Bernoulli actions and sofic entropy

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The Shannon entropy of a partition $\mathcal{P}$ of a probability space $(X, \mu)$ is defined as

$$H(\mathcal{P}) = - \sum_{i=1}^{n} \mu(P_i) \log \mu(P_i),$$

which can be viewed as the integral of the information function

$$I(x) = - \log \mu(P_i)$$

where $i$ is such that $x \in P_i$. 
Kolmogorov-Sinai entropy

For a measure-preserving action $G \curvearrowright (X, \mu)$ of an amenable group with Følner sequence $\{F_n\}$ define

$$h_\mu(T) = \sup_\mathcal{P} \lim_{n \to \infty} \frac{1}{|F_n|} H \left( \bigvee_{s \in F_n} s^{-1}\mathcal{P} \right).$$

Kolmogorov-Sinai theorem

*The supremum is achieved on every finite generating partition.*

For amenable $G$:

- the entropy of a Bernoulli action is equal to the Shannon entropy of the base
- Bernoulli actions are classified by entropy (Ornstein-Weiss)
- Factors of a Bernoulli action are Bernoulli (Ornstein-Weiss)
Sofic measure entropy

Basic idea

Replace internal averaging over partial orbits (information theory) by external averaging over abstract finite sets on which the dynamics is modeled (statistical mechanics).

Let $\mathcal{P}$ be a partition of $X$ whose atoms have measures $c_1, \ldots, c_n$. In how many ways can we approximately model this ordered distribution of measures by a partition of $\{1, \ldots, d\}$ for a given $d \in \mathbb{N}$? By Stirling’s formula, the number of models is roughly

$$c_1^{-c_1d} \cdots c_n^{-c_n d}$$

for large $d$, so that

$$\frac{1}{d} \log(\#\text{models}) \approx - \sum_{i=1}^{n} c_i \log c_i = H(\mathcal{P}).$$
Sofic measure entropy

Let $G \curvearrowright (X, \mu)$ be a measure-preserving action of a countable sofic group. Fix a sequence $\Sigma$ of maps $\sigma_i : G \to \text{Sym}(d_i)$ into finite permutation groups which are asymptotically multiplicative and free in the sense that

$$\lim_{i \to \infty} \frac{1}{d_i} \left| \left\{ k \in \{1, \ldots, d_i\} : \sigma_{i,s}^{st}(k) = \sigma_{i,s} \sigma_{i,t}(k) \right\} \right| = 1$$

for all $s, t \in G$, and

$$\lim_{i \to \infty} \frac{1}{d_i} \left| \left\{ k \in \{1, \ldots, d_i\} : \sigma_{i,s}(k) \neq \sigma_{i,t}(k) \right\} \right| = 1$$

for all distinct $s, t \in G$. 
Sofic measure entropy

Define \( \text{Hom}_\mu(\mathcal{P}, F, \delta, \sigma_i) \) to be the set of all homomorphisms from the algebra generated by \( \mathcal{P} \) to the algebra of subsets of \( \{1, \ldots, d_i\} \) which, to within \( \delta \),

- are approximately \( F \)-equivariant, and
- approximately pull back the uniform probability measure on \( \{1, \ldots, d_i\} \) to \( \mu \).

For a partition \( \mathcal{Q} \leq \mathcal{P} \), write \( |\text{Hom}_\mu(\mathcal{P}, F, \delta, \sigma_i)|_{\mathcal{Q}} \) for the cardinality of the set of restrictions of elements of \( \text{Hom}_\mu(\mathcal{P}, F, \delta, \sigma_i) \) to \( \mathcal{Q} \).

