishes unless the two bimodules have the same support, see Section 3.2 for the definition of the singular support of a Harish-Chandra bimodule. We also show that the
category of Harish-Chandra bimodules is equivalent to the category of representations of a finite-dimensional algebra. We would like to remark that, while this result is new, all the ingredients to prove it are already in the literature, specifically in work of Ginzburg, [Gi2], and Losev, [L3, L7].

Chapter 4 is technical. Its main purpose is to find sufficient conditions for an equivariant bimodule to be in the image of the equivariant restriction functor. This will be our main tool to produce new Harish-Chandra bimodules, see Theorem 4.1.1.

Chapter 5 studies the category $\mathcal{HC}$. There are two main ingredients in this chapter. One of them is localization to the regular locus, which relates Harish-Chandra bimodules to $D$-modules on $R^{reg}/W$ and therefore to the KZ functor and Hecke algebras. The other one is the Namikawa-Weyl group action. We introduce this group by means of a certain isomorphism of the spherical rational Cherednik algebras for cyclic groups and central reductions of the finite $W$-algebra associated to the subregular nilpotent element in $\mathfrak{sl}_\ell$. After carefully studying the action of the Namikawa-Weyl group on the set of parameters, we prove the results mentioned in Section 1.1 of this introduction.

Chapter 6 deals with rational Cherednik algebras of symmetric groups. There, we prove the results mentioned in Section 1.2. To prove an equivalence between the category of Harish-Chandra bimodules and a full subcategory of category $\mathcal{O}_c$, our main ingredients are the restriction functors for Harish-Chandra bimodules, even in their naive (non-equivariant) version. So the first thing we do is to study Harish-Chandra bimodules in the easiest non-trivial case, namely, when $c = r/n$ (for the symmetric group $S_n$). We find all the irreducible bimodules in the category $\mathcal{HC}(H_{r/n}(n), H_{r/n}(n))$, and after this we proceed to produce a desired equivalence. To find the image of this functor, our main ingredients are Theorem 4.1.1,
together with results of Wilcox, [Wi], on the support of irreducible modules in category $\mathcal{O}$, and a remarkable symmetry result of Calaque-Enriquez-Etingof, [CEE], see also [EGL].

Finally, in Chapter 7, we produce the duality mentioned in Section 1.3. In order to do this, we briefly recall the definition of Nakajima quiver varieties, as well as their quantizations. We also recall results of [BPW, BLPW] regarding localization theorems and quantizations of line bundles. After giving all necessary preliminaries, we proceed to define the duality for Harish-Chandra bimodules over quantized quiver varieties, and its particular case of type A rational Cherednik algebras.
Chapter 2

Rational Cherednik Algebras

2.1 Rational Cherednik Algebras.

2.1.1 Definition and examples.

Let $W$ be a complex reflection group, with reflection representation $R$ and set of reflections $S \subseteq W$. For each reflection $s \in S$, let $\alpha_s \in R^*$ be an eigenvector with eigenvalue $\lambda_s \neq 1$. In particular, the reflection hyperplane of $s$ is given by $\Gamma_s := \{\alpha_s = 0\} \subseteq R$. Also, let $\alpha_s^\vee \in R$ be an eigenvector of $s$ with eigenvalue $\lambda_s^{-1}$. We remark that $\alpha_s^\vee, \alpha_s$ are unique up to multiplication by a nonzero scalar, and we normalize so that $\langle \alpha_s, \alpha_s^\vee \rangle = 2$. To define the rational Cherednik algebra, we need a conjugation-invariant function $c : S \to \mathbb{C}$.

Definition 2.1.1 (Etingof and Ginzburg, [EG]). The rational Cherednik algebra $H_c := H_c(W, R)$ is the quotient of the smash product algebra $T(R \oplus R^*)#W$, where $T(\bullet)$ denotes the tensor algebra, by the relations:

$$\begin{align*}
[x, x'] &= 0, \\
[y, y'] &= 0 \\
[y, x] &= \langle y, x \rangle - \sum_{s \in S} c(s) \langle \alpha_s, y \rangle \langle x, \alpha_s^\vee \rangle s \quad x \in R^*, y \in R.
\end{align*}$$

(2.1)

Let us give a few examples of rational Cherednik algebras.
Assume the function $c$ is constant 0. Then, $H_0 = D(R)\#W$, where $D(R)$ denotes the algebra of differential operators on $R$.

Assume $W = S_n$ is the symmetric group, acting on $\mathbb{C}^n$ by permuting the coordinates. Then, $c$ may be thought of as a single complex number and $H_c(n) := H_c(S_n, \mathbb{C}^n)$ is the algebra

$$H_c(n) = \mathbb{C}\langle x_1, \ldots, x_n, y_1, \ldots, y_n \rangle \# S_n$$

where $(ij) \in S_n$ denotes the transposition $i \leftrightarrow j$.

Now let $W = S_n$ again, acting on its $(n-1)$-dimensional reflection representation $R := \{(x_1, \ldots, x_n) \in \mathbb{C}^n : \sum_{i=1}^n x_i = 0\}$. Then we have that a presentation for $\overline{H_c(n)} := H_c(S_n, R)$ is given by

$$\mathbb{C}\langle x_1, \ldots, x_n, y_1, \ldots, y_n \rangle \# S_n$$

where $H_c(n)$ is a subalgebra of $H_c(n)$ generated by $x_i := x_i - \frac{1}{n} \sum_{j=1}^n x_j$, $y_i := y_i - \frac{1}{n} \sum_{j=1}^n y_j$, $i = 1, \ldots, n$, and $S_n$. Furthermore, if $X = \frac{1}{n} \sum_{j=1}^n x_j \in H_c(n)$ and $Y = \frac{1}{n} \sum_{j=1}^n y_j \in H_c(n)$, then the algebra generated by $X, Y$ in $H_c(n)$ is isomorphic to the algebra of differential operators on $\mathbb{C}$ and, moreover,

$$H_c(n) \cong \overline{H_c(n)} \otimes D(\mathbb{C})$$

Now let $W = \mathbb{Z}/\ell\mathbb{Z}$, with generator $s$ which acts on $R = \mathbb{C}$ by multiplication by $\eta := \exp(2\pi \sqrt{-1}/\ell)$. Denote $c_i := c(s^i)$, $i = 1, \ldots, \ell - 1$. Then we have that $H_c(W, \mathbb{C})$ is given by

$$\mathbb{C}\langle x, y, s \rangle$$

where

$$s^\ell = 1, sx = \eta^{-1}xs, sy = \eta y s, [y, x] = 1 - \sum_{i=1}^{\ell-1} c_i s^i$$

### 2.1.2 Dunkl Operators

Let us mention that the rational Cherednik algebra $H_c := H_c(W, R)$ can be explicitly realized as a subalgebra of the smash-product algebra $D(R^{reg})\#W$, where $R^{reg}$ is the principal open
set of $R$ where the $W$-action is free. To produce a desired algebra we need the concept of Dunkl operators. Note that $R^{\text{reg}} = R \setminus \{ \delta = 0 \}$, where $\delta := \prod_{s \in S} \alpha_s \in S(R^*)$.

**Definition 2.1.6** (See [Du]). Let $c : S \to \mathbb{C}$ be a conjugation-invariant function. Then, for $y \in R$, we have the Dunkl operator

$$D_y := \partial_y - \sum_{s \in S} \frac{2c(s)(\alpha_s, y)}{(1 - \lambda_s)\alpha_s}(1 - s)$$

**Theorem 2.1.7** (See e.g. Chapter 7, [Et]). The rational Cherednik algebra $H_c$ is isomorphic to the subalgebra of $D(R^{\text{reg}})^\#W$ generated by $W, R^*$ and the Dunkl operators.

We remark that the hardest part of the theorem is proving that Dunkl operators actually commute. This is due to Dunkl, see e.g. [Du, Et].

Note that the action of $D_y \in D(R^{\text{reg}})^\#W$ on $\mathbb{C}[R^{\text{reg}}]$ preserves the space $\mathbb{C}[R]$ as, for any function $f \in \mathbb{C}[R]$, $(1 - s)f$ is divisible by $\alpha_s$. In particular, it follows that $\mathbb{C}[R]$ is naturally a module over $H_c$. We call this the polynomial representation of $H_c$.

### 2.1.3 Filtrations and gradings

The algebra $H_c$ comes equipped with several filtrations that we now mention. First of all, we have the Bernstein filtration, $H_c = \bigcup_{m \geq 0} B^m H_c$, which is given by $\deg W = 0$, $\deg R = \deg R^* = 1$.

**Theorem 2.1.8** (PBW Theorem for Rational Cherednik Algebras, [EG]). We have $\text{gr}^B H_c = \mathbb{C}[R \oplus R^*]^\#W$. Equivalently, the multiplication in $H_c$ induces a vector space isomorphism

$$S(R^*) \otimes \mathbb{C}W \otimes S(R) \xrightarrow{\cong} H_c$$

We also have the geometric filtration, $H_c = \bigcup_{m \geq 0} G^m H_c$, which is induced by the eponymous filtration on $D(R^{\text{reg}})^\#W$. Here, we have $\deg(R) = 1$ while $\deg(W) = \deg(R^*) = 0$. It follows from the PBW theorem that we also have an isomorphism $\text{gr}^G H_c = \mathbb{C}[R \oplus R^*]^\#W$, where in the latter algebra we take the grading with $\deg(R^*) = \deg(W) = 0, \deg(R) = 1$. 14
We remark that the Bernstein and geometric filtrations are related. To explain how this is so, we need a grading on the rational Cherednik algebra. It is easy to see from the relations that setting \( \deg(R) = -1, \deg(R^*) = 1, \deg(W) = 0 \) induces a grading on the rational Cherednik algebra. It is then easy to see (say from the PBW decomposition) that

\[
B^m H_c = \bigoplus_k \{ a \in G^k H_c : \deg(a) = m - 2k \}
\]

The grading on \( H_c \) described above is actually inner. To see that this is the case, pick a basis \( \{ y_i \} \) of \( R \) with dual basis \( \{ x_i \} \). Define the deformed Euler element:

\[
eu := \sum x_i y_i + \frac{\dim R}{2} - \sum_{s \in S} \frac{2c(s)}{1 - \lambda_s} s \in H_c
\]

Note that \( \eu \) is actually independent of the basis \( \{ y_i \} \). It is an easy calculation to check that \( [\eu, w] = 0 \) for \( w \in W \), that \( [\eu, x] = x \) for \( x \in R^* \), and that \( [\eu, y] = -y \) for \( y \in R \).

### 2.1.4 Spherical rational Cherednik algebra

Let \( e := \frac{1}{|W|} \sum_{w \in W} w \) be the trivial idempotent of \( \mathbb{C} W \). We define the spherical rational Cherednik algebra to be \( A_c := e H_c e \). While \( A_c \) is a subalgebra of \( H_c \), they do not have the same unit: the unit of \( A_c \) coincides with \( e \). Note that \( A_c \) inherits both filtrations from \( H_c \), and we have that the associated graded of \( A_c \) under any of these filtrations coincides with \( e(\mathbb{C}[R \oplus R^*] \# W)e = \mathbb{C}[R \oplus R^*]^W \) (of course, the grading on \( \mathbb{C}[R \oplus R^*]^W \) depends on which filtration on \( A_c \) one takes).

Often, but not always, the algebras \( H_c \) and \( A_c \) are Morita equivalent via the \((H_c, A_c)\)-bimodule \( H_c e \). When this is the case, we will say that the parameter \( c \) is spherical. Otherwise, \( c \) will be called aspherical.

**Example 2.1.9.** Consider the rational Cherednik algebra \( H_c(n) \) associated to the symmetric group on \( n \) letters. Then, the aspherical locus is \( (-1, 0) \cap \{ \frac{m}{r} : 1 < r \leq n, \gcd(m; r) = 1 \} \), see e.g [BE, Corollary 4.2]
2.1.5 Homogeneous rational Cherednik algebras

Below, we will also need to consider a homogeneous version of the rational Cherednik algebras. Let $S = \bigsqcup_{i=1}^{r} S_i$ be the decomposition of $S$ into conjugacy classes, and let $h, c_1, \ldots, c_r$ be independent variables. For $s \in S_i$, define $c(s) := c_i$. Let $\mathfrak{c}$ be the vector space with basis $h, c_1, \ldots, c_r$. Then, $H$ is the $S(\mathfrak{c})$-algebra defined by generators and relations analogous to those of the previous subsection, with the commutation relation between $R$ and $R^*$ replaced by

$$[y, x] = h\langle y, x \rangle - \sum_{s \in S} c(s)\langle \alpha_s, y \rangle\langle x, \alpha_s^\vee \rangle s,$$

note that the algebra $H$ is graded, with $W, R^*$ in degree 0 and $h, \mathfrak{c}$ in degree 1. We remark that $H$ is a flat $S(\mathfrak{c})$-algebra, and that $H/cH = \mathbb{C}[h \oplus \mathfrak{h}^*]\#W$ - this is just a reformulation of the PBW theorem.

Let us explain why the algebra $H$ is important for us. Let $R_h(H_c)$ denote the Rees algebra of $H_c$ with respect to the geometric filtration. We then have a graded surjection $H \rightarrow R_h(H_c)$, which is given by $w \mapsto w, x \mapsto x, y \mapsto hy, c_i \mapsto hc_i$, where $w \in W, x \in R^*, y \in R$ and $c_i := c(s)$ for $s \in S_i$. Thus, we can pass from filtered $H_c$-modules to graded $H$-modules by means of the Rees construction.

2.1.6 Sheafification

We remark that the construction of the rational Cherednik algebra can be sheafified over $R/W$, cf. [Et2]. First of all, note that if $f \in \mathbb{C}[R]^W$ is a $W$-invariant function, then the adjoint action of $f$ on $H_c$ is locally nilpotent. It follows, in particular, that the localization $H_c[f^{-1}]$ is well-defined. By the PBW theorem, this algebra coincides with the tensor product $\mathbb{C}[R/W][f^{-1}] \otimes_{\mathbb{C}[R/W]} H_c$. So we have a sheaf of algebras $\mathcal{H}_c := \mathfrak{S}_{R/W} \otimes_{\mathbb{C}[R/W]} H_c$, where $\mathfrak{S}_{R/W}$ denotes the structure sheaf of $R/W$. Thanks to the Dunkl presentation of $H_c$, it follows that for an open set $U \subseteq R/W$, the algebra of sections $\mathcal{H}_c(U)$ is generated by $\mathbb{C}[\pi^{-1}(U)], W$, and Dunkl operators, where $\pi : R \rightarrow R/W$ denotes the projection.
Note that we can also sheafify the spherical rational Cherednik algebra to $R/W$, we denote the sheafification by $\mathcal{A}_c$. We can also sheafify the homogeneous algebra $\mathcal{H}$ to get a sheaf of $S(c)$-algebras $\mathcal{H}$ on $R/W$. We will use this more geometric point of view on rational Cherednik algebras when defining restriction functors.

## 2.2 Category $\mathcal{O}_c$

### 2.2.1 Definition

We now review the category $\mathcal{O}$ for the algebra $H_c$, following [GGOR]. Recall the PBW decomposition $H_c = S(R^*) \otimes CW \otimes S(R)$. We then have the following definition, which is analogous to that of (the principal block of) category $\mathcal{O}$ for a semisimple Lie algebra $\mathfrak{g}$.

**Definition 2.2.1 ([GGOR]).** A module $M \in H_c$-mod is said to be in category $\mathcal{O}_c$ if the following conditions are satisfied.

1. $M$ is finitely generated.

2. $R$ acts on $M$ by locally nilpotent endomorphisms.

For example, if $M \in H_c$-mod is finite-dimensional, then $M \in \mathcal{O}_c$, this follows from the existence of the deformed Euler element $eu$.

**Example 2.2.2.** Assume $c = 0$, so that $H_c = D(R)^\# W$. Then, $\mathcal{O}_0$ is the category of $W$-equivariant $D(R)$-modules whose Fourier transform is supported at $0 \in R$. By Kashiwara’s Lemma, this is equivalent to the category of representations of $W$.

**Example 2.2.3.** Recall that via the Dunkl embedding we have the polynomial representation $\mathbb{C}[R]$ of $H_c$. It is easy to see that $\mathbb{C}[R] \in \mathcal{O}_c$. 

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2.2.2 Verma modules

We will now construct, for every irreducible representation \( \tau \) of \( W \), a module \( \Delta_c(\tau) \in \mathcal{O}_c \). First, let us note that the algebra \( S(R)\#W \) is a subalgebra of \( H_c \). If \( \tau \) is an irreducible representation of \( W \), then we can consider it as a representation of \( S(R)\#W \) by letting \( R \) act by 0.

**Definition 2.2.4 ([GGOR]).** Let \( \tau \) be an irreducible representation of \( W \), which we consider a representation of \( S(R)\#W \) as in the above paragraph. We define the Verma module \( \Delta_c(\tau) \) to be the induced module

\[
\Delta_c(\tau) := \text{Ind}_{S(R)\#W}^{H_c} \tau = H_c \otimes_{S(R)\#W} \tau = S(R^*) \otimes \tau
\]

where the last equality is just as vector spaces.

Note that \( \Delta_c(\text{triv}) \) is precisely the polynomial representation of \( H_c \). We remark that \( \Delta_c(\tau) \) is naturally graded by the action of \( eu \), with eigenvalues being of the form \( c_\tau + i \), \( i \geq 0 \) for some \( c_\tau \in \mathbb{C} \), and the eigenspace with eigenvalue \( c_\tau \) coincides with \( \tau \otimes 1 \subseteq \tau \otimes S(R^*) \). It easily follows that \( \Delta_c(\tau) \) admits a unique irreducible quotient, which we denote by \( L_c(\tau) \).

It is also not hard to see that \( \{ L(\tau) : \tau \in W\text{-irrep} \} \) forms a complete and irredundant collection of irreducible modules in category \( \mathcal{O}_c \). Let us also mention that category \( \mathcal{O}_c \) is a finite length category and, moreover, it is a highest weight category, in particular it has enough projectives. We remark that a consequence of this is that taking Verma modules gives an isomorphism between Grothendieck groups. More explicitly, consider the Grothendieck groups \([W\text{-rep}] := \mathbb{C} \otimes K_0(W\text{-rep}) \) and \([\mathcal{O}_c] \). Then, the map \( [\tau] \mapsto [\Delta_c(\tau)] \) is an isomorphism.

2.2.3 Supports

It is clear from the definition that a module in category \( \mathcal{O}_c \) is finitely generated over \( S(R^*) = \mathbb{C}[R] \). Thus, we may define its support \( \text{supp}(M) \subseteq R \) as the set-theoretic support of \( M \) as a coherent sheaf over \( R \). In other words, it is the zero set of the radical of the annihilator of \( M \).
in \( \mathbb{C}[R] \). Since \( M \) is, in particular, a \( \mathbb{C}[R] \)-module, \( \text{supp}(M) \) is a \( W \)-stable, closed subvariety of \( R \). Even more is true. For a subgroup \( W' \subseteq W \), let \( X_{W'} := \{ b \in R : W_b \text{ is } W\text{-conjugate to } W' \} \). Note that \( X_{W'} \) is a locally closed subvariety of \( R \), and the \( X_{W'} \) form a stratification of \( W \) when \( W' \) runs over the conjugacy classes of parabolic subgroups of \( W \) where, recall, a subgroup \( W' \subseteq W \) is called parabolic if it is the stabilizer of a point in \( R \).

**Proposition 2.2.5** (Proposition 3.2, [BE]). *For any module \( M \in \mathcal{O}_c \), \( \text{supp}(M) \) is a union of sets of the form \( X_{W'} \). Moreover, if \( M \) is irreducible, then \( \text{supp}(M) = \overline{X_{W'}} \) for some parabolic subgroup \( W' \subseteq W \).*

For a parabolic subgroup \( W' \subseteq W \), let us denote by \( \mathcal{O}_{c,W'} \) the full Serre subcategory of \( \mathcal{O}_c \) consisting of modules whose support is contained in \( \overline{X_{W'}} \). Let us also denote by \( \mathcal{O}^{\text{tor}}_{c,W'} \) the full subcategory consisting of modules whose support is contained in \( \partial X_{W'} := \overline{X_{W'}} \setminus X_{W'} \). This is a Serre subcategory of \( \mathcal{O}_{c,W'} \) and we can form the quotient

\[
\mathcal{O}^\circ_{c,W'} := \mathcal{O}_{c,W'}/\mathcal{O}^{\text{tor}}_{c,W'}
\]

For example, if \( W' = \{1\} \), then \( \mathcal{O}_{c,W'} = \mathcal{O}_c \), and \( \mathcal{O}^{\text{tor}}_{c,W'} \) consists of modules with proper support. We will see a description of the category \( \mathcal{O}^\circ_{c,\{1\}} \) in the next section. On the other extreme, \( \mathcal{O}_{c,W} = \mathcal{O}^\circ_{c,W} \) consists of the category of finite-dimensional modules in \( \mathcal{O}_c \).

Let us remark that the number of irreducible objects in \( \mathcal{O}^\circ_{c,W'} \) coincides with the number of irreducible modules in \( \mathcal{O}_c \) whose support coincides with \( \overline{X_{W'}} \). Let us now define the associated graded category:

\[
\text{gr} \mathcal{O}_c := \bigoplus_{W' \subseteq W} \mathcal{O}^\circ_{c,W'}
\]

Note that the number of irreducible objects of \( \mathcal{O}_c \) and \( \text{gr} \mathcal{O}_c \) coincide. Let us also remark that, for Weil generic \( c \), we have \( \mathcal{O}_c = \text{gr} \mathcal{O}_c = \mathcal{O}_{c,\{1\}} \).
2.3 The Knizhnik-Zamolodchikov functor

2.3.1 Localization

In this section, we follow Section 5 of [GGOR]. Recall that we denote \( \delta := \prod_{s \in S} \alpha_s \). This is a \( W \)-semiinvariant element of \( \mathbb{C}[R] \). Also recall that we denote by \( R^{reg} \) the principal open set in \( R \) determined by \( \delta \). Note that \( R^{reg} = R \setminus \bigcup_{s \in S} \Gamma_s \) coincides with the locus where the \( W \)-action is free. From the Dunkl embedding it is clear that the operator \([\delta, \cdot]\) is locally nilpotent on \( H_c \), so the localization \( H_c[\delta^{-1}] \) makes sense. Moreover, it is not hard to see that the Dunkl embedding induces an isomorphism \( H_c[\delta^{-1}] \cong D(R^{reg}) \#W \).

We then have a functor \([\delta^{-1}] : \mathcal{O}_c \longrightarrow (D(R^{reg}) \#W) \text{-mod}, M \mapsto M[\delta^{-1}] \). Note that this functor annihilates every module whose support is contained in the zero set of \( \delta \). Thanks to Proposition 2.2.5, the localization functor kills all modules in \( \mathcal{O}_c \) whose support is properly contained in \( R \), that is \( \mathcal{O}_{c,\otimes} \). We also remark that, since every module in \( \mathcal{O}_c \) is finitely generated over \( \mathbb{C}[R] \), the image of the localization functor is contained in the category of \( W \)-equivariant local systems on \( R^{reg} \). Since \( W \)-acts freely on \( R^{reg} \), taking \( W \)-invariants gives an equivalence of the latter category with that of local systems on the quotient \( R^{reg}/W \).

Taking then flat sections, we get a finite-dimensional representation of the braid group, \( B_W := \pi_1(R^{reg}/W) \).

\[
\begin{align*}
\mathcal{O}_c & \xrightarrow{M \mapsto M[\delta^{-1}]} \text{Loc}_W(R^{reg}) \xrightarrow{L \mapsto eL} \text{Loc}(R^{reg}/W) \xrightarrow{L \mapsto L^\nabla} B_W \text{-rep}
\end{align*}
\]

It turns out that the image of the composite functor \( \mathcal{O}_c \to B_W \text{-rep} \) falls inside the category of modules of a certain quotient of the group algebra \( \mathbb{C}B_W \). To motivate this definition, we are going to start with the case when \( W = \mathbb{Z}/\ell \mathbb{Z} \) acting on its reflection representation \( \mathbb{C} \) as in Example 2.1.5. Note that, here, \( R^{reg} = \mathbb{C}^\times \), while \( B_W = \mathbb{Z} \).
2.3.2 The cyclic group case

Let $W = \mathbb{Z}/\ell\mathbb{Z}$, with generator $s$ acting on $\mathbb{C}$ by multiplication by $\eta := \exp(2\pi\sqrt{-1}/\ell)$. For each $i = 0, \ldots, \ell - 1$, let $E_i$ be the 1-dimensional representation of $W$ where $s$ acts by $\eta^i$. In particular, $E_0 = \text{triv}$, the trivial representation. Note that, if $x, y \in \mathbb{R}^*$ are such that $\langle x, y \rangle = 1$, then $\Delta_c(E_i) = \mathbb{C}[x]$, with action given by $x.x^m = x^{m+1}$, $s.x^m = \eta^{i-m}x^m$, the action of $y$ can be recovered uniquely from these formulas and $y.1 = 0$.

Now, $\Delta_c(E_i)[\delta^{-1}] = \mathbb{C}[x, x^{-1}] \otimes E_i$. Since $y$ annihilates $E_i$, we have that the Dunkl operator also annihilates $E_i$. It follows that $\Delta_c(E_i) = \mathbb{C}[x, x^{-1}]$ with connection given by

$$\nabla = d - \sum_{m=1}^{\ell-1} \frac{2c_m}{1-\eta^{-m}} (\eta^{im} - 1) \frac{dx}{x}$$

where $c_m := c(s^m)$. It follows that, if we define

$$k_i := \sum_{m=1}^{\ell-1} \frac{2c_m}{1-\eta^{-m}} (\eta^{im} - 1)$$

then $[(\Delta_c(E_i))[\delta^{-1}]^W]^{\nabla}$ is the 1-dimensional $\mathbb{C}[t, t^{-1}]$-module where $t$ acts by the scalar

$$q_i := \exp\left(\frac{2\pi\sqrt{-1}(k_i - i)}{\ell}\right)$$

and, moreover, the image of the composite functor $\mathcal{O}_c \to B_W$-rep factors through the algebra $\mathcal{H}_q(W) := \mathbb{C}[t]/\prod_{i=0}^{\ell-1} (t - q_i)$. This is the simplest case of a so-called Hecke algebra, which we define next.

2.3.3 Hecke algebras

Recall that we denote $B_W := \pi_1(R^{reg}/W)$ the braid group associated to $W$. The group $B_W$ admits a system of generators indexed by the set $\mathcal{A}$ of reflection hyperplanes on $R$. For each $\Gamma \in \mathcal{A}$, the pointwise stabilizer $W_\Gamma$ is cyclic, of order say $\ell_\Gamma$. Let $s_\Gamma \in \mathcal{S} \cap W_\Gamma$ be the element with determinant $\exp(2\pi\sqrt{-1}/\ell_\Gamma)$, and let $T_\Gamma$ be a generator of the monodromy around $\Gamma$ such that a lift of $T_\Gamma$ to $R^{reg}$ is represented by a path from $x_0$ to $s_\Gamma(x_0)$, see [BMR, Appendix 1] for a precise definition. The set $\{T_\Gamma\}_{\Gamma \in \mathcal{A}}$ is a generating set for the group $B_W$. 

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To define the Hecke algebra, for each reflection hyperplane $\Gamma \in A$, fix nonzero complex numbers $q_{\Gamma,0}, \ldots, q_{\Gamma,\ell_{\Gamma}-1}$, in such a way that if $\Gamma, \Gamma'$ are $W$-conjugate then $q_{\Gamma,i} = q_{\Gamma',i}$ for each $i = 0, \ldots, \ell_{\Gamma} - 1 = \ell_{\Gamma'} - 1$. We denote this collection of complex numbers by $q$.

**Definition 2.3.1 ([BMR]).** The Hecke algebra $\mathcal{H}_q(W)$ is the quotient of the group algebra $\mathbb{C}B_W$ by the relations $\prod_{i=0}^{\ell_{\Gamma}-1} (T_{\Gamma} - q_{\Gamma,i})$, one for each $\Gamma \in A$.

For example, setting $q_{\Gamma,i} = \exp(2\pi\sqrt{-1}i/\ell_{\Gamma})$ we recover the group algebra $\mathbb{C}W$.

### 2.3.4 KZ functor

Let us go back to the rational Cherednik algebra $H_c := H_c(W)$. In [GGOR] it is shown that the image of the functor $\mathcal{O}_c \to B_W$-rep falls inside the category of finite dimensional $\mathcal{H}_q(W)$-modules, where the parameter $q$ explicitly depends on $c$ as follows. Recall that for each reflection $s \in S, \lambda_s$ denotes the unique non-trivial eigenvalue for the action of $s$ on $R^*$. For each reflection hyperplane $\Gamma \in A$, define

$$k_{\Gamma,i} := \sum_{s \in S_{\Gamma W_{\Gamma}}} \frac{2c(s)}{1 - \lambda_s} (\lambda_s^{-i} - 1), \quad i = 0, \ldots, \ell_{\Gamma} - 1. \quad (2.2)$$

Note that $k_{\Gamma,i}$ depends only on the conjugacy class of $\Gamma$, and that $k_{\Gamma,0} = 0$. Now the parameter $q$ is computed as follows:

$$q_{\Gamma,i} := \exp(2\pi\sqrt{-1}(k_{\Gamma,i} - i)/\ell_{\Gamma}). \quad (2.3)$$

Note that $q_{\Gamma,0} = 1$. For example, if $c = 0$ then $\mathcal{H}_q(W) = \mathbb{C}W$. We call the functor $\mathcal{O}_c \to \mathcal{H}_q$-mod the Knizhnik-Zamolodchikov (shortly, KZ) functor, and denote it $\text{KZ} : \mathcal{O}_c \to \mathcal{H}_q$-mod.

**Theorem 2.3.2 ([GGOR], [L5]).** The functor $\text{KZ} : \mathcal{O}_c \to \mathcal{H}_q$-mod is exact and induces an equivalence $\mathcal{O}_c^{\circ,\{1\}} \cong \mathcal{H}_q$-mod.
For example, let $\tau$ be a 1-dimensional representation of $W$. For a hyperplane $\Gamma \in \mathcal{A}$, let $s_\Gamma$ be a generator of $W_\Gamma$ with $\lambda_{s_\Gamma}^{-1} = \exp(2\pi\sqrt{-1}/\ell_\Gamma)$. Denote by $E_{\Gamma,i}$ the 1-dimensional representation of $W_\Gamma$ where $s_\Gamma$ acts by $\lambda_{s_\Gamma}^{-1}$. In particular, we have that $\text{Res}^W_{W_\Gamma} \tau = E_{\Gamma,\tau(\Gamma)}$ for some $\tau(\Gamma) \in \{0, \ldots, \ell_\Gamma - 1\}$. Then, we have that $\text{KZ}(\Delta(\tau))$ is the 1-dimensional $H_q$-module where $T_\Gamma$ acts by $q_{\Gamma,\tau(\Gamma)}$. In particular, $\text{KZ}(\text{triv})$ is the trivial representation of $H_q$, that is, the representation of $H_q$ where all $T_\Gamma$'s act by 1.

### 2.3.5 Integral parameters

Let us define the notion of integral parameters following [L7]. Let $p^* := c/h \cong \mathbb{C}^{[S/W]}$, a vector space with basis $c_1, \ldots, c_r$. We remark that the set of parameters ‘$c$’ for the Cherednik algebra $H_c$ can be naturally identified with $p$, the dual of $p^*$. So we can view $k_{\Gamma,i}$ as an element of $p^*$, its value on a parameter $c \in p$ is given by the formula (2.2). Define $p^*_Z$ to be the $\mathbb{Z}$-lattice inside $p^*$ spanned by elements $\ell^{-1}_{\Gamma} k_{\Gamma,i}$, and let $p_Z \subseteq p$ be the dual lattice.

The lattice $p_Z$ consists of all parameters $c$ such that $q_{\Gamma,i} = \eta_{\Gamma}^{-i}$, where $\eta_{\Gamma} := \exp(2\pi\sqrt{-1}/\ell_\Gamma)$, so $H_q = \mathbb{C}W$. Moreover, we have that $c - c' \in p_Z$ if and only if $q_{\Gamma,i} = q'_{\Gamma,i}$ for every $\Gamma \in \mathcal{A}$, $i = 0, \ldots, \ell_\Gamma - 1$ and thus the set of parameters for the Hecke algebra can be identified with $p/p_Z$, see [L7, Subsection 2.6]. For example, if $W$ is a Coxeter group, then $p_Z$ coincides with the set of parameters for which $c(s) \in \mathbb{Z}$ for all $s \in S$.

Let us give a spanning set for $p_Z$. First, we need to introduce some notation. For each $\Gamma \in \mathcal{A}$, the set of characters of $W_\Gamma$ is identified with $\mathbb{Z}/\ell_\Gamma \mathbb{Z}$, an isomorphism $\mathbb{Z}/\ell_\Gamma \mathbb{Z} \cong \text{Hom}(W_\Gamma, \mathbb{C}^\times)$ is given by $m \mapsto (s \mapsto \det(s)^m)$. We have a morphism $\text{Hom}(W, \mathbb{C}^\times) \rightarrow \prod_{\Gamma \in \mathcal{A}/W} \text{Hom}(W_\Gamma, \mathbb{C}^\times)$, given by restriction. According to [R, Subsection 3.3.1], this is an isomorphism. Thus, we have a correspondence between 1-dimensional characters $\chi$ of $W$ and $|\mathcal{A}/W|$-tuples of integers $(m_\Gamma)$ with $0 \leq m_\Gamma \leq \ell_\Gamma - 1$. So, to a character $\chi \in \text{Hom}(W, \mathbb{C}^\times)$ associated to the tuple $(m_\Gamma)$ we assign $\chi \in p$, given by
\[ \chi(k_{\Gamma,i}) = \begin{cases} \ell_{\Gamma} & \text{if } i \geq \ell - m_{\Gamma} \\ 0 & \text{if } i < \ell - m_{\Gamma}. \end{cases} \]

Clearly, \( \chi \in \mathfrak{p}_\mathbb{Z} \), and the elements \( \chi \) form a spanning set for \( \mathfrak{p}_\mathbb{Z} \).

Let us explain the reason why we are interested in the elements \( \chi \). According to [BC, Proposition 5.6], for each parameter \( c \in \chi \) we have an algebra isomorphism \( A_c \cong e_\chi H_{c+\chi} e_\chi \), where \( e_\chi := \frac{1}{|W|} \sum_{w \in W} \chi(w) w \) denotes the idempotent in \( \mathbb{C}W \) corresponding to \( \chi \). We will use this below, see Definition 3.1.5.

**Example 2.3.3.** Let us see what the lattice \( \mathfrak{p}_\mathbb{Z} \) is when \( W \) is a simply laced Weyl group. In this case, we have that \( \ell_{\Gamma} = 2 \) and all the reflection hyperplanes are conjugate. The parameter \( c \) may be thought of as a single complex number, so we have \( \mathfrak{p} = \mathbb{C} \). For every reflection hyperplane \( \Gamma \) we have \( k_{\Gamma,0} = 0 \), \( k_{\Gamma,1} = -2c \). It follows that \( \mathfrak{p}_\mathbb{Z} = \mathbb{Z} \subseteq \mathbb{C} \). By the same calculation, for a real reflection group \( W \), \( \mathfrak{p}_\mathbb{Z} \) coincides with the set of parameters \( c \) such that \( c(s) \in \mathbb{Z} \) for every reflection \( s \in W \).

### 2.3.6 Regular parameters

Recall that, if \( c \in \mathfrak{p}_\mathbb{Z} \), the Hecke algebra \( \mathcal{H}_q(W) \) is equal to the group algebra \( \mathbb{C}W \). In particular, \( \text{KZ} : \mathcal{O}_c \rightarrow \mathcal{H}_q \) is a quotient functor between two categories that have the same number of irreducible objects, and is therefore an equivalence. Moreover, a consequence of the highest weight structure on \( \mathcal{O}_c \) is that, for \( c \) outside of a countable collection of hyperplanes in \( \mathfrak{p}_\mathbb{Z} \), the category \( \mathcal{O}_c \) is semisimple with irreducible objects given by the Verma modules \( \Delta_c(\tau), \tau \in W \)-irrep.

**Definition 2.3.4.** We say that a parameter \( c \in \mathfrak{p} \) is regular if category \( \mathcal{O}_c \) is semisimple. Otherwise, we say that \( c \) is singular.

**Theorem 2.3.5** ([BE, GGOR, V]). Let \( c \in \mathfrak{p}_\mathbb{Z} \). Then, the following are equivalent.
1. $c$ is regular.

2. The algebra $H_c$ is simple.

3. Every module $M \in \mathcal{O}_c$ has full support.

4. The Hecke algebra $\mathcal{H}_q$ is isomorphic to the group algebra $\mathbb{C}W$.

5. The KZ functor $KZ : \mathcal{O}_c \to \mathcal{H}_q$ is an equivalence of categories.

Note that it follows from Theorem 2.3.5 that every regular parameter is spherical. Also note that, if $c$ and $c'$ are regular, we have that $\mathcal{O}_c \cong \mathcal{O}_{c'}$, as both categories are equivalent to the category of representations of $W$. As we will see, this is no longer the case for Harish-Chandra bimodules.

**Example 2.3.6 ([DJO]).** Let $W = S_n$ acting on its reflection representation $\mathbb{C}^{n-1}$. Then, a parameter $c \in \mathfrak{p} = \mathbb{C}$ is singular if and only if $c$ is a non-integral rational number of the form $r/m$ with $m \leq n$.

### 2.4 Restriction functors for category $\mathcal{O}$

#### 2.4.1 Restriction functors for Hecke algebras

Let us remark that, if $b \in R$ and $W' := W_b$ is a parabolic subgroup of $W$, then there is a natural inclusion of algebras $\iota : \mathcal{H}_q(W') \hookrightarrow \mathcal{H}_q(W)$ where, abusing the notation, we denote by $q$ the restriction of $q$ to $\mathcal{S} \cap W'$. This allows us to define a restriction functor, $\mathcal{H} \text{Res}_{W'}^W := \iota^* : \mathcal{H}_q(W)\text{-mod} \to \mathcal{H}_q(W')\text{-mod}$. The map $\iota : \mathcal{H}_q(W') \hookrightarrow \mathcal{H}_q(W)$ is actually induced from an inclusion $B_{W'} \hookrightarrow B_W$, see for example [BMR, Section 4] for details.
2.4.2 Bezrukavnikov-Etingof isomorphisms of completions

There is also a restriction functor on the level of category $O$, [BE]. This functor depends on the choice of a point $b \in R$ whose stabilizer $W_b$ coincides with $W'$. For distinct $b, b'$ with this property, the functors are isomorphic (but not in a canonical way) so we will just denote this functor by $\text{Res}_{W/W'}$. Let us describe the construction of this functor.

Recall the sheafification $\mathcal{H}_c$ of the rational Cherednik algebra. Let $H_c^{\wedge b} = \mathbb{C}[R/W][[b]] \otimes_{\mathcal{H}(R/W)} H_c$ be the algebra of sections of the sheaf $\mathcal{H}_c$ on the formal neighborhood of $[b] \in R/W$. Note that $H_c^{\wedge b}$ is generated by the algebra $\mathbb{C}[R]^{W_b}, W$, and Dunkl operators.

Bezrukavnikov and Etingof showed in [BE, Theorem 3.2] that $H_c^{\wedge b}$ is isomorphic to a matrix algebra of size $|W/W'|$ with coefficients in $H_c(W', \mathbb{R})^{\wedge 0}$. Let us give a more precise statement. We will need the centralizer construction, which we describe next. Let $H \subseteq G$ be finite groups, and let $C$ be an algebra containing $\mathbb{C}H$. In particular, both $\mathbb{C}G$ and $C$ are $H$-modules, where $H$ acts on both algebras by multiplication on the left. Consider $\text{Hom}_H(\mathbb{C}G, C)$, which is a free $C$-module of rank $|G/H|$. We define the centralizer algebra to be

$$Z(H, G, C) := \text{End}_C(\text{Hom}_H(\mathbb{C}G, C))$$

Of course, $Z(H, G, C) \cong \text{Mat}_{|G/H|}(C)$, but this isomorphism is not canonical. There is, however, a canonical way to recover $C$ from $Z(H, G, C)$, as follows. Let $e(H) \in Z(H, G, A)$ be defined by:

$$[(e(H).f)(g)] = \begin{cases} f(g) & \text{if } g \in H, \\ 0 & \text{otherwise} \end{cases}$$

then $e(H) \in Z(H, G, A)$ is an idempotent, and $e(H)Z(H, G, A)e(H) \cong A$.

**Theorem 2.4.1** ([BE]). Let $b \in R$ and $W' := W_b$ be the stabilizer of $b$. Then, there is an isomorphism
\[ \theta_b : H_c^{\wedge b} \xrightarrow{\cong} Z(W', W, H_c(W', R)^\wedge 0) \]

where we abuse the notation and also denote by \( c \) the restriction \( c|_{S\cap W'} \). This isomorphism is given by

\[
\begin{align*}
[\theta_b(w).f](u) & = f(wu), & w \in W \\
[\theta_b(x)f](u) & = (x + \langle ux, b \rangle)f(u), & x \in R^* \\
[\theta_b(y).f](u) & = u(y)f(w) + \sum_{s \in S\setminus W'} \frac{2e_{\alpha_s}(\alpha_s, w(y))}{1 - \alpha_s + \langle \alpha_s, b \rangle}(f(su) - f(u)), & y \in R
\end{align*}
\]

Let us give an intuitive way to see why Theorem 2.4.1 should hold. First of all, recall that the algebra \( H_c^{\wedge b} \) is generated by \( \mathbb{C}[R]^{W_b} \), \( W \) and Dunkl operators. Now, we have that \( \mathbb{C}[R]^{\wedge W_b} = \bigoplus_{y \in W_b} \mathbb{C}[R]^{\wedge y} \), and note that the algebra generated by \( \mathbb{C}[R]^{\wedge b} \), \( W_b \subseteq W \) and Dunkl operators is isomorphic to \( H_c(W_b, R)^{\wedge b} \) - here, we are using that elements of the form \( 1/\alpha_s \) for \( s \in W_b \) belong to \( \mathbb{C}[R]^{\wedge b} \). From here, it is easy to see that \( H_c^{\wedge b} \) should be a matrix algebra of size \( |W/W'| \) with coefficients in \( H_c(W', R)^{\wedge b} \), and the centralizer algebra construction is simply an invariant way of saying this. Finally, we remark that \( H_c(W', R)^{\wedge b} \cong H_c(W', R)^{\wedge 0} \).

### 2.4.3 Construction of the restriction functor

Let us now proceed to the construction of the restriction functor. First of all, we have a completion functor \( \bullet^{\wedge b} : \mathcal{O}_c \to H_c^{\wedge b} \text{-mod}, M \mapsto \mathbb{C}[R/W]^{\wedge b} \otimes_{\mathbb{C}[R/W]} M \).

**Definition 2.4.2.** Let \( \mathcal{O}_c^{\wedge b} \subseteq H_c^{\wedge b} \) denote the category of all \( H_c^{\wedge b} \)-modules which are finitely generated over \( \mathbb{C}[R]^{W_b} \).

