Extensions of amenable groups by recurrent groupoids

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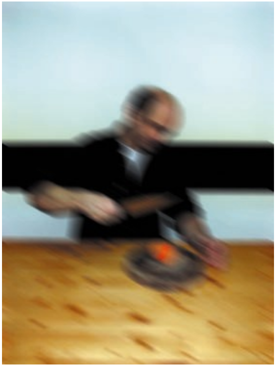
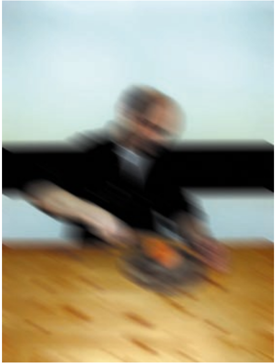
Banach-Tarski Paradox, 1924: There exists a decomposition of a ball into a finite number of non-overlapping pieces, which can be assembeled together into two identical copies of the original ball.





Amenable actions

Definition

An action of a discrete group G on a set X is **amenable** if there exists a map $\mu : \mathcal{P}(X) \to [0,1]$ such that

1. $\mu(X) = 1$, μ is finitely additive
2. $\mu(gE) = \mu(E)$ for all $E \subset X$ and $g \in G$.

Definition

G is **amenable** if the action of G on itself by left multiplication is amenable.

Fact

G is amenable iff there exists an amenable action of G on a set X such that $Stab_G(x)$ is amenable for all $x \in X$.

Examples

Amenable groups	Non-amenable groups
finite, abelian, solvable, nilpotent of subexponential growth	free groups $\mathbb{F}_n$, $n \geq 2$, free Burnside groups, Tarski monsters
Closed under: taking subgroups, extensions, quotients, inductive limits	All groups that contain a non-amenable subgroup

... and much more examples.

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Amenable actions of non-amenable groups

Let $W(\mathbb{Z})$ be **the wobbling group** of integers, i.e. $W(\mathbb{Z})$ consists of all bijections $g : \mathbb{Z} \to \mathbb{Z}$ such that

$$|g(j) - j| \text{ is uniformly bounded.}$$

Obviously, the action of $W(\mathbb{Z})$ on $\mathbb{Z}$ admits an invariant mean.

van Douwen: $\mathbb{F}_2 < W(\mathbb{Z})$.

We assume: G act on X transitively; G is generated by a finite set S.

Definition

An action of G on X is **recurrent** if the probability of returning to x_0 after starting at x_0 is equal to 1 for some (hence for any) $x_0 \in X$. An action is transient if it is not recurrent.

Note: The action of G on itself is recurrent $\iff$ G is virtually $\{0\}$, $\mathbb{Z}$ or $\mathbb{Z}^2$. Moreover, all recurrent actions are amenable.

Theorem

If there exists an increasing to X sequence of subsets X_i such that $\sum_i |\partial X_i|^{-1} = \infty$ then the action is recurrent.

Amenability of the actions of semidirect products and recurrent actions

Let $\mathcal{P}_f(X)$ be the set of finite subsets of X considered as a group with multiplication give by symmetric difference. Then G acts on $\mathcal{P}_f(X)$ by $g.\{n_1, \ldots, n_k\} = \{g(n_1), \ldots, g(n_k)\}$.

Lemma (Juschenko, Monod, 2011)

Fix $p \in X$ and let $L_2(\{0,1\}^X, \mu)$ be the standard Bernoulli space. TFAE:

(i) *The action of $G \ltimes \mathcal{P}_f(X)$ on $\mathcal{P}_f(X)$ is amenable*

(ii) *There exists a net of unit vectors $f_n \in L_2(\{0,1\}^X, \mu)$ such that for every $g \in G$*

$$\|gf_n - f_n\|_2 \to 0 \text{ and } \|f_n \cdot \chi_{\{(\omega_x) \in \{0,1\}^X : \omega_p = 0\}}\| \to 1.$$