Definition

\[
h_{\Sigma, \mu}(X, G) = \sup_{\mathcal{Q}} \inf_{\mathcal{P} \supseteq \mathcal{Q}} \inf_{F, \delta} \limsup_{i \to \infty} \frac{1}{d_i} \log |\text{Hom}_\mu(\mathcal{P}, F, \delta, \sigma_i)|_{\mathcal{Q}}
\]

Kolmogorov-Sinai-type theorem

If \( \mathcal{P} \) is generating then it suffices to compute \( \inf_{F, \delta} \) with \( \mathcal{Q} = \mathcal{P} \).
The topological entropy $h_\Sigma(X, G)$ of an action $G \curvearrowright X$ on a compact metrizable space can be defined similarly. It measures the exponential growth of the number of approximately equivariant maps $\{1, \ldots, d_i\} \rightarrow X$ that can be distinguished up to an observational error.
The entropy of a homeomorphism $T : X \to X$ of a compact metric space measures the exponential growth of the number of partial orbits up to an observational error. More precisely,

$$h_{\text{top}}(T) = \sup_{\varepsilon > 0} \limsup_{n \to \infty} \frac{1}{n} \log \text{sep}(n, \varepsilon)$$

where $\text{sep}(n, \varepsilon)$ is the maximal cardinality of an $\varepsilon$-separated set of partial orbits from 0 to $n - 1$. 
Topological entropy

We can view a partial orbit from 0 to \( n - 1 \) as a map

\[
\varphi : \{0, \ldots, n - 1\} \to X
\]

which is approximately equivariant with respect to the canonical cyclic permutation \( S \) of \( \{0, \ldots, n - 1\} \).

Notice that if \( k \) is proportionally small with respect to \( n \) then the map \( \varphi \) will be equivariant with respect to the actions of \( S^k \) and \( T^k \) on a proportionally large subset of \( \{0, \ldots, n - 1\} \).
Topological entropy

Let \( \rho \) be a continuous pseudometric on \( X \). For a finite \( F \subseteq G \) and a \( \delta > 0 \) define \( \text{Map}(\rho, F, \delta, \sigma_i) \) to be the set of all maps \( \{1, \ldots, d_i\} \to X \) which are approximately \((F, \delta)\)-equivariant w.r.t. 

\[
\rho_2(\varphi, \psi) = \left( \frac{1}{d_i} \sum_{v=1}^{d_i} \rho(\varphi(v), \psi(v))^2 \right)^{1/2}.
\]

Set

\[
h_\Sigma(\rho) = \sup_{\varepsilon > 0} \inf_{F, \delta} \limsup_{i \to \infty} \frac{1}{d_i} \log N_\varepsilon(\text{Map}(\rho, F, \delta, d_i))
\]

where \( N_\varepsilon(\cdot) \) denotes the max cardinality of a \((\rho_2, \varepsilon)\)-separated set.

**Definition**

\( h_\Sigma(X, G) \) is defined to be the common value of \( h_\Sigma(\rho) \) over all dynamically generating \( \rho \).
Theorem (variational principle)

Let $G \curvearrowright X$ be an action on a compact metrizable space. Then

$$h_\Sigma(X, G) = \sup_{\mu} h_{\Sigma, \mu}(X, G)$$

where $\mu$ ranges over all invariant Borel probability measures on $X$.

The sofic topological and measure entropies coincide with their classical counterparts when $G$ is amenable, and so this extends the classical variational principle.
Application to surjunctivity

Gottschalk’s surjunctivity problem asks which countable groups $G$ are **surjunctive**, which means that, for each $k \in \mathbb{N}$, every injective $G$-equivariant continuous map $\{1, \ldots, k\}^G \rightarrow \{1, \ldots, k\}^G$ is surjective.

**Theorem (Gromov)**

Every countable sofic group is surjunctive.

One can give an entropy proof of Gromov’s theorem by showing the following.

**Theorem (K.-Li)**

Let $G$ be a countable sofic group and $k \in \mathbb{N}$. Then with respect to every sofic approximation sequence the shift $G \curvearrowright \{1, \ldots, k\}^G$ has entropy $\log k$ and all proper subshifts have entropy less than $\log k$. 
Bernoulli actions

Like in the amenable case:

- the entropy of a Bernoulli action of a sofic group is equal to the Shannon entropy of its base (Bowen, K.-Li)
- Bernoulli actions of nontorsion sofic groups are classified by their entropy (Bowen)

Unlike in the amenable case:

- if $G$ is nonamenable then there are Bernoulli actions of $G$ which factor onto every Bernoulli action of $G$ (Ball)
- if $G$ contains $F_2$ then any two nontrivial Bernoulli actions of $G$ factor onto one another (Bowen)
- many nonamenable groups, including property (T) groups, have Bernoulli actions with non-Bernoulli factors (Popa)
Bernoulli actions

Definition
An action of a sofic group has **completely positive entropy** if every nontrivial factor has positive entropy with respect to every $\Sigma$.