Note that we have \( \bullet^{\wedge b} : \mathcal{O}_c \to \mathcal{O}_c^{\wedge b} \). Now, for \( N \in \mathcal{O}_c^{\wedge b} \), the pushforward \( (\theta_b)_* N \) lies in \( Z(W', W, H_c(W', R)^{\wedge 0}) \text{-mod} \), so thanks to the discussion above we have \( e(W')(\theta_b)_* N \in H_c(W', R)^{\wedge 0} \text{-mod} \). Moreover, we have \( e(W')(\theta_b)_*(N) \in \mathcal{O}_c(W', R)^{\wedge 0} \).
**Lemma 2.4.3** (Theorem 2.3, [BE]). The completion functor \( \bullet^{\wedge b} : \mathcal{O}_c \to \mathcal{O}_c^{\wedge b} \) has a right adjoint \( E^b : \mathcal{O}_c^{\wedge b} \to \mathcal{O}_c \) which is given by taking the space of \( R \)-locally nilpotent vectors. If \( b = 0 \), the functors \( \bullet^{\wedge 0} \) and \( E^0 \) are quasi-inverse equivalences of categories.

Thanks to the previous lemma, \( E_0(e(W')(\theta_b)(N)) \) is a module in category \( \mathcal{O} \) for the algebra \( H_c(W', R) \). To get a module in category \( \mathcal{O} \) for \( H_c(W', R_W) \), we take the set of elements that are annihilated by \( R^{W'} \subseteq H_c(W', R) \). We remark that this functor \( \mathcal{O}_c(W', R) \to \mathcal{O}_c(W', R_W) \) is an equivalence of categories. So we have the functor \( \text{Res}_b \), which is constructed as the composition of the several functors we have just defined.

\[
\begin{array}{cccccc}
\mathcal{O}_c(W, R) & \xrightarrow{M \mapsto M^{\wedge b}} & \mathcal{O}_c(W, R)^{\wedge b} & \xrightarrow{e(W')(\theta_b)^{\wedge b}} & \mathcal{O}_c(W', R)^{\wedge 0} & \xrightarrow{E_0} & \mathcal{O}_c(W', R) \\
\text{Res}_b & & & & & & \cong \downarrow \\
& & \mathcal{O}_c(W', R_W) & & & & \\
\end{array}
\]

Let us remark that the functor \( \text{Res}_b \) does not depend on \( b \), up to an isomorphism, which is not canonical. Since this is not going to be of great importance for us, we will simply denote \( \text{Res}_W^W := \text{Res}_b \). Let us remark that \( \text{Res}_W^W \) admits a right adjoint functor, \( \text{Ind}_W^W \). This is constructed by taking the quasi-inverse of all functors in the diagram above except the completion functor \( \bullet^{\wedge b} \). Here, we take the right adjoint \( E^b \). Just as for the restriction functor, the induction functor depends non-canonically on \( b \). Note also that both \( \text{Ind}_W^W \) and \( \text{Res}_W^W \) are exact functors.

**Theorem 2.4.4.** The following is true.

1. \( \text{Ind}_W^W \) is also a left adjoint of \( \text{Res}_W^W \). In particular, \( \text{Ind}_W^W \) and \( \text{Res}_W^W \) preserve projective and injective objects, [L4]

2. \( \text{Res}_W^W(M) \neq 0 \) if and only if \( X_W \subseteq \mathrm{SS}(M) \).

3. \( \text{Res}_W^W(M) \) is finite-dimensional if and only if \( \mathrm{SS}(M) = \overline{X_W} \), [BE].
4. At the level of Grothendieck groups, under the identification \([W\text{-rep}] \xrightarrow{\cong} \mathcal{O}_c\), \([\tau] \mapsto [\Delta_c(\tau)]\), the map \([\text{Res}_{W'}^W] : [\mathcal{O}_c(W, R)] \rightarrow [\mathcal{O}_c(W', R)]\) gets identified with the restriction functor for finite groups, \([\text{res}_{W'}^W] : [W\text{-rep}] \rightarrow [W'\text{-rep}], [BE]\).

5. The functor \(\text{Res}_{W'}^W\), preserves the categories of modules with a standard filtration, [Sh].

6. The restriction functor intertwines the KZ functors, that is, we have \(\text{KZ} \circ \text{Res}_{W'}^W = \mathcal{H}\text{Res}_{W'}^W \circ \text{KZ}\), where \(\text{KZ}\) denotes the KZ functor from category \(\mathcal{O}_c(W', R_{W'})\) to the category of finite-dimensional representations of \(\mathcal{H}_q(W')\), [Sh].

7. The induction functor \(\text{Ind}_{W'}^W\), does not kill nonzero modules, [SV].

Let us give two easy consequences of the previous result that will be needed later.

**Lemma 2.4.5.** Let \(P_c \in \mathcal{O}_c\) be a projective generator, and \(W'\) a parabolic subgroup of \(W\). Then, \(\text{Res}_{W'}^W(P_c) \in \mathcal{O}_c(W')\) is a projective generator.

**Proof.** The module \(\text{Res}_{W'}^W(P_c)\) is projective since \(\text{Res}_{W'}^W\) admits an exact biadjoint functor. Now let \(M \in \mathcal{O}_c(W')\) be irreducible. Then \(\text{Hom}_{\mathcal{O}_c(W')}(\text{Res}_{W'}^W(P_c), M)\) is naturally isomorphic to \(\text{Hom}_{\mathcal{O}_c(W)}(P_c, \text{Ind}_{W'}^W(M))\), which is nonzero since \(\text{Ind}_{W'}^W(M)\) is nonzero. We are done. \(\Box\)

**Lemma 2.4.6.** Let \(\tau\) be a 1-dimensional representation of \(W\). Then, for any parabolic subgroup \(W' \subseteq W\), \(\text{Res}_{W'}^W(\Delta_c(\tau)) = \Delta_c(\text{res}_{W'}^W(\tau))\).

**Proof.** By (5) of Theorem 2.4.4, \(\text{Res}_{W'}^W(\Delta_c(\tau))\) has a standard filtration. Since \(\tau\) is 1-dimensional, \(\text{res}_{W'}^W(\tau)\) is an irreducible representation of \(W'\). The result now follows from (4) of Theorem 2.4.4. \(\Box\)
Chapter 3

Preliminaries on Harish-Chandra bimodules

In this chapter, we start studying Harish-Chandra bimodules. These are a special class of bimodules over rational Cherednik algebras that were introduced in [BEG] associated to the study of the space of quasi-invariants for the action of a real reflection group on its reflection representation. Harish-Chandra bimodules naturally provide functors between distinct categories $\mathcal{O}$ which are analogous to the projective functors in Lie theory, and this is a reason to study Harish-Chandra bimodules. Let us remark, however, that unlike projective functors, the functors induced by Harish-Chandra bimodules are in general not exact. Here, we will see that the category of Harish-Chandra bimodules is equivalent to the category of representations of a finite-dimensional algebra, and so it is an interesting category on its own right.

3.1 Definition and basic results

Fix a complex reflection group $W$, and parameters $c, c' \in \mathfrak{p}$ to form the rational Cherednik algebras $H_c := H_c(W, R), H_{c'}(W, R)$. Note that the algebras $\mathbb{C}[R]^W, \mathbb{C}[R^*]^W$ are subalgebras
of both $H_c$ and $H_{c'}$. Thus, if $B$ is a $(H_c,H_{c'})$-bimodule and $a \in \mathbb{C}[R]^W \cup \mathbb{C}[R^*]^W$, we can consider the adjoint action of $a$ on $B$, $\text{ad}(a) : B \to B$, $b \mapsto ab - ba$.

**Definition 3.1.1 ([BEG]).** Let $B$ be a $(H_c,H_{c'})$-bimodule. We say that $B$ is Harish-Chandra (shortly, HC) if the following conditions are satisfied.

(HC1) $B$ is finitely generated as a bimodule.

(HC2) For any $a \in \mathbb{C}[R]^W \cup \mathbb{C}[R^*]^W$, $\text{ad}(a) : B \to B$ is locally nilpotent.

We denote the category of HC $(H_c,H_{c'})$-bimodules by $\text{HC}(H_c,H_{c'})$. Since both $H_c,H_{c'}$ are noetherian, it follows that the category $\text{HC}(H_c,H_{c'})$ is a full Serre subcategory of the category $(H_c,H_{c'})$-bimod.

**Example 3.1.2.** The regular bimodule $H_c$ is a Harish-Chandra $H_c$-bimodule.

The following are basic results about the category of HC bimodules.

**Proposition 3.1.3** (Lemma 3.3, [BEG]). (1) Any $B \in \text{HC}(H_c,H_{c'})$ is finitely generated as a left $H_c$-module, as a right $H_{c'}$-module, and as a $\mathbb{C}[R]^W \otimes \mathbb{C}[R^*]^W$-module (here, $\mathbb{C}[R]^W$ is considered inside $H_c$, while $\mathbb{C}[R^*]^W$ is considered inside $H_{c'}$).

(2) If $B \in \text{HC}(H_c,H_{c'})$, $B' \in \text{HC}(H_{c'},H_{c''})$ then $B \otimes_{H_{c'}} B' \in \text{HC}(H_c,H_{c''})$.

(3) If $B \in \text{HC}(H_c,H_{c'})$ and $M \in \mathcal{O}_{c'}$, then $B \otimes_{H_{c'}} M \in \mathcal{O}_{c}$.

**Proof.** Let us show (1). Equip $H_c,H_{c'}$ with the Bernstein filtration. It is easy to see that we can find a bimodule filtration on $B$ such that $\text{gr}(B)$ is a finitely generated $\mathbb{C}[R \oplus R^*]#W$-bimodule, and for every element $a \in \mathbb{C}[R]^W \cup \mathbb{C}[R^*]^W$, the adjoint action of $a$ on $\text{gr} B$ is nilpotent (rather than just locally nilpotent). Since, in particular, $\text{gr}(B)$ is a finitely generated $\mathbb{C}[R \oplus R^*]$-bimodule, this implies that $\text{gr}(B)$ is finitely generated:

(a) As a left $\mathbb{C}[R \oplus R^*]#W$-module,
(b) As a right \( \mathbb{C}[R \oplus R^*] \# W \)-module,

(c) And as a \( \mathbb{C}[R]^W \otimes \mathbb{C}[R^*]^W \)-module, where the action of \( \mathbb{C}[R]^W \) is on the left and that of \( \mathbb{C}[R^*]^W \) is on the right.

this implies (1). For (2), note that (1) implies that \( B \otimes_{H_{c'}} B' \) is finitely generated as a \((H_{c}, H_{c'})\)-bimodule, while the fact that the adjoint action of \( \mathbb{C}[R]^W \cup \mathbb{C}[R^*]^W \) is locally nilpotent is an easy exercise. Finally, for (3), note that (1) implies again that \( B \otimes_{H_{c'}} M \) is finitely generated over \( H_{c'} \). It is easy to see that the action of the augmentation ideal \( \mathbb{C}[R^*]^W_{+} \subseteq \mathbb{C}[R^*]^W \) is locally nilpotent. In particular, this implies that for every \( v \in B \otimes_{H_{c'}} M \), the \( \mathbb{C}[R^*]^W \)-module \( \mathbb{C}[R^*]^W v \) is finite-dimensional and supported at 0. But then the same is true for the \( \mathbb{C}[R^*] \)-module \( \mathbb{C}[R^*]v \). It follows that the action of \( R \) on \( B \otimes_{H_{c'}} M \) is locally nilpotent, so \( B \otimes_{H_{c'}} M \in \mathcal{O}_c \).

From (3) of Proposition 3.1.3 it follows that taking tensor product with a bimodule \( B \in \mathcal{H}C(H_{c}, H_{c'}) \) gives a right exact functor \( B \otimes_{H_{c'}} \cdot : \mathcal{O}_{c'} \rightarrow \mathcal{O}_c \), while from (2) of the same proposition it follows that the composition of two such functors is again of the same form.

We will use the functors \( B \otimes_{H_{c'}} \cdot \) extensively in this work.

### 3.1.1 Harish-Chandra bimodules for the spherical subalgebras

Recall that \( A_c, A_{c'} \) denote the spherical subalgebras of \( H_c, H_{c'} \), respectively. Note that we have \( \mathbb{C}[R]^W, \mathbb{C}[R^*]^W \subseteq A_c, A_{c'} \), so we can give the definition of a \( \mathcal{H}C \) \( (A_c, A_{c'}) \)-bimodule completely analogously to Definition 3.1.1. We will denote by \( \mathcal{H}C(A_c, A_{c'}) \) the category of \( \mathcal{H}C \) \( (A_c, A_{c'}) \)-bimodules. Statements (1) and (2) of Proposition 3.1.3 remain valid upon replacing \( H_c, H_{c'} \) with \( A_c, A_{c'} \). Moreover, the functors

\[
\begin{align*}
(H_c, H_{c'})\text{-bimod} & \rightarrow (A_c, A_{c'})\text{-bimod} \\
B & \mapsto eBe \\
(A_c, A_{c'})\text{-bimod} & \rightarrow (H_c, H_{c'})\text{-bimod} \\
B' & \mapsto H_c \otimes_{A_c} B \otimes_{A_{c'}} H_{c'}
\end{align*}
\]
preserve the categories of HC bimodules, and they induce inverse equivalences $\text{HC}(H_c, H_{c'}) \cong \text{HC}(A_c, A_{c'})$ provided both $c$ and $c'$ are spherical.

Let us give examples of Harish-Chandra bimodules over the spherical subalgebras. Let $\chi \in \text{Hom}(W, \mathbb{C}^\times)$ be a 1-dimensional character and consider the parameter $\chi \in \mathfrak{p}$ that was constructed in Section 2.3.5. Recall that we have an isomorphism $A_c \cong e_\chi H_{c+\chi} e_\chi$. Thus, the space $eH_{c+\chi}e_\chi$ is an $(A_{c+\chi}, A_c)$-bimodule.

**Proposition 3.1.4 ([BC, BL, L7]).** $eH_{c+\chi}e_\chi \in \text{HC}(A_{c+\chi}, A_c)$.

**Proof.** Let us remark that the isomorphism $A_c \rightarrow e_\chi H_{c+\chi} e_\chi$ is constructed from a filtered automorphism on $D(R^\text{reg})\#W$, where on the latter algebra we take the geometric filtration. It follows that this isomorphism preserves the filtration given by $\text{deg}(R) = 1, \text{deg}(R^*) = \text{deg} \ W = 0$. Moreover, the associated graded morphism $\text{gr} A_c \rightarrow \text{gr} e_\chi H_{c+\chi} e_\chi = \mathbb{C}[R \oplus R^*]^W$ is the identity and, under the induced bimodule filtration on $eH_{c+\chi} e_\chi$, the associated graded $\text{gr} eH_{c+\chi} e_\chi$ is the space of semi-invariants $\mathbb{C}[R \oplus R^*]^{W, \chi^{-1}}$. Note that the left and right actions of $\text{gr} A_c, \text{gr} A_{c'}$ on $\text{gr} eH_{c+\chi} e_\chi$ coincide. Since $\text{deg}(\mathbb{C}[R]^W) = 0$, this already shows that the adjoint action of $\mathbb{C}[R]^W$ on $eH_{c+\chi} e_\chi$ is locally nilpotent.

Now we show that the adjoint action of $\mathbb{C}[R^*]^W$ on $eH_{c+\chi} e_\chi$ is locally nilpotent. Let us abuse the notation and denote by $\text{ad} \text{eu} : eH_{c+\chi} e_\chi \rightarrow eH_{c+\chi} e_\chi$ the operator $x \mapsto \text{eu}_{c+\chi} x - xe_\chi$.

Viewing the space $eH_{c+\chi} e_\chi$ inside $D(R^\text{reg})\#W$, it is easy to see that the $\text{ad} \text{eu}$ action is diagonalizable. Now we twist the filtration on $eH_{c+\chi} e_\chi$ by this action:

$$G'_i(eH_{c+\chi} e_\chi) := \bigoplus_k G_k(eH_{c+\chi} e_\chi) \cap \{ a \in eH_{c+\chi} e_\chi : \text{ad}(\text{eu})(a) = (i - k)a \}$$

Note that now the left and right actions of $\mathbb{C}[R^*]^W$ preserve the filtration, while the associated graded does not change. It follows as in the previous paragraph that the adjoint action of $\mathbb{C}[R^*]^W$ is locally nilpotent. Thus, $eH_{c+\chi} e_\chi$ is HC.

**Definition 3.1.5 ([BC]).** Let $\chi \in \text{Hom}(W, \mathbb{C}^\times)$ be a 1-dimensional character. For $c \in \mathfrak{p}$ we
define the shift bimodule

\[ B_{c,c+\chi} := H_{c+\chi} \otimes_{A_{c+\chi}} e H_{c+\chi} e_{\chi} \otimes_{A_c} H_c \in HC(H_{c+\chi}, H_c) \]

and the shift functor \( \Phi_{c,c+\chi} := B_{c,c+\chi} \otimes_{H_c} : O_c \to O_{c+\chi} \).

### 3.1.2 Alternative definition

An alternative definition of a HC bimodule was found by Losev in [L3]. One direction of the following result follows similarly to the proof of Proposition 3.1.4, while the other one is considerably more technical. We refer the reader to [L3] for a proof.

**Theorem 3.1.6** (Section 5.4, [L3]). Let \( B \in (H_c, H_{c'})\)-bimod. Then, the following are equivalent.

1. \( B \) is HC.

2. There exists a bimodule filtration on \( B \), to be called a good filtration, such that the following conditions are satisfied.

   (a) \( \text{gr} B \) is a finitely generated bimodule over \( \mathbb{C}[R \oplus R^*]#W \).

   (b) The left and right actions of \( \mathbb{C}[R \oplus R^*]^W = Z(\mathbb{C}[R \oplus R^*]#W) \) on \( \text{gr} B \) coincide.

Let us remark that in (2) of Theorem 3.1.6 we can take either the Bernstein or the geometric filtration on \( H_c, H_{c'} \), this follows since we can get one filtration from the other by twisting with the adjoint Euler action, see e.g. Section 2 in [L7].

Thanks to Theorem 3.1.6 we can give a definition of HC bimodules for the homogeneous Cherednik algebra \( \mathbf{H} \). Let \( B \) be a HC \( (H_c, H_{c'})\)-bimodule with a good filtration, where we take the geometric filtration on \( H_c \) and \( H_{c'} \). Then, the Rees bimodule \( R_h(B) \) is a \( (R_h(H_c), R_h(H_{c'}))\)-bimodule. Recall from Section 2.1.5 that both algebras \( R_h(H_c), R_h(H_{c'}) \) are quotients of \( \mathbf{H} \). In particular, \( R_h(B) \) is a \( \mathbf{H} \)-bimodule. The following definition is then tailored so that \( R_h(B) \) is a HC \( \mathbf{H} \)-bimodule.
Definition 3.1.7 ([L3]). Let $B$ be a graded $H$-bimodule. We say that $B$ is HC if the following conditions are satisfied.

(i) $B$ is finitely generated as a $H$-bimodule.

(ii) The left and right actions of $\hbar$ on $B$ coincide.

(iii) $B$ is flat as a $\mathbb{C}[\hbar]$-module.

(iv) The left and right actions of $Z(H/\hbar H)$ on $B/\hbar B$ coincide.

3.2 Singular supports and annihilators

3.2.1 Singular supports

Let $B$ be a HC $(H_c, H_{c'})$-bimodule, equipped with a good filtration with respect to the geometric filtrations on $H_c, H_{c'}$. We can consider $\text{gr} \ B$ to be a $\mathbb{C}[R \oplus R^*]^W$-module. Note that, while the module $\text{gr} \ B$ itself depends on the chosen good filtration, its set-theoretic support (as a coherent sheaf on $(R \oplus R^*)/W$) does not.

Definition 3.2.1. We define the singular support of a HC $(H_c, H_{c'})$-bimodule $B$ to be the set-theoretic support of the $\mathbb{C}[R \oplus R^*]^W$-module $\text{gr} \ B$, where the associated graded is taken with respect to any good filtration on $B$.

$$SS(B) := \text{supp}(\text{gr} \ B) \subseteq (R \oplus R^*)/W$$

Similarly, for a HC $H$-bimodule $B$, we define its singular support

$$SS(B) := \text{supp}(B/cB) \subseteq (R \oplus R^*)/W$$

For a $(H_c, H_{c'})$-bimodule $B$, we denote by $\text{LAnn}(B) \subseteq H_c$ the left annihilator and by $\text{RAnn}(B) \subseteq H_{c'}$ the right annihilator.
Lemma 3.2.2 ([L8]). Let $B$ be a HC $(H_c, H_{c'})$-bimodule. Then $SS(B) = SS(H_c/\text{LAnn}(B)) = SS(H_{c'}/\text{RAnn}(B))$, where $H_c/\text{LAnn}(B)$ (resp. $H_{c'}/\text{RAnn}(B)$) is viewed as a HC $(H_c, H_c)$- (resp. $(H_{c'}, H_{c'})$-)bimodule.

Proof. Let us deal with the statement for the left annihilator, the other statement is analogous. First of all, since a good filtration on $B$ is compatible with the filtration on $H_c$ which induces a good filtration on $H_c/\text{LAnn}(B)$, it follows easily that $SS(B) \subseteq SS(H_c/\text{LAnn}(B))$.

To show the other inclusion, consider the $(H_c, H_c)$-bimodule $\tilde{B} := \text{Hom}_{H_c}(B, B)$. Note that a good filtration on $B$ induces a filtration on $\tilde{B}$, by setting

$$\tilde{F}^i \tilde{B} := \{ \varphi \in \tilde{B} : \varphi(F^j B) \subseteq F^{i+j} B \text{ for all } j \}$$

we remark that this filtration is exhausting because $B$ is finitely generated as a right $H_{c'}$-module, and it is compatible with the filtration on $H_c$ because so is the filtration on $B$. Note that, upon restricting to $\mathbb{C}[R \oplus R^*]^W$, we have

$$\text{gr} \tilde{B} \subseteq \text{Hom}_{\mathbb{C}[R \oplus R^*]^W}(\text{gr} B, \text{gr} B)$$

from where it follows that $\tilde{B}$ is HC, that $\tilde{F}$ is a good filtration, and that $SS(\tilde{B}) \subseteq SS(B)$. Now note that we have an inclusion $H_c/\text{LAnn}(B) \rightarrow \tilde{B}$, which is obviously a map of $(H_c, H_c)$-bimodules. The result follows.

Lemma 3.2.3. Let $B$ be an irreducible HC $(H_c, H_{c'})$-bimodule and $M \in O_c$ be irreducible. Then:

(i) $B \otimes_{H_c} M = 0$ unless $\text{Ann}(M) = \text{RAnn}(B)$.

(ii) If $B \otimes_{H_c} M \neq 0$, then the annihilator of every irreducible quotient of $B \otimes_{H_c} M$ coincides with $\text{LAnn}(B)$ and, moreover, $\text{Ann}(B \otimes_{H_c} M) = \text{LAnn}(B)$.

Proof. Assume $B \otimes_{H_c} M \neq 0$, and let $N$ be an irreducible quotient of $B \otimes_{H_c} M$. Note that, since $B$ is irreducible, we have an inclusion $B \hookrightarrow \text{Hom}_C(M, N)$. Let us first check that
RA\text{Ann}(B) = \text{Ann}(M), \text{ which will show } (i). \text{ Note that } \bigcap_{f \in B} \ker(f) \text{ is a proper submodule of } M. \text{ Since } M \text{ is irreducible, we must have } \bigcap_{f \in B} \ker(f) = 0. \text{ Now, if } a \in RA\text{Ann}(B), \text{ then } aM \subseteq \bigcap_{f \in B} \ker(f), \text{ so } a \in \text{Ann}(M). \text{ The other inclusion follows from the inclusion } \text{Ann}(M) \subseteq RA\text{Ann}(\text{Hom}_C(M,N)), \text{ which is clear. We are done with } (i).

To show (ii) we must show, first, that L\text{Ann}(B) = \text{Ann}(N). \text{ Observe that } \sum_{f \in B} f(M) \text{ is a nonzero submodule of } N, \text{ so we must have } \sum_{f \in B} f(M) = N. \text{ From here it follows similarly to the previous paragraph that } L\text{Ann}(B) = \text{Ann}(N). \text{ To prove the last statement of the lemma, note that we clearly have } L\text{Ann}(B) \subseteq \text{Ann}(B \otimes_{H_c} M). \text{ On the other hand, by what we just proved } \text{Ann}(B \otimes_{H_c} M) \subseteq \text{Ann}(N) = L\text{Ann}(B). \text{ So } L\text{Ann}(B) = \text{Ann}(B \otimes_{H_c} M). \ \\

Let us rephrase the previous lemma in terms of supports. First of all, let us remark that the \( \mathbb{C}[R \oplus R^*]^W \)-module gr B is a Poisson module, that is, it is equipped with a Poisson bracket \( \{\cdot, \cdot\} : \mathbb{C}[R \oplus R^*] \otimes \text{gr} B \to \text{gr} B \) satisfying the obvious compatibility conditions with the Poisson bracket on \( \mathbb{C}[R \oplus R^*]^W \). This is a standard result. It follows that the singular support of B is a union of symplectic leaves of \( (R \oplus R^*)/W \). We will need the following result of [BrGo].

**Lemma 3.2.4.** The symplectic leaves of \( (R \oplus R^*)/W \) are in bijection with conjugacy classes of parabolic subgroups of W. Namely, for every parabolic subgroup \( W' \subseteq W \), the set \( \mathcal{L}_{W'} := \pi\{ (x,y) \in R \oplus R^* : W_{(x,y)} = W' \} \) is a symplectic leaf, where \( \pi : R \oplus R^* \to (R \oplus R^*)/W \) is the quotient map.

Recall now, that for an irreducible module \( M \in \mathcal{O}_{\mathcal{O}'} \), \( \text{supp}(M) = X_{W'} \) for a parabolic subgroup \( W' \subseteq W \). It follows from Lemma 3.2.3 that if B is an irreducible HC bimodule such that \( B \otimes_{H_c} M \neq 0 \), then \( \text{SS}(B) = \overline{\mathcal{L}_{W'}} \). In particular, if B is an irreducible HC bimodule such that \( B \otimes_{H_c} M \neq 0 \) for some finite-dimensional \( M \in \mathcal{O}_{\mathcal{O}'} \), then B itself is finite-dimensional.
3.2.2 Filtration by supports

The notion of singular support allows us to define a filtration on the category $HC(H_c, H_{c'})$, as follows. Let $\mathcal{L} \subseteq (R \oplus R^*)/W$ be a symplectic leaf. We define the full Serre subcategory $HC_\mathcal{L}(H_c, H_{c'}) \subseteq HC(H_c, H_{c'})$ to be that whose objects are $HC(H_c, H_{c'})$-bimodules $B$ with $SS(B) \subseteq \mathcal{Z}$. Similarly, we define $HC_{\partial \mathcal{L}}(H_c, H_{c'})$, where $\partial \mathcal{L} := \mathcal{Z} \setminus \mathcal{L}$. Note that $HC_{\partial \mathcal{L}}(H_c, H_{c'})$ is a Serre subcategory of $HC_\mathcal{L}(H_c, H_{c'})$ and we can form the quotient

$$HC_{\mathcal{L}}(H_c, H_{c'}) := HC_\mathcal{L}(H_c, H_{c'}) / HC_{\partial \mathcal{L}}(H_c, H_{c'})$$

For example, when $\mathcal{L} = \mathcal{L}_{(1)}$ is the dense symplectic leaf, then $HC_{\mathcal{L}}(H_c, H_{c'})$ is the quotient of the category $HC(H_c, H_{c'})$ by the Serre subcategory consisting of bimodules with proper singular support. On the other extreme, when $\mathcal{L} = \mathcal{L}_W = \{0\}$, then $HC_0(H_c, H_{c'})$ is the category of finite-dimensional bimodules. Define the associated graded category

$$\text{gr} HC(H_c, H_{c'}) := \bigoplus_{\mathcal{L}} HC_{\mathcal{L}}(H_c, H_{c'})$$

Note that the irreducible objects of $HC_{\mathcal{L}}(H_c, H_{c'})$ are in bijective correspondence with the simple HC bimodules whose singular support coincides with $\mathcal{Z}$. In particular, the number of irreducibles in $\text{gr} HC(H_c, H_{c'})$ is no greater than the number of irreducibles in $HC(H_c, H_{c'})$. In the sequel, we will see that these numbers actually coincide (and are finite).

3.3 Restriction functors for HC bimodules, I: naive construction

3.3.1 Construction

Let $B$ be a $HC(H_c, H_{c'})$-bimodule. Note that, by its very definition, for any principal open set $U \subseteq R/W$, the localization $\mathbb{C}[U] \otimes_{\mathbb{C}[R/W]} B$ is actually a $(\mathcal{H}_c(U), \mathcal{H}_{c'}(U))$-bimodule, cf. Section 2.1.6. So we can sheafify the bimodule $B$ to get a $(\mathcal{H}_c, \mathcal{H}_{c'})$-bimodule $\mathcal{B}$. 38
Now let $b \in R$, with stabilizer $W_b$, and let $B^{\wedge b}$ be the sections of $\mathcal{B}$ on the formal neighborhood of $[b] \in R/W$. In other words, we have $B^{\wedge b} = \mathbb{C}[R/W]^{\wedge [b]} \otimes_{\mathbb{C}[R/W]} B$. Thanks to the considerations of the previous paragraph, this is a $(H_c^{\wedge b}, H_{c'}^{\wedge b})$-bimodule. So we can take the pushforward with respect to the Bezrukavnikov-Etingof isomorphism $\theta_b$, and we get a $(Z(W_b, W, H_c(W_b), R)^{\wedge 0}, Z(W_b, W, H_{c'}(W_b, R)^{\wedge 0}))$-bimodule $(\theta_b)_*(B^{\wedge b})$. Recall the idempotent $e(W_b) \in Z(W_b, W, H_c(W_b, R)^{\wedge 0})$. Abusing the notation, we also denote by $e(W_b)$ a similarly defined idempotent in the algebra $Z(W_b, W, H_{c'}(W_b, R)^{\wedge 0})$. So $e(W_b)B^{\wedge b}e(W_b)$ becomes a $(H_c(W_b, R)^{\wedge 0}, H_{c'}(W_b, R)^{\wedge 0})$-bimodule. Recall now the decomposition $R = R_{W_b} \oplus R_{W_b}$ and a similar decomposition for $R^*$. Let $B_{ib}$ be the subspace of $e(W_b)B^{\wedge b}e(W_b)$ consisting of vectors $v$ satisfying:

- They commute with $R_{W_b}^*, (R^*)_{W_b}$, that is, $xv = vx, yv = vy$ for $x \in (R^*)_{W_b}$, $y \in R_{W_b}$.

- For every $a \in \mathbb{C}[R_{W_b}]^{W_b} \cup \mathbb{C}[R_{W_b}^*]^{W_b}$ $\mathrm{ad}(a)^N(v) = 0$ for $N \gg 0$.

The assignment $B \mapsto B_{ib}$ is functorial, and $B_{ib}$ is a $(H_c(W_b, R_{W_b}), H_{c'}(W_b, R_{W_b}))$-bimodule.

**Theorem 3.3.1** (Section 3, [L3]). The assignment $B \mapsto B_{ib}$ defines a functor

$$\bullet_{ib} : \text{HC}(H_c, H_{c'}) \rightarrow \text{HC}(H_c(W_b, R_{W_b}), H_{c'}(W_b, R_{W_b})).$$

This functor is exact and admits a right adjoint

$$\bullet^{ib} : \text{HC}(H_c(W_b, R_{W_b}), H_{c'}(W_b, R_{W_b})) \rightarrow \widehat{\text{HC}}(H_c, H_{c'}),$$

where the latter category is the ind-completion of $\text{HC}(H_c, H_{c'})$.

Let us remark that the objects of $\widehat{\text{HC}}(H_c, H_{c'})$ are $(H_c, H_{c'})$-bimodules which are the union of their $\text{HC}$ (= finitely generated) sub-bimodules. For the sake of completeness, let us give a sketch of the construction of the functor $\bullet^{ib}$. Note first that the algebra $H_c(W_b, R)$ is naturally isomorphic to the tensor product $D(R_{W_b}^*) \otimes H_c(W_b, R_{W_b})$, this follows easily from the decomposition $R = R_{W_b} \oplus R_{W_b}$. So the completion of this algebra at 0 is $H_c(W_b, R)^{\wedge 0} = \mathbb{C}[R/W]^{\wedge [0]} \otimes_{\mathbb{C}[R/W]} B$. Thanks to the considerations of the previous paragraph, this is a $(H_c^{\wedge b}, H_{c'}^{\wedge b})$-bimodule.
It follows that, if \( B \in \text{HC}(H_c(W_b, R_{W_b}), H_c'(W_b, R_{W_b})) \), then \( D((R_{W_b})^{\Lambda_0}) \otimes H_c(W_b, R_{W_b})^{\Lambda_0} \) is a \((H_c(W_b, R), H_c'(W_b, R))^{\Lambda_0}\)-bimodule. Recall that the category of \((H_c(W_b, R), H_c'(W_b, R))^{\Lambda_0}\)-bimodules is equivalent to that of \((H_c^b, H_c'^b)^{\Lambda_0}\)-bimodules. Let us denote by \( B' \) the bimodule corresponding to \( D((R_{W_b})^{\Lambda_0}) \otimes B^{\Lambda_0} \) under this equivalence. Now \( B^{\Omega_b} \) is the subspace of \( B' \) consisting of vectors on which \( C[R, C[R^*]] \) act locally nilpotently.

We clearly have that \( B^{\Omega_b} \in \hat{H}(H_c, H_c') \). In Corollary 3.3.15, we will see that the image of \( \bullet^{\Omega_b} \) is actually contained in the smaller category \( HC(H_c, H_c') \), that is, \( B^{\Omega_b} \) is finitely generated for every \( B \in \text{HC}(H_c(W_b, R_{W_b}), H_c'(W_b, R_{W_b})) \). Let us remark that this will be a formal consequence of properties of the categories of HC bimodules and, in particular, we do not need the explicit description of the functor \( \bullet^{\Omega_b} \) to prove it.

### 3.3.2 Properties

Important properties of the functor \( \bullet^{\Omega_b} \) are summarized in the following result. Recall that symplectic leaves of \((R \oplus R^*)/W\) are labeled by conjugacy classes of parabolic subgroups of \( W \), and we denote by \( \mathcal{L}_{W'} \), the symplectic leaf corresponding to (the conjugacy class of) the parabolic subgroup \( W' \). Similarly, the symplectic leaves of \((R_{W_b} \oplus R_{W_b}^*)/W_b\) are labeled by conjugacy classes of parabolic subgroups of \( W_b \), and we denote these by \( \mathcal{L}_{W_b}^{W_b} \).

**Lemma 3.3.2 ([L3]).** The following is true.

1. Assume that \( \text{SS}(B) = \bigcup_{i=1}^n \mathcal{L}_{W_i} \). Then, \( \mathcal{L}_{W'}^{W_b} \subseteq \text{SS}(B^{\Omega_b}) \) if and only if \( W' \) is conjugate (in \( W \)) to one of \( W_1, \ldots, W_n \).

2. If \( \text{SS}(B) \subseteq \overline{\mathcal{L}_{W_b}} \) then \( B^{\Omega_b} \) is finite-dimensional. Moreover, \( B^{\Omega_b} = 0 \) if \( \text{SS}(B) \not\subseteq \overline{\mathcal{L}_{W_b}} \).

3. If \( B_1 \in \text{HC}(H_c, H_c'), B_2 \in \text{HC}(H_c', H_c') \) then we have a natural isomorphism \( (B_1 \otimes_{H_c'} B_2)^{\Omega_b} \xrightarrow{\cong} B_1^{\Omega_b} \otimes_{H_c(W_b, R_{W_b})} B_2^{\Omega_b} \).

4. If \( M \in \mathcal{O}_{c'} \), then we have a natural isomorphism \( \text{Res}_{W_b}^W(B \otimes_{H_c'} M)^{\Omega_b} \xrightarrow{\cong} B^{\Omega_b} \otimes_{H_c(W_b, R_{W_b})} \text{Res}_{W_b}^W(M) \).
5. Assume $B \in \Hom(H_c(W_b, R_{W_b}), H_{c'}(W_b, R_{W_b}))$ is finite-dimensional. Then $B^{\dagger_b}$ is finitely generated, that is, $B^{\dagger_b} \in \Hom(H_c, H_{c'})$.

### 3.3.3 Locally finite maps

Let us now give a way to construct HC bimodules. Consider modules $N \in \mathcal{O}_c$, $M \in \mathcal{O}_{c'}$. Then, $\Hom_C(M, N)$ is a $(H_c, H_{c'})$-bimodule.

**Definition 3.3.3** ([BEG]). By $\Hom_{\text{fin}}(M, N)$ we denote the $(H_c, H_{c'})$-sub-bimodule of the bimodule $\Hom_C(M, N)$ consisting of all those vectors that are locally nilpotent under the adjoint action of $\mathbb{C}[R]^W \cup \mathbb{C}[R^*]^W$.

Clearly, $\Hom_{\text{fin}}(M, N)$ is the direct limit (= union) of its HC sub-bimodules. We have, in fact, that $\Hom_{\text{fin}}(M, N)$ is HC.

**Lemma 3.3.4** (Proposition 5.7.1 in [L3]). For any $M \in \mathcal{O}_{c'}, N \in \mathcal{O}_c$, the $(H_c, H_{c'})$-bimodule $\Hom_{\text{fin}}(M, N)$ is finitely generated, and so it is HC.

We will use the following result, which is [L3, Lemma 5.7.2]. Alternatively, it follows from Lemma 3.2.3.

**Lemma 3.3.5.** Let $M \in \mathcal{O}_{c'}, N \in \mathcal{O}_c$ be irreducible. Then, $\Hom_{\text{fin}}(M, N) = 0$ unless $\text{supp}(M) = \text{supp}(N)$.

For us, the bimodules of locally finite maps are important because, in fact, every irreducible HC $(H_c, H_{c'})$-bimodule can be embedded into a bimodule of locally finite maps.

**Lemma 3.3.6** (Lemma 3.10 in [L7]). Let $P_c \in \mathcal{O}_c$ be a projective generator, and let $B \in \Hom(H_c, H_{c'})$ be a nonzero HC bimodule. Then, $B \otimes_{H_{c'}} P_c \neq 0$.

**Proof.** Let $W'$ be a parabolic subgroup such that $\mathcal{L}_{W'}$ is dense in $\text{SS}(B)$, and let $b \in R$ be such that $W_b = W'$. Then, $B^{\dagger_b}$ is a finite-dimensional $(H_c(W_b, R_{W_b}), H_{c'}(W_b, R_{W_b}))$-bimodule, and
Corollary 3.3.7. Let $B$ be an irreducible HC $(H_c, H_c')$-bimodule. Then, there exist irreducible modules $M \in \mathcal{O}_c$, $N \in \mathcal{O}_c$ and a monomorphism $B \hookrightarrow \text{Hom}_{\mathfrak{t}_{1\mathfrak{n}}}(M, N)$.

Proof. By Lemma 3.3.6 there exists an irreducible module $M \in \mathcal{O}_c$ with $B \otimes_{H_{c'}} M \neq 0$. Since the latter module is in category $\mathcal{O}_c$, there exists an irreducible module $N \in \mathcal{O}_c$ and a nonzero map $f : B \otimes_{H_c} M \rightarrow N$. Then, $v \mapsto (m \mapsto f(v \otimes_{H_c} m))$ defines a nonzero morphism $B \rightarrow \text{Hom}_{\mathfrak{t}_{1\mathfrak{n}}}(M, N)$.

Corollary 3.3.8. Let $B_1$ be an irreducible HC $(H_c, H_c')$-bimodule, and $B_2$ an irreducible HC $(H_{c'}, H_{c''})$-bimodule. Then, $B_1 \otimes_{H_{c'}} B_2 = 0$ unless $\text{SS}(B_1) = \text{SS}(B_2)$.

Proof. Assume that $\text{SS}(B_1) \neq \text{SS}(B_2)$, and denote $B := B_1 \otimes_{H_{c'}} B_2$. First, we assume that $\text{SS}(B_1) \subsetneq \text{SS}(B_2)$. By Lemma 3.3.6 it is enough to show that $B \otimes_{H_{c''}} N = 0$ for all irreducible modules $N \in \mathcal{O}_{c''}$. If $B_2 \otimes_{H_{c''}} N = 0$ we are done. So we may assume that $B_2 \otimes_{H_{c''}} N \neq 0$. By Lemma 3.2.3 this implies that $\text{Ann}(B_2 \otimes_{H_{c''}} N) = \text{LAnn}(B_2)$.

If $B_1 \otimes_{H_{c'}} (B_2 \otimes_{H_{c''}} N) \neq 0$, then $B_1 \otimes_{H_{c'}} M \neq 0$ for some irreducible subquotient $M$ of $B_2 \otimes_{H_{c''}} N$. So $\text{RAnn}(B_1) = \text{Ann}(M) \supseteq \text{Ann}(B_2 \otimes_{H_c} N) = \text{LAnn}(B_2)$. Thus, $\text{SS}(B_1) = \text{SS}(H_c / \text{RAnn}(B_1)) \subseteq \text{SS}(H_{c'} / \text{LAnn}(B_2)) = \text{SS}(B_2)$, a contradiction with our assumption. We conclude that $B_1 \otimes_{H_{c'}} B_2 = 0$.

Now assume that $\text{SS}(B_2) \not\subsetneq \text{SS}(B_1)$. Let $c^{\text{opp}}$ be defined by $c^{\text{opp}}(s) := -c(s^{-1})$. Then, it is easy to check that we have an isomorphism $H_c(W, R) \rightarrow H_{c^{\text{opp}}}(W, R^*)^{\text{opp}}$ given by $x \mapsto x, y \mapsto y, w \mapsto w^{-1}, x \in R^*, y \in R, w \in W$. We get an equivalence $\rho_{c,c'} : (H_c, H_{c'})^{\text{bimod}} \rightarrow (H_{(c')^{\text{opp}}}(W, R^*), H_{c^{\text{opp}}}(W, R^*))^{\text{bimod}}$. Similarly, we get equivalences $\rho_{c',c''}, \rho_{c,c''}$. Note that these equivalences preserve the categories of HC bimodules as well as the support of a HC bimodule. We have that $\rho_{c,c''}(B_1 \otimes_{H_c} B_2) = \rho_{c',c''}(B_2) \otimes_{H_{(c')^{\text{opp}}}(W, R^*)} \rho_{c,c''}(B_1)$. Thus, the result in this case follows from the previous paragraph. \qed
Corollary 3.3.9. Let $B$ be an irreducible $HC(H_c,H_{c'})$-bimodule. Then, there exists a parabolic subgroup $W' \subseteq W$ such that $SS(B) = \overline{L_{W'}}$.

Proof. Let $M \in O_{c'}$ be an irreducible module such that $B \otimes_{H_{c'}} M \neq 0$. By Proposition 2.2.5 there exists a parabolic subgroup $W' \subseteq W$ such that $\text{supp}(M) = \overline{X_{W'}}$. Now it follows from Lemma 3.2.3 that $SS(B) = \overline{L_{W'}}$. \qed

Corollary 3.3.10. The following holds.

1. The irreducible objects in $HC(H_c,H_{c'})$ and $\text{gr} HC(H_c,H_{c'})$ are in bijective correspondence.

2. Let $W'$ be a parabolic subgroup of $W$, and assume $HC_{L_{W'}}(H_c,H_{c'}) \neq 0$. Then, $O^{o}_{c,W'}$ and $O^{o}_{c',W'}$ are both nonzero.