Proof:
$f_n \in L^2(\{0,1\}^X, \mu)$ with $\|f_n\|_2 = 1$ and

$$\|g.f_n - f_n\|_2 \to 0, \text{ for every } g \in G,$$

$$\|f_n \cdot \chi_{\{\omega_j \in \{0,1\}^X : \omega_p = 0\}}\| \to 1.$$

We can identify the Pontryagin dual of $\mathcal{P}_f(X)$ with $\{0,1\}^X$ by pairing:

$$\phi(E, \omega) = exp(i\pi \sum_{j \in E} \omega_j)$$

Let $\hat{f}_n \in l_2(\mathcal{P}_f(X))$ be the Fourier transform of f_n:

$$\hat{f}_n(E) = \int_{\{0,1\}^X} f_n(\omega) exp(i\pi \sum_{j \in E} \omega_j) d\omega$$

1. $\hat{f}_n$ are G-almost invariant.
2. $\hat{f}_n$ are $\{p\}$-almost invariant. Thus there exists $\mathcal{P}_f(X) \rtimes G$-almost invariant mean on $\mathcal{P}_f(X)$.

(ii) There exists a net of unit vectors $f_n \in L_2(\{0,1\}^X, \mu)$ such that for every $g \in G$

$$\|gf_n - f_n\|_2 \to 0 \text{ and } \|f_n \cdot \chi_{\{(\omega_x)\in\{0,1\}^X : \omega_p = 0\}}\| \to 1.$$

We say that the function $f \in L_2(\{0,1\}^X, \mu)$ is *p.i.r.* if $f(\omega) = \prod_{x\in X} f_x(\omega_x)$

Theorem (J+N+dlS)

The action of G on X is recurrent iff there exists a net of p.i.r. which satisfies (ii). In particular, if the action is recurrent then the action of $G \ltimes \mathcal{P}_f(X)$ on $\mathcal{P}_f(X)$ is amenable.

Elementary amenable groups.

Definition

The class of elementary **amenable groups** is the smaller class which contain all finite and abelian groups and closed under taking subgroups, quotients, extensions and direct limits.

Day's problem, '57: find non-elementary amenable group.

Grigorchuk, '83: Grigorchuk's group of intermediate group

(Grigorchuk, Zuk '02)+(amenability proof of Bartholdi, Virag '05): Basilica group

Juschenko, Monod '11: the full topological group of Cantor minimal system

Actions on topological spaces

Let G be a group acting by homeomorphisms on a topological space $\mathcal{X}$.

The full topological group of the action, $[[G]]$, is the group of all homeomorphisms h of $\mathcal{X}$ such that for every $x \in \mathcal{X}$ there exists a neighborhood of x such that restriction of h to that neighborhood is equal to restriction of an element of G.

For $x \in \mathcal{X}$ **the group of germs** of G at x is the quotient of the stabilizer of x by the subgroup of elements acting trivially on a neighborhood of x.

Theorem (J+N+dlS)

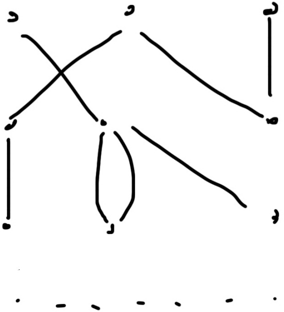
Let G and H be groups of homeomorphisms of a compact topological space $\mathcal{X}$, and G is finitely generated. Suppose that the following conditions hold:

1. *The full group $[[H]]$ is amenable.*
2. *For every element $g \in G$, the set of points $x \in X$ such that g does not coincide with an element of H on any neighborhood of x is finite.*
3. *For every point $x \in \mathcal{X}$ the Schreier graph of the action of G on the orbit of x is recurrent.*
4. *For every $x \in \mathcal{X}$ the group of germs of G at x is amenable.*

Then the group G is amenable. Moreover, the group $[[G]]$ is amenable.

Actions on Bratteli diagrams

A *Bratteli diagram* $D = ((V_i)_{i\geq 1}, (E_i)_{i\geq 1}, \mathrm{o}, \mathrm{t})$ is defined by two sequences of finite sets $(V_i)_{i=1,2,\ldots}$ and $(E_i)_{i=1,2,\ldots}$, and sequences of maps $\mathrm{o} : E_i \longrightarrow V_i$, $\mathrm{t} : E_i \longrightarrow V_{i+1}$.