A Bernoulli action $G \curvearrowright (Y, \nu)^G$ of an amenable $G$ has completely positive entropy because all factors are Bernoulli. One can see this more directly as follows.
Let $\mathcal{P}$ be a finite partition of $(Y, \nu)^G$ and $\varepsilon > 0$. Find a partition $\mathcal{Q}$ such that the members of $\mathcal{Q}$ are unions of cylinder sets over a finite set $K \subseteq G$ and $\max(H(\mathcal{P}|\mathcal{Q}), H(\mathcal{Q}|\mathcal{P})) < \varepsilon$. Given a finite set $F \subseteq G$ and an $F' \subseteq F$ for which the translates $s^{-1}K$ for $s \in F'$ are pairwise disjoint and belong to $F$, we then have

$$\frac{1}{|F|} H\left( \bigvee_{s \in F} s^{-1}\mathcal{P} \right) \geq \frac{1}{|F|} H\left( \bigvee_{s \in F'} s^{-1}\mathcal{P} \right) \geq \frac{1}{|F|} \left( H\left( \bigvee_{s \in F'} s^{-1}\mathcal{Q} \right) - |F'| \varepsilon \right) \geq \frac{|F'|}{|F|} (H(\mathcal{Q}) - \varepsilon).$$

If $\varepsilon$ is small, as $F$ runs through a Følner sequence this last quantity will be asymptotically bounded below by a positive number.
A Bernoulli action $G \curvearrowright (Y, \nu)^G$ of a sofic group has completely positive entropy.
Bernoulli actions

For a partition $\mathcal{Q}$ consisting of cylinder sets over $e$:

To show that the entropy is bounded below by $H_{\mu}(\mathcal{Q})$, enumerate the elements of $\mathcal{Q}$ as $A_1, \ldots, A_n$ and think of homomorphisms from the algebra generated by $\mathcal{Q}$ to the algebra of subsets of $\{1, \ldots, d_i\}$ as elements of $\{1, \ldots, n\}^{d_i}$, which we regard as a probability space under the measure $\nu^{d_i}$. One shows that with high probability an element of this space

1. is approximately equivariant with distribution like that of $\mathcal{Q}$,
2. has measure roughly $e^{-d_iH(\mathcal{Q})}$.

For an arbitrary partition:

Relativize the above argument using the positive density of independent translates like in the amenable case but over the sofic approximation space $\{1, \ldots, d_i\}$. 
Bowen’s $f$-invariant

Let $F_r \curvearrowright (X, \mu)$ be a measure-preserving action of a free group on $r$ generators $s_1, \ldots, s_r$. Write $B_n$ for the set of words in $s_1, \ldots, s_r$ of length at most $n$. For a finite partition $\mathcal{P}$ of $X$ set

$$F(\mathcal{P}) = (1 - 2r)H(\mathcal{P}) + \sum_{i=1}^{r} H(\mathcal{P} \lor s_i^{-1}\mathcal{P}),$$

$$f(\mathcal{P}) = \inf_{n \in \mathbb{N}} F\left(\bigvee_{s \in B_n} s^{-1}\mathcal{P}\right)$$

This last quantity is the same for all generating partitions $\mathcal{P}$, and in the case that there exists a generating partition we define the $f$-invariant of the action to be this common value.
Bowen’s $f$-invariant

Bowen showed that the $f$-invariant coincides with a version of sofic entropy which is locally computed by averaging over all sofic approximations on a finite set instead of using a given sofic approximation.

**Corollary**

*Every nontrivial factor of a Bernoulli action of $F_r$ possessing a finite generating partition has strictly positive $f$-invariant.*