3. We have $HC_{L_{W'}}(H_c,H_{c'}) \neq 0$ if and only if $O^{o}_{c,W'} \neq 0$.

We will see now that $HC(H_c,H_{c'})$ has a finite number of irreducible objects. We will use the following result due to Ginzburg, [Gi2, Corollary 6.7] and, independently, Losev, [L8, Section 4.1].

Proposition 3.3.11. Every object in $HC(H_c,H_{c'})$ has finite length.

Corollary 3.3.12. The category $HC(H_c,H_{c'})$ has finitely many irreducible objects.

Proof. Recall that every irreducible $HC(H_c,H_{c'})$-bimodule is contained in a bimodule of the form $\text{Hom}_{\text{fin}}(M,N)$ where $M \in O_{c'}, N \in O_{c}$ are irreducible modules with the same support. By Proposition 3.3.11, $\text{Hom}_{\text{fin}}(M,N)$ has finite length. This, together with the fact that the categories $O$ have finitely many irreducible modules, give the desired result. \qed

Let us now see that, in fact, the category $HC(H_c,H_{c'})$ is equivalent to the category of finite-dimensional representations of a finite-dimensional algebra. By Proposition 3.3.11 and Corollary 3.3.12, it is enough to check that the category $HC(H_c,H_{c'})$ has enough injective objects. In order to do so, we will use the following result.
Lemma 3.3.13 (Lemma 3.9, [L7]). Let $P_c' \in \mathcal{O}_{c'}$ be projective. Then, the functor $\bullet \otimes_{H_c} P_c' : \text{HC}(H_c, H_c') \to \mathcal{O}_c$ is exact.

Corollary 3.3.14. The category $\text{HC}(H_c, H_c')$ has enough injective objects. In particular, it is equivalent to the category of finite-dimensional representations of a finite-dimensional algebra.

Proof. Let $P_c'$ be a projective generator of $\mathcal{O}_{c'}$. Thanks to Lemma 3.3.13, the functor $\bullet \otimes_{H_c} P_c'$ is exact. Note that the right adjoint of this functor is precisely $\text{Hom}_{\text{fin}}(P_c', \bullet) : \mathcal{O}_c \to \text{HC}(H_c, H_c')$. Being a functor whose left adjoint is exact, the functor $\text{Hom}_{\text{fin}}(P_c', \bullet)$ sends injective objects to injective objects.

Now let $B$ be an irreducible $\text{HC}(H_c, H_c')$-bimodule. Recall that there exist simple $M \in \mathcal{O}_{c'}, N \in \mathcal{O}_c$, such that $B \hookrightarrow \text{Hom}_{\text{fin}}(M, N)$. In particular, $B \hookrightarrow \text{Hom}_{\text{fin}}(M, E)$, where $E \in \mathcal{O}_c$ is an injective module containing $N$. But now there exists $n > 0$ such that $P_c'^n \twoheadrightarrow M$, and therefore $B \hookrightarrow \text{Hom}_{\text{fin}}(M, E) \hookrightarrow \text{Hom}_{\text{fin}}(P_c'^n, E)$. By the observation in the first paragraph of this proof, the latter bimodule is injective. Thus, every irreducible $\text{HC}(H_c, H_c')$-bimodule may be embedded in an injective object. It follows from Proposition 3.3.11 that $\text{HC}(H_c, H_c')$ has enough injectives.

Corollary 3.3.15. Let $W' \subseteq W$ be a parabolic subgroup and $b \in R$ such that $W_b = W'$. Let $B \in \text{HC}(H_c(W', R_{W'}), H_c'(W', R_{W'}))$. Then, $B^b$ is finitely generated. In particular, we have $\bullet^b : \text{HC}(H_c(W', R_{W'}), H_c'(W', R_{W'})) \to \text{HC}(H_c, H_c')$.

Proof. Note that, since $\text{HC}(H_c, H_c')$ has finitely many simples and enough projective objects, it is enough to show that $\text{Hom}_{\text{HC}}(P, B^b)$ is finite-dimensional for every projective $P \in \text{HC}(H_c, H_c')$. This is immediate by adjunction.

In the sequel, we will need a much subtler version of the restriction functors. The main new feature of the upgraded restriction functors is that they see a certain equivariance on the image, which is not possible to see using the constructions in this section.
3.4 Restriction functors for HC bimodules, II: equivariance.

We will need an enhanced version of restriction functors for HC bimodules, introduced in [L3, Section 3], the exposition here follows [L7, Section 3]. The enhancement comes from an equivariance on the target category, which we now explain. Let $W \subseteq W$ be a parabolic subgroup, $N_W(W)$ its normalizer, and $\Xi := N_W(W)/W$. Recall that we have the decomposition $R := R_W \oplus R^W$, and note that the normalizer $N_W(W)$ preserves $R_W$. It follows that $N_W(W)$ acts on $H_c := H_c(W, R_W)$ by algebra automorphisms, in such a way that the action of $W \subseteq N_W(W)$ coincides with the adjoint action, $w : x \mapsto wxw^{-1}$.

**Definition 3.4.1.** A $\Xi$-equivariant $HC(H_c, H_{c'})$-bimodule is a bimodule $B \in HC(H_c, H_{c'})$ together with an action of $N_W(W)$-action on $B$ such that

- The structure map $H_c \otimes B \otimes H_{c'} \rightarrow B$ is $N_W(W)$-equivariant.
- For $w \in W$, $b \in B$, we have $w.b = wbw^{-1}$.

Let us denote by $HC^\Xi(H_c, H_{c'})$ the category of $\Xi$-equivariant $HC(H_c, H_{c'})$-bimodules, and by $HC^\Xi_0(H_c, H_{c'})$ the subcategory consisting of finite-dimensional equivariant bimodules.

3.4.1 Construction

Here, we follow [L7]. We work with the homogeneous Cherednik algebra $H$. Let us denote

$$R^{reg-W} := R \setminus \bigcup_{s \notin W} \Gamma_s = \{\delta_W \neq 0\}$$

where $\delta_W := \prod_{s \notin W} \alpha_s$. Note that $R^{reg-W}/W$ is the étale locus of the projection $R/W \rightarrow R/W$. The space $\mathbb{C}[R^{reg-W}/W] \otimes_{\mathbb{C}[R/W]} H_c$ is naturally an algebra, which can be identified with the subalgebra of $D(R^{reg-W}/W \times_{R/W} R)\#W$ generated by $\mathbb{C}[R^{reg-W}/W \times_{R/W} R]$, $\mathbb{C}W$, and the Dunkl operators. We denote this algebra by $H_{c,reg-W}$. The same proof of [BE, Lemma 3.2] gives the following.
Lemma 3.4.2. There is an isomorphism

$$\Theta : H_{c, \text{reg}} - W \to Z(W, W, \mathbb{C}[R_{\text{reg}} - W] \otimes_{\mathbb{C}[R/W]} H_c(W, R)) \quad (3.1)$$

Let us give a geometric intuition for the previous lemma. Note that there is a $W$-equivariant isomorphism

$$\overset{\cong}{\mathbb{R}_{\text{reg}} - W \times R/W R} \ni \bigcup_{w \in W/W} wR_{\text{reg}} - W \subseteq W/W \times R \quad (W \cdot, w \cdot) \mapsto (wW, w \cdot) \quad (3.2)$$

Let us denote by $X$ the variety $\bigcup_{w \in W/W} wR_{\text{reg}} - W$. So we can think of $H_{c, \text{reg}} - W$ as $H_c(W, X)$, see e.g. [Et2], [Wi, Section 2] for generalities on rational Cherednik algebras associated to the action of a complex reflection group on a smooth algebraic variety (not necessarily a vector space), in this work we will only use varieties that are disjoint unions of Zariski open sets inside a vector space. Similarly, we think of $\mathbb{C}[R_{\text{reg}} - W] \otimes_{\mathbb{C}[R/W]} H_c(W, R)$ as being the rational Cherednik algebra $H_c(W, R_{\text{reg}} - W)$. The isomorphism in Lemma 3.4.2 is an invariant way of expressing the decomposition in (3.2).

Now let $\mathcal{L}$ be the projection to $R_{\text{reg}} - W/W$ of $\{ x \in R : W_x = W \} \subseteq R_{\text{reg}} - W$. Note that $\mathcal{L}$ is closed in $R_{\text{reg}} - W$, so we can look at its formal neighborhood $\widehat{\mathcal{L}}$. Denote $H^\wedge_{c, \text{reg}} - W := \mathbb{C}[R_{\text{reg}} - W]^{\wedge \mathcal{L}} \otimes_{\mathbb{C}[R/W]} H_c$, which is naturally an algebra and the isomorphism in Lemma 3.4.2 can be restricted to an isomorphism

$$\Theta : H^{\wedge \mathcal{L}}_{c, \text{reg}} - W \overset{\cong}{\to} Z(W, W, \mathbb{C}[R_{\text{reg}} - W]^{\wedge \mathcal{L}} \otimes_{\mathbb{C}[R/W]} H_c(W, R)). \quad (3.3)$$

The isomorphism in (3.3) will take the role of the isomorphism $\Theta_b$ in the definition of the restriction functors. For technical reasons, let us introduce these functors for the homogeneous Cherednik algebras $H$. We remark that the isomorphisms (3.1), (3.3) are still valid at the level of these algebras, provided we define $H(W, R)$ as a $\mathbb{C}[c]$-algebra, even if the defining relations do not involve all the variables $c_1, \ldots, c_n$. 46
Now let $\mathcal{B}$ be a HC $\mathcal{H}$-bimodule. Consider the space $\mathcal{B}^{\mathcal{L}}_{\text{reg-}W} := \mathbb{C}[R^{\text{reg-}W}]^{\mathcal{L}} \otimes_{\mathbb{C}[R/W]} \mathcal{B}$. A priori, this is only a $(\mathcal{H}^{\mathcal{L}}_{\text{reg-}W}, \mathcal{H})$-bimodule.

**Lemma 3.4.3** (Lemma 3.6.3, [L3]). There is a unique right multiplication map making $\mathcal{B}^{\mathcal{L}}_{\text{reg-}W}$ a $\mathcal{H}^{\mathcal{L}}_{\text{reg-}W}$-bimodule such that the commutator $[\mathbb{C}[R^{\text{reg-}W}/W]^{\mathcal{L}}, \mathcal{B}^{\mathcal{L}}_{\text{reg-}W}]$ is contained in $h\mathcal{B}^{\mathcal{L}}_{\text{reg-}W}$.

**Proof.** Consider the projection $\eta_W : R^{\text{reg-}W}/W \to R/W$, which is étale. Note that it restricts to a covering $\eta_W : \mathcal{L} \to \eta_W(\mathcal{L})$ with Galois group $\Xi$. So the formal neighborhood $(R/W)^{\eta_W(\mathcal{L})}$ is a quotient by the action of $\Xi$ on the formal neighborhood $(R^{\text{reg-}W}/W)^{\mathcal{L}}$.

Recall now that we may sheafify $\mathcal{B}$ and $\mathcal{H}$, to get a sheaf $\mathcal{B} := \mathcal{S}_{R/W} \otimes_{\mathbb{C}[R/W]} \mathcal{B}$ of bimodules over $\mathcal{H}$. In particular, the restriction $\mathcal{B}|_{(R/W)^{\eta_W(\mathcal{L})}}$ is a sheaf of bimodules over $\mathcal{H}|_{(R/W)^{\eta_W(\mathcal{L})}}$. Note that $\mathcal{B}^{\mathcal{L}}_{\text{reg-}W} = \mathbb{C}[R^{\text{reg-}W}/W]^{\mathcal{L}} \otimes_{\mathbb{C}[R/W]}^{\mathcal{L}} \otimes_{\mathbb{C}[R/W]} \Gamma(\mathcal{B}|_{(R/W)^{\eta_W(\mathcal{L})}})$, and a similar equation holds for $\mathcal{H}^{\mathcal{L}}_{\text{reg-}W}$.

Now note that, since $\mathbb{C}[R/W] \subseteq \mathcal{H}$ lives in degree 0, the sheaf $\mathcal{B}$ is graded, and the grading is bounded below. In particular, $\Gamma(\mathcal{B}|_{(R/W)^{\eta_W(\mathcal{L})}})$ is a graded $\Gamma(\mathcal{H}|_{(R/W)^{\eta_W(\mathcal{L})}})$-bimodule, with the grading bounded below. Note also that $\mathcal{B}^{\mathcal{L}}_{\text{reg-}W}$ is a graded $(\mathcal{H}^{\mathcal{L}}_{\text{reg-}W}, \mathcal{H})$-bimodule with the grading bounded below.

By the first paragraph of this proof, we can find a free basis $f_1, \ldots, f_k$ of $\mathbb{C}[R^{\text{reg-}W}/W]^{\mathcal{L}}$ over $\mathbb{C}[R/W]^{\eta_W(\mathcal{L})}$ and elements $a_{ij} \in \mathbb{C}[R/W]^{\eta_W(\mathcal{L})}$, $i = 1, \ldots, k$, $j = 0, \ldots, k - 1$, such that

$$P_i(f_i) := f_i^k + a_{i,k-1}f_i^{k-1} + \cdots + a_{i,0} = 0,$$

but $P_i'(f_i) = kf_i^{k-1} + (k-1)a_{i,k-1}f_i^{k-2} + \cdots + a_{i,1}$ is invertible in $\mathbb{C}[R^{\text{reg-}W}/W]^{\mathcal{L}}$. In particular, for every $x \in \mathcal{B}^{\mathcal{L}}_{\text{reg-}W}$, $xP_i(f_i) = 0$. Assume, for a moment, that we have already defined a right multiplication with the properties required by the statement of the lemma. Then it is easy to see that we must have

$$0 = [P_i(f_i), x] = P_i'(f_i)[f_i, x] + Q_i(f_i, x) + F(f_i, a_{i,j}, x)$$

(3.4)

where $Q_i(f_i, x) := f_i^{k-1}[a_{i,k-1}, x] + \cdots + f_i[a_{i,1}, x] + [a_{i,0}, x]$, and $F(f_i, a_{i,j}, x)$ is an expression involving several brackets and products. Note that $Q_i(f_i, x), F(f_i, a_{i,j}, x) \in h\mathcal{B}^{\mathcal{L}}_{\text{reg-}W}$. 47
Thanks to (3.4), we must have \([f_i, x] = -P_i(f_i)^{-1}(Q_i(f_i, x) + F(f_i, a_{i,j}, x)) \in \h B^\wedge \reg_W^\wedge \). This shows uniqueness of the right product map. Existence is shown by induction, starting from the fact that \(\mathbb{C}[R^{\reg-W}/W]^{\wedge \ell} \) must commute with the lowest degree component of \(B^\wedge \reg_W^\wedge \). \[\square\]

Note that, actually, \(B^\wedge \reg_W^\wedge \) becomes a \(\Xi\)-equivariant \(H^\wedge \reg_W^\wedge \) bimodule, the action of \(N_W(W)\) comes from the adjoint action on \(B\) and the action on \(\mathbb{C}[R^{\reg-W}/W]^{\wedge \ell}\). Now we recall the isomorphism (3.3) in its homogeneous version. Since \(R^{\reg-W}\) is stable under the action of \(N_W(W)\), the idempotent \(e(W) \in Z(W, W, H^\wedge \reg_W^\wedge )\) is \(N_W(W)\)-invariant. It follows that \(e(W)B^\wedge \reg_W^\wedge e(W)\) is a \(\Xi\)-equivariant \(H(W, R)^\wedge \reg_W^\wedge \) bimodule. According to [L7], there exists a, unique up to isomorphism, \(\Xi\)-equivariant \(H\)-bimodule \(B\) such that

\[
e(W)B^\wedge \reg_W^\wedge e(W) = \mathbb{C}[\mathcal{L} \times R_W/W]^{\wedge \ell} \otimes_{\mathbb{C}[\mathcal{L} \times R_W/W]} (D_h(\mathcal{L}) \otimes \mathbb{C}[\h] B)
\]

Let us be more precise. First, note that we have an isomorphism \(H(W, R)^\wedge \reg_W^\wedge = D_h(\mathcal{L}) \otimes \mathbb{C}[\h]\) \(H^{\wedge 0}\) that is induced from the natural isomorphism \(H(W, R) = D_h(R_W) \otimes \mathbb{C}[\h] H\). Define a functor \(\mathcal{G} : HC^\Xi(H) \rightarrow HC^\Xi(W, R)^\wedge \reg_W^\wedge \) by \(\mathcal{G}(B) := \mathbb{C}[\mathcal{L} \times R^{\reg-W}/W]^{\wedge \ell} \otimes_{\mathbb{C}[\mathcal{L} \times R_W/W]} (D_h(\mathcal{L}) \otimes \mathbb{C}[\h] B)\). Note that \(\mathcal{G}\) is a full embedding, with left inverse \(\mathcal{F}\) given by first taking the centralizer of \(D_h(\mathcal{L})\) and then taking elements which are locally finite with respect to the adjoint action of the Euler element. Then, \(e(W)B^\wedge \reg_W^\wedge e(W)\) is in the image of \(\mathcal{G}\), and \(B := \mathcal{F}(e(W)B^\wedge \reg_W^\wedge e(W))\). Thus, we have.

**Theorem 3.4.4 ([L3, L7]).** The assignment \(B \mapsto B\) gives a functor \(HC(H) \rightarrow HC^\Xi(H)\).

Let us now proceed to construct the restriction functors for specialized parameters. Let \(B \in HC(H_e, H_{e'})\). Find a good filtration for \(B\) and consider the Rees bimodule, \(R_{\h}(B)\). This is a HE-bimodule, so we can consider \(R_{\h}(B)\). Then, \(B := R_{\h}(B) / (\h - 1)\). According to [L3, Subsection 3.9], \(B\) does not depend, up to a distinguished isomorphism, on the chosen good filtration. Thus, we get a functor \(\bullet : HC(H_e, H_{e'}) \rightarrow HC^\Xi(H_e, H_{e'})\).
**Theorem 3.4.5** (Theorem 3.4.6, [L3]). Let $W \subseteq W$ be a parabolic subgroup. The following is true.

1. Let $b \in R$ be such that $W_b = W$. Then, the functors $\bullet$ and $F \circ \bullet_b$ are isomorphic, where $F : HC^\Xi(H_c, H_{c'}) \to HC(H_c, H_{c'})$ is the functor that forgets the $\Xi$-equivariance.

2. Let $\mathcal{L} := \mathcal{L}^W$ so that, in particular, $\bullet |_{HC^\Xi(H_c, H_{c'})}$ factors through $HC_{\mathcal{L}}(H_c, H_{c'})$. Then $\bullet$ induces an equivalence between $HC_{\mathcal{L}}(H_c, H_{c'})$ and a full subcategory of $HC^\Xi_0(H_c, H_{c'})$ that is closed under taking subquotients.

3. The functor $\bullet : HC(H_c, H_{c'}) \to HC^\Xi(H_c, H_{c'})$ admits a right adjoint $\bullet^\dagger : HC^\Xi(H_c, H_{c'})$.

4. There is a functor embedding of $\bullet^\dagger \circ F$ into $\bullet^\dagger b$.

5. For $B \in HC^\Xi(H_c, H_{c'})$, the kernel and cokernel of the adjunction morphism $B \to (B^\dagger)^\dagger$ belong to $HC_{\partial\mathcal{L}}(H_c, H_{c'})$.

### 3.4.2 Applications

**Semisimplicity of the head**

Let us see some applications of the restriction functors. The first one of these tells us that, if $\mathcal{L} = \mathcal{L}_{\{1\}}$ is the dense symplectic leaf, then $HC_{\mathcal{L}}(H_c, H_c)$ is semisimple. Let us denote $HC(H_c, H_c) := HC_{\mathcal{L}_{\{1\}}}(H_c, H_c)$.

**Proposition 3.4.6.** The category $HC(H_c, H_c)$ is semisimple.

**Proof.** Here we take the restriction functor for $W = \{1\}$. Note that $\Xi = W$, and $H_c = \mathbb{C}$, so $HC^\Xi_0(H_c, H_c)$ is precisely the category of finite dimensional representations of $W$. So thanks to Theorem 3.4.5, $HC(H_c, H_c)$ can be embedded as a full subcategory of the category of representations of $W$. Moreover, this subcategory is closed under subquotients (= direct summands) and tensor products. It follows that $HC(H_c, H_c)$ is equivalent to the category
of representations of \( W/N \) for a normal subgroup \( N \subseteq W \). In particular, it is a semisimple category.

In Chapter 5, we will find an explicit description of the subgroup \( N \) that appears in the proof of Proposition 3.4.6, see Section 5.4. We will also see, Corollary 5.3.10, that \( \overline{HC}(H_c, H_{c'}) \) is semisimple for different parameters \( c, c' \).

**Injectivity of the regular bimodule**

Recall that the category \( HC(H_c, H_c) \) has enough injectives.

**Proposition 3.4.7.** The regular bimodule \( H_c \) is injective in the category of \( HC \) \( H_c \)-bimodules.

**Proof.** In view of Proposition 3.3.11, we need to show that \( \text{Ext}(B, H_c) = 0 \) for any irreducible HC bimodule \( B \), where \( \text{Ext} \) denotes \( \text{Ext}^1_{H_c\text{-bimod}} \). We separate in two cases.

**Case 1:** \( B \) has proper support. This case is due to Bezrukavnikov-Losev and it is contained in an old version of the paper [BL]. We provide a proof for convenience of the reader. Consider an exact sequence \( 0 \rightarrow H_c \rightarrow X \rightarrow B \rightarrow 0 \). Let \( \mathcal{L} \subseteq (R \oplus R^*)/W \) be the open symplectic leaf, and consider the corresponding restriction functor \( \bullet_\dagger \). Note that \( B_\dagger = 0 \). Since the restriction functor is exact, we must then have \( ((H_c)_\dagger)_\dagger = (X_\dagger)_\dagger \). We have the adjunction map \( X \rightarrow ((H_c)_\dagger)_\dagger \). The latter bimodule admits a filtration whose associated graded is contained in \( ((\mathbb{C}[R \oplus R^*]\#W)_\dagger)_\dagger \). By construction, \( ((\mathbb{C}[R \oplus R^*]\#W)_\dagger)_\dagger \) is the global sections of the restriction of \( \mathbb{C}[R \oplus R^*]\#W \) to \( \mathcal{L} \). But the complement of this leaf has codimension 2. Hence, \( ((\mathbb{C}[R \oplus R^*]\#W)_\dagger)_\dagger = \mathbb{C}[R \oplus R^*]\#W \), and this implies that \( ((H_c)_\dagger)_\dagger = H_c \). Now the adjunction map \( X \rightarrow H_c \) is a splitting of the exact sequence \( 0 \rightarrow H_c \rightarrow X \rightarrow B \rightarrow 0 \).

**Case 2:** \( B \) has full support. Assume \( 0 \rightarrow H_c \xrightarrow{\varphi} X \rightarrow B \rightarrow 0 \) is an exact sequence. Pick again the dense symplectic leaf \( \mathcal{L} \subseteq (R \oplus R^*)/W \) and consider the corresponding restriction functor \( \bullet_\dagger \). We have an exact sequence \( 0 \rightarrow (H_c)_\dagger \rightarrow X_\dagger \rightarrow B_\dagger \rightarrow 0 \). Since the category of \( \Xi \)-equivariant Harish-Chandra \( H_c \)-bimodules is semisimple, Proposition 3.4.6, this exact sequence splits, \( X_\dagger = (H_c)_\dagger \oplus B_\dagger \). Now, recall from the previous case that \( (H_c)_\dagger)_\dagger = H_c \), and
that we have the adjunction morphism $X \to (X_\dagger)^\dagger = H_c \oplus (B_\dagger)^\dagger$. By [L3, Theorem 3.7.3], the kernel of this morphism is a HC bimodule with proper support, so the morphism must be injective. Thus, we can consider $X \subseteq H_c \oplus (B_\dagger)^\dagger$, and $\varphi = (\varphi_1, \varphi_2)$, where $\varphi_1 : H_c \to H_c$, $\varphi_2 : H_c \to (B_\dagger)^\dagger$. Let us remark that, since the center of $H_c$ is trivial, every nonzero endomorphism of $H_c$ is an automorphism. So, if $\varphi_1 \neq 0$, we can find a splitting for $\varphi$. Thus, we may assume $\varphi_1 = 0$, and $\varphi_2 : H_c \to (B_\dagger)^\dagger$ is an inclusion.

Now recall that we have the adjunction morphism $B \to (B_\dagger)^\dagger$. Since $B$ is irreducible, this is actually an injection. The cokernel of this morphism is a HC bimodule with proper support, cf. Theorem 3.4.5. Thus, we must have $B \subseteq \varphi_2(H_c)$, so $B$ is isomorphic to an ideal of $H_c$. But this implies that $B_\dagger = (H_c)_\dagger$. So the exact sequence $0 \to H_c \to X \to B \to 0$ induces an inclusion $X \subseteq (X_\dagger)^\dagger = H_c \oplus H_c$ and, reasoning as in the previous paragraph, we can find a splitting for $\varphi$. Thus, $\text{Ext}(H_c, B) = 0$. \qed
Chapter 4

Reduction to corank 1.

4.1 Introduction

Let $W \subseteq W$ be a parabolic subgroup, and let $\mathcal{L} \subseteq (R \oplus R^*)/W$ be the corresponding symplectic leaf. Recall that, if we denote $\Xi := N_W(W)/W$, then we have the restriction functor

$$\bullet : HC_\mathcal{L}(H_c, H_{c'}) \hookrightarrow HC_0^\Xi(H_c, H_{c'})$$

A natural question, then, is to describe the image of this functor or, equivalently, of the functor $\bullet : HC_\Xi(H_c, H_{c'}) \to HC_0^\Xi(H_c, H_{c'})$. We would like to know, for example, the number of irreducible objects in the image. In this chapter, we reduce this question to the case where $W$ sits inside $W$ in corank 1, that is, when $\text{codim}_R(R_W) = 1$, equivalently, when $W$ is a maximal parabolic subgroup of $W$. More precisely, we prove the following result.

**Theorem 4.1.1.** Let $B \in HC_0^\Xi(H_c, H_{c'})$. Assume that, for every parabolic subgroup $W' \subseteq W$ containing $W$ in corank 1, there exists a $HC(H'_c, H'_{c'})$-bimodule $B_{W'}$ such that $(B_{W'})_{\Xi_{W'}} = B$. Here, $H'_c = H_c(W', R_{W'})$ and the $N_{W'}(W)/W$-equivariant structure on $B$ is restricted from the $\Xi$-equivariant structure. Then, there exists a $HC(H_c, H_{c'})$-bimodule $\overline{B}$ with $\overline{B}_{\Xi_{W}} = B$.

The proof of Theorem 4.1.1 passes through its homogeneous version, which is not surprising.
giving the construction of the functor \( \bullet \). In the first two sections of this chapter we give preliminary technical results that will go into the proof of Theorem 4.1.1. In particular, we study supports and annihilators of bimodules over localized Cherednik algebras. Then, we state and prove a variant of the homogeneous version of Theorem 4.1.1. After that, we use the Rees construction to get Theorem 4.1.1.

### 4.2 Technical lemmas

#### 4.2.1 Supports and symplectic leaves

Let \( W \subseteq W \) be a parabolic subgroup. Recall the \( W \)-equivariant isomorphism

\[
R^{reg-W}/W \times_{R/W} R \cong \bigsqcup_{w \in W/W} wR^{reg-W} \subseteq W/W \times R \tag{4.1}
\]

Also recall that we denote by \( \mathcal{X} \) the variety \( \bigsqcup_{w \in W/W} wR^{reg-W} \). So \( H_{reg-W} = H(W, \mathcal{X}) \). Note that \( \mathcal{X} \times R^* = T^*\mathcal{X} \) is a symplectic algebraic variety, and the action of \( W \) on \( T^*\mathcal{X} \) is by symplectomorphisms. So \( (T^*\mathcal{X})/W \) is a Poisson variety. Moreover, it follows from the isomorphism (4.1) that \( \mathcal{X}/W = R^{reg-W}/W \) and \( (T^*\mathcal{X})/W = (T^*R^{reg-W})/W \). Let us define a HC \( H_{reg-W} \)-bimodule in a manner completely analogous to Definition 3.1.7. As before, for a HC \( H_{reg-W} \)-bimodule \( B \), its support \( SS(B) \subseteq (T^*R^{reg-W})/W \) is a union of symplectic leaves.

We can describe the symplectic leaves inside \( (T^*R^{reg-W})/W \) using the results in [BrGo, 7.4]. We remark that [BrGo] works with actions on a vector space, but the proofs work in our setting. Namely, let \( W'' \subseteq W \) be a parabolic subgroup. Let \( L_{R^{reg-W}}^{W} := \pi_{R^{reg-W}}(\{ x \in T^*R^{reg-W} : W_x = W'' \}) \), where \( \pi_{R^{reg-W}} : T^*R^{reg-W} \to (T^*R^{reg-W})/W \) is the quotient by the \( W \)-action. Note that \( L_{R^{reg-W}}^{W} \) depends only on the conjugacy class of \( W'' \) in \( W \). The symplectic leaves in \( (T^*R^{reg-W})/W \) are precisely the sets \( L_{R^{reg-W}}^{W} \), where \( W'' \subseteq W \) is a parabolic subgroup. Note that \( L_{R^{reg-W}}^{W} = \bigcup_{W'' \subseteq W'' \subseteq W} L_{R^{reg-W}}^{W''} \). It follows that the unique closed leaf inside
\( (T^* R^{reg-W})/W \) is \( L^W/W \). We remark that \( L^W/W = R^W_{reg} \times (R^*)^W \), where \( R^W_{reg} := \{ b \in R : W_b = W \} \).

Similarly, we can think of \( H^{reg-W} \) as being the rational Cherednik algebra \( H(W, R^{reg-W}) \)
(recall, we are taking all variables \( h, c_1, \ldots, c_n \) even if the defining relations do not involve all of them). The singular support of a HC \( H^{reg-W} \)-bimodule \( B \) is again a union of symplectic leaves of \( (T^* R^{reg-W})/W \). Let us describe a relation between supports of HC \( H^{reg-W} \) and \( H^{reg-W} \)-bimodules. Note that \( \mathbb{C}[\mathcal{X}] = \bigoplus_{w \in W/W} \mathbb{C}[w R^{reg-W}] \). We may think of the idempotent \( e(W) \in Z(W, H^{reg-W}) \cong H^{reg-W} \) introduced in Section 2.4 as being the primitive idempotent in \( \mathbb{C}[\mathcal{X}] \subseteq H^{reg-W} \) corresponding to the direct summand \( \mathbb{C}[R^{reg-W}] \). It follows that for a HC \( H^{reg-W} \)-bimodule \( B \), \( SS(B) = SS(e(W)Be(W)) \).

The description of the singular support of a HC bimodule has the following consequence for the support of the elements of \( B/cB \).

**Lemma 4.2.1.** Let \( B \) be a HC \( H^{reg-W} \)-bimodule. Consider \( B/cB \) as a \( \mathbb{C}[\mathcal{X}/W] \otimes \mathbb{C}[R^*/W] \)-module where, recall, \( \mathcal{X} = \bigsqcup_{w \in W/W} w R^{reg-W} \), and \( \mathcal{X}/W = R^{reg-W}/W \). Then, for every nonzero element \( v \in B/cB \), its support \( X_v \subseteq R^{reg-W}/W \times R^*/W \) contains \( R^W_{reg} \times (R^*)^W \).

In view of the description of the symplectic leaves inside \( (\mathcal{X} \times R^*)/W \), Lemma 4.2.1 is a consequence of the following result.

**Lemma 4.2.2.** Let \( A \) be a commutative, Noetherian Poisson algebra, and let \( M \) be a finitely generated Poisson \( A \)-module. Then, for every element \( m \in M \), its set-theoretic support \( X_m \subseteq \text{Spec}(A) \) is a Poisson subvariety.

**Proof.** First of all, let \( I \subseteq A \) be any ideal. For \( k \geq 0 \), let \( M_{I^k} := \{ n \in M : I^k n = 0 \} \). Note that \( M_{I^k} \subseteq M_{I^{k+1}} \), so that \( M(I) := \bigcup_{k \geq 0} M_{I^k} \) is a submodule of \( M \). We claim that it is a Poisson submodule. Take \( m \in M_{I^k} \) and \( a \in A \). Let \( a_1, \ldots, a_{2k} \in I \), so that \( a_1 \cdots a_{2k} m = 0 \). It follows that \( 0 = \{ a, a_1 \cdots a_{2k} m \} = a_1 \cdots a_{2k} \{ a, m \} + \{ a, a_1 \cdots a_{2k} \} m \). Thanks to the Leibiniz identity again, \( \{ a, a_1 \cdots a_{2k} \} = a_1 \cdots a_k \{ a, a_{k+1} \cdots a_{2k} \} + a_k+1 \cdots a_{2k} \{ a, a_1 \cdots a_k \} \in I^k \). Since
m ∈ M_I^k, this implies that a_1 · · · a_{2k}\{a, m\} = 0. Thus, \(\{A, m\} \subseteq M_{I^{2k}}\). So \(M(I)\) is a Poisson submodule and thus \(\text{supp}(M(I)) \subseteq \text{Spec}(A)\) is a Poisson subvariety.

Now specialize to the case where \(I = \text{Ann}_A(m)\). Since \(A\) is Noetherian and \(M\) is finitely generated, \(M(I) = M_{I^k}\) for some \(k > 0\). Then, \(I^k \subseteq \text{Ann}(M(I))\). On the other hand, since \(m \in M(I)\) and \(I = \text{Ann}_A(m), \text{Ann}(M(I)) \subseteq I\). So \(\sqrt{\text{Ann}(M(I))} = \sqrt{I}\) and the result follows.

Let us remark that, thanks to the correspondence between supports of HC \(H_{\text{reg-}W}\) and \(H_{\text{reg-}W}\)-bimodules, we get from Lemma 4.2.1 the following result.

**Corollary 4.2.3.** Let \(B\) be a HC \(H_{\text{reg-}W}\)-bimodule. Consider \(B/cB\) as a \(\mathbb{C}[R^{\text{reg-}W}/W] \otimes \mathbb{C}[R^*/W]\)-module. Then, for every nonzero \(v \in B/cB\), its support \(X_v \subseteq R^{\text{reg-}W}/W \times R^*/W\) contains \(R^{W}_{\text{reg}} \times (R^*)^W\).

### 4.2.2 Annihilators and liftings

We will describe the annihilator of a HC \(H_{\text{reg-}W}\)-bimodule as a left \(\mathbb{C}[R^{\text{reg-}W}/W]\)-module. In order to do so, we need the following finiteness result.

**Lemma 4.2.4.** Let \(B\) be a HC \(H_{\text{reg-}W}\)-bimodule. Then, \(B\) is finitely generated over the algebra \(\mathbb{C}[R^{\text{reg-}W}/W][c] \otimes_{S(c)} \mathbb{C}[R^*/W][c]^\text{opp}\). Similarly, \(B\) is finitely generated over the algebra \(\mathbb{C}[R^*/W][c] \otimes_{S(c)} \mathbb{C}[R^{\text{reg-}W}/W][c]^\text{opp}\), where the superscript \text{opp} means that the corresponding algebra acts on the right.

**Proof.** Since \(B\) is HC, we have that \(B/cB\) is a module over \(\mathbb{C}[\mathcal{X} \times R^*]^W\), which is the center of the algebra \(H_{\text{reg-}W}/cH_{\text{reg-}W}\). This latter algebra is finite over its center, so \(B/cB\) is a finitely generated module over \(\mathbb{C}[\mathcal{X} \times R^*]^W\). Now, the natural map \((\mathcal{X} \times R^*)/W \rightarrow \mathcal{X}/W \times R^*/W\) is finite, so \(B/cB\) is a finitely generated module over \(\mathbb{C}[\mathcal{X}]^W \otimes \mathbb{C}[R^*]^W\). Let \(v_1, \ldots, v_m\) be generators of \(B/cB\) under the action of \(\mathbb{C}[\mathcal{X}]^W \otimes \mathbb{C}[R^*]^W\). We can assume that these elements are homogeneous with respect to the grading on \(B/cB\) inherited from the one on \(B\). Let
$v_1, \ldots, v_m$ be homogeneous lifts of $v_1, \ldots, v_m$. It is now standard to see that $v_1, \ldots, v_m$ are generators of $B$ under the action of $\mathbb{C}[\mathcal{X}]^W[c] \otimes_{S(\mathfrak{c})} \mathbb{C}[R^*]^W[c]^\text{opp}$. \hfill \Box

**Lemma 4.2.5.** Let $B$ be a HC $H_{\text{reg}-W}$-bimodule. Assume that $SS(B) = \mathcal{L}_W^W$ is the minimal symplectic leaf. Then, as a (left or right) $\mathbb{C}[R^{\text{reg}}-W/W]$-module, $B$ is annihilated by a power of the ideal $I$ of functions vanishing on $\mathcal{L}_W^W \subseteq R^{\text{reg}}-W/W$ where $\mathcal{L}_W^W := \{ x \in R : Wx = W \}$.

**Proof.** First, we show that any element in $B$ is annihilated by a large enough power of $I$. Recall that $L_W^W = R_{\text{reg}}^W \times (R^*)^W$. In particular, $\mathcal{L}_W^W \times 0 \subseteq \mathcal{L}_W^W$. It follows by our assumption on $SS(B)$ that $I^n \subseteq \mathbb{C}[R^{\text{reg}}-W/W] \subseteq \mathbb{C}[R^{\text{reg}}-W/W] \otimes \mathbb{C}[R^*/W] \subseteq \mathbb{C}[\mathcal{X} \times R^*]^W$ annihilates $B/cB$ for $n \gg 0$. So for any $i \in \mathbb{Z}$, $I^nB^i \subseteq cB^{i-1}$. Now the claim follows because the grading on $B$ is bounded below.

Now let $v_1, \ldots, v_m$ be generators of $B$ as a $\mathbb{C}[R^{\text{reg}}-W/W][\mathfrak{c}] \otimes_{S(\mathfrak{c})} \mathbb{C}[R^*/W][\mathfrak{c}]^\text{opp}$-module, and let $N \gg 0$ be such that $I^NV_i = 0$ for all $i = 1, \ldots, m$. It is easy to see that $I^NB = 0$. We are done. \hfill \Box

Let us discuss some consequences of Lemma 4.2.5. Recall that we have a natural action of the group $N_W(W)$ on the algebra $H_{\text{reg}-W}$ by algebra automorphisms, in such a way that the action of $W \subseteq N_W(W)$ coincides with the adjoint action. Recall also that we denote $\Xi = N_W(W)/W$. The map $\eta_W : R^{\text{reg}}-W/W \to R/W$ is étale and it restricts to a covering $\eta_W : \mathcal{L} \to \eta_W(\mathcal{L}) = \mathcal{L}/\Xi$ with Galois group $\Xi$, where $\mathcal{L} := \mathcal{L}_W^W$. This implies that the formal neighborhood $(R/W)^{\wedge \eta_W(\mathcal{L})}$ may be identified with the quotient by the action of $\Xi$ on the formal neighborhood $(R^{\text{reg}}-W/W)^{\wedge \mathcal{L}}$. Now let $B$ be a $\Xi$-equivariant HC $H_{\text{reg}-W}$-bimodule supported on $\mathcal{L}_W^W$. Thanks to Lemma 4.2.5, $B$ may be thought of as a quasi-coherent sheaf on an infinitesimal neighborhood of $\mathcal{L} \subseteq R^{\text{reg}}-W/W$. Thus, the space of invariants $B^\Xi$ is a quasi-coherent sheaf on an infinitesimal neighborhood of $\eta_W(\mathcal{L}) \subseteq R/W$, and we may think of it as a quasi-coherent sheaf on $WR^{\text{reg}}-W/W$.

We claim that, moreover, $B = \mathbb{C}[R^{\text{reg}}-W/W] \otimes_{\mathbb{C}[R/W]} B^\Xi$. Recall that the map $\eta_W : (R^{\text{reg}}-W/W)^{\wedge \mathcal{L}} \to (WR^{\text{reg}}-W/W)^{\wedge \eta_W(\mathcal{L})}$ is the quotient by the free $\Xi$-action on the for-
mal neighborhood of $L$, and $\mathbb{C}[WR^{reg-W}/W]^{\eta_W}(L)$ may be identified with the algebra of $\Xi$ invariants in $\mathbb{C}[R^{reg-W}/W]^\xi$. So the desired equality will follow if we show that the right-hand side is equal to $\mathbb{C}[WR^{reg-W}/W]^{\eta_W}(L) \otimes \mathbb{C}[WR^{reg-W}/W]^{\eta_W}(L)$. But this is clear by our description of the annihilator of $B$ (and of $B^\Xi$).

4.3 Main result for homogeneous algebras

We are now ready to state our main result. Let $W$ be a parabolic subgroup of $W$, and let $B$ be a $\Xi = N_W(W)/W$-equivariant HC $H_{reg-W}$-bimodule. We require that $SS(B) = L_{W}$, the minimal symplectic leaf in $R_{reg-W}/W$. In particular, this implies thanks to Lemma 4.2.5 that $B$ may be considered as a HC $H_{reg-W}$-bimodule. Recall that we denote $\eta_W : R_{reg-W}/W \rightarrow R/W$. From the previous subsection it follows that $\eta_W^*(B^\Xi)$ is an $H$-bimodule satisfying $(\eta_W^*(B^\Xi))^\xi_{reg-W} = B$. However, $\eta_W^*(B^\Xi)$ need not be finitely generated over $H$ so it is not, in general, a HC $H$-bimodule. Similarly, if $W'$ is a parabolic subgroup of $W$ containing $W$, $(\eta_W^*(B^\Xi))^\xi_{reg-W'}$ does not need to be a HC $H_{reg-W'}$-bimodule. However, we can further localize to the punctured formal neighborhood $\hat{L}_{W'}$. The algebra of functions of $\hat{L}_{W'}$ is a localization of the algebra $\mathbb{C}[R^{reg-W'}/W']^{\eta_W}$ at a $W'$-invariant subset, and so we have the algebra $H_{\hat{L}_{W'}} := \mathbb{C}[\hat{L}_{W'}] \otimes \mathbb{C}[R^{reg-W'}/W']^{\eta_W} H_{\hat{L}_{W'}}^{\xi_{W'}}$. Similarly, we can form the bimodule $(\eta_W^*(B^\Xi))^\xi_{\hat{L}_{W'}}$, which is now a HC $H_{\hat{L}_{W'}}$-bimodule.

**Theorem 4.3.1.** Let $B$ be a $\Xi$-equivariant HC $H_{reg-W}$-bimodule. Assume that $SS(B) = L_{W}$ and that for all parabolic subgroups $W'$ with $W \subseteq W'$ in corank $1$, there is a HC $H_{reg-W'}$-bimodule $B_{W'}$ whose localization to $\hat{L}_{W'}$ coincides with $(\eta_W^*(B^\Xi))^\xi_{\hat{L}_{W'}}$. Then, there exists a HC $H$-bimodule $B$ such that $B = B_{reg-W}$.