$\mathcal{X} = \Omega(\mathrm{D})$ is the set of all infinite paths in D.

Let v and w be paths of length n that end in the same vertex. $T_{v,w}$ be a homeomorphism of $\Omega(\mathrm{D})$ that maps a path of the form $(v, x_{n+1}, x_{n+2} \ldots)$ to $(w, x_{n+1}, x_{n+2} \ldots)$.

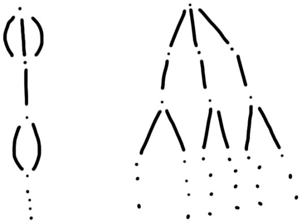
Let $F : \Omega \longrightarrow \Omega$ be a homeomorphism. For $t \in V_i$ denote by $\alpha_t(F)$ the number of paths w that end in t such that $F|_{w\Omega}$ is not equal to a transformation of the form $T_{w,u}$ for some u that ends in t.

The homeomorphism F is said to be *of bounded type* if $\alpha_v(F)$ is uniformly bounded and the set of the points $x \in \Omega(\mathrm{D})$ such that F does not coincide with T_{ω_1,ω_2} on any neighborhood of x is finite.

Theorem (J+N+dlS)

Let D *be a Bratteli diagram. Let G be a group acting faithfully by homeomorphisms of bounded type on* $\Omega(\mathrm{D})$*. Suppose that all groups of germs of G are amenable. Then the group G is amenable.*

Automorphisms of rooted trees



Denote by X^* a rooted homogeneous tree and X^n is its n-th level. For every $g \in \mathrm{Aut}\,\mathsf{X}^*$ and $v \in \mathsf{X}^n$ there exists an automorphism $g|_v \in \mathrm{Aut}\,\mathsf{X}^*$ such that

$$g(vw) = g(v)g|_v(w)$$

An automorphism $g \in \mathrm{Aut}\,\mathsf{X}^*$ is *finite-state* if the set $\{g|_v \;:\; v \in \mathsf{X}^*\} \subset \mathrm{Aut}\,\mathsf{X}^*$ is finite.

Denote by $\alpha_n(g)$ the number of paths $v \in X^n$ such that $g|_v$ is non-trivial. We say that $g \in \mathrm{Aut}\, X^*$ is *bounded* if the sequence $\alpha_n(g)$ is bounded.

Corollary (Bartholdi, Nekrashevych, Kaimanovich, Duke '09)

The group of bounded automata of finite state is amenable.

Corollary

The group of bounded automata are amenable.

Corollary (Amir, Angel, Virag, JEMS '10)

The group of automata of linear growth are amenable.

Corollary

The group of automata of quadratic growth are amenable.

Cantor minimal systems

C - Cantor space, $T : C \to C$ be a homeomorphism.

The system (T, C) is **minimal** if there is no non-trivial closed T-invariant subsets in C.

Corollary (Juschenko, Monod, Annals of Math. '13)

The full topological group of Cantor minimal system is amenable.

The *Basilica group* is generated by two automorphisms a, b of the binary rooted tree given by the rules

$$\begin{aligned} a(0v) &= 1v, & a(1v) &= 0b(v), \\ b(0v) &= 0v, & b(1v) &= 1a(v). \end{aligned}$$

Theorem (Grigorchuk, Zuk, IJAC, '02)

The Basilica group is not elementary amenable

Corollary (Bartholdi, Virag, Duke '05)

The Basilica group is amenable

Summary

The following are amenable

- Groups of bounded, linear and quadratic growth
- uncountably many modifications of Grigorchuk's group
- Basilica group
- Penrose tiling group
- Neuman-Segal groups (non-uniformly exponential growth)
- Groups naturally appearing in holomorphic dynamics: iterated monodromy group of polynomial iterations, holonomy group of the stable foliation of the Julia of Hénon maps
- ?

The End