The proof of Theorem 4.3.1 is inspired by [L5, Section 3], where a similar result is shown at the level of category $\mathcal{O}$ (for the stratum corresponding to the dense symplectic leaf.) The strategy is as follows. We will define a bimodule that is coherent over $H|_{U}$, where $U$ is
an open subset in \( R/W \) whose complement has codimension 2. Then, we can take global sections. This open subset will be the image in \( R/W \) of

\[
R^{sr-W} := \bigcup_{W \subseteq W' \text{ in corank } 1} R^{reg-W'}.
\]

It is clear that the complement of \( R^{sr-W} \) in \( R \) has codimension 2. Moreover, \( R^W \cap R^{sr-W} \) is an open subset of \( R^W \) whose complement has codimension at least 2. Indeed, we have

\[
R^W \cap R^{sr-W} = R^W \setminus \bigcup_{\Gamma_s \cap \Gamma_s' \neq \emptyset} \Gamma_s \cap \Gamma_s'.
\] (4.2)

The way to get a desired bimodule is as follows. First, for each parabolic subgroup \( W' \) containing \( W \) in corank 1, we will construct a HC \( H_{reg-W'} \)-bimodule with the property that its lift to \( R^{reg-W}/W \) coincides with \( B \). Then we will get our bimodule by, roughly speaking, gluing the bimodules defined over \( R^{reg-W'}/W' \).

**Proof of Theorem 4.3.1. Part 1: Constructing HC \( H_{reg-W'} \)-bimodules.** For each parabolic subgroup \( W' \) containing \( W \) in corank 1, let \( B_{W'} \) be a HC \( H^{\Lambda \mathcal{L}_{W'}}_{reg-W'} \)-bimodule that localizes to \( (\eta_{W*}(B^\Xi))_{\mathcal{L}_{W'}^{\infty}} \). Note that we may assume that \( B_{W'} \subseteq \eta_{W*}(B^\Xi)_{\mathcal{L}_{W'}^{\infty}} \), if this is not the case we can just replace \( B_{W'} \) by its quotient by the maximal sub-bimodule that is killed by the localization, we can find such a sub-bimodule because \( B_{W'} \) is a finitely generated bimodule over the noetherian algebra \( H^{\Lambda \mathcal{L}_{W'}}_{reg-W'} \).

On the other hand, let \( \eta_{W'} : R^{reg-W'}/W' \to R/W \) be the natural projection, and consider \( \eta_{W'}^*(\eta_{W*}(B^\Xi)) = C[R^{reg-W'}/W'] \otimes_{C[R/W]} B^\Xi \). The inclusion \( C[R^{reg-W'}/W'] \hookrightarrow C[\mathcal{L}_{W'}^{\infty}] \) induces a map \( \eta_{W',*}(B^\Xi) \to \eta_{W*}(B^\Xi)_{\mathcal{L}_{W'}^{\infty}} \) that we claim to be injective. Indeed, this follows because inside \( (R \oplus R^*)/W \) we have \( \mathcal{L}_{W'} \subseteq \mathcal{L}_{W} \) and the singular support of every finitely generated \( H \)-sub-bimodule of \( \eta_{W*}(B^\Xi) \) (which is the union of its HC sub-bimodules) contains \( \mathcal{L}_{W} \). The claim is now a consequence of the fact that \( \mathcal{L}_{W}^{\infty} \) is the minimal symplectic leaf inside \( T^*\mathcal{X}_W/W \), cf. Subsection 4.2.1.
Define \( \overline{B}_{W'} := (\eta_{W'}^*(\eta_{W'}(B^\Xi))) \cap B_{W'} \subseteq \eta_{W'}(B^\Xi) \). Note that this is an \( H_{\text{reg-}W'} \)-bimodule.

Let us see that it is finitely generated. By a suitable straightforward adaptation of Lemma 4.2.4, \( \eta_{W'}(B^\Xi) \) is finitely generated over the algebra \( \mathbb{C}[\mathcal{L}_{W'}^\times ][c] \otimes_{S(i)} \mathbb{C}[R^*/W][c]^{\text{opp}} \). Note that \( B_{W'} \) is a \( \mathbb{C}[\mathcal{L}_{W'}][c] \otimes_{S(i)} \mathbb{C}[R^*/W][c]^{\text{opp}} \)-lattice inside of \( \eta_{W'}(B^\Xi) \). So what we need to show is that \( \eta_{W'}(\eta_{W'}(B^\Xi)) \cap L \) is finitely generated over \( \mathbb{C}[R_{\text{reg-}W'}/W'][c] \otimes_{S(i)} \mathbb{C}[R^*/W][c]^{\text{opp}} \) for some lattice \( L \). We can produce such a lattice as follows. Again thanks to Lemma 4.2.4, \( B \) is finitely generated over \( \mathbb{C}[R_{\text{reg-}W'}/W'][c] \otimes_{S(i)} \mathbb{C}[R^*/W][c]^{\text{opp}} \), so we have an epimorphism \( \Upsilon : (\mathbb{C}[R_{\text{reg-}W'}/W'][c] \otimes_{S(i)} \mathbb{C}[R^*/W][c]^{\text{opp}})^\oplus_n \to B \), which in turn induces an epimorphism \( \overline{\Upsilon} : (\mathbb{C}[\mathcal{L}_{W'}][c] \otimes_{S(i)} \mathbb{C}[R^*/W][c]^{\text{opp}})^\oplus_n \to \eta_{W'}(B^\Xi) \). We take \( L \) to be the image of the restriction of \( \overline{\Upsilon} \) to \( (\mathbb{C}[\mathcal{L}_{W'}][c] \otimes_{S(i)} \mathbb{C}[R^*/W][c]^{\text{opp}})^\oplus_n \). This is clearly a lattice. Since \( \mathbb{C}[R_{\text{reg-}W'}/W'] \cap \mathbb{C}[\mathcal{L}_{W'}] = \mathbb{C}[R_{\text{reg-}W'}/W'] \) we have that \( L \cap \eta_{W'}^*(\eta_{W'}(B^\Xi)) \) coincides with the intersection of \( \eta_{W'}^*(\eta_{W'}(B^\Xi)) \) with the image of the restriction of \( \overline{\Upsilon} \) to \( (\mathbb{C}[R_{\text{reg-}W'}/W'][c] \otimes_{S(i)} \mathbb{C}[R^*/W][c]^{\text{opp}})^\oplus_n \). So \( L \cap \eta_{W'}^*(\eta_{W'}(B^\Xi)) \) is finitely generated over \( \mathbb{C}[R_{\text{reg-}W'}/W'][c] \otimes_{S(i)} \mathbb{C}[R^*/W][c]^{\text{opp}} \). Note that it follows that \( \overline{B}_{W'} \) is a \( \mathbb{H}_0 \)-bimodule with \( SS(\overline{B}_{W'}) = \mathcal{L}_{W'}^W \).

It remains to show that \( (\overline{B}_{W'})_{\text{reg-}W} = B \). Since \( B = \eta_{W'}(B^\Xi)_{\text{reg-}W} = (\eta_{W'}(B^\Xi)_{\text{reg-}W'})_{\text{reg-}W} \) it is enough to check that the lift of \( \overline{B}_{W'} \) to \( R_{\text{reg-}W'}/W' \) coincides with that of \( (\eta_{W'}(B^\Xi))_{\text{reg-}W'} \).

By definition, \( \overline{B}_{W'} \subseteq (\eta_{W'}(B^\Xi))_{\text{reg-}W'} \subseteq (\eta_{W'}(B^\Xi))_{\mathcal{L}_{W'}^\times} \). Now, for every \( b \in (\eta_{W'}(B^\Xi))_{\mathcal{L}_{W'}^\times} \), there exists \( f \in \mathbb{C}[\mathcal{L}_{W'}] \) vanishing on \( \mathcal{L}_{W'} \), with \( fb \in B_{W'} \). If, moreover, \( b \in (\eta_{W'}(B^\Xi))_{\text{reg-}W'} \), then \( f \in \mathbb{C}[R_{\text{reg-}W'}/W'] \) and \( fb \in \overline{B}_{W'} \). This implies the desired result.

**Part 2: Glueing.** First we will define a sheaf on \( R \), then we will take \( W \)-invariants to pass to \( R/W \). For each parabolic subgroup \( W' \) containing \( W \) in corank 1, let \( \pi_{W'} : R_{\text{reg-}W'} \to R_{\text{reg-}W'}/W' \) be the projection, and \( \iota_{W'} : R_{\text{reg-}W'} \to R \) the inclusion. So we can consider \( \iota_{W'} \pi_{W'}^* \overline{B}_{W'} \). We will take the intersection of these sheaves, so we need to find a sheaf containing all of them. Since \( R_{\text{reg-}W'} \subseteq R_{\text{sr-}W} \) for all \( W' \), this will be a sheaf defined on
So let $\pi : R \to R/W$ and $\iota : R_{sr-W} \to R$ be the natural projection and inclusion, respectively. By construction, viewing $\eta_{W*}(B^\Xi)$ as a quasicoherent sheaf on $R/W$, we may think of $\iota_{W*}\pi_{W*}^* B_{W'}$ as being contained inside of $\iota_*\pi_{W*}^*\eta_{W*}(B^\Xi)$. So the intersection

$$\tilde{B} := \bigcap_{W \subseteq W' \text{ in corank 1}} \iota_{W*}\pi_{W*}^* B_{W'}.$$ 

makes sense and is a sheaf on $R_{sr-W}$. 

Note that $W$ acts naturally on $\iota_*\pi_{W*}^*\eta_{W*}(B^\Xi)$. Now notice that, for a parabolic subgroup $W' \subseteq W$ and $w \in W$, we have a canonical, graded isomorphism $H_{reg-W'} \cong H_{reg-wW'w^{-1}}$. Indeed, recall that $H_{reg-W'}$ is the rational Cherednik algebra for the action of $W$ on $X_{W'}$, a disjoint union of Zariski open subsets of $R$, cf. Subsection 4.2.1. It is clear that $X_{W'} = X_{wW'w^{-1}}$ and the isomorphism between the algebras follows. So tracing back the construction, we see that we can pick our bimodules $B\hat{L}_{W'}$ in such a way that, for $w \in W$, $w(\iota_{W*}\pi_{W*}^* B_{W'}) = \iota_{wW'w^{-1}}\pi_{wW'w^{-1}}^* B_{wW'w^{-1}}$. So $\tilde{B}$ is $W$-stable. Finally, define

$$\hat{B} := (\pi_*\tilde{B})^W,$$

where $\pi : R \to R/W$ is the projection. We claim that $\hat{B}$ is stable under the bimodule action of $\mathcal{H}|_{\pi(R_{sr-W})}$. To see this first note that, by definition, $\hat{B} = \pi_*\tilde{B} \cap j_* B^\Xi$, where $j : R_{reg-W} \to R/W$ is the projection. Each one of the bimodules on the right-hand side of the previous equality is stable under the (left or right) action of $\mathcal{H}|_{\pi(R_{sr-W})}$. So $\hat{B}$ is also $\mathcal{H}|_{\pi(R_{sr-W})}$-stable.

Now set $\overline{B} := \Gamma(\tilde{B}(R_{sr-W}), \hat{B})$. We have that $\overline{B}$ is a $\mathcal{H}$-bimodule. We claim that it is HC. First of all, since $\hat{B} \subseteq j_* B^\Xi$, we have that $\overline{B}$ is $\mathbb{C}[h]$-flat. It is also clear that $\overline{B}/h\overline{B}$ is a module (rather than a bimodule) over $Z(\mathcal{H}/h\mathcal{H})$ and that $\overline{B}$ is graded. So, to finish the claim that $\overline{B}$ is HC, we need to show that it is finitely generated. We will show that, in fact, $\overline{B}/c\overline{B}$ is finitely generated over the algebra $\mathbb{C}[R/W] \otimes \mathbb{C}[R^*/W] \subseteq \mathbb{C}[(R \oplus R^*)/W]$. The following is an easy consequence of [Wi, Lemma 3.6].
Lemma 4.3.2. Let $X$ be an affine Noetherian scheme, and let $U \subseteq X$ be an open subset of $X$ whose complement has codimension at least 2. Let $M$ be a coherent sheaf on $U$, and assume that the support of any global section $m \in \Gamma(U, M)$ contains an irreducible component of $U$. Then, $\Gamma(U, M)$ is finitely generated over $\mathbb{C}[X]$.

Proof. By [Wi, Lemma 3.6], we get that $\Gamma(U, M)/I\Gamma(U, M)$ is finitely generated over the algebra $\mathbb{C}[X]/I$, where $I$ is the nilradical of $\mathbb{C}[X]$. The result follows.

Note that we can look at $\hat{\mathfrak{B}}/c\hat{\mathfrak{B}}$ as a coherent sheaf on an infinitesimal neighborhood $U$ of $\pi(R_{sr}^W \cap R^W) \times R^*/W$, this follows from our assumptions on the singular support of $\mathfrak{B}$, $\text{SS}(\mathfrak{B}) = \mathcal{L}^W_W$, the construction of $\hat{\mathfrak{B}} \subseteq \mathfrak{J}_c^*\mathfrak{B}^\mathbb{Z}$ and Lemma 4.2.5. This infinitesimal neighborhood may be regarded as an open set inside an infinitesimal neighborhood $X$ of $\pi(R_{sr}^W \cap R^W) \times R^*/W$. Now, it follows from Lemma 4.2.1 that the support of any global section $m \in \Gamma(U, \hat{\mathfrak{B}}/c\hat{\mathfrak{B}})$ contains $\pi(R_{reg}^W_W) \times W(R^*)^W/W$. Then, it follows from Lemma 4.3.2 that $\hat{\mathfrak{B}}/c\hat{\mathfrak{B}}$ is finitely generated over $\mathbb{C}[X]$. In particular, it is finitely generated over $\mathbb{C}[R/W] \otimes \mathbb{C}[R^*/W]$, this follows because the codimension of the complement of $R_{sr}^W$ in $R$ is 2. Then, $\overline{\mathfrak{B}}$ is a HC $H$-bimodule. By construction, $\overline{\mathfrak{B}}_{reg-W} = \mathfrak{B}$. This finishes the proof of Theorem 4.3.1. □

Let us remark one important feature of the bimodule $\overline{\mathfrak{B}}$ we have constructed: it has no sub-bimodules whose singular support is properly contained inside $\mathcal{L}^W_W$. Indeed, this follows from Corollary 4.2.3 and the fact that $\hat{\mathfrak{B}} \subseteq \mathfrak{B}^\mathbb{Z}$.

4.4 Specializing parameters.

Let $c, c': S \to \mathbb{C}$ be conjugation invariant functions. Recall that $R_h(H_c), R_h(H_{c'})$ are quotients of $H$ and that, if $B$ is a HC $H_c\cdot H_{c'}$-bimodule with a good filtration, then $R_h(B)$ is a HC $H$-bimodule. An analogous result holds for HC $(H_{c,reg-W}, H_{c',reg-W})$-bimodules. We remark that, since $R^*$ is in degree 0, the Rees construction commutes with localiza-
tion: for an affine, open subset $U \subseteq R/W$, $R_h(B)|_U = R_h(B|_U)$. We also remark that the Bezrukavnikov-Etingof isomorphisms hold in the specialized setting. So we can take a $\Xi$-equivariant HC $(H_{c,\text{reg-}W}, H_{c',\text{reg-}W})$-bimodule $B$ such that $SS(B) = L^W_W$ and for every parabolic subgroup $W'$ containing $W$ in corank 1, the $(H_c(W', R)\hat{\otimes}_W, H_{c'}(W', R)\hat{\otimes}_W)$-bimodule $B^{\Xi}_{\hat{\otimes}_{W'}}$ is the localization of a HC $H_c(W', R)\hat{\otimes}_W, H_{c'}(W', R)\hat{\otimes}_W$-bimodule. By Theorem 4.3.1, we can find a HC $H$-bimodule $\overline{B}$ that lifts to $R_h(B)$. Since $\overline{B}_{\hat{\otimes}_W}$ is a bimodule over $R_h(H_c(W', R)\hat{\otimes}_W)-R_h(H_{c'}(W', R)\hat{\otimes}_W)$, we see that the bimodule $\overline{B}$ factors through $R_h(H_c)-R_h(H_{c'})$, so $\overline{B}/(h - 1)\overline{B}$ is a HC $H_c-H_{c'}$-bimodule that lifts to $B$. We summarize this discussion in the following theorem, which is a specialized version of Theorem 4.3.1.

**Theorem 4.4.1.** Let $W$ be a parabolic subgroup of $W$, and let $B$ be a $\Xi$-equivariant HC $H_{c,\text{reg-}W}-H_{c',\text{reg-}W}$-bimodule, where $c, c' \in C[S]$ are conjugation invariant functions. Assume that $SS(B) = L^W_W$ and that for all minimal parabolic subgroups $W' \subseteq W$ containing $W$, the bimodule $(\eta_{W'}(B^{\Xi}))_{\hat{\otimes}_{W'}}$ is the localization of a HC $H_c(W', R)\hat{\otimes}_{W'}, H_{c'}(W', R)\hat{\otimes}_{W'}$-bimodule. Then, there exists a HC $H_c-H_{c'}$-bimodule $\overline{B}$ such that $\overline{B}_{\text{reg}} = B$.

**Proof of Theorem 4.1.1.** Recall the functor $\mathcal{G}$ defined in Section 3.4.1. Since $B$ is finite dimensional, $SS(\mathcal{G}(R_h(B))) = L^W_W$. Our assumptions on $B$ imply, since $\mathcal{G}$ is a fully faithful embedding, that $\mathcal{G}(R_h(B))$ satisfies the conditions of Theorem 4.4.1. So we can find a HC $H$-bimodule $\overline{B}$ with $\overline{B}_{\text{reg-}W} = \mathcal{G}(R_h(B))$. Note that since $SS(\mathcal{G}(R_h(B))) = L^W_W$, $(\overline{B}_{\text{reg-}W})^\mathcal{L} = \overline{B}_{\text{reg-}W}$. Thus, by the construction of the restriction functor, $\overline{B}_{\text{reg-}W} = R_h(B)$. It remains to put $\overline{B} := \overline{B}/(h - 1)\overline{B}$. □
Chapter 5

Bimodules with full support.

5.1 Introduction

In this chapter, we give a description of the category $\overline{HC}(H_c, H_{c'})$. Recall that this is the quotient of the category of all $HC (H_c, H_{c'})$-bimodules by the pull subcategory consisting of bimodules with proper support. Let us remark that, when $W$ is a Coxeter group and $c \in p_Z$ then, [BEG2], we have

$\overline{HC}(H_c, H_{c'}) \cong \begin{cases} W\text{-rep}, & c' \in p_Z \\ 0 & \text{else} \end{cases}$

Let us remark that, in this case, $\overline{HC} = HC$, and that the above equivalence is an equivalence of monoidal categories when $c = c'$. The construction of [BEG2] is based on the study of the module of $c$-quasi-invariants when $c$ is integral. This notion can be generalized to the case of an arbitrary complex reflection group, see [BC], but it only makes sense for parameters in $p_Z$. Here, we take a different approach and use bimodules of locally finite maps and the KZ functor. We generalize the result of [BEG2] by constructing a normal subgroup $W_c \subseteq W$ satisfying the following properties.

1. $W_c = \{1\}$ if and only if $\mathcal{H}_q = CW$. 

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2. $W_c = W$ for $c$ outside of a countable collection of hyperplanes.

3. $W_c = W_{c'}$ provided $\mathcal{H}_q = \mathcal{H}_{q'}$.

Let us remark that, if $W$ is a Coxeter group, then the condition $\mathcal{H}_q = \mathbb{C}W$ simply means that $c \in p_{\mathbb{Z}}$. For a more general complex reflection group, we have to take into consideration a certain symmetric group action, the Namikawa-Weyl group of $W$. We will elaborate on this on Section 5.3.

**Theorem 5.1.1.** For any $c \in p$, we have an equivalence of monoidal categories $\overline{\mathcal{H}C}(H_c, H_c) \cong (W/W_c)\text{-rep}$.

We also provide a description of the category $\overline{\mathcal{H}C}(H_c, H_{c'})$ when $c \neq c'$. In fact, in the case when this category is nonzero, we have $\overline{\mathcal{H}C}(H_{c'}, H_{c'}) \cong \overline{\mathcal{H}C}(H_c, H_{c'}) \cong \overline{\mathcal{H}C}(H_c, H_c)$. The Namikawa-Weyl group acts on the space of parameters, and we have the following result.

**Theorem 5.1.2.** The category $\overline{\mathcal{H}C}(H_c, H_{c'})$ is nonzero if and only if we can get from $c$ to $c'$ by integral translations and the action of elements of the Namikawa-Weyl group. Moreover, if this is the case, then the categories $\overline{\mathcal{H}C}(H_c, H_{c'})$, $\overline{\mathcal{H}C}(H_{c'}, H_c)$, $\overline{\mathcal{H}C}(H_c, H_c)$ and $\overline{\mathcal{H}C}(H_{c'}, H_{c'})$ are all equivalent.

This chapter is structured as follows. In Section 5.2, we study fully supported Harish-Chandra bimodules via the KZ functor. In particular, we obtain a description of irreducible HC bimodules with full support by means of bimodules of locally finite maps, and obtain some restrictions on when can this be nonzero. We remark that the main result of this section, Lemma 5.2.7, has already appeared in the literature, with basically the same proof, on a slightly less general version, see [Sp]. In Section 5.3 we construct an action of a product of symmetric groups on the space $p$ that has several good properties, in particular, it preserves categories of HC bimodules. To define this action we need the theory of finite $W$-algebras, and this section includes a few facts about it. In Section 5.3 we also prove Theorem 5.1.2. Finally, in Section 5.4 we construct the group $W_c$ and prove Theorem 5.1.1.
5.2 Localization of Harish-Chandra bimodules

5.2.1 Bimodules of differential maps

We are going to use localization to the regular locus to relate the \((H_c, H_c')\)-bimodules of locally finite maps with the \(D(R^{reg})/W\)-bimodules of differential maps. So let us recall some basic facts about the latter.

First, we recall Grothendieck's definition of differential operators. Let \(X\) be a smooth, affine algebraic variety. If \(M\) and \(N\) are \(\mathbb{C}[X]\)-modules, then the space of \(\mathbb{C}[X]\)-differential operators is a subspace of \(\text{Hom}_{\mathbb{C}}(M, N)\), defined via an increasing filtration \(\text{Diff}(M, N) = \bigcup_{n \geq 0} \text{Diff}(M, N)_n\), where the components \(\text{Diff}(M, N)_n\) are inductively defined as follows:

\[
\text{Diff}(M, N)_{-1} := 0, \\
\text{Diff}(M, N)_{n+1} := \{ f \in \text{Hom}_{\mathbb{C}}(M, N) : [a, f] \in \text{Diff}(M, N)_n \text{ for all } a \in \mathbb{C}[X] \}.
\]

If \(M, N\) are \(D(X)\)-modules, then \(\text{Diff}(M, N)\) is a \(D(X)\)-bimodule. We remark that if \(N\) is a local system then we have a \(D(X)\)-bimodule isomorphism, \(N \otimes_{\mathbb{C}[X]} D(X) \cong \text{Diff}(\mathbb{C}[X], N)\), where \(\mathbb{C}[X]\) is equipped with the trivial flat connection and the flat connection on \(N \otimes_{\mathbb{C}[X]} D(X)\) is as in \([HTT, \text{Proposition 1.2.9}]\). An explicit isomorphism is given by \(n \otimes_{\mathbb{C}[X]} d \mapsto (f \mapsto d(f)n)\). Note that this implies that \(\text{Diff}(\mathbb{C}[X], N)\) is finitely generated both as a right and as a left \(D(X)\)-module whenever \(N\) is a local system. As a right \(D\)-module, an explicit set of generators is \(n_1 \otimes_{\mathbb{C}[X]} 1, \ldots, n_i \otimes_{\mathbb{C}[X]} 1\), where \(n_1, \ldots, n_i\) are generators of the \(\mathbb{C}[X]\)-module \(N\). This set also generates \(N \otimes_{\mathbb{C}[X]} D(X)\) as a left \(D(X)\)-module. The following lemma describes a basic property of the bimodules of the form \(\text{Diff}(\mathbb{C}[X], N)\) whenever \(N\) is a local system.

**Lemma 5.2.1.** Let \(N\) be an irreducible local system on the smooth, affine algebraic variety \(X\). Then, the \(D(X)\)-bimodule \(\text{Diff}(\mathbb{C}[X], N)\) is irreducible.
5.2.2 Harish-Chandra bimodules and bimodules of differential maps

Recall that, if $B$ is an irreducible HC $(H_c, H_{c'})$-bimodule, then there exist irreducible modules $M \in \mathcal{O}_{c'}, N \in \mathcal{O}_c$, such that $B$ can be embedded into $\text{Hom}_{\text{fin}}(M, N)$. Moreover, the supports of $M$ and $N$ coincide and, if $\text{supp}(M) = \text{supp}(N) = \overline{X_{W'}}$ for some parabolic subgroup $W' \subseteq W$ then $\text{SS}(B) = L_{W'}$. When $\text{SS}(B) = (\mathbb{R} \oplus \mathbb{R}^*)/W = L_{\{1\}}$, we see that the choice of $M$ can be arbitrary.

**Lemma 5.2.2.** Let $B \in \text{HC}(H_c, H_{c'})$ be an irreducible bimodule. Assume that $\text{SS}(B) = (\mathbb{R} \oplus \mathbb{R}^*)/W$. Let $M \in \mathcal{O}_{c'}$ be a (not necessarily irreducible) module with full support. Then, there exists $N \in \mathcal{O}_c$ irreducible with full support, such that $B$ can be embedded into $\text{Hom}_{\text{fin}}(M, N)$.

**Proof.** It is enough to check that $B \otimes_{H_{c'}} M \neq 0$. We have that $\text{Res}_{\{1\}}^W (B \otimes_{H_{c'}} M) = B_{\{1\}} \otimes_{\mathbb{C}} \text{Res}_{\{1\}}^W (M) \neq 0$, since the rational Cherednik algebra of the group $\{1\}$ is simply $\mathbb{C}$. We are done. \hfill \Box

In particular, for $M$ we can take the polynomial representation $\Delta_c'(\text{triv})$. We remark that $\Delta_c'(\text{triv})$ has an irreducible socle, say $S_c'(W)$, and this is the unique subquotient of $\Delta_c'(\text{triv})$ that has full support. These claims follow from the fact that $\text{KZ}(\Delta_c'(\text{triv})) = \mathbb{C}$, the trivial representation of the Hecke algebra $H_{q'}$. In particular, $e\Delta_c'(\text{triv})[\delta^{-1}] = \mathbb{C}[R^{\text{reg}}/W]$ is the trivial local system where, recall, $\delta := \prod_{s \in S} \alpha_s$ and $e \in CW$ is the trivial idempotent.

Note that for an irreducible module $N \in \mathcal{O}_c$, we have that $e \text{Hom}_{\text{fin}}(\Delta_c'(\text{triv}), N)[\delta^{-1}]e$ is a $D(R^{\text{reg}}/W)$-bimodule. We claim that this bimodule is isomorphic to $\text{Diff}(\mathbb{C}[R^{\text{reg}}/W], eN[\delta^{-1}])$ whenever the former bimodule is nonzero. This claim follows from Lemma 5.2.1 and the following result.

**Lemma 5.2.3.** Let $N \in \mathcal{O}_c$ be an irreducible module with full support. For any standard module $\Delta_c'(\tau)$, the bimodule $e \text{Hom}_{\text{fin}}(\Delta_c'(\tau), N)[\delta^{-1}]e$ is isomorphic to a sub-bimodule of $\text{Diff}(e\Delta_c'(\tau)[\delta^{-1}], eN[\delta^{-1}])$. 66
Proof. Set $M := \Delta_c(\tau)$. Let $f \in \text{Hom}_{\text{fin}}(M, N)$. Since for some $m$, $\delta^m$ is $W$-invariant, we have that $(\text{ad}(e\delta^m))^k f = 0$, so for every $x \in M$,

$$
\sum_{j=0}^{k} (-1)^j \binom{k}{j} (e\delta^m)^{(k-j)} f((e\delta^m)^j x) = 0.
$$

Then, since $M$ is free as a $\mathbb{C}[R]$-module we can extend $f$ to $eM[\delta^{-1}]$ by

$$
f(\delta^{-m} x) = -(\delta^m)^{-k} \sum_{j=1}^{k} (-1)^j \binom{k}{j} (e\delta^m)^{(k-j)} f((e\delta^m)^{(j-1)} x).
$$

To see that this actually defines an inclusion, assume that $f \neq 0$. Then, $f(x) \neq 0$ for some element $x \in M$. Since $N$ is torsion-free (see e.g. [GGOR, Proposition 5.21]), the element $f(x)$ is not a zero divisor. This implies that the image of $f$ in $\text{Diff}(e\Delta_c(\tau)[\delta^{-1}], eN[\delta^{-1}])$ is nonzero.

**Corollary 5.2.4.** Let $N \in \mathcal{O}_c$ be an irreducible bimodule with full support. Assume that $\text{Hom}_{\text{fin}}(\Delta_c(\text{triv}), N) \neq 0$. Then, $e \text{Hom}_{\text{fin}}(\Delta_c(\text{triv}), N)[\delta^{-1}]e$ is the bimodule of differential maps $\text{Diff}(\mathbb{C}[R^{\text{reg}}/W], eN[\delta^{-1}])$.

Using the ideas in the proof of Lemma 5.2.3, together with the fact that the regular bimodule $H_c$ is injective in the category of HC bimodules, cf. Proposition 3.4.7, we prove the following result.

**Corollary 5.2.5.** For any $c \in \mathfrak{p}$, we have an isomorphism of the regular bimodule $H_c$ with the space of locally finite maps $\text{Hom}_{\text{fin}}(\Delta_c(\text{triv}), \Delta_c(\text{triv}))$.

Proof. Reasoning as in the proof of Lemma 5.2.3, we have that $\text{Hom}_{\text{fin}}(\Delta_c(\text{triv}), \Delta_c(\text{triv}))$ is a HC $H_c$-bimodule whose localization to $R^{\text{reg}}$ is an irreducible $\mathcal{D}(R^{\text{reg}})\#W$-bimodule. So $\text{Hom}_{\text{fin}}(\Delta_c(\text{triv}), \Delta_c(\text{triv}))$ contains a unique irreducible bimodule with full support. It is easy to see that any subbimodule of $\text{Hom}_{\text{fin}}(\Delta_c(\text{triv}), \Delta_c(\text{triv}))$ has full support, so this bimodule has an irreducible socle. In particular, it is indecomposable.

On the other hand, we have a natural map $H_c \rightarrow \text{Hom}_{\text{fin}}(\Delta_c(\text{triv}), \Delta_c(\text{triv}))$, $x \mapsto (m \mapsto$
Since the representation $\Delta_c(triv)$ is faithful, this is an inclusion. Then, by Proposition 3.4.7, we have that $H_c$ must be isomorphic to a direct summand of $\text{Hom}_{\text{fin}}(\Delta_c(triv),\Delta_c(triv))$. By the previous paragraph, we must have $H_c \cong \text{Hom}_{\text{fin}}(\Delta_c(triv),\Delta_c(triv))$. 

**Remark 5.2.6.** We remark that the isomorphism in Corollary 5.2.5 is also an algebra isomorphism with respect to the composition structure on $\text{Hom}_{\text{fin}}(\Delta_c(triv),\Delta_c(triv))$. This generalizes [BEG, Proposition 8.10 (i)].

### 5.2.3 Harish-Chandra bimodules and the KZ functor.

Since for any HC $(H_c,H_{c'})$-bimodule $B$ and any module $M \in \mathcal{O}_{c'}$, the module $B \otimes_{H_{c'}} M$ is a module in category $\mathcal{O}_c$, it makes sense to ask what is the image of a module of the form $B \otimes_{H_{c'}} M$ under the KZ functor. In this subsection, we answer this question when $B$ has the form $\text{Hom}_{\text{fin}}(\Delta_{c'}(triv),M)$ for an irreducible module with full support $M \in \mathcal{O}_c$. Namely, we have the following result.

**Lemma 5.2.7.** Let $c,c' \in \mathfrak{p}_\mathbb{Z}$ be parameters and consider the rational Cherednik algebras $H_c,H_{c'}$. Let $q,q'$ be the associated sets of parameters for the Hecke algebras $\mathcal{H}_q,\mathcal{H}_{q'}$, so that we have $\text{KZ}_c: \mathcal{O}_c \to \mathcal{H}_q$-mod, $\text{KZ}_{c'}: \mathcal{O}_{c'} \to \mathcal{H}_{q'}$-mod. Let $M \in \mathcal{O}_c$ be an irreducible module with full support. Assume that $\text{Hom}_{\text{fin}}(\Delta_{c'}(triv),M) \neq 0$. Then, for every finite dimensional module $N \in \mathcal{H}_{q'}$-mod, the $\pi_1(\mathcal{R}^{reg}/\mathcal{W})$-module $\text{KZ}_c(M) \otimes_{\mathcal{C}} N$ factors through $\mathcal{H}_q$.

**Proof.** We show that for every $\tilde{N} \in \mathcal{O}_{c'}$:

$$\text{KZ}_c(\text{Hom}_{\text{fin}}(\Delta_{c'}(triv),M) \otimes_{H_{c'}} \tilde{N}) = \text{KZ}_c(M) \otimes_{\mathcal{C}} \text{KZ}_{c'}(\tilde{N}).$$

Since $\mathcal{H}_{q'}$-mod is a quotient of $\mathcal{O}_{c'}$ via the KZ functor, this implies the result. Now, by Lemma 5.2.3 the localization to $\mathcal{R}^{reg}/\mathcal{W}$ (this means, first localize to $\mathcal{R}^{reg}$ and then take $\mathcal{W}$-invariants) of $\text{Hom}_{\text{fin}}(\Delta_{c'}(triv),M) \otimes_{H_{c'}} N$ is $\text{Diff}(\mathbb{C}[\mathcal{R}^{reg}/\mathcal{W}],eM[\delta^{-1}]) \otimes_{D(\mathcal{R}^{reg}/\mathcal{W})} eN[\delta^{-1}]$. Since $eM[\delta^{-1}]$ is a local system on $\mathcal{R}^{reg}/\mathcal{W}$, $\text{Diff}(\mathbb{C}[\mathcal{R}^{reg}/\mathcal{W}],eM[\delta^{-1}]) \cong eM[\delta^{-1}] \otimes_{\mathbb{C}} [\mathcal{R}^{reg}/\mathcal{W}]$. 

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\( \mathcal{D}(R^{reg}/W) \). Then,

\[
e(\text{Hom}_{\text{fin}}(\Delta_{c'}(\text{triv}), M) \otimes_{H_c} \tilde{N})[\delta^{-1}] = eM[\delta^{-1}] \otimes_{\mathbb{C}[R^{reg}/W]} D(R^{reg}/W) \otimes_{D(R^{reg}/W)} e\tilde{N}[\delta^{-1}] = eM[\delta^{-1}] \otimes_{\mathbb{C}[R^{reg}/W]} e\tilde{N}[\delta^{-1}].
\]

By [HTT, Proposition 4.7.8], the flat sections \( (eM[\delta^{-1}] \otimes_{\mathbb{C}[R^{reg}/W]} e\tilde{N}[\delta^{-1}])^\nabla \) coincide with \( (eM[\delta^{-1}])^\nabla \otimes_{\mathbb{C}} (e\tilde{N}[\delta^{-1}])^\nabla \), with diagonal action of the braid group \( \pi_1(R^{reg}/W) \). The lemma is proved. \( \square \)

**Remark 5.2.8.** Recall that \( \Delta_{c'}(\text{triv}) \) has an irreducible socle, which we denote by \( S_{c'}(W) \). Moreover \( \text{KZ}(S_{c'}(W)) = \text{KZ}(\Delta_{c'}(\text{triv})) \). Then, Lemma 5.2.7 holds, with the same proof, if we substitute \( \Delta_{c'}(\text{triv}) \) by \( S_{c'}(W) \). We will mostly use this form of the lemma.

Lemma 5.2.7 gives necessary conditions on a module \( N \) (or, rather, on \( \text{KZ}_{c}(N) \)) for the existence of a nonzero locally finite map in \( \text{Hom}_{\text{fin}}(\Delta_{c'}(\text{triv}), N) \). We will see that these conditions are also sufficient. The proof of this is based on the shift bimodules of Section 3.1.1 and the action of a certain product of symmetric groups on the space of parameters \( p \).

In the next section, we introduce this action.

### 5.3 The Namikawa-Weyl group

#### 5.3.1 A reparametrization of \( H_c \)

In the sequel, a reparametrization of the rational Cherednik algebra \( H_c \) by parameters which are more amenabe with the KZ functor will be convenient. Let us, first, recall some notation. By \( \mathcal{A} \) we denote the set of reflection hyperplanes in \( R \), and for each reflection hyperplane \( \Gamma \) we denote by \( W_{\Gamma} \) its stablizer, this is a cyclic group of order \( \ell_{\Gamma} \). Let \( s_{\Gamma} \in W_{\Gamma} \) be a generator with \( \det(s_{\Gamma}|_{R}) = \exp(2\pi \sqrt{-1}/\ell_{\Gamma}) =: \eta_{\Gamma} \). We may and will assume that \( \alpha_s = \alpha_{s_{\Gamma}}, \alpha^\vee_s = \alpha^\vee_{s_{\Gamma}} \) for every \( s \in W_{\Gamma} \cap S \). We will denote these elements by \( \alpha_{\Gamma} \in R^*, \alpha^\vee_{\Gamma} \in R \), respectively.
Now, for each $i = 0, \ldots, \ell - 1$ we have the idempotent
\[ e_{i,\Gamma} := \frac{1}{\ell} \sum_{j=0}^{\ell-1} \eta^{-ij}_{\Gamma} s_j^i \in \mathbb{C}W_{\Gamma} \]
so that, for example, $e_{0,\Gamma} \in \mathbb{C}W_{\Gamma}$ is the trivial idempotent. For each reflection hyperplane $\Gamma \in \mathcal{A}$, pick a collection of numbers $k_{\Gamma,0}, \ldots, k_{\Gamma,\ell - 1}$ such that $k_{\Gamma,i} = k_{\Gamma',i}$ for every $i = 0, \ldots, \ell - 1 = \ell' - 1$ if $\Gamma, \Gamma'$ are in the same $W$-orbit. Then define the algebra $H_k$ by generators and relations similar to (2.1), with the last relation replaced by
\[ [y, x] = \langle y, x \rangle - \frac{1}{2} \sum_{\Gamma \in \mathcal{A}} \langle y, \alpha_{\Gamma} \rangle \langle \alpha_{\Gamma}^\vee, x \rangle \sum_{i=0}^{\ell-1} (k_{\Gamma,i} - k_{\Gamma',i}) e_{i,\Gamma} \] (5.1)
where $k_{\Gamma,-1} := k_{\Gamma,\ell-1}$. We remark that $H_k = H_c$, where the parameter $c : \mathcal{S} \to \mathbb{C}$ is recovered from $k$ as follows. For each reflection $s \in \mathcal{S}$, let $\Gamma_s$ be the reflection hyperplane of $s$, that is, $\Gamma_s = \ker(id_R - s)$. Then,
\[ c(s) = \frac{1}{2\ell_{\Gamma_s}} \sum_{j=0}^{\ell_{\Gamma_s}-1} (k_{\Gamma_s,j} - k_{\Gamma_s,j-1}) \lambda_s^{-j} \] (5.2)
Note that the $k_{\Gamma,i}$ are only defined up to a common summand. In this work, we will always assume that $k_{\Gamma,0} = 0$. In this case, we can recover the parameter $k$ from the parameter $c$ via (2.2), that is,
\[ k_{\Gamma,i} = \sum_{s \in \mathcal{S} \cap W_{\Gamma}} \frac{2c(s)}{1 - \lambda_s} (\lambda_s^{-i} - 1), \quad i = 0, \ldots, \ell - 1. \]
We will still denote the space of $k$-parameters by $p$. In particular, we have the notion of integral parameters: a parameter $k$ is integral if the corresponding $c$-parameter is, or equivalently, if $k_{\Gamma,i}/\ell \in \mathbb{Z}$ for every $\Gamma \in \mathcal{A}$, $i = 0, \ldots, \ell - 1$, cf. Section 2.3.5.

**Remark 5.3.1.** Note that for $k, k' \in p$ we have $k - k' \in p\mathbb{Z}$ if and only if, for each $\Gamma \in \mathcal{A}$ and $i = 0, \ldots, \ell - 1$, we have
\[ q(k)_{\Gamma,i} = \exp \left( \frac{2\pi \sqrt{-1} (k_{\Gamma,i} - i)}{\ell} \right) = \exp \left( \frac{2\pi \sqrt{-1} (k'_{\Gamma,i} - i)}{\ell} \right) = q(k')_{\Gamma,i} \]
Below, we will extensively use this without further mention.
For the rest of this section, we will work with the $k$-parameters. The reason for this will become clear in Section 5.3.2, cf. Theorem 5.3.2.

### 5.3.2 Finite W-algebras and rational Cherednik algebras of cyclic groups

Consider the cyclic group $W = \mathbb{Z}/\ell\mathbb{Z}$ with generator $s$ acting on $R = \mathbb{C}$ by multiplication by $\exp(2\pi \sqrt{-1}/\ell)$. For each collection of complex numbers $k_0 = 0, k_1, \ldots, k_{\ell-1}$ we have the rational Cherednik algebra $H_k$, and its spherical subalgebra $A_k = e_0 H_k e_0$.

It turns out that the spherical subalgebra $A_k$ is isomorphic to a finite $W$-algebra. We are not going to give a full definition of a finite $W$-algebra here, we are simply going to state its main properties that will be important for us. For more on $W$-algebras, the reader can consult the surveys [L, Wa].

A finite $W$-algebra is a filtered associative algebra $\mathcal{U}(\mathfrak{g}, e)$ associated to a complex semisimple Lie algebra $\mathfrak{g}$ and a nilpotent element $e \in \mathfrak{g}$. When $e = 0$, then $\mathcal{U}(\mathfrak{g}, e) = \mathcal{U}(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$. On the other hand, when $e \in \mathfrak{g}$ is a regular nilpotent, $\mathcal{U}(\mathfrak{g}, e) = \mathfrak{z}$, the center of $\mathcal{U}(\mathfrak{g})$ which, by the Harish-Chandra isomorphism, is isomorphic to the algebra of $W(\mathfrak{g})$-invariant functions on the Cartan subalgebra, $\mathbb{C}[h]^{W(\mathfrak{g})}$, where $W(\mathfrak{g})$ denotes the Weyl group of $\mathfrak{g}$. In general, $\mathcal{U}(\mathfrak{g}, e)$ falls between these cases: it is an associative algebra whose center coincides with $\mathfrak{z}$. In particular, for $\lambda \in \mathfrak{h}$ we can consider the central reduction $\mathcal{U}(\mathfrak{g}, e)_\lambda = \mathcal{U}(\mathfrak{g}, e)/I_\lambda \mathcal{U}(\mathfrak{g}, e)$, where $I_\lambda \subseteq \mathfrak{z} = \mathbb{C}[h]^{W(\mathfrak{g})}$ is the ideal of all functions vanishing on $[\lambda] \in \mathfrak{h}/W(\mathfrak{g})$. Obviously, $\mathcal{U}(\mathfrak{g}, e)_\lambda = \mathcal{U}(\mathfrak{g}, e)_{w(\lambda)}$ for every $w \in W(\mathfrak{g})$.

Now let $\mathfrak{g} = \mathfrak{sl}_\ell$, the Lie algebra of traceless $(\ell \times \ell)$-matrices. Let $e := e_{(\ell-1,1)} \in \mathfrak{sl}_\ell$ be the subregular nipotent, that is, the nilpotent element that has a Jordan block of size $(\ell-1)$ and a Jordan block of size 1. Identify the Cartan subalgebra $\mathfrak{h}$ with $\{(x_1, \ldots, x_\ell) \in \mathbb{C}^\ell : \sum x_i = 0\}$.

Let $\alpha_i := \epsilon_i - \epsilon_{i+1}$, $i = 1, \ldots, \ell - 1$ be the simple roots, where $\epsilon_i$ is the $i$-th coordinate function, and let $\pi_1, \ldots, \pi_{\ell-1}$ be the corresponding fundamental weights.
Theorem 5.3.2 ([L2]). The algebra $A_k$ is isomorphic, as a filtered algebra, to the algebra $U(\mathfrak{sl}_\ell, e_{(\ell-1,1)})_\lambda$ where

$$\lambda = \sum_{i=1}^{\ell-1} \lambda_i \pi_i, \quad \lambda_i = \frac{1}{\ell} (1 - k_i + k_{i-1})$$

(5.3)

We denote this isomorphism by $\varphi_k : A_k \rightarrow U(\mathfrak{sl}_\ell, e_{(\ell-1,1)})_\lambda$.

A consequence of Theorem 5.3.2 is that the symmetric group $S_\ell$ acts on the space of parameters $k$, in such a way that there is a filtered isomorphism between $A_k$ and $A_{\sigma(k)}$ for every $\sigma \in S_\ell$. Indeed, we can just define $\sigma(k)$ to be the parameter associated to $\sigma(\lambda)$ and we have the filtered isomorphism

$$A_k \xrightarrow{\varphi_k} U(\mathfrak{sl}_\ell, e_{(\ell-1,1)})_\lambda \xrightarrow{\varphi_{\sigma(k)}^{-1}} U(\mathfrak{sl}_\ell, e_{(\ell-1,1)})_{\sigma(\lambda)} \xrightarrow{\varphi_{\sigma(k)}} A_{\sigma(k)}$$

Since these are filtered isomorphisms, an easy consequence of the definitions is the following result.

Lemma 5.3.3. Let $W = \mathbb{Z}/\ell\mathbb{Z}$ be a cyclic group, with reflection representation $R$. Then, for every $\sigma \in S_\ell$ transfer of the structure gives equivalences of categories

$$HC(A_k, A_k) \cong HC(A_k, A_{\sigma(k)}) \cong HC(A_{\sigma(k)}, A_{\sigma(k)}) \cong HC(A_{\sigma(k)}, A_k)$$

which preserve the support of a bimodule.

Example 5.3.4. Let us consider the group $W = \mathbb{Z}/2\mathbb{Z}$. In this case, we may think of the parameter $k$ as a single complex number. Then, the associated parameter for the finite $W$-algebra is $(1 - k)\rho/2$, where $\rho = \alpha/2$ and $\alpha$ is the positive root of the root system $A_1$. Let $\sigma \in \text{Nam}$ be the unique non-trivial element. Then, we have $\sigma(k) = -k + 2$. Since in this case $k = -2c$, in terms of the $c$-parameter we have $\sigma(c) = -c - 1$.

Note that, in general, the equivalences in Lemma 5.3.3 do not lift to equivalences between categories of $HC H_k$-bimodules. They do, however, if we only consider the categories of bimodules with full support. In this case, we have an equivalence.
\[
\overline{\mathbb{C}}(H_k, H_k) \cong \overline{\mathbb{C}}(H_k, H_{\sigma(k)}) \cong \overline{\mathbb{C}}(H_{\sigma(k)}, H_{\sigma(k)}) \cong \overline{\mathbb{C}}(H_{\sigma(k)}, H_k) \tag{5.4}
\]

We would like to have similar equivalences for every complex reflection group, not just cyclic groups. This is what we are going to achieve next. So, first, we need to find a proper substitute for the group \(S_\ell\).

### 5.3.3 The Namikawa-Weyl group

Let \(W\) be a complex reflection group, \(R\) its reflection representation, \(A\) the set of reflection hyperplanes. Remember that for each reflection hyperplane \(\Gamma \in A\), its stabilizer is a cyclic group, of order say \(\ell_\Gamma\).

**Definition 5.3.5.** The Namikawa-Weyl group of \(W\) is the group

\[
\text{Nam} := \text{Nam}(W) := \prod_{\Gamma \in A/W} S_{\ell_\Gamma}
\]

Let us remark that \(\text{Nam}(W)\) acts on the space of parameters \(k\) for the rational Cherednik algebra \(H_k(W)\). Indeed, for \(\Gamma \in A\), the factor \(S_{\ell_\Gamma}\) acts on the subset \(\{k_{\Gamma,0}, \ldots, k_{\Gamma,\ell_\Gamma-1}\}\) as in the previous subsection, and leaves \(\{k_{\Gamma',0}, \ldots, k_{\Gamma',\ell_{\Gamma'}-1}\}\) fixed if \(\Gamma\) and \(\Gamma'\) are not in the same \(W\)-orbit.

**Example 5.3.6.** Let \(W\) be the Weyl group of an irreducible root system. Then, \(\text{Nam} = \mathbb{Z}/2\mathbb{Z}\), if the root system is simply-laced, or \(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\), if the root system is of type \(B_n\), \(F_4\) or \(G_2\). The action of \(\text{Nam}\) on the space of parameters is given by Example 5.3.4.

**Remark 5.3.7.** When \(W = S_n\) or, more generally, when \(W\) is the wreath-product \(S_n \ltimes (\mathbb{Z}/\ell\mathbb{Z})^n\), the variety \((R \oplus R^*)/W\) admits a symplectic resolution and \(\text{Nam}\) coincides with the Namikawa-Weyl group of the symplectic resolution, see [BPW, Section 2.2]. For general \(W\), the variety \((R \oplus R^*)/W\) does not admit a symplectic resolution. It does, however, admit a \(\mathbb{Q}\)-factorial terminalization, the notion of the Namikawa-Weyl group can still be extended to this setting, and this notion still coincides with Definition 5.3.5, see [L10, Section 2.3].
5.3.4 Equivalences from the Namikawa-Weyl group

We now generalize (5.4) for every complex reflection group, where the role of the symmetric group is now played by the Namikawa-Weyl group Nam(W). We start with the following preparatory lemma.

**Lemma 5.3.8.** Let $W$ be a complex reflection group, Nam := Nam(W) its Namikawa-Weyl group, and $k$ a parameter for the Cherednik algebra. Then, for every $\sigma \in$ Nam, the category $\text{HC}(H_k, H_\sigma)$ is nonzero.

**Proof.** We use Theorem 4.1.1, with $W = \{1\}$, so $\Xi = W$ and

$$\text{HC}_0(W, H_\sigma(W)) = W\text{-rep}$$

Let $\Gamma \in A$ be a reflection hyperplane and $W_\Gamma$ its setwise stabilizer. By (5.4), the category $\text{HC}(H_k(W_\Gamma, R_{W_\Gamma}), H_\sigma(W_\Gamma, R_{W_\Gamma}))$ is nonzero. So we can find an irreducible (= 1-dimensional) representation $\tau_\Gamma$ of $W_\Gamma$ such that $\text{Hom}_{\text{fin}}(S_\sigma(W_\Gamma), \Delta_k(\tau_\Gamma))$ is nonzero where, recall, $S_\sigma(W_\Gamma)$ denotes the socle of the polynomial representation $\Delta_\sigma(\text{triv}_{W_\Gamma})$. Note that, by the $W$-invariance of the parameter $k$, we may and will assume that $\tau_\Gamma = \tau_{\Gamma'}$ if $\Gamma, \Gamma'$ are in the same $W$-orbit.

Now, we have that $\text{Hom}(W, \mathbb{C}^\times) \cong \prod_{\Gamma \in A/W} \text{Hom}(W_\Gamma, \mathbb{C}^\times)$, cf. [R]. So our choice of 1-dimensional representations $\{\tau_\Gamma : \Gamma \in A\}$ determines a 1-dimensional representation $\tau$ of $W$. We claim that $\text{Hom}_{\text{fin}}(S_\sigma(W), \Delta_k(\tau)) \neq 0$. To see this, we will use Theorem 4.1.1. Assume for the moment that for every reflection hyperplane $\Gamma \in A$, the bimodule of locally finite maps $\text{Hom}_{\text{fin}}(S_\sigma(W_\Gamma), \text{Res}_{W_\Gamma}(\Delta_k(\tau)))$ is nonzero. Then, the restriction $\text{Hom}_{\text{fin}}(S_\sigma(W_\Gamma), \text{Res}_{W_\Gamma}(\Delta_k(\tau)))_{W_\Gamma} \mid_{\{1\}}$ is a nonzero sub-bimodule of the 1-dimensional bimodule $\text{Hom}_{\mathbb{C}}(\mathbb{C}, \text{KZ}_k(\Delta_k(\tau)))$, so the conditions of Theorem 4.1.1 are satisfied and we get a HC $(H_k, H_\sigma)$-bimodule $B$ that localizes to $\text{Hom}_{\mathbb{C}}(\mathbb{C}, \text{KZ}_k(\Delta_k(\tau)))$. Using Lemma 5.2.7 it is easy to see that, up to subquotients with proper support, $B = \text{Hom}_{\text{fin}}(S_\sigma(W), \Delta_e(\tau))$. 

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So what we need to show now is that, for every $\Gamma \in \mathcal{A}$, $\text{Hom}_{\text{fin}}(S_{\sigma(k)}(W_{\Gamma}), \text{Res}_{W_{\Gamma}}^{W} \Delta_{k}(\tau))$ is nonzero. But this follows immediately from Lemma 2.4.6.

We remark that, with the same technique as in the proof of Lemma 5.3.8, we can show the following result.

**Lemma 5.3.9.** Assume that the category $\overline{\text{HC}}(H_k, H_{k'})$ is nonzero. Then, there exists a 1-dimensional character $\tau$ of $W$ such that $\text{Hom}_{\text{fin}}(S_{k'}(W), \Delta_{k}(\tau)) \neq 0$ where, recall $S_{k'}(W) \in \mathcal{O}_{k'}$ is the irreducible module that gets sent to the trivial representation under the KZ functor.

**Proof.** Assume that $\overline{\text{HC}}(H_k, H_{k'}) \neq 0$. Then, there exists an irreducible module $N \in \mathcal{O}_k$ with full support such that $\text{Hom}_{\text{fin}}(S_{k'}(W), N) \neq 0$. Now, for each reflection hyperplane $\Gamma \in \mathcal{A}$, consider the pointwise stabilizer $W_{\Gamma} \subseteq W$. This is a cyclic group. Note that, since $\text{Hom}_{\text{fin}}(S_k(W), N) \neq 0$, we have that $\text{Hom}_{\text{fin}}(\text{Res}_{W_{\Gamma}}^{W}(S_k(W)), \text{Res}_{W_{\Gamma}}^{W}(N)) \neq 0$, this follows from Lemma 3.3.2 (4). Since KZ commutes with restriction, we have that $S_{k'}(W_{\Gamma})$ is the unique subquotient of $\text{Res}_{W_{\Gamma}}^{W}(S_k(W))$ with full support, which implies that $\text{Hom}_{\text{fin}}(S_{k'}(W_{\Gamma}), \text{Res}_{W_{\Gamma}}^{W}(N)) \neq 0$. Now, in category $\mathcal{O}$ for the rational Cherednik algebra of $W_{\Gamma}$, for every irreducible representation (= 1-dimensional character) $\tau$ of $W_{\Gamma}$, we have that either $L_{k}(\tau) = \Delta_{k}(\tau)$, or $L_{k}(\tau)$ is finite dimensional. Since $\text{Res}_{W_{\Gamma}}^{W}(N)$ has full support, we conclude that there exists an irreducible representation $\tau_{\Gamma}$ of $W_{\Gamma}$ with $\text{Hom}_{\text{fin}}(S_{k'}(W_{\Gamma}), \Delta_{k}(\tau_{\Gamma})) \neq 0$. We remark that we can take $\tau_{\Gamma} = \tau_{\Gamma'}$ if $\Gamma, \Gamma' \in \mathcal{A}$ are conjugate, this follows from the conjugation invariance of $k$. Now proceed as in the proof of Lemma 5.3.8. \hfill \Box

**Corollary 5.3.10.** Assume that $\overline{\text{HC}}(H_k, H_{k'}) \neq 0$. Then, the categories $\overline{\text{HC}}(H_k, H_{k'})$ and $\overline{\text{HC}}(H_{k'}, H_{k})$ are equivalent. Moreover, they are equivalent to the category of representations of $W/W'$ for some normal subgroup $W'$ of $W$.

**Proof.** Let $B$ be a HC $(H_k, H_{k'})$-bimodule with full support. By the previous lemma, we may assume that $B = \text{Hom}_{\text{fin}}(S_{k'}(W), \Delta_{k}(\tau))$ for a 1-dimensional character $\tau$ of $W$, so that
\[ eB[\delta^{-1}]e = \text{Diff}(\mathbb{C}[R_{\text{reg}}/W], N). \] Here, \( N = e\Delta_k(\tau)[\delta^{-1}] \) is a rank 1 local system on \( R_{\text{reg}}/W \).

Then, the tensor product functor \( eB[\delta^{-1}]e \otimes_{D(R_{\text{reg}}/W)} \bullet \) induces a self-equivalence in the category of \( D(R_{\text{reg}}/W) \)-bimodules. Indeed, this follows because \( eB[\delta^{-1}]e = N \otimes_{\mathbb{C}[R_{\text{reg}}/W]} D(R_{\text{reg}}/W) \) and \( N \) is a line bundle on \( R_{\text{reg}}/W \). This implies that \( B \otimes_{H_k} \bullet : \text{HC}(H_{k'}, H_{k'}) \rightarrow \text{HC}(H_k, H_k') \) induces an equivalence between \( \overline{\text{HC}}(H_{k'}, H_{k'}) \) and \( \overline{\text{HC}}(H_k, H_k) \). The last assertion is immediate from Proposition 3.4.6.

\[ \text{Corollary 5.3.11. Assume that either } k' = \sigma(k) \text{ for some element } \sigma \in \text{Nam}, \text{ or that } k - k' \in p_Z. \text{ Then, the categories } \overline{\text{HC}}(H_k, H_k), \overline{\text{HC}}(H_k, H_{k'}), \overline{\text{HC}}(H_{k'}, H_k) \text{ and } \overline{\text{HC}}(H_{k'}, H_{k'}) \text{ are all equivalent.} \]

\[ \text{Proof. Let us do the first case. That } \overline{\text{HC}}(H_k, H_k) \cong \overline{\text{HC}}(H_k, H_{k'}) \text{ and } \overline{\text{HC}}(H_{k'}, H_k) \cong \overline{\text{HC}}(H_{k'}, H_{k'}) \text{ follow immediately from Lemma 5.3.8 and Corollary 5.3.10. Note that these equivalences come from tensoring with a HC bimodule, and this becomes an equivalence after localizing to } R_{\text{reg}}. \text{ Let } B_1 \in \text{HC}(H_k, H_{k'}) \text{ and } B_2 \in \text{HC}(H_{k'}, H_k) \text{ be such bimodules. Then, } B_1 \otimes_{H_k} \bullet \otimes_{H_k} B_2 : \text{HC}(H_k, H_k) \rightarrow \text{HC}(H_{k'}, H_k) \text{ induce an equivalence } \overline{\text{HC}}(H_k, H_k) \rightarrow \overline{\text{HC}}(H_{k'}, H_k). \text{ The case } k - k' \in p_Z \text{ is similar, instead of Lemma 5.3.8 we have to use the shift bimodules, cf. Definition 3.1.5.} \]

5.3.5 Action on the set of Hecke parameters

Let \( k \) be a parameter for the rational Cherednik algebra \( H_k \). Recall that we are always assuming that \( k_{\Gamma,0} = 0 \) for every reflection hyperplane \( \Gamma \), and under this assumption the Hecke parameter \( q := q(k) \) is simply given by

\[ q_{\Gamma,i} = \exp \left( \frac{2\pi \sqrt{-1}(k_{\Gamma,i} - i)}{t_{\Gamma}} \right) \] (5.5)

For \( \sigma \in \text{Nam} \), we will find the parameter \( q(\sigma(k)) \) in terms of \( q(k) \). Since both the action of the Namikawa-Weyl group and the calculation of the Hecke parameters are done by restricting
to a single reflection hyperplane $\Gamma$, it is enough to do this in the case of a cyclic group $W = \mathbb{Z}/\ell\mathbb{Z}$.

Now let $i = 1, \ldots, \ell - 1$ and let $\sigma_i := (i, i + 1) \in S_\ell$. For $k = (k_0 = 0, k_1, \ldots, k_{\ell-1})$, we are going to calculate the parameter $\sigma_i(k) = (k'_0 = 0, k'_1, \ldots, k'_{\ell-1})$.

First, assume that $i \neq 1, \ell - 1$. Recall that $\sigma_i(\pi_j) = \pi_j$, if $j \neq i$, and $\sigma_i(\pi_i) = \pi_{i-1} - \pi_i + \pi_{i+1}$.

An easy calculation now shows that

$$k'_j = \begin{cases} k_j, & j \neq i, i - 1 \\ k_i - 1, & j = i - 1 \\ k_{i-1} + 1, & j = i \end{cases}$$

Note that a similar formula holds for $i = \ell - 1$, here we use that $\sigma_{\ell-1}(\pi_j) = \pi_j$ if $j \neq \ell - 1$, $\sigma_{\ell-1}(\pi_{\ell-1}) = \pi_{\ell-2} - \pi_{\ell-1}$. Note that it follows that

$$q(k'_j) = \begin{cases} q(k)_j, & j \neq i - 1, i \\ q(k)_i, & j = i - 1 \\ q(k)_{i-1} & j = 1 \end{cases}$$

For the case $i = 1$, we have $\sigma_1(\pi_1) = \pi_2 - \pi_1$, while $\sigma_1(\pi_j) = \pi_j$ if $j \neq 1$. In this case, we have $k'_0 = 0$, $k'_1 = 2 - k_1$, and $k'_j = 1 + k_j - k_1$ if $j \neq 0, 1$. Thus, we get

$$q(k'_j) = \begin{cases} 1, & j = 0 \\ q(k)_{i-1}^{-1} & j = 1 \\ q(k)_j q(k)_{i-1}^{-1} & j \neq 0, 1 \end{cases}$$

Let us denote by $S_{\{2, \ldots, \ell\}}$ the group of permutations of the set $\{2, \ldots, \ell\}$. Of course, $S_{\{2, \ldots, \ell\}}$ is isomorphic to $S_{\ell-1}$, an isomorphism $S_{\{2, \ldots, \ell\}} \to S_{\ell-1}$, $\sigma \mapsto \tilde{\sigma}$ is given by $\tilde{\sigma}(i) = \sigma(i+1) - 1$, $i = 1, \ldots, \ell - 1$.

This discussion has the following consequence.
Lemma 5.3.12. Let $k = (k_0 = 0, k_1, \ldots, k_{\ell-1})$, and let $\sigma \in S_{\ell}$. Then, $q(\sigma(k))$ is determined by the following:

- $q(\sigma(k))_0 = 1$.
- $q(\sigma(k))_i = q(k)_{\sigma(i)}$ if $\sigma \in S_{\{2, \ldots, \ell\}}$, $i = 1, \ldots, \ell - 1$.
- $q(\sigma(k))_i = q(k)_i^{-1}q(k)_i$ if $\sigma = s_1$, $i = 1, \ldots, \ell - 1$.

Finally, let us remark that there is also an action of the group of characters $\text{Hom}(W, \mathbb{C}^\times)$ on the set of parameters $p$ in such a way that $H_k$ is isomorphic to $H_{\varepsilon(k)}$ for any $\varepsilon \in \text{Hom}(W, \mathbb{C}^\times)$.

Since this action is, again, defined by restricting to a single reflection hyperplane, let us define it only in the case when $W$ is a cyclic group, say $W = \mathbb{Z}/\ell\mathbb{Z} = \langle s : s^\ell = 1 \rangle$. In this case, the group of characters is identified with $\mathbb{Z}/\ell\mathbb{Z}$, $j \mapsto \eta_j$, $\eta_j := \exp(2\pi\sqrt{-1}/\ell)$. We denote the character $s \mapsto \eta_j$ by $\varepsilon_j$. Then, we have for $k = (k_0 = 0, k_1, \ldots, k_{\ell-1})$, $\varepsilon_j(k)_i = k_{i-j} - k_{\ell-j}$, where the subscripts are taken modulo $\ell$. An isomorphism $H_k \to H_{\varepsilon_k}$ is given by $x \mapsto x, y \mapsto y, w \mapsto \varepsilon(w)w$, $x \in R^*, y \in R, w \in W$.

Lemma 5.3.13. Let $\varepsilon : W \to \mathbb{C}^\times$ be a 1-dimensional character, and let $k$ be a parameter. Then, there exist $\sigma \in \text{Nam}(W)$ and $k' \in p_{\mathbb{Z}}$ such that $\varepsilon(k) = \sigma(k) + k'$.

Proof. It is again enough to show this when $W = \mathbb{Z}/\ell\mathbb{Z}$ is a symmetric group. Assume $\varepsilon = \varepsilon_j$ for some $j = 0, \ldots, \ell - 1$. Then, we have

$$q(\varepsilon_j(k))_i = \exp\left(\frac{2\pi\sqrt{-1}(k_{i-j} - k_{\ell-j})}{\ell}\right) = \exp\left(\frac{2\pi\sqrt{-1}(k_{i-j} - i + j)}{\ell}\right) \exp\left(\frac{2\pi\sqrt{-1}(-k_{\ell-j} - j)}{\ell}\right) = q(k)_{i-j}q(k)_{\ell-j}^{-1}$$

(5.6)

Thanks to Lemma 5.3.12, we can find an element $\sigma \in \text{Nam}$ such that $q(\varepsilon_j(k))_i = q(\sigma(k))_i$ for every $i = 0, \ldots, \ell - 1$. But this means that $\varepsilon_j(k) - \sigma(k) \in p_{\mathbb{Z}}$. We are done. \qed
Example 5.3.14. Take $W = \mathbb{Z}/2\mathbb{Z}$. Then, the action of the only nontrivial character $\varepsilon$ of $\mathbb{Z}_2$ is given by $\varepsilon(k) = -k = \sigma(k) + 2$, cf. Example 5.3.4, and the integral $k$-parameters are precisely the even numbers, cf. Section 2.3.5. In the c-parameters, we have $\varepsilon(c) = -c = \sigma(c) + 1$, and here the lattice of integral parameters are the integers.

We are ready to show for which pairs of parameters $k, k'$ we have $\mathcal{H}(H_k, H_{k'}) \neq 0$. Recall that, if this is the case, then we have $\mathcal{H}(H_k, H_{k'}) \cong \mathcal{H}(H_{k'}, H_{k'})$, cf. Corollary 5.3.10. Thus, the following result will reduce the description of the category $\mathcal{H}(H_k, H_{k'})$ to the case where $k = k'$. We will study this in Section 5.4.

Theorem 5.3.15. The category $\mathcal{H}(H_k, H_{k'})$ is nonzero if and only if there exists a sequence $k^0 = k, k^1, \ldots, k^m = k'$, where $k^i$ can be obtained from $k^{i-1}$ from an integral translation or by applying an element of the Namikawa-Weyl group for every $i = 1, \ldots, m$.

Proof. First, assume that such a sequence exist. Then, thanks to Corollary 5.3.11, the categories $\mathcal{H}(H_{k^i-1}, H_{k^i})$ are nonzero for every $i = 1, \ldots, m$. In particular, we can find $B^i \in \mathcal{H}(H_{k^i-1}, H_{k^i})$ such that $eB^i[\delta^{-1}]e \cong N^i \otimes D(R^{reg}/W) D(R^{reg}/W)$ for a line bundle $N^i$ on $R^{reg}/W$ with a flat connection, cf. Lemma 5.3.9. Then, $B^1 \otimes_{H_{k^1}} B^2 \otimes_{H_{k^2}} \cdots \otimes_{H_{k^{m-1}}} B^m \in \mathcal{H}(H_{k^0}, H_{k^m})$ is not killed upon localization to $R^{reg}$, and thus $\mathcal{H}(H_k, H_{k'}) \neq 0$.

Now assume $\mathcal{H}(H_k, H_{k'}) \neq 0$. Thanks to Lemma 5.3.9, we can find a character $\varepsilon : W \to \mathbb{C}^\times$ such that $\text{Hom}_{\text{fin}}(\Delta_{k'}(\text{triv}), \Delta_k(\varepsilon)) \neq 0$. Under the equivalence $\varphi_* : \mathcal{O}_k \to \mathcal{O}_{\varepsilon^{-1}(k)}$, coming from the isomorphism $\varphi : H_k \to H_{k^{-1}(k)}$ we have $\varphi_*(\Delta_k(\varepsilon)) = \Delta_{\varepsilon^{-1}(k)}(\text{triv})$. Thus, Theorem 5.3.15 is a consequence of Lemma 5.3.13 and the following result.

Lemma 5.3.16. Assume $\text{Hom}_{\text{fin}}(\Delta_{k'}(\text{triv}), \Delta_k(\text{triv})) \neq 0$. Then, there exists $\sigma \in \text{Nam}$ such that $\sigma(k') - k \in \mathfrak{p}_W$.

Proof. First of all, note that a parameter $k$ is integral if and only if $k|_{W^\Gamma} \in \mathfrak{p}_W(W^\Gamma)$ for every reflection hyperplane $\Gamma \in \mathcal{A}$. Also, since $\text{Res}_{W^\Gamma}^W(\Delta_{k'}(\text{triv})) = \Delta_{\varepsilon}(\text{triv}(W^\Gamma))$ where $\varepsilon = k, k'$, cf. Lemma 2.4.6, we have that $\text{Hom}_{\text{fin}}(\Delta_{k'}(\text{triv}(W^\Gamma)), \Delta_k(W^\Gamma)) \neq 0$ for every
Finally, since the action of the Namikawa-Weyl group is defined by restricting to stabilizers of reflection hyperplanes, we may assume that $W$ is a cyclic group $\mathbb{Z}/\ell\mathbb{Z}$.

So assume $\text{Hom}_{\text{fin}}(\Delta_k'(\text{triv}), \Delta_k(\text{triv})) \neq 0$ and $W$ is a cyclic group. Since $KZ_k(\Delta_k(\text{triv})) = \mathbb{C}$, the trivial representation of $H_q(k)$, Lemma 5.2.7 implies that $H_q(k') - \text{mod} \subseteq H_q(k) - \text{mod}$ as full subcategories of $\mathbb{C}[t, t^{-1}] - \text{mod}$. But $H_q(k), H_q(k')$ are commutative algebras of the same dimension. This implies that $H_q(k) = H_q(k')$, in other words, the numbers $q(k)_0 = 1, q(k)_1, \ldots, q(k)_{\ell-1}$ and $q(k')_0 = 1, q(k')_1, \ldots, q(k')_{\ell-1}$ coincide up to a permutation of the indices that fixes 0. Now the lemma is an immediate consequence of Lemma 5.3.12 and the definition of having integral difference. 

Let us examine the condition of Theorem 5.3.15 in the case of a Coxeter group $W$. In this case, the stabilizer of every reflection hyperplane has order 2, and we have that the Namikawa-Weyl group is $\mathbb{Z}_2^{A/W}$. We can think of a $k$-parameter as a $|A/W|$-tuple of complex numbers.

The action of an element of the Namikawa-Weyl group has the effect of changing the sign and adding even integers to some components, cf. Example 5.3.4. Adding an integral parameters amount to adding even integers in every component. Now for every reflection hyperplane $\Gamma$, let $s_\Gamma \in W$ be the reflection such that $\Gamma = \ker(id - s_\Gamma)$. In terms of the $c$-parameters, we have $k_{\Gamma,0} = 0, k_{\Gamma,1} = 2c(s_\Gamma)$. Thus, translating to the $c$-parameters we have.

**Corollary 5.3.17.** Let $W$ be a Coxeter group, and let $c, c' \in p$. Then, $\text{HC}(H_c, H_{c'}) \neq 0$ if and only if there exists a character $\varepsilon : W \to \mathbb{C}^\times$ such that $c - \varepsilon c' : S \to \mathbb{C}$ is an integer-valued function.

Finally, let us mention a few words on how to check the condition in Theorem 5.3.15. Recall that two parameters $k$ and $k'$ have integral difference if and only if $q(k)_{\Gamma,i} = q(k')_{\Gamma,i}$ for every $i = 0, \ldots, \ell_{\Gamma} - 1$. It follows from Lemma 5.3.12 that the parameters $k, k'$ satisfy the condition in Theorem 5.3.15 if and only if, for every reflection hyperplane $\Gamma \in A$ there exists $i_0 \in \{0, 1, \ldots, \ell_{\Gamma} - 1\}$ such that
\[
\{q(k')_{\Gamma,0}, \ldots, q(k')_{\Gamma, \ell_{\Gamma}-1}\} = \{q(k)^{-1}_{\Gamma,i_0} q(k)_{\Gamma,0}, \ldots, q(k)^{-1}_{\Gamma,i_0} q(k)_{\Gamma, \ell_{\Gamma}-1}\}
\]  
(5.7)

as multisets.

Let us also mention that, in the statement of Theorem 5.3.15, we may always assume that \(m \leq 2\). This is a consequence of the following result.

**Lemma 5.3.18.** Let \(k \in p, \sigma \in \text{Nam} \) and \(k' \in p_Z\). Then

1. There exists \(k_1 \in p_Z\) such that \(\sigma(k) + k' = \sigma(k + k_1)\).
2. There exists \(k_2 \in p_Z\) such that \(\sigma(k + k') = \sigma(k) + k_2\).

**Proof.** Since (2) is a formal consequence of (1), we only need to show (1). Note that we have \(q(\sigma(k) + k') = q(\sigma(k))\). It follows that \(\sigma^{-1}(\sigma(k) + k') \in k + p_Z\). We are done. \(\square\)

**Corollary 5.3.19.** The following are equivalent.

1. The category \(\overline{\text{HC}}(H_k, H_{k'})\) is nonzero.
2. There exist \(\sigma \in \text{Nam}, k_1 \in p_Z\) such that \(k' = \sigma(k) + k_1\).
3. There exist \(\sigma' \in \text{Nam}, k_2 \in p_Z\) such that \(k' = \sigma'(k + k_2)\).
4. For every reflection hyperplane \(\Gamma \in A\), there exists \(i_0 \in \{0, 1, \ldots, \ell_{\Gamma} - 1\}\) such that (5.7) holds.

### 5.4 Subgroup \(W_c\)

#### 5.4.1 Definition

For the rest of this section, it will be more convenient to return to the ‘\(c\)-parametrization’ of the rational Cherednik algebra. Of course, we still have an action of the Namikawa-Weyl
group, as well as of the group of characters, on the space of parameters, and Lemmas 5.3.12, 5.3.13 remain valid.

Recall that the category $HC(H_c, H_c)$ is equivalent to the category of representations of $W/W'$ for some normal subgroup $W' \subseteq W$. Here, we describe the group $W'$. To motivate our description, we first look at the case where $W$ is a cyclic group.

So assume $W = \mathbb{Z}/\ell\mathbb{Z}$, with generator $s$. The Hecke algebra $H_{q(c)}$ is the quotient of the polynomial algebra $\mathbb{C}[t]$ by the ideal generated by the polynomial $(t - 1) \prod_{i=1}^{\ell-1}(t - q(c)_i)$. We remark that $q(c)_i$ is the scalar by which $t$ acts on $KZ(C_i)$, where $C_i$ is the irreducible representation of $W$ where $s$ acts by multiplication by $\exp(2\pi \sqrt{-1}/\ell)$. Now, if $\text{Hom}_{\text{fin}}(\Delta(\text{triv}),\Delta(C_i))$ is nonzero then, thanks to Lemma 5.2.7, multiplication by $q(c)_i$ induces a map $q(c)_i \to q(c)_i$, where we think of $q(c)_i$ as a multiset $q(c)_i = \{q(c)_0 = 1, q(c)_1, \ldots, q(c)_\ell-1\}$. It is not hard to see that this map is actually a bijection, i.e. it preserves multiplicities. In particular, $q(c)_i$ is an $\ell$-root of 1.

So set $\eta := \exp(2\pi \sqrt{-1}/\ell)$. Note that the group $W$ acts on the set of Hecke parameters, the element $s^i$ acts on a multiset $q' = \{q'_0, \ldots, q'_{\ell-1}\}$ by multiplying each element by $\eta^i$. The stabilizer of $q(c)$, the Hecke parameter associated to the Cherednik parameter $c$, is cyclic, so it is generated by $s^m$, where $m$ divides $\ell$, say $mp = \ell$. By definition, $W_c := \langle s^p \rangle$. Note that for generic $c$ we have that $m = \ell$, so $W_c = W$.

**Example 5.4.1.** Let $W = \mathbb{Z}/4\mathbb{Z}$, with generator $s$ acting on $\mathbb{C}$ by $s \mapsto \sqrt{-1}$. Let the parameter $c$ be given by $c(s) = 0$, $c(s^2) = -1/2$, $c(s^3) = 0$. Then, we have $k_0 = 0, k_1 = 1, k_2 = 0, k_3 = 1$, so $\{q_0, q_1, q_2, q_3\} = \{1, 1, -1, -1\}$ and $W_c = \langle s^2 \rangle$.

Let us generalize the definition of $W_c$ for the case where $W$ is any complex reflection group.

Set $q := q(c)$. Fix a reflection hyperplane $\Gamma \in \mathcal{A}$. Let $\eta_\Gamma := \exp(2\pi \sqrt{-1}/\ell_\Gamma)$, and $s_\Gamma \in W_\Gamma$ be the unique element with $\det(s_\Gamma|_H) = \eta_\Gamma$, the element $s_\Gamma$ is a generator of $W_\Gamma$. Now consider the set $X_\Gamma := \{i \in \{1, \ldots, \ell_\Gamma\} : \eta_\Gamma^{q_\Gamma,j} \in \{q_{\Gamma,0} = 1, \ldots, q_{\Gamma,\ell_\Gamma-1}\}$ with the same multiplicity as $q_{\Gamma,j}$ for every $j = 0, \ldots, \ell_\Gamma - 1\}$. For example, $\ell_\Gamma \in X_\Gamma$. Now let $m_\Gamma := \min X_\Gamma$. It is
clear that $m_\Gamma$ is a divisor of $\ell_\Gamma$, say $m_\Gamma p_\Gamma = \ell_\Gamma$. We define $W_c := \langle s_\Gamma^{p_\Gamma} : \Gamma \in \mathcal{A} \rangle \subseteq W$. By definition, this is a reflection group. Note that the conjugation invariance of $c$ implies that $W_c$ is a normal subgroup of $W$.

Note that $W_c = \{1\}$ if and only if $m_\Gamma = 1$ for every reflection hyperplane $\Gamma \in \mathcal{A}$. This happens if and only if $\{q_{\Gamma,0} = 1, \ldots, q_{\Gamma,\ell_\Gamma-1}\} = \{1, \eta_{\Gamma,1}, \ldots, \eta_{\Gamma,\ell_\Gamma-1}\}$, that is, if and only if $c \in \text{Nam}(p_\mathbb{Z})$. On the other hand $W_c = W$ if and only if $m_\Gamma = \ell_\Gamma$ for every $\Gamma \in \mathcal{A}$, and this is a generic condition.

Example 5.4.2. Let $W$ be a Coxeter group, so that the setwise stabilizer of every reflection hyperplane is cyclic of order 2. For a reflection hyperplane $\Gamma$, let $s_\Gamma$ be a reflection such that $\Gamma = \ker(\text{id} - s_\Gamma)$. Note that the group $W_c$ will be generated by those $s_\Gamma$ for which $m_\Gamma = 2$.

We have that $m_\Gamma = 1$ if and only if $\{q_{\Gamma,0}, q_{\Gamma,1}\} = \{1, q_{\Gamma,1}\}$, that is, if and only if $q_{\Gamma,1} = -1$. Since $q_{\Gamma,1} = -\exp(2\pi \sqrt{-1}c(s_\Gamma))$, we get $W_c = \langle s : c(s) \notin \mathbb{Z} \rangle$.

It is clear that $W_c = W_c'$ provided there exists $\sigma \in \text{Nam}$ such that $c - \sigma(c') \in p_\mathbb{Z}$, as integral translations do not affect the Hecke parameter $q$. Let us check that the subgroup $W_c$ is also stable under the action of the Namikawa-Weyl group.

Lemma 5.4.3. Let $c \in p$ and let $\sigma \in \text{Nam}$. Then, $W_c = W_{\sigma(c)}$.

Proof. It is enough to check this when $W$ is a cyclic group, so let $W = \mathbb{Z}/\ell\mathbb{Z} = \langle s : s^\ell = 1 \rangle$, and $\eta := \exp(2\pi \sqrt{-1}/\ell)$. Assume that $W_c = \langle s^p \rangle$ with $mp = \ell$. This means that there exist $Q_0 = 1, Q_1, \ldots, Q_{m-1} \in \mathbb{C}^\times$ such that

$$q(c) = \{Q_j \eta^{mi} : j = 0, \ldots, m - 1, i = 0, \ldots, p - 1\}$$

It is enough to check that $W_c = W_{\sigma(c)}$ for $i = 1, \ldots, \ell - 1$, where, recall, $\sigma_i = (i, i + 1) \in S_\ell = \text{Nam}(W)$. Thanks to Lemma 5.3.12, $q(c) = q(\sigma_i(c))$ as a multi-set for $i = 2, \ldots, \ell - 1$, so that in this case we have $W_c = W_{\sigma(c)}$. For $\sigma_1$, Lemma 5.3.12 implies that there exist $j_0 \in \{0, \ldots, m - 1\}, i_0 \in \{0, \ldots, p - 1\}$ such that

$$q(\sigma_1(c)) = \{Q_{j_0}^{-1}Q_j \eta^{m(i-i_0)} : j = 0, \ldots, m - 1, i = 0, \ldots, p - 1\}$$
Setting $Q'_j := Q^{-1}_{j_0}Q_j$, we get that $q(\sigma_1(c)) = \{Q'_j\eta^{mi} : j = 0, \ldots, m-1, i = 0, \ldots, p-1\}$, so $W_{\sigma_1(c)} = \langle s^{p'} \rangle$ with $p'$ dividing $p$. But since $c = \sigma_1\sigma_1(c)$, we also have that $p$ divides $p'$. Since $p, p' \in \{1, \ldots, \ell\}$, this implies that $W_c = \langle s^p \rangle = W_{\sigma_1(c)}$. 

5.4.2 Main result

**Theorem 5.4.4.** The category $\overline{HC}(c, c)$ is equivalent, as a monoidal category, to the category of representations of $W/W_c$.

To prove Theorem 5.4.4, we will check that there exists a parameter $c' \in p$ and an element $\sigma \in \text{Nam}$ such that $\sigma(c) - c' \in p_Z$ and the algebra $H_{c'}(W)$ decomposes as $H_{\mathfrak{c}}(W_c)\#_{W_c} W$, for some parameter $\mathfrak{c} \in \mathbb{C}[S \cap W_c]^{W_c}$ which is naturally computed from $c'$. Since $\sigma(c) - c' \in p_Z$, the categories $\overline{HC}(c, c)$ and $\overline{HC}(c', c')$ are equivalent, cf. Corollary 5.3.11. The result will now follow if we check that $H_{\mathfrak{c}}(W_c)$ has a unique irreducible HC bimodule with full support.

Assume, for the moment, that the parameter $c$ is such that, for $\Gamma \in A$, $c(s^\Gamma_i) = 0$ unless $i = p_\Gamma, 2p_\Gamma, \ldots, (m_\Gamma - 1)p_\Gamma$. Then, it is clear from the relations (2.1) that the subalgebra of $H_c$ generated by $R, R^*$ and $W_c$ is isomorphic to $H_{\mathfrak{c}}(W_c)$, where $\mathfrak{c}$ simply denotes the restriction of the parameter $c$ to $W_c$. So $H_c$ is generated by $H_{\mathfrak{c}}(W_c)$ and $W$. Moreover, the subalgebra $H_{\mathfrak{c}}(W_c)$ is stable under the adjoint action of $W$, this follows because $W_c$ is a normal subgroup of $W$. It follows that $H_c \cong H_{\mathfrak{c}}(W_c)\#_{W_c} W$, where the latter algebra is $H_{\mathfrak{c}}(W_c) \otimes_{W_c} CW$ with product defined analogously to the smash-product algebra, using the action of $W$ on $H_{\mathfrak{c}}(W_c)$. Thus, HC $H_c$-bimodules with full support correspond to $W$-equivariant HC $H_{\mathfrak{c}}(W_c)$-bimodules with full support, where the action of $W_c \subseteq W$ coincides with that coming from the inclusion $W_c \subseteq H_{\mathfrak{c}}(W_c)$.

Denote $q := q(c)$ and let us now examine the Hecke parameters $q_{\Gamma,i}$, still under the assumption that $c(s^\Gamma_i) = 0$ unless $i = p_\Gamma, 2p_\Gamma, \ldots, (m_\Gamma - 1)p_\Gamma$. It follows easily from (2.2) that $k_{\Gamma,i} = k_{\Gamma,i+m_\Gamma}$ for all $i$. But then it follows that:

$$q_{\Gamma,i+m_\Gamma} = \exp(2\pi \sqrt{-1}(k_{\Gamma,i} - i - m_\Gamma)/\ell_\Gamma) = \eta^{-m_\Gamma}_{\Gamma} q_{\Gamma,i}$$
Note that, given numbers $Q_{\Gamma,0} = 1, Q_{\Gamma,1}, \ldots, Q_{\Gamma,m_{\Gamma} - 1} \in \mathbb{C}^\times$ we can always find a parameter $c \in \mathfrak{p}$ with $c(s^i_{\Gamma}) = 0$ unless $i$ is a multiple of $p_{\Gamma}$ and such that $q_{\Gamma,i} = Q_{\Gamma,i}$. This implies the following.

**Lemma 5.4.5.** Let $c \in \mathfrak{p}$ be a parameter, and let $m_{\Gamma}, p_{\Gamma}$ have the same meaning as in Section 5.4.1. Then, there exists a parameter $c' \in \mathfrak{p}$ such that for every $\Gamma \in \mathcal{A}$, $c'(s^i_{\Gamma}) = 0$ unless $i = p_{\Gamma}, 2p_{\Gamma}, \ldots, (m_{\Gamma} - 1)p_{\Gamma}$ and $\mathcal{H}_{q(c)} = \mathcal{H}_{q(c')}$. In particular, $\sigma(c) - c' \in \mathfrak{p}_\mathbb{F}$ for some $\sigma \in \text{Nam}$, and so the categories $\mathcal{H}_c(\mathcal{H}_c, \mathcal{H}_c)$ and $\mathcal{H}_c(\mathcal{H}_{c'}, \mathcal{H}_{c'})$ are equivalent.

**Proof of Theorem 5.4.4.** Thanks to Lemmas 5.4.5, 5.4.3 and Corollary 5.3.11, we may assume that $c(s^i_{\Gamma}) = 0$ unless $i$ is a multiple of $p_{\Gamma}$, i.e. that $H_c \cong H_c(W_c)\#_{W_c} W$. We claim now that $H_c(W_c)$ has a unique irreducible $\mathcal{H}$ bimodule with full support. For $\Gamma \in \mathcal{A}$, let $q_{\Gamma,0} = 1, \ldots, q_{\Gamma,m_{\Gamma} - 1}$ be the parameters for the Hecke algebra $\mathcal{H}_q(W_c)$ associated to $c$, and denote $\eta_{\Gamma} := \exp(2\pi \sqrt{-1}/m_{\Gamma}) = \eta_{\Gamma}^p$. We also denote $m_{\Gamma}' := \min\{i \in \{1, \ldots, m_{\Gamma}\} : \eta_{\Gamma}^i q_{\Gamma,j} \in q_{\Gamma}$ with the same multiplicity as $q_{\Gamma,j}$ for every $j = 0, \ldots, m_{\Gamma} - 1\}$. Thanks to Lemma 5.2.7, our claim will follow if we check the following.

**Claim:** For every hyperplane $\Gamma \in \mathcal{A}$, $m_{\Gamma}' = m_{\Gamma}$.

We proceed by contradiction. Assume there exists $0 < i < m_{\Gamma}$ such that, for every $j = 0, \ldots, m_{\Gamma} - 1$, $\eta_{\Gamma}^i q_{\Gamma,j}$ is in the multiset $q_{\Gamma}$ with the same multiplicity as $q_{\Gamma,j}$. Note that we have

$$q_{\Gamma,j} = \exp\left(\frac{2\pi \sqrt{-1}(k_i - i)}{m_{\Gamma}}\right) = q_{\Gamma,j}^{p_{\Gamma}}$$

Thus, $\eta_{\Gamma}^i q_{\Gamma,j} \in q_{\Gamma}$ implies that $\eta_{\Gamma}^i q_{\Gamma,j} \in \{q_{\Gamma,0}, \ldots, q_{\Gamma,m_{\Gamma} - 1}\}$, with the same multiplicity as $q_{\Gamma,j}$. But

$$q_{\Gamma} = \{q_{\Gamma,0}, \ldots, q_{\Gamma,m_{\Gamma} - 1}, \eta_{\Gamma} m_{\Gamma} q_{\Gamma,0}, \ldots, \eta_{\Gamma}^m q_{\Gamma,m_{\Gamma} - 1}, \ldots, \eta_{\Gamma}^{(p_{\Gamma} - 1)m_{\Gamma}} q_{\Gamma,0}, \ldots, \eta_{\Gamma}^{(p_{\Gamma} - 1)m_{\Gamma}} q_{\Gamma,m_{\Gamma} - 1}\}$$

Thus, we see that $\eta_{\Gamma}^i q_{\Gamma,j} \in q$ with the same multiplicity as $q_{\Gamma,j}$ for every $j = 0, \ldots, \ell_{\Gamma} - 1$. 

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This contradicts the choice of $m_\Gamma$. Thus, $H_c(W_c)$ has a unique irreducible HC bimodule with full support. Since $H_c = H_c(W_c) \#_{W_c} W$, this proves Theorem 5.4.4. □
Chapter 6

Type A

6.1 Introduction and preliminary results.

6.1.1 Main results

We now turn our attention to type A, that is, \( W = S_n \), with reflection representation \( R = \{(x_1, \ldots, x_n) \in \mathbb{C}^n : \sum x_i = 0\} \). Throughout this chapter, we denote \( H_c(n) := H_c(S_n, R) \).

Similarly, we denote \( \mathcal{H}_q(n) := \mathcal{H}_q(S_n) \), the Hecke algebra associated to \( S_n \) with parameter \( q \in \mathbb{C}^\times \). More explicitly, \( \mathcal{H}_q(n) \) is the algebra with generators \( T_1, \ldots, T_{n-1} \) and relations

\[
T_{i+1}T_i = T_iT_{i+1}, \quad i = 1, \ldots, n - 2;
\]
\[
T_iT_j = T_jT_i, \quad |i - j| > 1;
\]
\[
(T_i - 1)(T_i + q) = 0, \quad i = 1, \ldots, n - 1
\]

Note the change in the sign of \( q \), in particular, if \( q = \exp(2\pi \sqrt{-1}c) \) then the KZ functor for \( \mathcal{O}_c \) has its image in the category of \( \mathcal{H}_q \)-modules. There are two main results in this chapter. The first one of them is an embedding of the category of Harish-Chandra bimodules \( \mathcal{HC}(H_c(n), H_c(n)) \) into category \( \mathcal{O}_c \), for any parameter \( c \in \mathbb{C} \).

**Theorem 6.1.1.** Assume \( c \notin \mathbb{R}_{<0} \), and consider the functor \( \Phi_c := \Phi_{\Delta_c(\text{triv})} : \mathcal{HC}(H_c, H_c) \rightarrow \mathcal{O}_c, \ B \mapsto B \otimes_{H_c} \Delta_c(\text{triv}) \). Then
1. The functor $\Phi_c$ is a fully faithful embedding whose image is closed under subquotients.

2. The functor $\Phi_c$ preserves supports in the sense that, for a parabolic subgroup $W' \subseteq S_n$, we have $\overline{L_{W'}} \subseteq SS(B)$ if and only if $X_{W'} \subseteq \text{supp}(\Phi_c(B))$. In particular, it sends a finite-dimensional bimodule to a finite-dimensional module.

Let us remark that the restriction $c \not\in \mathbb{R}_{<0}$ is not very important. Indeed, we have an isomorphism $H_c(n) \to H_{-c}(n)$, which implies that in this case Theorem 6.1.1 remains valid upon substituting the trivial representation with the sign representation.

We also remark that Theorem 6.1.1 does not generalize to an arbitrary complex reflection group. For example, let $W = \mathbb{Z}/4\mathbb{Z} = \langle s : s^4 = 1 \rangle$ with parameter given by $c(s) = 0, c(s^2) = -1/2, c(s^3) = 0$. Then, there are two irreducible bimodules with full support in $HC(H_c, H_c)$, cf. Example 5.4.1, and there are 4 irreducible finite-dimensional bimodules, this follows because category $\mathcal{O}_c$ has 2 irreducible finite-dimensional modules. Then, there are 6 irreducible bimodules in $HC(H_c, H_c)$, so there cannot exist an embedding $HC(H_c, H_c) \to \mathcal{O}_c$ with the properties of the one in Theorem 6.1.1.

Of course, a question that comes after Theorem 6.1.1 is to describe the image of the functor $\Phi_c$. Recall from Example 2.3.6 that the singular locus of $S_n$ consists of those rational numbers that can be written in the form $r/m$, $\gcd(r; m) = 1$, $1 < m \leq n$. The following result is from [BEG]. Alternatively, it can be easily deduced from the results in Chapter 4 of this work, more precisely Theorem 5.1.1, cf. Example 5.4.2.

**Proposition 6.1.2.** Assume $c \in \mathbb{C}$ is regular. Then, $\Phi_c : HC(H_c, H_c) \to \mathcal{O}_c$ is an equivalence if $c \in \mathbb{Z}$. If $c \not\in \mathbb{Z}$, then $HC(H_c, H_c)$ is equivalent to the category of finite-dimensional vector spaces, and the only irreducible in the image of $\Phi_c$ is $\Delta_c(\text{triv})$.

To explain what happens in the singular case, we need some notation regarding partitions. A partition $\lambda$ is a non-increasing sequence $\lambda = (\lambda^1, \lambda^2, \cdots)$ of non-negative integers that is eventually 0. The size of the partition $\lambda$ is $|\lambda| := \sum_{i \geq 0} \lambda^i$, this is a non-negative integer. We
write $\lambda \vdash |\lambda|$ to indicate that $\lambda$ is a partition of size $|\lambda|$. We add partitions componentwise, $(\lambda + \mu)^i = \lambda^i + \mu^i$. Similarly, we can multiply a partition by a positive integer in an obvious way.

For a positive integer $m$, we say that a partition $\lambda$ is $m$-regular if $\lambda^i - \lambda^{i+1} < m$ for every $i = 1, 2, \ldots$. It is easy to see from the division algorithm that for every partition $\lambda$, there exist unique partitions $\mu, \nu$ such that $\lambda = m\mu + \nu$ and $\nu$ is $m$-regular. We say that an $m$-regular partition $\nu$ is $m$-trivial if $\nu = ((m-1), (m-1), \ldots, (m-1), d, 0, \ldots)$ with $0 \leq d \leq m - 1$, where the number of components equal to $m - 1$ may be zero. Obviously, for each positive integer $k$ there exists a unique $m$-trivial partition $\nu$ with $|\nu| = k$.

Now assume that $c \in \mathbb{C}$ is positive and singular, so $c = r/m > 0$, $\gcd(r; m) = 1$, $1 < m \leq n$. For each partition $\lambda \vdash n$, we will also denote by $\lambda$ the irreducible representation of the symmetric group $S_n$ labeled by the partition $\lambda$. So we have the irreducible representation $L_c(\lambda) \in \mathcal{O}_c$.

**Theorem 6.1.3.** Let $c = r/m > 0$, $\gcd(r; m) = 1$, $1 < m \leq n$. Let $\lambda \vdash n$ be a partition, and let $\lambda = m\mu + \nu$ be its decomposition so that $\nu$ is $m$-regular. Then, $L_c(\lambda) \in \text{im}(\Phi_c)$ if and only if $\nu$ is $m$-trivial.

Note that, while Theorem 6.1.3 describes the irreducible objects in the image of $\Phi_c$, it does not describe the image of $\Phi_c$. Indeed, we will see through an example that the image of $\Phi_c$ is, in general, not closed under extensions, see Example 6.2.4.

Let us now turn out our attention to the study of the category $HC(H_c, H_{c'})$ for distinct $c, c' \in \mathbb{C}$. We remark that if either $c$ or $c'$ is regular, then $HC(H_c, H_{c'}) = HC(H_c, H_c)$ and in this case the description of $HC(H_c, H_{c'})$ follows from Chapter 4, cf. Corollary 5.3.17, Example 5.4.2. So we will assume that $c, c'$ are singular.

**Theorem 6.1.4.** Let $c = r/m, c' = r'/m'$, $\gcd(r, m) = \gcd(r', m') = 1$, $1 < m, m' \leq n$.

1. If $HC(H_c, H_{c'}) \neq 0$, then $m = m'$. 

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2. Assume $m = m'$ is not a divisor of $n$. Then, $\text{HC}(H_c, H_{c'}) \neq 0$ if and only if either $c + c'$ or $c - c'$ is an integer. In this case, for each symplectic leaf $\mathcal{L} \subseteq (R \oplus R^*)/S_n$, there is a category equivalence $\text{HC}_\mathcal{L}(H_c, H_{c'}) \cong \text{HC}_\mathcal{L}(H_c, H_c) \cong \text{HC}_\mathcal{L}(H_{c'}, H_c) \cong \text{HC}_\mathcal{L}(H_{c'}, H_{c'}).$

Note that if $m = m'$ is a divisor of $n$, Theorem 6.1.4 fails. Indeed, we can take for example $m = m' = n$. In this case, both algebras $H_c$ and $H_{c'}$ have a finite-dimensional module, so $\text{HC}(H_c, H_{c'}) \neq 0$, regardless of the integrality of $c - c'$ or $c + c'$. For this case, see Theorem 6.4.3.

### 6.1.2 Preliminary results

Now we gather some results on the structure of the algebra $H_c(n)$ and of category $\mathcal{O}_c$ that we will use in our arguments. It is known, cf. [BE, Example 3.25], [L3, Theorem 5.8.1], that the algebra $H_c := H_c(n)$ is simple unless $c = r/m$ with $r, m \in \mathbb{Z}$, $\gcd(r; m) = 1$ and $1 < m \leq n$.

In this case, [L3, Theorem 5.8.1 (2)], the algebra $H_c$ has $\lfloor n/m \rfloor$ proper nonzero two-sided ideals that are linearly ordered by inclusion, say $\mathcal{J}_1 \subset \mathcal{J}_2 \subset \cdots \subset \mathcal{J}_{\lfloor n/m \rfloor}$. Moreover, $\mathcal{J}_i^2 = \mathcal{J}_i$ for any $i = 1, \ldots, \lfloor n/m \rfloor$. We set $\mathcal{J}_0 := \{0\}$, $\mathcal{J}_{\lfloor n/m \rfloor + 1} := H_c$.

The classification of two-sided ideals gives a characterization of the possible supports of HC bimodules. For $i = 0, 1, \ldots, \lfloor n/m \rfloor$, consider the subgroup $S_m^{x_i} \subseteq S_n$, and consider the set $\mathcal{X}_i := \{x \in R \oplus R^* : W_x = S_m^{x_i}\}$. Let $\mathcal{L}_i$ be the image of $\mathcal{X}_i$ under the natural projection $R \oplus R^* \to (R \oplus R^*)/S_n$. This is a symplectic leaf. The support of $H_c/\mathcal{J}_i$ is $\mathcal{L}_i$. Now recall Lemma 3.2.2, that says that for a HC $(H_c, H_{c'})$-bimodule $B$, $\text{SS}(B) = \text{SS}(H_c/\text{LAnn}(B)) = \text{SS}(H_{c'}/\text{RAnn}(B))$. This already implies (1) of Theorem 6.1.4, namely, that $\text{HC}(H_c, H_{c'}) = 0$ unless $c, c'$ have the same denominator when expressed as an irreducible fraction. Thus, throughout the rest of this chapter we will assume that $c = r/m$, $c' = r'/m$, with $\gcd(r; m) = \gcd(r'; m) = 1$ and $1 < m \leq n$.

We now give a description of the supports of irreducible modules in $\mathcal{O}_c$. For every $i = 0, 1, \ldots, \lfloor n/m \rfloor$, let $X'_i = \{(x_1, \ldots, x_n) \in \mathbb{C}^n : \sum x_i = 0, x_1 = x_2 = \cdots = x_m, x_{m+1} = \cdots = \}$
Let us recall how [Wi, Theorem 1.8] is proved, as this will be important for our arguments. So let $i$ and $p$ be as in the previous paragraph. Consider the subgroup $S^x_i \subseteq S_n$. Let $R := \{ x \in R : S^x_m \subseteq \text{Stab}_S(x) \} (= X'_i)$ and $\mathbb{R}^{reg} := \{ x \in R : \text{Stab}_S(x) = S^x_m \}$. Then, Wilcox proves that we have a localization functor, $\text{Loc}^i : \mathcal{O}^i \to D(\mathbb{R}^{reg})\#(S_i \times S_p)-\text{mod}, M \mapsto \mathbb{C}[\mathbb{R}^{reg}] \otimes \mathbb{C}[R] M$ that factors through $\mathcal{O}^i/\mathcal{O}^{i+1}$ and that identifies this quotient category with a subcategory of the category of $(S_i \times S_p)$-equivariant $D(\mathbb{R}^{reg})$-modules with regular singularities. Then, he checks that under the Riemann-Hilbert correspondence that identifies the latter category with the category of finite dimensional representations of $\pi_1(\mathbb{R}^{reg}/(S_i \times S_p))$, the image of $\mathcal{O}^i/\mathcal{O}^{i+1}$ gets identified with the subcategory $(\mathbb{C}S_i \otimes \mathcal{H}_q(p))$-mod of $\pi_1(\mathbb{R}^{reg}/(S_i \times S_p))-\text{rep}$ where, recall, $q = \exp(2\pi \sqrt{-1}c)$. We denote by $\text{KZ}^i : \mathcal{O}^i \to (\mathbb{C}S_i \otimes \mathcal{H}_q(p))$-mod the composition of the localization functor $\text{Loc}^i$ with the Riemann-Hilbert correspondence.

This construction has the following consequence for HC bimodules. Let $S \in \mathcal{O}_c$ be the irreducible module supported on $X_i$ that gets sent to the trivial $\mathbb{C}S_i \otimes \mathcal{H}_q(p)$-module under $\text{KZ}^i$ so that, in particular, $\text{Loc}^i(S) = \mathbb{C}[\mathbb{R}^{reg}]$. Then, the proofs in Section 5.2.3 can be carried out in this setting and we see that, whenever $T$ is a simple module with $\text{Hom}_{\text{fin}}(S, T) \neq 0$, and $N$ is another simple module in $\mathcal{O}^i$, then $\text{KZ}^i(\text{Hom}_{\text{fin}}(S, T) \otimes_{H_c} N) = \text{KZ}^i(S) \otimes_{\mathbb{C}} \text{KZ}^i(T)$. This is already in [BEG2].
Lemma 6.1.5. Let $S \in O^i_c$ be the irreducible module satisfying $KZ^i_c(S) = \mathbb{C}$, the trivial $\mathbb{C}S_i \otimes \mathcal{H}_q(p)$-module. Let $T \in O_c$ (necessarily supported on $X_i$) be a simple module satisfying $\text{Hom}_{\text{fin}}(S, T) \neq 0$. Then, for every $M \in \mathbb{C}S_i \otimes \mathcal{H}_q(p)$-mod, the $\pi_1(R^{reg}/(S_i \times S_p))$-module $KZ^i(T) \otimes \mathbb{C}M$ factors through $\mathbb{C}S_i \otimes \mathcal{H}_q(p)$.

We will see, Lemma 6.2.5, that in fact every irreducible HC $(H_c, H_c)$-bimodule supported on $\mathcal{Z}_i$ has the form $\text{Hom}_{\text{fin}}(S, T)$, where $S \in O^i_c$ is the module such that $KZ^i_c$ is the trivial representation of $\mathbb{C}S_i \otimes \mathcal{H}_q(p)$, and $T$ is such that $KZ^i_c(T) = \lambda \otimes \mathbb{C}$, where $\lambda$ is any representation of $S_i$, and $\mathbb{C}$ is the trivial representation of $\mathcal{H}_q(p)$, i.e., that on which all $T_i$ act by $1$.

Let us show now that the category $HC_{\mathcal{L}_i}(H_c, H_c)$ is actually semisimple for $i = 0, \ldots, \lfloor n/m \rfloor$. Consider the subgroup $W := S^{x_i}_m \subseteq S_n$. Note that $\Xi := N_{S_n}(W)/W$ may be identified with $S_i \times S_{n-\cdot m}$. Now recall the restriction functor $\bullet|_{W}^{S_{n}} : HC_{\mathcal{L}_i}(H_c, H_c) \rightarrow HC_{N}^{\Xi}(H_c, H_c)$, that identifies the source category with a full subcategory of the target category closed under taking subquotients and tensor products, cf. Section 3.4. Here, $H_c$ is the rational Cherednik algebra of $W$. But the algebra $H_c$ is isomorphic to $H_c(m)^{\otimes 1}$, and this algebra has a unique irreducible finite-dimensional module, hence also a unique irreducible finite-dimensional bimodule. So $HC_{N}^{\Xi}(H_c, H_c) \cong \Xi$-rep, and similarly to the proof of Proposition 3.4.6 we get the following result.

Lemma 6.1.6. Let $c = r/m$, $\text{gcd}(r; m) = 1$, $1 < m \leq n$. Then, for every $i = 0, \ldots, \lfloor n/m \rfloor$, the category $HC_{\mathcal{L}_i}(H_c, H_c)$ is equivalent to the category of representations of $\Xi/N$ for some normal subgroup $N \subseteq \Xi$. In particular, it is semisimple.

In Theorem 6.3.8, we will see that, upon identifying $\Xi = S_i \times S_{n-\cdot m}$, the normal subgroup $N$ in Lemma 6.1.6 is precisely the factor $S_{n-\cdot m}$, and so $HC_{\mathcal{L}_i}(H_c, H_c) \cong S_i$-rep.
6.2 Functor $\Phi_c$

6.2.1 Case $c = r/n$

We now start studying the functor $\Phi_c := \bullet \otimes_{H_c} \Delta_c(\text{triv}) : \text{HC}(H_c, H_c) \to \mathcal{O}_c$ where $c > 0$ is a singular parameter. We start with the easiest case, which is when $c = r/n$, $\gcd(r; n) = 1$. In this case, thanks to Theorem 5.1.1, we have that there exists a unique irreducible HC $H_c$-bimodule with full support, which necessarily has to coincide with the unique non-trivial proper two-sided ideal $\mathcal{J}$ of $H_c$. Since there is a unique irreducible finite-dimensional $H_c$-module, cf. [BEG2], we have that there is a unique irreducible finite-dimensional HC $H_c$-bimodule, which is $M := H_c/\mathcal{J}$. Note that these are all the irreducible HC $H_c$-bimodules, this follows from the description of the possible supports of HC bimodules that was recalled in Section 6.1.2. Our first task will be to compute the indecomposable bimodules. After that, we will prove Theorem 6.1.1 in this case.

Indecomposable bimodules

Note that the bimodule $\mathcal{J}$ does not have self-extensions, this follows because $\text{HC}(H_c, H_c)$ is a semisimple category, Theorem 5.1.1. Also note that, since the category of finite-dimensional $H_c$-modules is semisimple, cf. [BEG2, Proposition 1.12], $M$ does not have self-extensions. It is clear that $\text{Ext}(M, \mathcal{J}) \neq 0$, as the regular bimodule $H_c$ is a non-split extension of $\mathcal{J}$ by $M$. Here and for the rest of this section, we denote $\text{Ext} := \text{Ext}_{H_c}^1$. On the other hand, Bezrukavnikov and Losev construct in [BL, Section 7.6] a non-split extension of $M$ by $\mathcal{J}$, the so-called double wall-crossing bimodule $D$.

Our goal now is to show that $M, \mathcal{J}, H_c$ and $D$ form a complete list of indecomposable HC $H_c$-bimodules. This is a consequence of the following result.

Proposition 6.2.1. The following is true:
(i). \( \text{Ext}(H_c, M) = 0 \). (ii). \( \text{Ext}(M, H_c) = 0 \). (iii). \( \dim(\text{Ext}(M, J)) = 1 \).

(iv). \( \text{Ext}(H_c, J) = 0 \). (v). \( \text{Ext}(J, H_c) = 0 \). (vi). \( \text{Ext}(M, D) = 0 \).

(vii). \( \text{Ext}(D, J) = 0 \). (viii) \( \text{Ext}(J, D) = 0 \).

(ix) \( \dim(\text{Ext}(J, M)) = 1 \).

Proof. We show that (i) holds more generally, namely, we have the following result.

**Lemma 6.2.2.** Let \( H_c \) be any rational Cherednik algebra of type A (we do not put restrictions on the parameter \( c \)), and let \( M \) be an irreducible Harish-Chandra \( H_c \)-bimodule with minimal support. Then, \( \text{Ext}(H_c, M) = 0 \).

**Proof.** We know that \( \text{Ext}^\bullet(H_c, M) = \text{HH}^\bullet(H_c, M) \), where \( \text{HH}^\bullet \) denotes Hochschild cohomology, so we need to compute \( \text{HH}^1(H_c, M) \). It is well known that this is the space of outer derivations (i.e. the space of derivations modulo the space of inner derivations). Now, let \( \delta : H_c \to M \) be a derivation. Since \( J^2 = J \), Subsection 6.1, the Leibniz rule implies that \( \delta(J) = 0 \), so \( \delta \) factors through the quotient algebra \( H_c/J \). This implies that \( \text{HH}^1(H_c, M) = \text{HH}^1(H_c/J, M) \) (note that \( M \) is an \( H_c/J \)-bimodule since \( \text{RAnn}(M) = \text{LAnn}(M) = J \), so this last Hochschild cohomology does make sense). Now, both \( H_c/J \) and \( M \) are irreducible HC bimodules with minimal support. Recall from Lemma 6.1.6 that the category of HC bimodules with minimal support is semisimple. Then, \( \text{Ext}(H_c/J, M) = 0 \), which implies that \( \text{HH}^1(H_c/J, M) = \text{Ext}_{H_c/J\text{-bimod}}(H_c/J, M) = 0 \). \( \Box \)

Then (i) is a special case of Lemma 6.2.2. Note that (ii) and (v) are consequences of Proposition 3.4.7. Now (iii) is a consequence of (ii): we have a long exact sequence

\[ 0 \to \text{Hom}(M, J) \to \text{Hom}(M, H_c) \to \text{Hom}(M, M) \to \text{Ext}(M, J) \to \text{Ext}(M, H_c) \to \cdots \]

Now, both \( \text{Hom}(M, H_c) \) and \( \text{Ext}(M, H_c) \) are 0, so \( \text{Hom}(M, M) \to \text{Ext}(M, J) \) must be an isomorphism and the claim follows. Again using long exact sequences, we can see that \( \dim(\text{Ext}(H_c, J)) = -\dim(\text{Hom}(J, J)) + \dim(\text{Ext}(M, J)) = 0 \), so (iv) is proved. Statements (vi), (ix) are consequences of the following.
**Lemma 6.2.3.** Assume \( c = r/n > 0 \), with \( \gcd(r; n) = 1 \). The bimodule \( D \) is injective in the category of \( HC \, H_c(n) \)-bimodules.

**Proof.** Here we make use of the functor \( \Phi_c \) and its right adjoint, \( G := \text{Hom}_{\text{fin}}(\Delta_c(\text{triv}), \bullet) \).

Note that the object \( \Delta_c(\text{triv}) \) is projective in \( O_c \) so, thanks to Lemma 3.3.13, \( \Phi_c \) is exact and \( \text{Hom}_{\text{fin}}(\Delta_c(\text{triv}), \bullet) \) sends injectives to injectives. So the lemma will follow if we find an injective object \( E \in O_c \) with \( D \cong G(E) \).

First, we remark that \( \Delta_c(\text{triv}) \) has a unique nonzero proper submodule, say \( I \), see [BEG2, Theorem 1.3]. The costandard module \( \nabla_c(\text{triv}) \) is injective in category \( O_c \), it has a unique proper submodule isomorphic to \( L_c(\text{triv}) \) and \( \nabla_c(\text{triv})/L_c(\text{triv}) \cong I \), all of these properties follow from the construction of \( \nabla_c(\text{triv}) \), see e.g. [GGOR, Subsection 2.3]. So \( G(\nabla_c(\text{triv})) \) is injective and contains \( G(L_c(\text{triv})) = \text{Hom}_{\text{fin}}(\Delta_c(\text{triv}), L_c(\text{triv})) = \text{Hom}_{\text{fin}}(L_c(\text{triv}), L_c(\text{triv})) = M \). It follows that we have an injection \( D \hookrightarrow G(\nabla_c(\text{triv})) \). Note, however, that we have an exact sequence

\[
0 \to G(L_c(\text{triv})) \to G(\nabla_c(\text{triv})) \to G(I)
\]  

(6.1)

Now, we have that \( G(I) \subseteq G(\Delta_c(\text{triv})) = H_c \), cf. Corollary 5.2.5. It is easy to see that the inclusion \( G(I) \subseteq G(\Delta_c(\text{triv})) \) is proper, so we must have \( G(I) = J \). Thanks to the exact sequence (6.1), we conclude that the composition length of \( G(\nabla_c(\text{triv})) \) is at most 2. So \( D \cong \text{Hom}_{\text{fin}}(\Delta_c(\text{triv}), \nabla_c(\text{triv})) \) and is therefore injective. \( \square \)

Now we show that \( \text{Ext}(D, J) = 0 \). Assume we have a short exact sequence

\[
0 \to J \to X \xrightarrow{\pi} D \to 0
\]  

(6.2)

Consider the induced exact sequence \( 0 \to J \to \pi^{-1}(M) \to M \to 0 \). So either \( \pi^{-1}(M) = H_c \) or \( \pi^{-1}(M) = J \oplus M \). If \( \pi^{-1}(M) = H_c \), then the exact sequence \( 0 \to \pi^{-1}(M) \to X \to J \to 0 \) gives \( X = H_c \oplus J \), cf. (iv), which contradicts the existence of the exact sequence (6.2). Then,
we must have $\pi^{-1}(M) = \mathcal{J} \oplus M$. Using again the exact sequence $0 \to \pi^{-1}(M) \to X \to J \to 0$, we get that $X = \mathcal{J} \oplus V$, where $V$ is an extension of $M$ by $\mathcal{J}$. Then (6.2) forces $V = D$ and the sequence splits.

The proof of (viii) is similar: say that we have a short exact sequence

$$0 \to M \to X \to D$$

(6.3)

So we see that $\text{Soc}(X) = M \oplus M$ and we have an exact sequence

$$0 \to M \oplus M \to X \to J \to 0$$

An extension of $M \oplus M$ by $\mathcal{J}$ must be of the form $B \oplus M$, where $B$ is an extension of $M$ by $\mathcal{J}$. Using the short exact sequence (6.3) we see that $X \cong D \oplus M$. Finally, (x) is an easy consequence of the previous statements. \qed

Note that the previous proposition implies that both $H_c$ and $D$ are injective-projective in the category $\text{HC}(H_c, H_c)$. The injective hull of $\mathcal{J}$ coincides with the projective cover of $M$, which is $H_c$, while $D$ is both the injective hull of $M$ and the projective cover of $\mathcal{J}$. In other words, the category $\text{HC}(H_c(n), H_c(n))$ is equivalent to the category of representations of the quiver

$$\bullet \xleftrightarrow{\alpha} \bullet \xleftrightarrow{\beta} \bullet$$

with relations $\alpha \beta = \beta \alpha = 0$. It follows, in particular, that the homological dimension of $\text{HC}(H_c, H_c)$ is infinite.

**Proof of Theorem 6.1.1, case $c = r/n$**

We now show Theorem 6.1.1 in the case where $c = r/n > 0$, $\gcd(r; n) = 1$. Since we know the indecomposable HC bimodules and the Krull-Schmidt theorem holds in the category $\text{HC}(H_c, H_c)$ (this is an easy consequence of Corollary 3.3.14) we can actually do this \"by
hand”. First of all, note that the functor $\Phi_c$ is faithful. Indeed, $\Phi_c(M) = L_c(\text{triv})$, $\Phi_c(J) = I$ and, since $M$ and $J$ are the only irreducible HC bimodules up to isomorphism, it follows that $\Phi(B) \neq 0$ for any HC $(H_c,H_c)$-bimodule $B$. Faithfulness of $\Phi_c$ now follows from the general fact that an exact functor that does not kill any nonzero object is faithful. By standard properties of adjunctions, it follows that the unit of adjunction $B \rightarrow G\Phi_c(B)$ is injective for every $B \in \text{HC}(H_c,H_c)$.

Now, we have that $\Phi_c(H_c) = \Delta_c(\text{triv})$, from where it follows by exactness that $\Phi_c(J) = I$, $\Phi_c(M) = L_c(\text{triv})$. The module $\Phi_c(D)$ is an extension of $L_c(\text{triv})$ by $I$. This extension is non-split since $D$ embeds into $G(\Phi_c(D))$. So $\Phi_c(D) = \nabla_c(\text{triv})$. Note that it follows that the unit of adjunction $\text{id}_{\text{HC}(H_c,H_c)} \rightarrow G\Phi_c$ is an isomorphism, so $\Phi_c$ is a fully faithful embedding. The claim about its image being closed under subquotients follows immediately from our computations, as well as the claim about the supports.

**Example 6.2.4.** Let $W = S_2$, $c = 1/2$. So there are two irreducible modules in category $O_c$, $L_c(\text{triv})$, which is finite-dimensional, and $L_c(\text{sign})$, which coincides with the socle of $\Delta_c(\text{triv})$. Both irreducibles are in the image of $\Phi_c$. However, $\Phi_c$ is not an equivalence, for $O_c$ being a highest weight category has finite homological dimension, while we have seen that $\text{HC}(H_c,H_c)$ has infinite homological dimension. Alternatively, this claim can be seen from the fact that $O_c$ is equivalent to the principal block of category $O(\mathfrak{sl}_2)$. This implies that the image of $\Phi_c$ in this case is not closed under extensions.

**6.2.2 General case**

$\Phi_c$ is faithful.

Let us show that $\Phi_c$ is faithful in the general case. Since $\Phi_c$ is exact, it is enough to show that $\Phi_c(B) \neq 0$ for any irreducible HC bimodule $B$.

**Lemma 6.2.5.** Let $c = r/m > 0$, $1 < m \leq n$, $\gcd(r,m) = 1$. Let $B \in \text{HC}(H_c,H_c)$. Then, $\Phi_c(B) = B \otimes_{H_c} \Delta_c(\text{triv})$ is nonzero.
Proof. Let \( i \in \{0, \ldots, \lfloor n/m \rfloor \} \) be such that \( \text{SS}(B) = \mathcal{L}_i \). If \( i = 0 \), then the result follows from Lemma 5.2.2. So we may assume \( i > 0 \). Let \( \mathcal{W} := S_m^i \subseteq S_n \), so that \( B|_{\mathcal{W}} \) is finite-dimensional. Now \( H_c = H_c(m)^{\otimes i} \), so thanks to [BEG2] \( \Delta_c(\text{triv}_\mathcal{W}) \) has a finite-dimensional quotient. So \( B|_{\mathcal{W}} \otimes \Delta_c(\text{triv}_\mathcal{W}) \) is nonzero. But this is \( \text{Res}_{\mathcal{W}}(B \otimes_{H_c} \Delta_c(\text{triv})) \), see Lemmas 2.4.6 and 3.3.2. We are done. \( \square \)

It follows that \( \Phi_c \) is faithful.

\( \Phi_c \) is full.

Now we show that \( \Phi_c \) is full. In order to do this, we will show that the unit map \( \text{id}_{\text{HC}(H_c, H_c)} \rightarrow G \Phi_c \) is an isomorphism where, recall, \( G : \mathcal{O}_c \rightarrow \text{HC}(H_c, H_c) \) is \( \text{Hom}_{\mathfrak{f}1\text{in}}(\Delta_c(\text{triv}), \bullet) \). Since \( \Phi_c \) is faithful, the unit is always injective.

**Lemma 6.2.6.** Let \( B \in \text{HC}(H_c, H_c) \). Then, the adjunction map \( B \rightarrow \text{Hom}_{\mathfrak{f}1\text{in}}(\Delta_c(\text{triv}), B \otimes_{H_c} \Delta_c(\text{triv})) \) is an isomorphism.

Proof. Denote \( N := \Phi_c(B) \in \mathcal{O}_c \). Since \( \Phi_c \) is exact and faithful, to check that the map \( B \rightarrow \text{Hom}_{\mathfrak{f}1\text{in}}(\Delta_c(\text{triv}), N) \) is an isomorphism, it is enough to check that \( \text{Hom}_{\mathfrak{f}1\text{in}}(\Delta_c(\text{triv}), N) \otimes_{H_c} N \) is isomorphic to \( N \). Note that, since \( N \) is in the image of \( \Phi_c \), we have an epimorphism \( \text{Hom}_{\mathfrak{f}1\text{in}}(\Delta_c(\text{triv}), N) \twoheadrightarrow N \). We show that its kernel is zero. To do so, let \( \ell := \lfloor n/m \rfloor \) and let \( \mathcal{W} := S_m^\ell \subseteq S_n \), and consider \( \text{Res}_{\mathcal{W}}^S N \). This functor is exact and, by our choice of \( \ell \), does not kill nonzero modules. So it is enough to check that the induced epimorphism

\[
\text{Res}_{\mathcal{W}}^S(\text{Hom}_{\mathfrak{f}1\text{in}}(\Delta_c(\text{triv}), N) \otimes_{H_c} \Delta_c(\text{triv})) \twoheadrightarrow \text{Res}_{\mathcal{W}}^S(N) \quad (6.4)
\]

is an isomorphism, i.e. it is injective. Recall that we have \( \text{Res}_{\mathcal{W}}^S(\text{Hom}_{\mathfrak{f}1\text{in}}(\Delta_c(\text{triv}), N) \otimes_{H_c} \Delta_c(\text{triv})) = \text{Hom}_{\mathfrak{f}1\text{in}}(\Delta_c(\text{triv}), N) \otimes_{\mathcal{W}} \Delta_c(\text{triv}_\mathcal{W}) \). By construction of the restriction functor, we have that \( \text{Hom}_{\mathfrak{f}1\text{in}}(\Delta_c(\text{triv}), N) \otimes_{\mathcal{W}} \Delta_c(\text{triv}_\mathcal{W}) \) embeds into \( \text{Hom}_{\mathfrak{f}1\text{in}}(\Delta_c(\text{triv}_\mathcal{W}), \text{Res}_{\mathcal{W}}^S(N)) \), so by
exactness of the functor • ⊗_{H_c} \Delta_c(\text{triv}_W), it is enough to show that the morphism

$$\text{Hom}_{\text{fin}}(\Delta_c(\text{triv}_W), \text{Res}_{W}^{S_n}(N)) \otimes_{H_c} \Delta_c(\text{triv}_W) \to \text{Res}_{W}^{S_n}(N)$$

is an isomorphism. Since \(H_c = H_c(m)^{\otimes \ell}\), this follows by the results of Section 6.2.1, provided we show that \(\text{Res}_{W}^{S_n}(N)\) belongs to the image of the functor • ⊗_{H_c} \Delta_c(\text{triv}_W). This follows from (6.4) and results of Section 6.2.1, namely, that for \(c = r/m\) the image of \(\Phi_c\) is closed under subquotients. We are done.

**Remark 6.2.7.** Note that it follows from the proof of Lemma 6.2.6 that, whenever \(N \in \mathcal{O}_c\) is such that there exists a HC bimodule \(B\) and an epimorphism \(B \otimes_{H_c} \Delta_c(\text{triv})\) then we have

$$\text{Hom}_{\text{fin}}(\Delta_c(\text{triv}), N)_{|_{W}} = \text{Hom}_{\text{fin}}(\Delta_c(\text{triv}_W), \text{Res}_{W}^{S_n}(N))$$

where, recall, \(W = S_m^\times \subseteq S_n\), where \(c = r/m > 0\) and \(\ell := \lfloor n/m \rfloor\).

**The image of \(\Phi_c\) is closed under subquotients**

Let us now show that the image of the functor \(\Phi_c\) is closed under subquotients. First, we show that it is closed under quotients.

**Lemma 6.2.8.** Let \(M := B \otimes_{H_c} \Delta_c(\text{triv})\) for some \(B \in \text{HC}(H_c, H_c)\), and assume that \(f : M \twoheadrightarrow N\) is an epimorphism. Then, there exists a quotient \(B'\) of \(B\) such that \(B' \otimes_{H_c} \Delta_c(\text{triv}) = N\).

**Proof.** Note that by Lemma 6.2.6 we may asumme that \(B = \text{Hom}_{\text{fin}}(\Delta_c(\text{triv}), M)\). Consider the induced morphism

$$B = \text{Hom}_{\text{fin}}(\Delta_c(\text{triv}), M) \xrightarrow{f} \text{Hom}_{\text{fin}}(\Delta_c(\text{triv}), N)$$

and let us denote by \(B'\) its image. So we have a map \(B' \otimes_{H_c} \Delta_c(\text{triv}) \to N\) which is an epimorphism since \(f\) is an epimorphism. Let us show that it is injective. So let \(W := S_m^\times \subseteq S_n\) where \(\ell := \lfloor n/m \rfloor\). It is enough to check that the induced epimorphism \(\text{Res}_{W}^{S_n}(B' \otimes_{H_c} \Delta_c(\text{triv})) \to \text{Res}_{W}^{S_n}(N)\) is an isomorphism. Since \(H_c = H_c(m)^{\otimes \ell}\), this follows by the results of Section 6.2.1, provided we show that \(\text{Res}_{W}^{S_n}(N)\) belongs to the image of the functor • ⊗_{H_c} \Delta_c(\text{triv}_W). This follows from (6.4) and results of Section 6.2.1, namely, that for \(c = r/m\) the image of \(\Phi_c\) is closed under subquotients. We are done. \(\square\)
\[ \Delta_c(\text{triv}) \to \text{Res}_{H_c}^N(N) \] is an isomorphism. This follows exactly as in the proof of Lemma 6.2.6.

Let us now show that the image of \( \Phi_c \) is closed under submodules. So let \( M = B \otimes_{H_c} \Delta_c(\text{triv}) \) and \( N' \subseteq M \). By Lemma 6.2.8 we know that there exists a quotient \( B' \) of \( B \) such that \( B' \otimes_{H_c} \Delta_c(\text{triv}) = N \). If we let \( B'' \) be the kernel of the epimorphism \( B \to B' \), exactness of \( \Phi_c \) shows that \( N' = B'' \otimes_{H_c} \Delta_c(\text{triv}) \).

Finally, we remark that (2) of Theorem 6.1.1 is a consequence of Lemma 3.2.3. This finishes the proof of Theorem 6.1.1. Let us state an easy consequence.

**Corollary 6.2.9.** Assume \( c \not\in \mathbb{R}_{<0} \). For any \( B \in \text{HC}(H_c, H_c) \), there is an order-preserving bijection between sub-bimodules of \( B \) and submodules of \( \Phi_c(B) \). In particular, there is an order-preserving bijection between the set of ideals of \( H_c \) and the set of submodules of \( \Delta_c(\text{triv}) \), which is given by \( \mathcal{J} \mapsto \mathcal{J}\Delta_c(\text{triv}) \).

The statement about ideals in \( H_c \) and submodules of the polynomial representation \( \Delta_c(\text{triv}) \) is not new. The set of ideals of \( H_c \) was calculated by Losev in [L3], while the set of submodules of the \( \Delta_c(\text{triv}) \) was calculated by Etingof-Stoica in [ES].

Another consequence of Theorem 6.1.1 is that we can give an alternative description of the double wall-crossing bimodule \( D \) of [BL, Section 7]. There, the double wall-crossing bimodule is constructed by taking the derived tensor product of two wall-crossing bimodules (one that crosses a wall and one that goes back) and observing that the homology of this derived tensor product is concentrated in degree 0. The main property of the double wall-crossing bimodule is that it is indecomposable and has a composition series in which the composition factors are those appearing in the regular bimodule \( H_c \), but the order is the opposite, see [BL, Theorem 7.7].

**Proposition 6.2.10.** Let \( c \not\in \mathbb{R}_{<0} \). Then, the double wall-crossing bimodule \( D \) is isomorphic to \( \text{Hom}^\text{fin}_{H_c}(\Delta_c(\text{triv}), \nabla_c(\text{triv})) \). If \( c \in \mathbb{R}_{<0} \) the result is still valid after replacing \( \text{triv} \) with \( \text{sign} \).
Proof. The proposition is only interesting when $c$ is singular, so we assume $c = r/m > 0$, $\gcd(r; m) = 1$, $1 < m \leq n$. The bimodule $\text{Hom}_{\mathfrak{t} \mathfrak{l} \mathfrak{n}}(\Delta_c(\text{triv}), \nabla_c(\text{triv}))$ is injective and its socle coincides with $H_c/J_{[n/m]}$. So there exists a monomorphism $D \to \text{Hom}_{\mathfrak{t} \mathfrak{l} \mathfrak{n}}(\Delta_c(\text{triv}), \nabla_c(\text{triv}))$. That this is an isomorphism now follows because the composition length of the bimodule $\text{Hom}_{\mathfrak{t} \mathfrak{l} \mathfrak{n}}(\Delta_c(\text{triv}), \nabla_c(\text{triv}))$ is at most the composition length of $H_c$, which coincides with the composition length of $D$. \hfill \Box

**Corollary 6.2.11.** The double wall-crossing bimodule $D$ is injective in $\text{HC}(H_c(n), H_c(n))$.

### 6.3 Irreducible Harish-Chandra bimodules.

In this section, we compute the irreducible modules in the image of the functor $\Phi_c$. Before, let us fix some notation. Recall that we are taking $c = r/m > 0$, with $1 < m \leq n$, $\gcd(r; m) = 1$, and $\ell := \lfloor n/m \rfloor$. It follow from results in [ES] that for every $i = 0, \ldots, \ell$, the polynomial representation $\Delta_c(\text{triv})$ has a unique irreducible subquotient supported on $X_i$, let us denote this irreducible module by $S_i$. In fact, we have $S_i = J_{i+1} \Delta_c(\text{triv})/J_i \Delta_c(\text{triv})$ where, recall, $\{0\} = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_{[n/m]+1} = H_c$ are the ideals in the rational Cherednik algebra $H_c$. It follows from Theorem 6.1.1 and Lemma 3.2.3 that every irreducible $\text{HC} H_c$-bimodule has the form $\text{Hom}_{\mathfrak{t} \mathfrak{l} \mathfrak{n}}(S_i, L)$ for some $i = 0, \ldots, \ell$, where $L \in \mathcal{O}_c$ is an irreducible module in category $\mathcal{O}_c$ supported on $X_i$.

#### 6.3.1 Results of Wilcox and consequences

Let us recall more precisely the results of Wilcox [Wi], that says that the quotient category $\mathcal{O}_c^i/\mathcal{O}_c^{i+1}$ is equivalent to the category of representations of $\mathbb{C} S_i \otimes \mathcal{H}_q(p)$, where $p = n - mi$ and $q = \exp(2\pi \sqrt{-1}c)$. We are interested in a more concrete description of this, namely, which irreducible modules in category $\mathcal{O}_c$ have a given support, and, for an irreducible $L \in \mathcal{O}_c$ supported on $X_i$, what is its image under the functor $\text{KZ}_i : \mathcal{O}_c^i \to (\mathbb{C} S_i \otimes \mathcal{H}_q(p))\text{-mod}$.
First, recall that irreducible modules over the Hecke algebra $\mathcal{H}_q(p)$ are parametrized by $m$-regular partitions of $p$, see for example [M, Chapter 3]. We denote by $D_\nu$ the irreducible representation of $\mathcal{H}_q(p)$ corresponding to the $m$-regular partition $\nu$. The representation $D_\nu$ is trivial (i.e., it is 1-dimensional and all $T_i$ act by 1) precisely when $\nu$ is the unique $m$-trivial partition of $p$, this is an easy special case of the LLT algorithm, cf. [M, Chapter 6].

Now let $\lambda$ be a partition of $n$, and decompose it as $\lambda = m\mu + \nu$, where $\nu$ is $m$-regular. The following is part of [Wi, Theorem 1.8].

**Proposition 6.3.1.** Assume $c = r/m > 0$, $\gcd(r; m) = 1$, $1 < m \leq n$. The support of the irreducible module $L_c(\lambda)$ is $X^{[\mu]}$. Moreover, the functor $KZ^{[\mu]}$ sends $L_c(\lambda)$ to the representation $\mu \otimes D_\nu$ of $S^{[\mu]} \otimes \mathcal{H}_q(n - m|\mu|)$.

Together with Lemma 6.1.5, Proposition 6.3.1 already implies one direction of Theorem 6.1.3.

**Corollary 6.3.2.** Let $c = r/m > 0$, $\gcd(r; m) = 1$, $1 < m \leq n$. Let $\lambda$ be a partition of $n$, and let $\lambda = m\mu + \nu$ be it decomposition so that $\nu$ is $m$-regular. Assume that $\text{Hom}_{\text{fin}}(S^{[\mu]}, L_c(\lambda)) \neq 0$ (equivalently, $L_c(\lambda) \in \text{im}(\Phi_c)$). Then, $\nu$ is $m$-trivial.

**Proof.** According to Lemma 6.1.5, for every representation $N \in (S^{[\mu]} \otimes \mathcal{H}_q(n - m|\mu|))$-mod, the $\pi_1((R^{reg}/(S^{[\mu]} \times S_{n-m|\mu|}))-\text{representation} \ N \otimes \text{KZ}^{[\mu]}(L_c(\lambda))$ factors through $S^{[\mu]} \otimes \mathcal{H}_q(n - m|\mu|)$. But this can only happen if $D_\nu$ is the trivial representation of $\mathcal{H}_q(n - m|\mu|)$. \qed

We will see that, in fact, the converse of the previous result holds, thus proving Theorem 6.1.3. In order to do this, we are going to count the number of irreducible objects of $HC_{\mathcal{L}_i}(H_c, H_c)$. In particular, we will see that it equals the number of partitions of $i$. This will show our desired result.
6.3.2 Bimodules with minimal support.

We give a complete description of the category of HC $H_c(n)$-bimodules with minimal support when the parameter $c$ has the form $c = r/m$, for $m$ a divisor of $n$. In particular, we show that it is equivalent to the category of representations of $S_{n/m}$. Throughout this section, we denote $\ell := n/m$. For convenience, we assume first that $c > 0$, we will deal with the case $c < 0$ at the end of this section. The following is our main result in this section.

**Proposition 6.3.3.** Let $c := r/m$, where $\gcd(r; m) = 1$ and $m$ is a divisor of $n$. Let $\ell := n/m$. Then, the category $HC_{L_\ell}(H_c(n), H_c(n))$ of HC $H_c$-bimodules with minimal support is equivalent, as a monoidal category, to the category of representations of $S_{\ell}$.

The proof of Proposition 6.3.3 will be done by induction on $r$. The proof for the case $r = 1$ is based on a remarkable symmetry result obtained in [CEE], see also [EGL]. There is a symmetry of parameters for the simple quotients of spherical rational Cherednik algebras, namely, for positive integers $n, N$ (not necessarily coprime) consider the Cherednik algebras $H_{n/n}(n)$ and $H_{n/N}(N)$, with maximal ideals $J_{\text{max}}$ and $J'_{\text{max}}$, respectively. Both parameters are spherical so $eJ_{\text{max}}e$, $e'J'_{\text{max}}e'$ are the maximal ideals of the spherical Cherednik algebras $A_{n/n}(n)$ and $A_{n/N}(N)$, respectively. Here, $e \in \mathbb{C}S_n$ and $e' \in \mathbb{C}S_N$ denote the trivial idempotents in their respective group algebras.

**Proposition 6.3.4** (Proposition 9.5, [CEE] and Proposition 7.7 [EGL]). There is an isomorphism between the algebras $A_{n/n}(n)/eJ_{\text{max}}e$ and $A_{n/N}(N)/e'J'_{\text{max}}e'$. Moreover, an isomorphism maps (the images of) the subalgebras $\mathbb{C}[R_n]^{S_n}, \mathbb{C}[R_n^*]^{S_n}$ to (the images of) the subalgebras $\mathbb{C}[R_N]^{S_N}, \mathbb{C}[R_N^*]^{S_N}$, respectively.

Since HC $A_c$-bimodules with minimal support are precisely the ones whose annihilator is the maximal ideal in $A_c$, we have the following easy consequence of Proposition 6.3.4.

**Proposition 6.3.5.** The isomorphism $A_{n/n}(n)/eJ_{\text{max}}e \cong A_{n/N}(N)/e'J'_{\text{max}}e'$ induces a tensor equivalence between the categories of minimally supported HC $A_{n/n}(n)$-bimodules and
minimally supported HC $A_{n/N}(N)$-bimodules.

Now, the parameter $c = r/m > 0$, with $\gcd(r; m) = 1$ and $m\ell = n$ for $k \in \mathbb{Z}_{>1}$, is spherical for the rational Cherednik algebra associated to $S_n$. Then, Proposition 6.3.5 has the following consequence.

**Corollary 6.3.6.** The categories of minimally supported HC $H_{r/m}(n)$-bimodules and minimally supported HC $H_{m/r}(r\ell)$-bimodules are equivalent as monoidal categories.

Now the case $r = 1$ of Proposition 6.3.3 is an easy consequence of Corollary 6.3.6 and Theorem 5.4.4, that asserts that the category of HC $H_{m}(k)$-bimodules is equivalent, as a monoidal category, to the category of representations of $S_\ell$, see also [BEG, Theorem 8.5]. To complete the proof of Proposition 6.3.3 we use an inductive argument for which we will need the theory of shift functors for rational Cherednik algebras, cf. Definition 3.1.5, which in type A originally appeared in [GS, Section 3]. Consider the $(A_{c+1}(n), A_c(n))$-bimodule $Q_c^{c+1} := eH_{c+1}(n)e_{\text{sign}}$. Here, $e_{\text{sign}}$ denotes the sign idempotent, $e_{\text{sign}} = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sign}(\sigma)\sigma$. The bimodule $Q_c^{c+1}$ is HC, cf. Proposition 3.1.4. The functor $F : A_c(n)\text{-mod} \to A_{c+1}(n)\text{-mod}$ given by $F(M) = Q_c^{c+1} \otimes_{A_c} M$ is then an equivalence of categories, see [BE, Corollary 4.3]. A quasi-inverse functor is given by tensoring with the $(A_{c+1}(n), A_c(n))$-bimodule $P_c^{c+1} := \delta^{-1}e_{\text{sign}}H_{c+1}(n)e$, see Section 3 in [GS] (we remark that [GS] assumes that $c \not\in \frac{1}{2} + \mathbb{Z}$, an assumption that was later removed in [BE, Corollary 4.3]). The bimodule $P_c^{c+1}$ is also HC. It then follows that we have an equivalence of monoidal categories $F : HC(A_c(n), A_c(n)) \to HC(A_{c+1}(n), A_{c+1}(n)), F(B) = Q_c^{c+1} \otimes_{A_c(n)} B \otimes_{A_c(n)} P_c^{c+1}$. Clearly, this equivalence preserves the filtrations of the categories of HC bimodules by the support.

We now proceed to finish the proof of Proposition 6.3.3. So let $r, m, n, \ell$ be as in the statement of that proposition. We work over spherical subalgebras, and we make the following inductive assumption:

*For every $0 < r' < r$ and every $m', \ell' \in \mathbb{Z}_{>0}$ with $\gcd(r', m') = 1$, the category of minimally supported bimodules $HC_{L_{r'}}(A_{r'/m'}(m'\ell'), A_{r'/m'}(m'\ell'))$ is equivalent, as a monoidal category,
to the category of representations of $S_\ell$.

Clearly, Proposition 6.3.5, together with Theorem 5.4.4, give the base of induction. Now, using Proposition 6.3.5 again, we have that the categories $HC_{\mathcal{L}_\ell}(A_{r/m}(n), A_{r/m}(n))$ and $HC_{\mathcal{L}_\ell}(A_{m/r}(\ell r), A_{m/r}(\ell r))$ are equivalent as monoidal categories. Using shift functors, we get a tensor equivalence between $HC_{\mathcal{L}_\ell}(A_{m/r}(\ell r), A_{m/r}(\ell r))$ and $HC_{\mathcal{L}_\ell}(A_{r'/r}(\ell r), A_{r'/r}(\ell r))$, where $0 < r' < r$. By our inductive assumption, this is tensor equivalent to $S_\ell$-rep. Proposition 6.3.3 now follows by sphericity, since we are assuming our parameter $c$ is positive.

The description of the irreducible objects in $HC_{\mathcal{L}_\ell}(H_{r/m}(n), H_{r/m}(n))$ follows at once from Theorem 6.1.1. We have that $S^\ell = L_c(m \text{triv}_\ell) = L_c(\text{triv})$, while the irreducible modules in $\mathcal{O}_c$ are those of the form $L_c(m\lambda)$ for a partition $\lambda \vdash \ell$. Thus, the irreducible HC bimodules with minimal support have the form $\text{Hom}_{\text{fin}}(L_c(\text{triv}), L_c(m\lambda)), \lambda \vdash \ell$.

**Remark 6.3.7.** We remark that, while $\{\text{Hom}_{\text{fin}}(L(\text{triv}), L(m\lambda)) : \lambda \vdash \ell\}$ forms a complete and irredundant list of irreducible HC bimodules with minimal support, we have that $\text{Hom}_{\text{fin}}(L(m\mu), L(m\lambda)) \neq 0$ where $\lambda, \mu$ are any partitions of $\ell$. This follows from, for example, Theorem 8.16 in [BEG], which gives a description of $\text{Hom}_{\text{fin}}(L(m\mu), L(m\lambda))$ as a direct sum of bimodules of the form $\text{Hom}_{\text{fin}}(L(\text{triv}), L(m\xi))$.

To finish this subsection, let us explain what happens when we have $c = r/m < 0$, with $\gcd(r; m) = 1$ and $m$ divides $n$, say $\ell m = n$. In this case, the category $HC_{\mathcal{L}_\ell}(H_c(n), H_c(n))$ is still equivalent to the category of representations of $S_\ell$. This follows because there is an equivalence $HC_{\mathcal{L}_\ell}(H_c(n), H_c(n)) \cong HC_{\mathcal{L}_\ell}(H_{-c}(n), H_{-c}(n))$ induced by an isomorphism $H_c(n) \rightarrow H_{-c}(n)$, mapping $R^* \ni x \mapsto x, R \ni y \mapsto y, S_n \ni \sigma \mapsto \text{sign}(\sigma)\sigma$.

### 6.3.3 Description of irreducibles

We use the results of the previous subsection and Theorem 4.1.1 to give a classification of all irreducible HC $H_c(n)$-bimodules where we assume that $c$ has the form $c = r/m > 0$, with $1 < m \leq n$ and $\gcd(r; m) = 1$. The following is the main result of this section.
**Theorem 6.3.8.** Let \( c = r/m > 0 \), with \( 1 < m \leq n, \gcd(r;m) = 1, \) and let \( i = 1, \ldots, \lfloor n/m \rfloor \). Then, the category \( \mathcal{HC}_{\xi_i}(H_{c}(n), H_{c}(n)) \) is equivalent, as a monoidal category, to the category of representations of \( S_i \).

Before proceeding to the proof of Theorem 6.3.8 we set \( W := S_i^m \subseteq S_n \) and describe the objects in the category \( \mathcal{HC}^\Xi_0(H_{c}, H_{c}) \) where, recall, \( \Xi = N_{S_n}(W)/W \). This category is equivalent to the category of representations of \( S_i \times S_{n-mi} \), this follows because the algebra \( H_c \) has a unique irreducible finite dimensional bimodule (that does not admit non-trivial self-extensions). This bimodule is \( B := \text{Hom}_C(L_{c}(\text{triv}_W), L_{c}(\text{triv}_W)) \), this is a consequence of the fact that \( L_{c}(\text{triv}_W) \) is the unique irreducible finite dimensional module over the algebra \( H_c \). Moreover, since \( c = r/m \) and \( W = S_i^m \), we have that \( H_c = H_c(m)^{\otimes i} \), and \( B = B^{\otimes i} \), where \( B \) is the unique irreducible finite dimensional bimodule over \( H_c(m) \), so \( B \) admits a \( \Xi \)-equivariant structure, where \( S_i \) permutes the tensor factors and \( S_{n-mi} \) acts trivially. Under the equivalence \( \mathcal{HC}^\Xi_0(H_{c}, H_{c}) \to (S_i \times S_{n-mi})\text{-rep}, B \) corresponds to the trivial representation. So we have the following result.

**Lemma 6.3.9.** The irreducible objects in \( \mathcal{HC}^\Xi_0(H_{c}, H_{c}) \) have the form \( B^{\otimes i} \otimes \xi \), where \( \xi \) runs over the set of irreducible representations of \( S_i \times S_{n-mi} \), which acts diagonally. The irreducibles where \( S_{n-mi} \) acts trivially correspond precisely to those representations \( B^{\otimes i} \otimes \xi \) where \( \xi \) factors through \( S_i \).

**Proof of Theorem 6.3.8.** We need to check that, if \( \xi \) is an irreducible representation of \( S_i \), then the equivariant bimodule \( B^{\otimes i} \otimes \xi \) belongs to the image of \( \bullet_{W}^{S_i} \). By Theorem 4.1.1, for every parabolic subgroup \( W' \) containing \( W \) in corank 1 we need to produce a HC \( H_{c}(W') \)-bimodule \( B' \) with \( B'_W = B \otimes \xi \), with the restricted \( N_{W'}(W)/W \)-equivariant structure. The subgroups \( W' \) have three different types. Either \( W' \cong S_m^{\times (i-2)} \times S_{2m}, W' \cong S_m^{\times (i-1)} \times S_{m+1} \) or \( W' \cong S_m^{\times i} \times S_2 \).

**Case 1.** \( W' \cong S_m^{\times (i-2)} \times S_{2m} \). So that \( N_{W'}(W)/W \cong S_2 \) acting on \( H_c = H_{r/m}(m)^{\otimes i} \) by permuting two of the tensor factors. Thanks to the results of Section 6.3.2 the functor \( \bullet_{W'}^{S_i} :
HC_{L}(H_{c}(W), H_{c}(W')) \rightarrow HC_{0}^{S_{2}}(H_{c}, H_{c}) is essentially surjective, were HC_{L}(H_{c}(W'), H_{c}(W')) denotes the category of minimally supported HC $H_{c}(W')$-bimodules. So we can certainly find a bimodule $B'$ with $B_{W'} = B \otimes \xi$.

**Case 2.** $W' \cong S_{m}^{(i-1)} \times S_{m+1}$, so that $N_{W'}(W)/W \cong \{1\}$. Thus, what we have to check here is that $B$ belongs to the image of the functor $B_{W'}$. But this follows because the image of $B_{W'}$ is closed under sub-bimodules.

**Case 3.** $W' \cong S_{m}^{i} \times S_{2}$, so that $N_{W'}(W)/W \cong S_{2}$, acting trivially on $H_{c}$. Thanks to our assumptions on $\xi$, $S_{2}$ also acts trivially on $\xi$. It also acts trivially on $B$. So we need to check that $B$, with trivial action of $S_{2}$, belongs to the image of $B_{W'}$. Upon the identification $HC_{0}^{S_{2}}(H_{c}, H_{c}) \rightarrow S_{2}$-rep, $B$ corresponds to the trivial representation. The image of the restriction functor is closed under tensor products and sub-bimodules. Since the trivial representation of $S_{2}$ is contained in $S^{\otimes 2}$ for any representation $S$ of $S_{2}$, the result follows.

Now the proof of Theorem 6.3.8 follows since, by Corollary 6.3.2, the number of irreducible objects in $HC_{L}(H_{c}(n), H_{c}(n))$ is no greater than the number of irreducible representations of $S_{1}$. □

Note that Theorem 6.3.8 and Corollary 6.3.2 imply Theorem 6.1.3. As a consequence, we have that the irreducible HC $H_{c}(n)$-bimodules have the form $\text{Hom}_{\text{fin}}(\Delta_{c}(\text{triv}), L_{c}(m\mu + \nu))$, where $\mu$ and $\nu$ are partitions such that $m|\mu| + |\nu| = n$ and $\nu$ is $m$-trivial.

### 6.4 Two-parametric case.

#### 6.4.1 Proof of Theorem 6.1.4

We study the category $HC(H_{c}(n), H_{c}(n))$ when the parameters $c, c'$ are distinct. First, we remark that if either $c$ or $c'$ is a regular parameter, then every HC $(H_{c}, H_{c'})$-bimodule has full support, and this case has been described in Theorem 5.4.4. So we may assume that both parameters $c, c'$ are singular. Since for a HC bimodule $B$ we have $SS(B) = 1$
SS(H_c/ LAnn(B)) = SS(H_c/ RAnn(B)), the description of the two-sided ideals of H_c given in Section 6.1, together with Theorem 5.4.4 imply that a necessary condition for HC(H_c, H_c') to be nonzero is that c and c' have the same denominator when expressed as irreducible fractions. Then, throughout this subsection we assume that c = r/m, c' = r'/m, gcd(r; m) = gcd(r'; m) = 1, 1 < m ≤ n.

Recall that, for i = 1, . . . , [n/m] we have the functor KZ_c^i : O_c^i → (CS_i ⊗ H_q(S_{n−mi}))-mod, where q' = exp(2π√−1c'). Let N = S_c^i ∈ O_c^i be the irreducible module with KZ_c^i(N) = triv, this is the unique irreducible subquotient of ∆_c(τ) supported on X_i, where τ = triv is c' > 0 and τ = sign if c' < 0. Then, reasoning completely analogously to the proof of Lemma 6.2.5, we have that every irreducible HC (H_c(n), H_c'(n))-bimodule supported on the closure of the symplectic leaf L_i is contained in a bimodule of the form Homfin(S_c^i, M) for an irreducible module M ∈ O_c^i and, moreover, that whenever Homfin(S_c^i, M) is nonzero then, for every module L ∈ (CS_i ⊗ H_q(n))-mod the B_i × B_{n−mi}-module KZ_c^i(M) ⊗ C L factors through the algebra CS_i ⊗ H_q(n−mi).

**Proposition 6.4.1.** Let i ∈ {1, . . . , [n/m]} and assume that n−mi ≠ 0. Then, the category HC_{L_i}(H_c, H_c') is 0 unless c − c' ∈ Z or c + c' ∈ Z.

**Proof.** Assume HC_{L_i}(H_c, H_c') ≠ 0. Let M ∈ O_c^i be irreducible such that Homfin(S_c^i, M) ≠ 0. Let KZ_c^i(M) = M_1 ⊗ M_2, where M_1 is an irreducible representation of S_i and M_2 an irreducible representation of H_q(n−mi). Then, for every representation L = L_1 ⊗ L_2 of CS_i ⊗ H_q(n−mi), we have that the module (L_1 ⊗ M_1) ⊗ (L_2 ⊗ M_2) is a representation of CS_i ⊗ H_q(n−mi).

If M_2 is the trivial representation of H_q(n−mi), this implies that {1, −q'} = {1, −q} and therefore c − c' ∈ Z. Otherwise, we get {−q, qq'} = {1, −q}, so c + c' ∈ Z. □

Now assume that c' = c + k, with c > 0 and k ∈ Z_{>0}. Then, using shift functors we have an equivalence of categories HC(H_c, H_c) ∼ HC(H_c, H_c') ∼ HC(H_c', H_c') preserving the filtration by supports so that, in particular, they descend to equivalences HC_{L_i}(H_c, H_c) ∼ HC_{L_i}(H_c, H_c') ∼ HC_{L_i}(H_c', H_c'). Since we have an isomorphism H_c(n) → H_{−c}(n) fixing
the subalgebras $\mathbb{C}[R]^{S_n}, \mathbb{C}[R^n]^{S_n}$, we also have an equivalence $HC(H_c, H_{-c-k}) \cong HC(H_c, H_c)$ preserving the filtration by supports. Similar results hold if $c < 0$ and $k \in \mathbb{Z}_{<0}$.

Assume now that $c = c' + 1$, with $-1 < c' < 0$. In this case, the shift functor is not an equivalence. However, it does induce a derived equivalence $\rho_{c'} \otimes_{H_c} : D^b(O_{c'}) \to D^b(O_c)$, see e.g. [GL, Section 5]. It follows that, if we denote by $D^b_{HC}(H_c, H_c)$ the subcategory of $D^b((H_c, H_c)-bimod)$ consisting of complexes with HC homology, then we have a derived equivalence $\rho_{c'} \otimes_{H_{c'}} : D^b_{HC}(H_{c'}, H_c) \to D^b_{HC}(H_c, H_c)$. Thus, the categories $HC(H_c, H_c)$ and $HC(H_{c'}, H_c)$ have the same number of irreducibles. We remark here that an analogous of Lemma 6.1.5 implies that the category $HC_{\mathbb{C}}(H_c, H_{c'})$ is embedded, using the restriction functor, in the category of representations of $S_i$. Hence, $HC_{\mathbb{C}}(H_c, H_{c'}) \cong S_i\text{-rep}$ for $i = 1, \ldots, [n/m]$. The same holds for the category $HC_{\mathbb{C}}(H_{c'}, H_c)$.

Note that Theorem 6.1.4 follows from Proposition 6.4.1 together with the discussion above.

### 6.4.2 Bimodules with minimal support

We now come to the case that is not covered by Proposition 6.4.1. Note that this proposition may fail when $m$ divides $n$ and we look at the categories of bimodules with minimal support. As an easy example, if $c = r/n, c' = r'/n$, $\gcd(r; n) = \gcd(r'; n) = 1$, then $HC_0(H_c, H_{c'}) \neq 0$, so it does not matter whether $c + c'$ or $c - c'$ are integers.

So assume $m$ divides $n$, say $n = m\ell, 1 < m \leq n$. Let $c = r/m, c' = r'/m$ as irreducible fractions. We have the following result.

**Proposition 6.4.2.** The category $HC_{L}(H_c(n), H_{c'}(n))$ is equivalent to the category of representations of $S_\ell$.

Note that in Proposition 6.4.2 we do not impose any other conditions on $c$ and $c'$. We just require that they are expressed as irreducible fractions with the same denominator which is a factor of $n$, with quotient $\ell$.  

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Proof. We proceed in several steps.

Step 1. We remark that, using the isomorphisms \( H_c(n) \rightarrow H_{-c}(n) \), we may assume that both \( c, c' \) are positive. Moreover, using shift functors, we may assume that \( 0 < c, c' < 1 \). So we have \( 0 < r, r' < m \leq n \). Since both \( c \) and \( c' \) are positive, they are spherical. So we can work over the spherical subalgebras \( A_c(n), A_{c'}(n) \).

Step 2. Let us introduce the following notation. For a positive integer \( N \), set \( R_N = \{ (z_1, \ldots, z_N) \in \mathbb{C}^N : \sum_{i=1}^N z_i = 0 \} \). This is, of course, the reflection representation of \( S_N \).

Let \( x_1, \ldots, x_N \) be the coordinate functions on \( \mathbb{C}^N \), and for a positive integer \( k \) let \( p_{k,N}(x) = x_1^r + \cdots + x_N^r \). So the invariant algebra \( \mathbb{C}[R_N]^{S_N} \) is generated by \( p_{2,N}(x), \ldots, p_{N,N}(x) \). Similarly, the invariant algebra \( \mathbb{C}[R_N^*]^{S_N} \) is generated by \( p_{2,N}(y), \ldots, p_{N,N}(y) \).

Step 3. For \( c > 0 \), let us denote by \( \overline{A}_c(n) \) the quotient of the spherical subalgebra \( A_c(n) \) by its unique maximal ideal. Recall the isomorphism \( \varphi_{N,M} : \overline{A}_{M/N}(N) \rightarrow \overline{A}_{N/M}(M) \) from Proposition 6.3.4. It is known that \( \varphi_{N,M} \) maps \( p_{k,N}(x) \mapsto (N/M)p_{k,M}(x) \), while \( p_{k,N}(y) \mapsto (M/N)^{k-1}p_{k,M}(y) \). Both of these assertions follow from [CEE, Section 8], see also [EGL, Section 7].

Step 4. In Steps 4-8 we are going to produce an equivalence from the category of minimally supported bimodules \( HC_{L_c}(A_c(n), A_c'(n)) \) to \( HC(A_{N_1}(\ell), A_{N_2}(\ell)) \) for some integers \( N_1, N_2 \in \mathbb{Z}_{>0} \), Proposition 6.4.2 follows from here. First of all note that, since we are taking bimodules with minimal support, \( HC_{L_c}(A_c(n), A_c'(n)) = HC_{\mathbb{Z}_{\ell}}(A_c(n), A_c'(n)) \), so we are going to think of objects in \( HC_{L_c}(A_c(n), A_c'(n)) \) as honest bimodules. So let \( B \in HC_{L_c}(A_{r/m}(n), A_{r'/m}(n)) \).

Since \( B \) has minimal support it is, in particular, a \( (\overline{A}_{r/m}(n), \overline{A}_{r'/m}(n)) \)-bimodule. Using the isomorphisms \( \varphi_{N,\ell r}, \varphi_{N,\ell r'} \), we may think of \( B \) as an \( (\overline{A}_{m/r}(\ell r), \overline{A}_{m/r'}(\ell r')) \)-bimodule, equivalently, as an \( (A_{m/r}(\ell r), A_{m/r'}(\ell r')) \)-bimodule whose left (resp. right) annihilator coincides with the maximal ideal of \( A_{m/r}(\ell r) \) (resp. of \( A_{m/r'}(\ell r') \)). By Step 3, the following operators
act locally nilpotently on $B$:

\[
    a_k(x) : b \mapsto (m/r)p_{k,\ell r}(x)b - (m/r')bp_{k,\ell r}(x) \quad (6.5)
\]

\[
    d_k(y) : b \mapsto (r/m)^{k-1}p_{k,\ell r}(y)b - (r'/m)^{k-1}bp_{k,\ell r}(y) \quad (6.6)
\]

**Step 5.** Now let $m = rk_1 + m_1$, with $0 \leq m_1 < r$, $k_1 \in \mathbb{Z}$. So we have the shift $(A_{m_1/r}(\ell r), A_{m/r}(\ell r))$-bimodule, say $P_{m_1/r,m/r}(\ell r)$. Consider $B' := P_{m_1/r,m/r}(\ell r) \otimes_{A_{m_1/r}(\ell r)} B$, which is an $(A_{m_1/r}(\ell r), A_{m'/r'}(\ell r'))$-bimodule. We claim that the operators (6.5), (6.6) act locally nilpotently on $B'$. For (6.5), this follows because $a_k(x)(b_1 \otimes b_2) = \operatorname{ad}((m/r)p_{k,\ell r}(x))(b_1) \otimes b_2 - b_1 \otimes a_k(x)(b_2)$ and the operator $\operatorname{ad}((m/r)p_{k,\ell r}(x))$ acts locally nilpotently on $P_{m_1/r,m/r}$.

The reasoning for (6.6) is the same. Now let $m' = r'k_1' + m_1'$ be division with remainder, and consider the shift $(A_{m'/r'}(\ell r'), A_{m'/r'}(\ell r'))$-bimodule $P_{m'/r',m'/r'}(\ell r')$. Let $B_1 := B' \otimes_{A_{m'/r'}(\ell r')} P_{m'/r',m'/r'}(\ell r')$. This is an $(A_{m_1/r}(\ell r), A_{m'/r'}(\ell r'))$-bimodule. It is clear that its left (resp. right) annihilator is the maximal ideal in $A_{m_1/r}(\ell r)$ (resp. $A_{m'/r'}(\ell r')$), that it is finitely generated as a left or right module, and that the operators (6.5), (6.6) act locally nilpotently on $B_1$.

**Step 6.** Now we use the isomorphisms $\varphi_{\ell r,\ell m_1}$ and $\varphi_{\ell r',\ell m_1'}$ to view $B_1$ as bimodule over $(A_{r/m_1}(\ell m_1), A_{r'/m_1'}(\ell m_1'))$. Note that the operators that act locally nilpotently on $B_1$ now are

\[
    a_k^1(x) : b \mapsto (m_1/m)p_{k,\ell m_1}(x)b - (m_1/m)b p_{k,\ell m_1}(x) \quad (6.7)
\]

\[
    d_k^1(y) : b \mapsto (m_1/m)^{k-1}p_{k,\ell m_1}(y)b - (m_1'/m)^{k-1}bp_{k,\ell m_1}(y) \quad (6.8)
\]

And we repeat the same procedure of multiplying by shift bimodules on the left and right, to get an $(A_{r_1/m_1}(\ell m_1), A_{r_1'/m_1'}(\ell m_1'))$-bimodule $B_2$, which is finitely generated as a left or right module, and on which the operators (6.7), (6.8) act locally nilpotently.

**Step 7.** Continuing with this procedure, since $\gcd(r,m) = 1 = \gcd(r',m)$, the Euclidean algorithm tells us that we are going to get to a $(A_{N_1}(\ell), A_{N_2}(\ell))$-bimodule $\tilde{B}$, where $N_1, N_2$
are integers, $\tilde{B}$ is finitely generated as either a left or right module, and the operators

$$b \mapsto mp_{k,\ell}(x)b - mbp_{k,\ell}(x) \quad (6.9)$$

$$b \mapsto (1/m)^{k-1}p_{k,\ell}(y)b - (1/m)^{k-1}bp_{k,\ell}(y) \quad (6.10)$$

act locally nilpotently. From (6.9), it follows that $\mathbb{C}[R_\ell]^{S_\ell}$ acts locally nilpotently on $\tilde{B}$. From (6.10) it follows that $\mathbb{C}[R_\ell^*]^{S_\ell}$ acts locally nilpotently on $\tilde{B}$, too. Thus, $\tilde{B} \in HC(A_{N_1}(\ell), A_{N_2}(\ell))$.

**Step 8.** It is clear that everything we have done in Steps 4-7 can be reversed. So we get a category equivalence $HC_{\mathcal{L}_i}(A_c(n), A_{c'}(n)) \cong HC(A_{N_1}(\ell), A_{N_2}(\ell))$. Since all our parameters are positive, hence spherical, we get $HC_{\mathcal{L}_i}(H_c(n), H_{c'}(n)) \cong HC(H_{N_1}(\ell), H_{N_2}(\ell))$. The latter category is equivalent to the category of representations of $S_\ell$, this follows from Theorem 5.1.1. We are done.

Let us summarize our results in the two-parametric setting in the following theorem.

**Theorem 6.4.3.** Let $c = r/m, c' = r'/m$ be such that $\gcd(r; m) = 1 = \gcd(r'; m')$, and $1 < m \leq n$. Then.

1. Let $i = 0, \ldots, \lfloor n/m \rfloor$. Then, $HC_{\mathcal{L}_i}(H_c(n), H_{c'}(n)) = 0$ unless $c - c' \in \mathbb{Z}$ or $c + c' \in \mathbb{Z}$. If $c - c' \in \mathbb{Z}$ or $c + c' \in \mathbb{Z}$, then $HC_{\mathcal{L}_i}(H_c(n), H_{c'}(n))$ is equivalent to the category of representations of $S_i$.

2. If $n - mi = 0$, then $HC_{\mathcal{L}_i}(H_c(n), H_{c'}(n))$ is equivalent to the category of representations of $S_i$, without further restrictions on the parameters $c, c'$.

Note that, in particular, if $c = r/m, c' = r'/m$ with $m\ell = n$ but neither $c - c'$ nor $c + c'$ is an integer, then $HC(H_c(n), H_{c'}(n)) = HC_{\mathcal{L}_i}(H_c(n), H_{c'}(n))$. 

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Chapter 7

Duality

In this chapter, we construct a duality on the category of Harish-Chandra bimodules for type $A$ (or, more generally, cyclotomic) rational Cherednik algebras. Our construction, however, is done at the level of algebras quantizing Nakajima quiver varieties. These algebras are special cases of the algebras that are studied in [BPW, BLPW], that is, they arise as the global sections of a quantization of a smooth conical variety. The representation theory of quantized quiver varieties has been studied in more detail in [BL, L9], where Harish-Chandra bimodules play a crucial role. The main result in this chapter is the construction of a duality functor in the category of Harish-Chandra bimodules. To obtain this functor we need to introduce categories of twisted HC bimodules. The twist here is provided by an automorphism of the quiver variety.

Since the spherical subalgebra of a cyclotomic rational Cherednik algebra is a special case of a quantized quiver variety, cf. [EGGO, Go, L2, O], we obtain, in particular, a duality in the category of HC bimodules for cyclotomic Cherednik algebras. For type $A$ Cherednik algebras, we show that this duality fixes every irreducible module.
7.1 Nakajima quiver varieties

In this section, we define Nakajima quiver varieties. These are defined as the G.I.T. Hamiltonian reduction of a reductive group acting on a representation space of a quiver. Thus, first we study this action and the respective moment map. Later, we study the G.I.T. properties of the action, giving in particular a concrete description of the semistable points (for a particular choice of stability condition). After that, we define both the affine and projective Nakajima quiver varieties, as well as their universal deformations. This section does not contain new results. Nakajima quiver varieties were first studied in [Nak], and our exposition here mostly follows [Nak2, Gi3].

7.1.1 Representation spaces and moment maps.

Let $Q = (Q_0, Q_1, s, t)$ be a quiver: $Q_0$ is the set of vertices; $Q_1$ the set of arrows; and $s, t : Q_1 \to Q_0$ the maps that to each arrow assign its starting and terminating vertex, respectively. We will consider the co-framed quiver:

\[ Q^\triangledown := (Q_0^\triangledown = Q_0 \sqcup Q'_0, Q_1^\triangledown := Q_1 \sqcup Q'_1, s^\triangledown, t^\triangledown) \]

where $Q'_0 := \{k' : k \in Q_0\}$ is a copy of $Q_0$; $Q'_1 := \{\alpha_i : i \in Q_0\}$ is a set of arrows indexed by $Q_0$; $s^\triangledown|_{Q'_1} = s$, $s(\alpha_k) = k$, $t^\triangledown|_{Q_1} = t$, and $t^\triangledown(\alpha_k) = k'$. In pedestrian terms, the quiver $Q^\triangledown$ is obtained from $Q$ by attaching to each vertex $k \in Q$ a coframing vertex $i'$ with an arrow $k \to k'$.

Now let $v, w \in \mathbb{Z}_{<0}$ be dimension vectors. We can consider the space of representations:

\[ R(v, w) := \text{Rep}(Q^\triangledown, (v, w)) = \bigoplus_{\alpha \in Q_1} \text{Mat}(v_{s(\alpha)}, v_{t(\alpha)}) \oplus \bigoplus_{k \in Q_0} \text{Mat}(v_k, w_k) \]

We will denote an element of $R(v, w)$ by $(X_\alpha, i_k)_{\alpha \in Q_1, k \in Q_0}$. Note that the reductive group $G := \text{GL}(v) := \prod_{k \in Q_0} \text{GL}(v_k)$ acts on the space $R(v, w)$ by changing basis. For $g = \prod_{k \in Q_0} g_k$, where $g_k = (g_{V_k}, g_{W_k})$, and $g_{V_k}$ and $g_{W_k}$ are matrices acting on $v_k$ and $w_k$, respectively, we have:

\[ g \cdot (X_\alpha, i_k) = (g_{V_{s(\alpha)}}, g_{V_{t(\alpha)}}) X_\alpha (g_{W_k}^{-1}, g_{W_{i_k}}) \]

This action induces an action on the space of representations $R(v, w)$, and the goal is to find the semistable points of this action, which are the points that are fixed by a stabilizer group of a reductive group acting on the space $R(v, w)$.
We take the induced action of $G$ on $T^*R$. Since this is an induced action, it is Hamiltonian, let us describe the moment map. First, we will describe this action in linear-algebraic terms. For any $m > 0$ we have a $\text{GL}(n)$-equivariant isomorphism $\text{Mat}(n,m)^* \cong \text{Mat}(m,n)$ which is given by the trace form. Thus, we have a $G$-equivariant identification of $T^*R$ with the space of representations of the double quiver $Q^\vee$

$$T^*R = \bigoplus_{\alpha \in Q_1} (\text{Mat}(v_{s(\alpha)}, v_{t(\alpha)}) \oplus \text{Mat}(v_{t(\alpha)}, v_{s(\alpha)})) \oplus \bigoplus_{k \in Q_0} (\text{Mat}(v_k, w_k) \oplus \text{Mat}(w_k, v_k))$$

We will denote an element of $T^*R$ by $(X_\alpha, Y_\alpha, i_k, j_k)$. Thus, the action of $G$ is given by

$$g.(X_\alpha, Y_\alpha, i_k, j_k) = (g_{t(\alpha)} X_\alpha g_{s(\alpha)}^{-1}, i_k g_k^{-1}, g_k j_k)$$

And the moment map is given by

$$\mu : T^*R \to \mathfrak{gl}(v)^* = \mathfrak{gl}(v) =: g, \quad (X_\alpha, Y_\alpha, i_k, j_k) \mapsto \sum_{\alpha \in Q_1} (Y_\alpha X_\alpha - X_\alpha Y_\alpha) - \sum_{k \in Q_0} j_k i_k$$ (7.1)

Equation (7.1) should be interpreted as follows. First, we have identified $\mathfrak{gl}(v)$ with its dual via the trace form. We have that $\mu(X_\alpha, Y_\alpha, i_k, j_k)$ is a $(Q_0)$-tuple of matrices. So what Equation 7.1 says is that, for $k \in Q_0$ in the $k$-th position we have

$$\sum_{\alpha \in Q_1 \atop s(\alpha) = k} Y_\alpha X_\alpha - \sum_{\alpha \in Q_1 \atop t(\alpha) = k} X_\alpha Y_\alpha - j_k i_k$$

The dual to this map is the comoment map $\mu^* : g \to \mathbb{C}[T^*R]$. Since the moment map $\mu$ is $G$-equivariant, for every $\lambda \in (\mathfrak{g}^*)^G$ the group $G$ acts on $\mu^{-1}(\lambda)$. For $\lambda \in (\mathfrak{g}^*)^G$, we will be interested in the affine variety
\[ \mathcal{M}_0^\lambda := \mathcal{M}_0^\lambda(v, w) := \mu^{-1}(\lambda) / G = \text{Spec}(\mathbb{C}[\mu^{-1}(\lambda)]^G) = \text{Spec}([\mathbb{C}[T^*R]/(\{\mu^*(\xi) - (\lambda, \xi) : \xi \in \mathfrak{g}\})]^G) \]

Note that the variety \( \mathcal{M}_0^\lambda \) comes equipped with a \( \mathbb{C}^\times \)-action induced by the \( \mathbb{C}^\times \) action on \( T^*R \) by dilations. In particular, the Poisson bracket on \( \mathcal{M}_0^\lambda \) has degree \(-2\). In general, \( \mathcal{M}_0^\lambda \) is singular. In some cases, we can construct resolutions of singularities by looking at G.I.T. quotients of \( T^*R \). This is what we will do next.

7.1.2 G.I.T. quotients

Note that the group of characters \( \chi(\text{GL}(v)) \) may be identified with \( \mathbb{Z}^{Q_0} \), to \( \theta = (\theta_k)_{k \in Q_0} \) we associate the character \( (g_k) \mapsto \prod_{k \in Q_0} \det(g_k)^{\theta_k} \). For \( \theta \in Q_0 \), we have the graded algebra of semi-invariants

\[ \mathbb{C}[T^*R]^{\theta - \text{si}} := \bigoplus_{n \geq 0} \mathbb{C}[T^*R]^n\theta \quad (7.2) \]

where, recall, \( \mathbb{C}[T^*R]^n\theta := \{ f \in \mathbb{C}[T^*R] : f(g^{-1}x) = \theta^n(g)f(x) \text{ for every } x \in T^*R \} \).

Recall that a point \( x \in T^*R \) is said to be \( \theta \)-semistable if there exist \( n > 0 \) and \( f \in \mathbb{C}[T^*R]^n\theta \) such that \( f(x) \neq 0 \). Let us describe the semistable points with respect to certain stability conditions. First of all, for a \( \overline{Q}^\circ \)-representation \( x = (X_\alpha, Y_\alpha, i_k, j_k) \in T^*R = \text{Rep}(\overline{Q}^\circ, (v, w)) \) we denote by \( \underline{x} = (X_\alpha, Y_\alpha) \in \text{Rep}(\overline{Q}, (v)) \) the representation of \( \overline{Q} \) that is obtained by forgetting the framing and coframing maps.

**Proposition 7.1.1** (Proposition 5.1.5, [Gi3]). 1. Assume \( \theta \in \mathbb{Z}^{Q_0}_{>0} \). Then, a representation \( x = (X_\alpha, Y_\alpha, i_k, j_k) \in T^*R \) is \( \theta \)-semistable if and only if the only subrepresentation of \( \underline{x} \) which contains \( (\text{im}(j_k))_{k \in Q_0} \) is the entire representation \( \underline{x} \).

2. Assume \( \theta \in \mathbb{Z}^{Q_0}_{<0} \). Then, a representation \( x = (X_\alpha, Y_\alpha, i_k, j_k) \in T^*R \) is \( \theta \)-semistable if and only if the only subrepresentation of \( \underline{x} \) contained in \( (\text{ker}(i_k))_{k \in Q_0} \) is the 0 representation.
We will denote $\theta^+ := (1,1,\ldots,1)$ and $\theta^- := -\theta^+$. In particular, $\theta^+$ falls under (1) of the previous proposition, while $\theta^-$ falls under (2).

Now we proceed to define the G.I.T. Hamiltonian reduction of $T^*R$ by the action of $G$. First of all, obviously $G$ acts on $\mu^{-1}(\lambda)$ for $\lambda \in (g^*)^G$. So, similarly to (7.2) we may define the algebra of semi-invariants $C[\mu^{-1}(\lambda)]^{\theta-si}$, and the set of $\theta$-semistable points $\mu^{-1}(\lambda)^{\theta-ss}$. We remark that $\mu^{-1}(\lambda)^{\theta-ss} = (T^*R)^{\theta-ss} \cap \mu^{-1}(\lambda)$, this is a consequence of the Hilbert-Mumford criterion. Now we can define the G.I.T. Hamiltonian reduction

$$\mathcal{M}_\lambda^\theta := \text{Proj}(C[\mu^{-1}(\lambda)]^{\theta-si}) = \mu^{-1}(\lambda)^{\theta-ss} // G \quad (7.3)$$

Let us remark, first, that the formalism of Hamiltonian reduction implies that the symplectic structure on $T^*R$ gives $\mathcal{M}_\lambda^\theta$ the structure of an algebraic Poisson variety. We also remark that the 0th graded component of the algebra $C[\mu^{-1}(\lambda)]^{\theta-si}$ is precisely $C[\mu^{-1}(\lambda)]^G$, so we have a projective morphism $\varpi : \mathcal{M}_\lambda^\theta \to \mathcal{M}_\lambda^0$. We will state sufficient conditions for this morphism to be a resolution of singularities.

Of course, for $\varpi : \mathcal{M}_\lambda^\theta \to \mathcal{M}_\lambda^0$ to be a resolution of singularities we need, first, that $\mathcal{M}_\lambda^\theta$ is smooth. The variety $\mathcal{M}_\lambda^\theta$ will be smooth and symplectic provided the $G$-action on $\mu^{-1}(\lambda)^{\theta-ss}$ is free. When this happens, we will say that the pair $(\theta, \lambda)$ is _generic_. The following result gives a sufficient condition for a pair $(\theta, \lambda)$ to be generic. We denote by $g(Q)$ the Kac-Moody algebra associated to the quiver $Q$. By $\alpha^i$, we mean the simple root of $g(Q)$ corresponding to the vertex $i \in Q_0$.

**Proposition 7.1.2.** The pair $(\theta, \lambda)$ is generic if there is no $v' \in \mathbb{Z}_{\geq 0}^{Q_0}$ such that:

1. Componentwise, $v' \leq v$.

2. $\sum_{i \in Q_0} v'_i \alpha^i$ is a root of $g(Q)$.

3. $\sum_{i \in Q_0} v'_i \lambda_i = \sum_{i \in Q_0} v'_i \theta_i = 0$. 

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For example, both $\theta^+$ and $\theta^-$ are generic, in the sense that $(\theta^\pm, \lambda)$ is generic for any $\lambda \in (\mathfrak{g}^*)^G$. This will be important for us later.

From now on, we assume that the pair $(\theta, \lambda)$ is generic. Let us denote by $\mathcal{M}_\lambda$ the affine variety $\text{Spec}[\Gamma(\mathcal{M}_\lambda^0, \mathcal{G}_{\mathcal{M}_\lambda^0})]$. In particular, we have a projective morphism $\rho : \mathcal{M}_\lambda^0 \to \mathcal{M}_\lambda$.

**Proposition 7.1.3** (Proposition 2.3, [BL]). The map $\rho$ is a resolution of singularities. Moreover, the variety $\mathcal{M}_\lambda$ is independent of the stability condition $\theta$.

So the map $\varpi$ will be a resolution of singularities provided we have $\mathcal{M}_\lambda = \mathcal{M}_\lambda^0$ (assuming $(\theta, \lambda)$ is generic). This is the case when the moment map $\mu$ is flat, [BL, Proposition 2.5]. Sufficient conditions for this to happen were found by Crawley-Boevey in [CB] see, for example, Theorem 1.1 in loc. cit. A consequence of this is that $\mu$ is flat whenever $Q$ is a finite or affine quiver and $\nu := \sum_{i \in Q_0} (w_i\omega_i - v_i\alpha_i)$ is a dominant weight for $\mathfrak{g}(Q)$, where $\omega_i, \alpha_i$ denote the fundamental weights and simple roots for $\mathfrak{g}(Q)$, respectively. In general, all constructions that follow remain valid if we replace the variety $\mathcal{M}_\lambda^0$ by $\mathcal{M}_\lambda$.

**7.1.3 Universal quiver varieties**

Now assume that the stability condition $\theta$ is such that $(\theta, \lambda)$ is a generic pair for every $\lambda$. For example, we can take $\theta = \theta^+$ or $\theta^-$. In this case, the action of $G$ on $\mu^{-1}((\mathfrak{g}^*)^G)^{\theta-ss}$ is free, and so the “universal” quiver varieties

$$
\mathcal{M}_p^0 := \mu^{-1}((\mathfrak{g}^*)^G)^{\theta-ss} \backslash \!/ G, \quad \mathcal{M}_p^0 := \mu^{-1}((\mathfrak{g}^*)^G) \backslash \!/ G, \quad \mathcal{M}_p := \text{Spec}(\Gamma(\mathcal{M}_p^0, \mathcal{G}_{\mathcal{M}_p^0}))
$$

are smooth and symplectic. We remark that $\mathcal{M}_p^0$ is a scheme over $\mathfrak{p} := (\mathfrak{g}^*)^G$ and its specialization to $\lambda \in \mathfrak{p}$ coincides with $\mathcal{M}_\lambda^0$. In particular, $\mathcal{M}_p^0$ is a deformation of $\mathcal{M}_\lambda^0$ over $\mathfrak{p}$. Note that we have an action of $\mathbb{C}^*$ on $\mathcal{M}_p^0$ that restricts to the usual action on $\mathcal{M}_0^0$ on the fiber over 0. On the other hand, Namikawa, cf. [Nam], has proved that the variety $\mathcal{M}_0^0$ admits a universal deformation $\mathcal{M}^0$ over the space $H^2_{\text{DR}}(\mathcal{M}_0^0, \mathbb{C})$. Let us see the relation between the universal deformation $\mathcal{M}^0$ and the universal quiver variety $\mathcal{M}_p^0$. We have a natural map.
\[ p \to H^2_{\text{DR}}(\mathcal{M}_\theta^0, \mathbb{C}) \] given as follows. Let \( \chi \in \mathcal{X}(G) \) be a character. Consider the line bundle \( V_\chi \) on \( T^*R \), which is trivial as a line bundle and with a \( G \)-action given by \( \chi^{-1} \). Since \( V_\chi \) is \( G \)-equivariant, its restriction \( V_\chi |_{\mu^{-1}(0)} \) descends to a line bundle \( \mathcal{S}(\chi) \) on \( \mathcal{M}_\theta^0 \). For example, the line bundle \( \mathcal{S}(\theta) \) is ample by definition. This defines a map \( \iota : \mathcal{X}(G) \to H^2_{\text{DR}}(\mathcal{M}_\theta^0, \mathbb{C}) \), \( \chi \mapsto c_1(\mathcal{S}(\chi)) \) (the first Chern class) which extends by linearity to \( \iota : p \to H^2_{\text{DR}}(\mathcal{M}_\theta^0, \mathbb{C}) \). We will assume the following.

**Assumption 7.1.4.** The map \( \iota : p \to H^2_{\text{DR}}(\mathcal{M}_\theta^0, \mathbb{C}) \) is an isomorphism and therefore the universal deformation \( \mathcal{M}^\theta \) of \( \mathcal{M}_\theta^0 \) coincides with \( \mathcal{M}_p^\theta \).

We remark that Assumption 7.1.4 is known to be true when \( Q \) is a finite or affine quiver, and it is conjectured to hold in all cases, [BPW]. If we do not make Assumption 7.1.4 we simply have \( \mathcal{M}_\theta^0 = p \times_{H^2_{\text{DR}}(\mathcal{M}_\theta^0, \mathbb{C})} \mathcal{M}^\theta \).

### 7.1.4 Isomorphisms between quiver varieties

Let us remark that we have the following \( \mathbb{C}^\times \)-equivariant symplectomorphism of \( T^*R \)

\[ \Upsilon : T^*R \to T^*R \]

\[ (X_\alpha, Y_\alpha, i_k, j_k) \mapsto (-Y_\alpha^t, X_\alpha^t, -j_k^t, i_k^t) \]

where \( \bullet^t \) denotes matrix transposition. Note, however, that this map is not \( G \)-equivariant. Rather, we have that \( \Upsilon(g,x) = (g)^{-1}\Upsilon(x) \). This implies the following.

**Lemma 7.1.5.** 1. For any character \( \theta \in \mathcal{X}(G) \), the map \( \Upsilon \) induces a graded isomorphism

\[ \Upsilon^* : [T^*R]^\theta-\text{si} \to [T^*R]^{-\theta-\text{si}}. \]

2. The following diagram commutes

\[ \begin{array}{ccc}
T^*R & \xrightarrow{\Upsilon} & T^*R \\
\downarrow{\mu} & & \downarrow{\mu} \\
g & \xrightarrow{\xi \to -\xi^t} & g
\end{array} \]

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Corollary 7.1.6. For any $\lambda \in (g^*)^G$, we have an induced, graded isomorphism

$$\mathbb{C}[\mu^{-1}(\lambda)]^{\theta-si} \rightarrow \mathbb{C}[\mu^{-1}(-\lambda)]^{-\theta-si}$$

and consequently we have an isomorphism of projective varieties $M^\theta_\lambda \rightarrow M^{-\theta}_\lambda$. These isomorphisms glue together to an isomorphism $M^\theta_p \rightarrow M^{-\theta}_p$ such that the following diagram commutes

$$
\begin{array}{ccc}
M^\theta_p & \rightarrow & M^{-\theta}_p \\
\downarrow & & \downarrow \\
p & \overset{\lambda \rightarrow -\lambda}{\rightarrow} & p
\end{array}
$$

Let us remark that $\Upsilon$ also induces a $\mathbb{C}^*$-equivariant Poisson automorphism of the affine quiver variety $\Upsilon : M^0_0 \rightarrow M^0_0$. We will denote by $\Upsilon^* : \mathbb{C}[M^0_0] \rightarrow \mathbb{C}[M^0_0]$ the induced automorphism on its algebra of functions.

7.2 Quantizations

Let us proceed to quantizations of the quiver varieties $M^0_0, M^\theta_0$ and $M^\theta_p$. Since $M^0_0$ is an affine variety, its quantizations are straightforward to define. It is slightly harder to define quantizations of the varieties $M^\theta_0$ and $M^\theta_p$. We follow, mostly, [BL]. The only new result of this section is in Section 7.2.4, see (7.7), but is not really original.

7.2.1 Quantizations of $M^0_0$

Recall that $M^0_0$ is an affine, Poisson variety with a $\mathbb{C}^*$-action. Moreover, note that the Poisson bracket has degree $-2$ with respect to the $\mathbb{C}^*$-action.

Definition 7.2.1. A quantization of $M^0_0$ is a pair $(\mathcal{A}, \iota)$ where

1. $\mathcal{A}$ is an associative, filtered algebra $\mathcal{A} = \bigcup_{i \geq 0} \mathcal{A}^i$ such that, for $a \in \mathcal{A}^i, b \in \mathcal{A}^j$, $[a, b] \in \mathcal{A}^{i+j-2}$ (in particular, this implies that $\text{gr} \mathcal{A}$ is equipped with a Poisson bracket of degree $-2$.)
2. $\iota : \text{gr} \mathcal{A} \to \mathbb{C}[\mathcal{M}_0^0]$ is an isomorphism of graded Poisson algebras.

By an isomorphism of quantizations we mean a filtered isomorphism $f : \mathcal{A} \to \mathcal{A}'$ that induces the identity on $\mathbb{C}[\mathcal{M}_0^0]$ on the associated graded level. Next, we review how to get quantizations of the $\mathcal{M}_0^0$ using quantum Hamiltonian reduction.

Quantum Hamiltonian reduction: algebra level

Let us, first, describe the general case. Then we will specialize to our situation with quiver varieties.

Let $G$ be a reductive algebraic group acting on an algebra $\mathcal{A}$ by algebra automorphisms. In particular, the Lie algebra $\mathfrak{g}$ acts on $\mathcal{A}$ by derivations. For $\xi \in \mathfrak{g}$, let us denote by $\xi_A : \mathcal{A} \to \mathcal{A}$ the corresponding derivation. We say that a map $\Phi : \mathfrak{g} \to \mathcal{A}$ is a quantum comoment map if it is $G$-equivariant and, for $\xi \in \mathfrak{g}$, $\xi_A = [\Phi(\xi), \cdot]$. Note, in particular, that $\Phi([\xi, \eta]) = [\Phi(\xi), \Phi(\eta)]$. So the quantum comoment map extends to an algebra map $\Phi : \mathbb{U}(\mathfrak{g}) \to \mathcal{A}$.

For a character $\lambda \in (\mathfrak{g}^*)^G$, we consider the ideal $\mathcal{I}_\lambda := \mathbb{U}(\mathfrak{g})\{\xi - \langle \lambda, \xi \rangle : \xi \in \mathfrak{g}\}$. Now we define the quantum Hamiltonian reduction

$$\mathcal{A}_\lambda := \mathcal{A}/\mathcal{A}\Phi(\mathcal{I}_\lambda)^G$$

We remark that $\mathcal{A}_\lambda$ has an algebra structure induced from the algebra structure on $\mathcal{A}_\lambda$. Moreover, if we denote by $M_\lambda$ the cyclic $\mathcal{A}$-module $\mathcal{A}/\mathcal{A}\Phi(\mathcal{I}_\lambda)$ then $\mathcal{A}_\lambda = \text{End}_\mathcal{A}(M_\lambda)^{opp}$. As in the classical case, we may define a “universal” quantum Hamiltonian reduction, as follows. Denote $\mathfrak{P} := \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \cong (\mathfrak{g}^*)^G$. Consider the ideal $\mathcal{I} := \mathbb{U}(\mathfrak{g})[\mathfrak{g}, \mathfrak{g}] \subseteq \mathbb{U}(\mathfrak{g})$. The universal quantum Hamiltonian reduction is

$$\mathcal{A}_\mathfrak{P} := [\mathcal{A}/\mathcal{A}\Phi(\mathcal{I})]^G$$

we remark that this is an $S(\mathfrak{P})$-algebra, that is, the image of $\mathfrak{P}$ under the natural map
Let us now specialize to the case of Nakajima quiver varieties. Here, the algebra $\mathcal{A}$ is the algebra of global differential operators on $R$, $D(R)$. The quantum comoment map is given as follows: the action of $G$ on $R$ induces a map $\mathfrak{g} \to \text{Vect}_R$, $\xi \mapsto \xi_R$. But $\text{Vect}_R \subseteq D(R)$, so we may consider this as a map to $D(R)$, and this is a quantum comoment map. For $\lambda \in (\mathfrak{g}^*)^G$, we will denote by $\hat{\mathcal{A}}_\lambda$ the quantum Hamiltonian reduction of $D(R)$ at $\lambda$. If the moment map $\mu$ is flat, this is a quantization of $M^\theta_0$.

### 7.2.2 Quantizations of $M^\theta_0$

Now we want to define quantizations of the variety $M^\theta_0$. Since this is not an affine variety, a quantization of it will not be specified by a single algebra of functions. Rather, we need a sheaf on $M^\theta_0$ whose associated graded coincides with the structure sheaf $\mathcal{S}_{M^\theta_0}$. Note, however, that the last sentence does not make sense as stated: for an open set $U \subseteq M^\theta_0$, the algebra $\Gamma(U, \mathcal{S}_{M^\theta_0})$ is only naturally graded when $U$ is $\mathbb{C}^\times$-stable. So, before, we need to change the topology.

**Definition 7.2.2.** The conical topology on $M^\theta_0$ is that topology whose open set are the Zariski open $\mathbb{C}^\times$-stable subsets of $M^\theta_0$.

We remark that every point $x \in M^\theta_0$ has a Zariski open neighborhood which is affine and $\mathbb{C}^\times$-stable, this is known as Sumihiro’s theorem, [Su]. So the conical topology is still fine enough for most purposes. A quantization of $M^\theta_0$ will then be a filtered sheaf of algebras in the conical topology whose associated graded coincides with $\mathcal{S}_{M^\theta_0}$. For technical reasons, this sheaf of algebras is supposed to satisfy some conditions which we now make precise.

**Definition 7.2.3.** A quantization of $M^\theta_0$ is a pair $(\mathcal{A}^\theta, \iota)$ where $\mathcal{A}^\theta$ is a filtered sheaf of algebras in the conical topology and $\iota : \text{gr} \mathcal{A}^\theta \to M^\theta_0$ is a Poisson isomorphism, such that the filtration on $\mathcal{A}^\theta$ satisfies the following conditions.
1. It is complete, meaning that every Cauchy sequence in the topology determined by the filtration converges. More explicitly, we require that for every sequence \( \{x_i\}_{i=1}^{\infty} \) of sections of \( A^\theta \) such that for every \( n \in \mathbb{Z} \) there exist \( i_0 \) such that \( x_i - x_j \in (A^\theta)^{\leq n} \) for \( i, j > i_0 \), there exists a section \( x \in A^\theta \) such that for every \( m \in \mathbb{Z} \) there exist \( j_0 \) such that \( x - x_i \in (A^\theta)^{\leq m} \) for every \( i > j_0 \).

2. It is separated, meaning that \( \bigcap_{i \in \mathbb{Z}} (A^\theta)^{\leq i} = 0 \). Equivalently, the limit \( x \) of the previous paragraph is unique.

Quantum Hamiltonian reduction: sheaf level

Quantizations of \( \mathcal{M}_0^\theta \) may be produced similarly those of \( \mathcal{M}_0^0 \). Instead of using the algebra \( D(R) \), we use the microlocalization of the sheaf of differential operators on \( R \). This is a sheaf \( D_R \) on the conical topology of \( T^* R \), where the action of \( \mathbb{C}^* \) is by dilations on the cotangent fibers, see, for example, [Gi], [K] for details on microlocalization. The global sections of \( D_R \) coincide with \( D(R) \).

Now let \( f \in \mathbb{C}[T^* R] \) be an \( n^\theta \)-semiinvariant element, where \( n > 0 \). Then, have the open set \( X_f \subseteq \mathcal{M}_0^\theta \) that is \( \mu^{-1}(0) \cap \{ f \neq 0 \} / G \). That the open sets \( X_f \) form a base of the topology of \( \mathcal{M}_0^\theta \) follows by the definition of Proj. It is easy to see that if, moreover, \( f \) is homogeneous with respect to the \( \mathbb{C}^* \)-action on \( T^* R \), then \( X_f \) is open in the conical topology, and the sets \( X_f \) form a basis for the conical topology of \( \mathcal{M}_0^\theta \).

So, for \( \lambda \in (g^*)^G \) we define the sheaf \( D_\lambda^\theta \) by setting, on an open set \( X_f \):

\[
D_\lambda^\theta(X_f) := D_R((T^* R)_f) // \lambda G
\]

It is possible to see that \( D_\lambda^\theta \) defined in this way is a quantization of \( \mathcal{M}^\theta \). Let us denote by \( D_\lambda \) the algebra of global sections of \( D_\lambda^\theta \). This is a quantization of \( \mathcal{M}_0 \). If the moment map \( \mu \) is flat, then we actually have that \( D_\lambda = \hat{A}_\lambda \). For proofs of these statements, see [BL, Section 2].
Let us remark that we also have the notion of quantizations of the universal quiver variety $\mathcal{M}_\theta^\rho$. This is a sheaf of $\mathbb{C}[p]$-algebras in the conical topology (recall that we have an action of $\mathbb{C}^\times$ on $\mathcal{M}_\theta^\rho$) satisfying conditions analogous to those of Definition 7.2.3, see [BPW, Section 3].

### 7.2.3 The period map

Isomorphism classes of quantizations of $\mathcal{M}_0^\theta$ and $\mathcal{M}_p^\theta$ have been parametrized in [BK], [L2] by the vector spaces $H^2_{DR}(\mathcal{M}_0^\theta, \mathbb{C})$ and $H^2_{DR}(\mathcal{M}_p^\theta/p, \mathbb{C})$, respectively. For $\lambda \in H^2_{DR}(\mathcal{M}_0^\theta, \mathbb{C}) = p$, we denote by $\mathcal{A}_\lambda^\theta$ the quantization of $\mathcal{M}_0^\theta$ with period $\lambda$. We remark that it is not the case that $\mathcal{A}_\lambda^\theta$ coincides with $\mathcal{D}_\lambda^\theta$, for this we would have to take a symmetrized quantum comoment map, see [BPW, Section 3.4], [L2, Section 3.2].

The quantization of $\mathcal{M}_0^\theta$ (or of $\mathcal{M}_p^\theta$) with period 0 is called the canonical quantization. It is characterized by the fact that it is isomorphic, as a quantization, to its opposite, this is [L2, Corollary 2.3.3]. In fact we have, $\mathcal{A}_{-\lambda}^\theta = (\mathcal{A}_{\lambda}^\theta)^{opp}$, cf. [L2].

### 7.2.4 Isomorphisms of quantizations

Now let $\mathcal{A}^\theta$ be the canonical quantization of the universal deformation $\mathcal{M}^\theta$ (= $\mathcal{M}_p^\theta$ by Assumption 7.1.4) of $\mathcal{M}_0^\theta$. Recall that $\mathcal{A}^\theta$ is characterized by it being isomorphic to its opposite. This implies, in particular, that we have an isomorphism $\Upsilon^*\mathcal{A}^{-\theta} \cong \mathcal{A}^\theta$, where $\Upsilon : \mathcal{M}_p^\theta \rightarrow \mathcal{M}_p^{-\theta}$ is the isomorphism introduced in Subsection 7.1.4. Thus, we have an induced isomorphism

$$\Gamma(\mathcal{M}^\theta, \mathcal{A}^\theta) \cong \Gamma(\mathcal{M}^{-\theta}, \mathcal{A}^{-\theta})$$ (7.5)

We remark, however, that (7.5) is not an isomorphism of $\mathbb{C}[p]$-algebras. Rather, it induces the automorphism on $\mathbb{C}[p]$ given by $f(\lambda) \mapsto f(-\lambda)$. 

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Now let $\lambda \in p$ be a period of quantization. According to [BLPW], we have an isomorphism of quantizations of $\mathcal{M}_0^0$:

$$\mathcal{A}_\lambda = \Gamma(\mathcal{M}_0^0, \mathcal{A}_0^0) \cong \Gamma(\mathcal{M}^0, \mathcal{A}^0)/\mathcal{I}_\lambda$$  

(7.6)

where $\mathcal{I}_\lambda$ is the ideal generated by the maximal ideal of $\lambda$, $m_\lambda \subseteq \mathbb{C}[p] \subseteq \Gamma(\mathcal{M}^0, \mathcal{A}^0)$. It follows from (7.5) and (7.6) that we have an isomorphism

$$\Phi_\lambda : \mathcal{A}_\lambda \xrightarrow{\cong} \mathcal{A}_{-\lambda}$$  

(7.7)

This is a filtered isomorphism that is, however, not an isomorphism of quantizations. Indeed, it follows by construction that the associated graded of (7.7) coincides with the isomorphism $\Upsilon^* : \mathbb{C}[\mathcal{M}_0^0] \rightarrow \mathbb{C}[\mathcal{M}_0^0]$ constructed in Subsection 7.1.4.

### 7.3 Harish-Chandra bimodules

In this section we proceed to define Harish-Chandra bimodules. The definition is, basically, the same as with rational Cherednik algebras. However, for technical reasons (see (7.7)) we define a wider class of bimodules, which we call twisted HC bimodules. The twist here is provided by a $\mathbb{C}^*\times$-equivariant automorphism of $\mathcal{M}_0^0$. When this automorphism is the identity, we recover the usual definition of HC bimodules.

#### 7.3.1 Harish-Chandra bimodules: algebra level

**Definition 7.3.1.** Let $\mathcal{A}, \mathcal{A}'$ be filtered quantizations of the same graded Poisson algebra $A$, and let $f : A \rightarrow A$ be a graded Poisson automorphism of $A$. We say that a $(\mathcal{A}, \mathcal{A}')$-bimodule $\mathcal{B}$ is $f$-twisted Harish-Chandra (shortly, $f$-HC) if it admits a bimodule filtration such that:

1. $\text{gr} \mathcal{B}$ is a finitely generated $A$-bimodule.

2. For any $a \in A$, $b \in \text{gr} \mathcal{B}$, $ab = bf(a)$.  

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We denote the category of $f$-HC $(\mathcal{A}, \mathcal{A}')$-bimodules by $^f\text{HC}(\mathcal{A}, \mathcal{A}')$.

We remark that, when $f$ is the identity, we recover the usual category of HC bimodules that is considered in [BL, BLPW]. We will abbreviate $\text{HC}(\mathcal{A}, \mathcal{A}') := \text{idHC}(\mathcal{A}, \mathcal{A}')$, and call these bimodules simply HC. The following proposition is clear.

**Proposition 7.3.2.** 1. The category $^f\text{HC}(\mathcal{A}, \mathcal{A}')$ is a Serre subcategory of the category of all $(\mathcal{A}, \mathcal{A}')$-bimodules.

2. Every $f$-HC $(\mathcal{A}, \mathcal{A}')$-bimodule $\mathcal{B}$ is finitely generated as a left $\mathcal{A}$-module and as a right $\mathcal{A}'$-module.

3. The tensor product $\otimes_{\mathcal{A}'}$ gives a bifunctor:

\[ g^\text{HC}(\mathcal{A}, \mathcal{A}') \times ^f\text{HC}(\mathcal{A}', \mathcal{A}'') \rightarrow g^\circ f\text{HC}(\mathcal{A}, \mathcal{A}'') \]

Let us specialize to the case where $\mathcal{A} = \mathcal{A}_\lambda$ is a quantization of the Nakajima quiver variety $\mathcal{M}_0^0$. In this case, we write $^f\text{HC}(\lambda, \lambda') := ^f\text{HC}(\mathcal{A}_\lambda, \mathcal{A}_{\lambda'})$. The following result is clear.

**Proposition 7.3.3.** Twisting the right action by the isomorphism $\Phi_{-\mu}$ (see (7.7)) gives a category equivalence $^f\text{HC}(\lambda, \mu) \xrightarrow{\sim} \text{Y}^f \circ \text{HC}(\lambda, -\mu)$. Similarly, twisting the left action by the isomorphism $\Phi_\lambda$ gives a category equivalence $^f\text{HC}(\lambda, \mu) \xrightarrow{\sim} ^f \circ \text{Y}^\star \text{HC}(\lambda, \mu)$.

### 7.3.2 Harish-Chandra bimodules: sheaf level

We proceed to give the definitions of sheaf-theoretic HC bimodules. For the sake of concreteness, we will work with the varieties $\mathcal{M}_0^0$, although the definitions can be stated for any symplectic resolution. So let $\mathcal{A}_\lambda^0, \mathcal{A}_\mu^0$ be quantizations of $\mathcal{M}_0^0$. Note that the external tensor product $\mathcal{A}_\lambda^0 \boxtimes (\mathcal{A}_\mu^0)^{\text{opp}}$ is a quantization of the product $\mathcal{M}_0^0 \times \mathcal{M}_0^0$.

Now let $f : \mathcal{M}_0^0 \rightarrow \mathcal{M}_0^0$ be a $\mathbb{C}^\times$-equivariant Poisson automorphism. Let $\Sigma_f \subseteq \mathcal{M}_0^0 \times \mathcal{M}_0^0$ denote the graph of $f$, with its reduced scheme structure, and let $\mathcal{S}_f$ be the scheme-theoretic preimage of $\Sigma_f$ under the natural map $\mathcal{M}_0^0 \times \mathcal{M}_0^0 \rightarrow \mathcal{M}_0^0 \times \mathcal{M}_0^0$. For example, when $f$ is the
identity then $\Sigma_f$ is just the diagonal and $S_f$ is the usual Steinberg variety. So we call $S_f$ the $f$-Steinberg variety.

**Definition 7.3.4.** An $A_\lambda^\theta \boxtimes (A_\mu^\theta)^{\text{opp}}$-module $B$ is said to be $f$-twisted Harish-Chandra (shortly, $f$-HC) if it admits a filtration such that $\text{gr} B$ is coherent and scheme-theoretically supported on the $f$-Steinberg variety $S_f$. We denote this category of $f$-HC $A_\lambda^\theta \boxtimes (A_\mu^\theta)^{\text{opp}}$-modules by $f \mathcal{HC}(\lambda, \mu)$.

Note that, by definition, $\dim S_f = \dim \Sigma_f = \dim M^0_0 = (1/2) \dim (M^0_0 \times M^0_0)$. By standard results in homological duality (see, for example, [BL, Section 4.2]) we get the following.

**Proposition 7.3.5.** The functor $\mathbb{D} : B \mapsto \mathcal{E}xt^\dim M^0_0 A_\lambda^\theta \boxtimes (A_\mu^\theta)^{\text{opp}}(B, A_\lambda^\theta \boxtimes (A_\mu^\theta)^{\text{opp}})$ gives an equivalence:

$$\mathbb{D} : f \mathcal{HC}(\lambda, \mu) \xrightarrow{\cong} f^{-1} \mathcal{HC}(\mu, \lambda)^{\text{opp}}$$

**Proof.** Standard results in homological duality say that the functor $\mathbb{D}$ gives an equivalence $f \mathcal{HC}(A_\lambda^\theta \boxtimes (A_\mu^\theta)^{\text{opp}}) \rightarrow f \mathcal{HC}((A_\lambda^\theta)^{\text{opp}} \boxtimes A_\mu^\theta)$. We have now to compose with the isomorphism $(A_\lambda^\theta)^{\text{opp}} \boxtimes A_\mu^\theta \rightarrow A_\mu^\theta \boxtimes (A_\lambda^\theta)^{\text{opp}}$. At the associated graded level, this induces the automorphism of $M^0_0 \times M^0_0$ that interchanges the factors. So the image of $S_f$ is $S_{f^{-1}}$ and the result is proved. \qed

**Quantization of line bundles**

Let us give an example of a Harish-Chandra $\mathcal{HC}(\lambda, \lambda+\theta)$-bimodule. Recall that on the variety $M^\theta_0$ we have the $\mathbb{C}^\times$-equivariant ample line bundle $\mathcal{S}(\theta)$. According to [BPW, Section 5.1], there exists a unique $(A^\theta_\lambda \boxtimes (A^\theta_\mu)^{\text{opp}})$-module $\lambda A^\theta_{\lambda+\theta}$ admitting a filtration whose associated graded coincides with the pushforward of the line bundle $\mathcal{S}(\theta)$ to $M^0_0 \times M^0_0$ under the diagonal embedding. Thus, $\lambda A^\theta_{\lambda+\theta} \in \mathcal{HC}(\lambda, \lambda + \theta)$. The following proposition now follows from uniqueness of the quantizations of line bundles and from the fact that, it being a quantization of a line bundle, taking tensor product with $\lambda A^\theta_{\lambda+\theta}$ does not affect the support of a module.
**Proposition 7.3.6.** For any \( n, m \in \mathbb{Z} \), taking tensor product with quantizations of line bundles gives equivalences

\[
\mathcal{F}\mathcal{H}\mathcal{C}(\lambda + n\theta, \mu) \leftrightarrow \mathcal{F}\mathcal{H}\mathcal{C}(\lambda, \mu) \to \mathcal{F}\mathcal{H}\mathcal{C}(\lambda, \mu + m\theta)
\]

### 7.3.3 Localization theorems

Note that we have the global sections and localization functors

\[
\Gamma^\theta_\lambda : A^\theta_\lambda -\text{mod} \leftrightarrow A_\lambda -\text{mod} : \text{Loc}^\theta_\lambda
\]

When these \( \Gamma^\theta_\lambda \) and \( \text{Loc}^\theta_\lambda \) are quasi-inverse equivalences of categories, we say that *abelian localization holds at* \( \lambda \). In general, it is not easy to find the locus where abelian localization holds. However, the following result, [BPW], tells us that abelian localization holds for \( \lambda \) sufficiently dominant.

**Theorem 7.3.7** (Corollary B.1, [BPW]). Let \( \lambda \in \mathfrak{p} \). Then, abelian localization holds at \( \lambda + n\theta \) for \( n \gg 0 \).

Since \( S_f \) is defined to be precisely the scheme-theoretic preimage of the graph \( \Sigma_f \), [BLPW, Proposition 2.13] implies.

**Proposition 7.3.8.** Global sections and localization induce functors

\[
\Gamma : \mathcal{F}\mathcal{H}\mathcal{C}(\lambda, \mu) \leftrightarrow \mathcal{F}\mathcal{H}\mathcal{C}(\lambda, \mu) : \text{Loc}
\]

moreover, if abelian localization holds at \( \lambda \) and \(-\mu\), these are quasi-inverse equivalences of categories.

### 7.3.4 Duality

We use our previous work to construct a duality functor between categories of twisted Harish-Chandra bimodules. These functors will be constructed as composition of several equivalences we have already seen. The duality step happens at the level of sheaves, this is just
the homological duality given by Proposition 7.3.5. To pass from sheaves to bimodules we use, of course, localization theorems. The problem here is that, in general, localization does not hold at $\lambda$ and $-\lambda$ simultaneously. So we need to first use an equivalence provided by Proposition 7.3.3. This will introduce a twist by $\Upsilon$ that will be cancelled at the end using an equivalence of the same form.

**Theorem 7.3.9.** Let $\lambda$ be a period of quantization such that abelian localization holds at $\lambda$. Then, for any $\mathbb{C}^\times$-equivariant Poisson automorphism $f : M_0^0 \to M_0^0$ and $n \gg 0$ there is an equivalence of categories

$$D : f^*HC(\lambda, \lambda) \xrightarrow{\cong} (f^*)^{-1}HC(\lambda - n\theta, \lambda - n\theta)^{\text{opp}}$$

**Proof.** As we have said above, this equivalence is just a composition of several equivalences that we have introduced before. Let us list these.

1. $f^*HC(\lambda, \lambda) \xrightarrow{\cong} \Upsilon \circ f^*HC(\lambda, -\lambda)$.  
2. $\Upsilon \circ f^*HC(\lambda, -\lambda) \xrightarrow{\cong} f \circ \Upsilon HC(\lambda, -\lambda)$.  
3. $f \circ \Upsilon HC(\lambda, -\lambda) \xrightarrow{\cong} \Upsilon \circ f^{-1}HC(-\lambda, \lambda)^{\text{opp}}$.  
4. $\Upsilon \circ f^{-1}HC(-\lambda, \lambda)^{\text{opp}} \xrightarrow{\cong} \Upsilon \circ f^{-1}HC(-\lambda + n\theta, \lambda - n\theta)^{\text{opp}}$.  
5. $\Upsilon \circ f^{-1}HC(-\lambda + n\theta, \lambda - n\theta)^{\text{opp}} \xrightarrow{\cong} (f^*)^{-1} \circ \Upsilon^* HC(-\lambda + n\theta, \lambda - n\theta)^{\text{opp}}$.  
6. $(f^*)^{-1} \circ \Upsilon^* HC(-\lambda + n\theta, \lambda - n\theta)^{\text{opp}} \xrightarrow{\cong} (f^*)^{-1}HC(\lambda - n\theta, \lambda - n\theta)^{\text{opp}}$.

Equivalences (1) and (6) are provided by Proposition 7.3.3; the equivalence (2) is simply the localization theorem, cf. Proposition 7.3.8. The localization theorem also provides the equivalence (5): here, we need to take $n$ big enough so that localization will hold at $-\lambda + n\theta$, cf. Theorem 7.3.7. The equivalence (4) is given by the translation equivalences of Proposition 7.3.6. Finally, the duality (3) is the homological duality of Proposition 7.3.5.
Remark 7.3.10. Theorem 7.3.9 holds, with the same proof, if we take two parameters \( \lambda, \mu \) such that localization holds at \( \lambda \) and \(-\mu\). In this case, we obtain an equivalence \( \mathbb{D} : \text{HC}(\lambda, \mu) \to \text{HC}(\mu - n\theta, \lambda - n\theta)^{\text{opp}} \), for \( n \gg 0 \).

### 7.4 Connection to rational Cherednik algebras

Let us remark that the spherical rational Cherednik algebras of the groups \( G(\ell, 1, n) = S_n \ltimes (\mathbb{Z}/\ell \mathbb{Z})^n \) can be realized as quantized quiver varieties. Let \( Q \) be the cyclic quiver with \( \ell \) vertices (in particular, if \( \ell = 1 \), then \( Q \) is a point with a loop). Pick a vertex in \( Q \) and call it 0 (so this is the extending vertex of the affine type A Dynkin diagram underlying \( Q \)). For a dimension vector \( v \), we take the vector \( n(1,1,\ldots,1) \) (note that \((1,1,\ldots,1)\) is the affine root for the affine type A Dynkin diagram underlying \( Q \)) while for a dimension vector \( w \) we the vector that takes the value 1 at the vertex 0 and 0 everywhere else. For example, for \( \ell = 1 \) the quiver \( \overline{Q^\vee} \) is

\[
\begin{array}{c}
\vdots \\
1 \rightarrow n \rightarrow 1
\end{array}
\]

Let us denote by \( A := e(H/(h-1))e \). This is a \( \mathbb{C}[p'] \)-algebra, where we now denote the space of parameters for the Cherednik algebra by \( p' \). Let us denote by \( \mathfrak{P} \) the set of parameters for the quantum Hamiltonian reduction, \( \mathfrak{P} = \mathfrak{g}_v/[[\mathfrak{g}_v, \mathfrak{g}_v]]. \)

**Theorem 7.4.1 ([EGGO, Go, L2, O]).** There is a filtered isomorphism \( A \to A_{\mathfrak{P}} \), which induces a linear isomorphism between the spaces of parameters \( \omega : p' \to \mathfrak{P} \). In particular, for every \( c \in p' \) we have that the algebras \( A_c \) and \( \hat{A}_{\omega(c)} \) are isomorphic as filtered algebras.

Let us give a description of the map \( \omega \) in the case of rational Cherednik algebra of type A. Upon identifying \( p' = \mathbb{C} c \mathfrak{P} = \mathbb{C} z \), this map is simply \( c \mapsto -z-1 \), see [L2]. In particular, we have that the categories \( HC(A_c, A_{c'}) \) and \( HC(\hat{A}_{c-1}, \hat{A}_{c'-1}) \) are equivalent.
Let us now say that, for $c \in \mathbb{C}$, abelian localization holds for the rational Cherednik algebra $A_c$ if it holds for $\lambda$, where $\lambda$ is such that $A_\lambda = \widehat{A}_{-c}$. It is known, see [GS, Sections 5 and 6] and [EG, Corollary 4.2], that abelian localization fails at $c$ if and only if $c = -r/m$, with $1 < m \leq n$ and $1 \leq r < m$. Note that this is the case if and only if $c$ is aspherical. Let us denote the locus of aspherical parameters for the type $A$ Cherednik algebra by $\mathfrak{A}$. Thus, setting $f = \text{id}$ in Theorem 7.3.9 we get the following result.

**Corollary 7.4.2.** Let $c, c' \in \mathbb{C}$ and let $N \in \mathbb{Z}$ be such that $(-1)^{1-\chi_\mathfrak{A}(c)}c + N, (-1)^{1-\chi_\mathfrak{A}(c')}c' + N \notin \mathfrak{A}$, where $\chi_\mathfrak{A}$ is the indicator function of $\mathfrak{A}$ (for example, take $N > |\Re(c)|, |\Re(c')|$). Then, there is an equivalence of categories

$$\mathbb{D} : \text{HC}(H_c(n), H_{c'}(n)) \to \text{HC}(H_{-1}^{1-\chi_\mathfrak{A}(c)}c + N(n), H_{-1}^{1-\chi_\mathfrak{A}(c')}c' + N(n))^\text{opp}.$$  

**Proof.** If neither $c$ nor $c'$ are aspherical, then this is clear from Theorem 7.3.9, see Remark 7.3.10. In the case where one of them is aspherical, say $c'$, first use the equivalence $\text{HC}(H_c(n), H_{c'}(n)) \to \text{HC}(H_{-c}(n), H_{c'}(n))$ given by the isomorphism $H_c \to H_{-c}$. \qed

Let us now compare the double wall-crossing bimodule $D$ with the dual of the regular bimodule $H_c$. So let $N$ be sufficiently big so that localization holds at $c$ and $N - c$. We will denote by $D_{N-c}(n)$ the double wall-crossing bimodule for the algebra $H_{N-c}(n)$.

**Proposition 7.4.3.** Let $c = r/m, \gcd(r; m) = 1$, $1 < m \leq n$ and $c \notin (-1, 0)$. Let $N \in \mathbb{Z}$ be such that $-c + N \notin (-1, 0)$, and consider the duality functor $\mathbb{D} : \text{HC}(H_c(n), H_c(n)) \to \text{HC}(H_{-c+N}(n), H_{-c+N}(n))$. Then, $\mathbb{D}(D_c(n)) = H_{-c+N}(n)$.

**Proof.** First of all, note that the duality preserves supports. Since both $H_c(n)$ and $H_{-c+N}(n)$ have a unique irreducible HC bimodule with full support, cf. Theorem 5.4.4, the duality has to send the unique irreducible bimodule with full support over $H_c(n)$ to the unique irreducible fully supported bimodule over $H_{-c+N}(n)$. Now, $\mathbb{D}(D_c(n))$ will be an $H_{-c+N}(n)$-bimodule whose socle coincides with the unique minimal ideal of $H_{-c+N}(n)$. Since $H_{-c+N}(n)$ is the
injective hull of its unique minimal ideal, we have an embedding $D(D_c(n)) \to H_{-c+N}(n)$. The result now follows by comparing the composition lengths.

Of course, the previous proposition has its corresponding result when $c$ is aspherical. Here, we need to take $N$ such that localization holds at $-c$ and $N + c$ (note that $N = 1$ suffices). Then, we have the duality functor $D : (H_c(n), H_c(n)) \to D(H_{c+N}(n), H_{c+N}(n))^{\text{opp}}$, and we get $D(D_c(n)) = H_{c+N}(n)$.

**Corollary 7.4.4.** For any $c \in \mathbb{C}$, the regular bimodule $H_c(n)$ and the double wall-crossing bimodule $D_c(n)$ are injective-projective in the category $HC(H_c(n), H_c(n))$.

**Proof.** We have seen that both $H_c(n)$ and $D_c(n)$ are injective in $HC(H_c(n), H_c(n))$, see Proposition 3.4.7 and Corollary 6.2.11. The result now follows immediately from Proposition 7.4.3. \qed
Bibliography


