Bernoulli actions and sofic entropy

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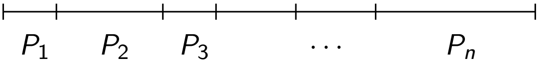
Shannon entropy

The *Shannon entropy* of a partition $\mathcal{P}$



of a probability space (X, μ) is defined as

$$H(\mathcal{P}) = -\sum_{i=1}^{n} \mu(P_i) \log \mu(P_i),$$

which can be viewed as the integral of the information function

$$I(x) = -\log \mu(P_i)$$

where i is such that $x \in P_i$.

Kolmogorov-Sinai entropy

For a measure-preserving action $G \curvearrowright (X, \mu)$ of an amenable group with Følner sequence $\{F_n\}$ define

$$h_\mu(T) = \sup_{\mathcal{P}} \lim_{n\to\infty} \frac{1}{|F_n|} H\Big(\bigvee_{s \in F_n} s^{-1}\mathcal{P} \Big).$$

Kolmogorov-Sinai theorem

The supremum is achieved on every finite generating partition.

For amenable G:

- the entropy of a Bernoulli action is equal to the Shannon entropy of the base
- Bernoulli actions are classified by entropy (Ornstein-Weiss)
- Factors of a Bernoulli action are Bernoulli (Ornstein-Weiss)

Sofic measure entropy

Basic idea

Replace **internal** *averaging over partial orbits (information theory) by* **external** *averaging over abstract finite sets on which the dynamics is modeled (statistical mechanics).*

Let $\mathcal{P}$ be a partition of X whose atoms have measures $c_1, \dots, c_n$. In how many ways can we approximately model this ordered distribution of measures by a partition of $\{1, \dots, d\}$ for a given $d \in \mathbb{N}$? By Stirling's formula, the number of models is roughly

$$c_1^{-c_1 d} \cdots c_n^{-c_n d}$$

for large d, so that

$$\frac{1}{d} \log(\#\text{models}) \approx -\sum_{i=1}^{n} c_i \log c_i = H(\mathcal{P}).$$

Sofic measure entropy

Let $G \curvearrowright (X, \mu)$ be a measure-preserving action of a countable sofic group. Fix a sequence Σ of maps $\sigma_i : G \to \mathrm{Sym}(d_i)$ into finite permutation groups which are asymptotically multiplicative and free in the sense that

$$\lim_{i \to \infty} \frac{1}{d_i} \big| \{ k \in \{1, \dots, d_i\} : \sigma_{i,st}(k) = \sigma_{i,s}\sigma_{i,t}(k) \} \big| = 1$$

for all $s, t \in G$, and

$$\lim_{i \to \infty} \frac{1}{d_i} \big| \{ k \in \{1, \dots, d_i\} : \sigma_{i,s}(k) \neq \sigma_{i,t}(k) \} \big| = 1$$

for all distinct $s, t \in G$.

Sofic measure entropy

Define $\mathrm{Hom}_\mu(\mathcal{P}, F, \delta, \sigma_i)$ to be the set of all homomorphisms from the algebra generated by $\mathcal{P}$ to the algebra of subsets of $\{1, \ldots, d_i\}$ which, to within δ,

- are approximately F-equivariant, and
- approximately pull back the uniform probability measure on $\{1, \ldots, d_i\}$ to μ.

For a partition $\mathcal{Q} \leq \mathcal{P}$, write $|\mathrm{Hom}_\mu(\mathcal{P}, F, \delta, \sigma_i)|_\mathcal{Q}$ for the cardinality of the set of restrictions of elements of $\mathrm{Hom}_\mu(\mathcal{P}, F, \delta, \sigma_i)$ to $\mathcal{Q}$.

Definition

$$h_{\Sigma,\mu}(X, G) = \sup_{\mathcal{Q}} \inf_{\mathcal{P} \geq \mathcal{Q}} \inf_{F,\delta} \limsup_{i \to \infty} \frac{1}{d_i} \log |\mathrm{Hom}_\mu(\mathcal{P}, F, \delta, \sigma_i)|_\mathcal{Q}$$

Kolmogorov-Sinai-type theorem

If $\mathcal{P}$ is generating then it suffices to compute $\inf_{F,\delta}$ with $\mathcal{Q} = \mathcal{P}$.

Topological entropy

The topological entropy $h_\Sigma(X, G)$ of an action $G \curvearrowright X$ on a compact metrizable space can be defined similarly. It measures the exponential growth of the number of approximately equivariant maps $\{1, \ldots, d_i\} \to X$ that can be distinguished up to an observational error.

Topological entropy

The entropy of a homeomorphism $T : X \to X$ of a compact metric space measures the exponential growth of the number of partial orbits up to an observational error. More precisely,

$$h_{\mathrm{top}}(T) = \sup_{\varepsilon>0} \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{sep}(n, \varepsilon)$$

where $\mathrm{sep}(n, \varepsilon)$ is the maximal cardinality of an ε-separated set of partial orbits from 0 to $n-1$.

Topological entropy

We can view a partial orbit from 0 to $n-1$ as a map

$$\varphi : \{0, \ldots, n-1\} \to X$$

which is approximately equivariant with respect to the canonical cyclic permutation S of $\{0, \ldots, n-1\}$.

Notice that if k is proportionally small with respect to n then the map φ will be equivariant with respect to the actions of S^k and T^k on a proportionally large subset of $\{0, \ldots, n-1\}$.

Topological entropy

Let ρ be a continuous pseudometric on X. For a finite $F \subseteq G$ and a $\delta > 0$ define $\mathrm{Map}(\rho, F, \delta, \sigma_i)$ to be the set of all maps $\{1, \ldots, d_i\} \to X$ which are approximately (F, δ)-equivariant w.r.t.

$$\rho_2(\varphi, \psi) = \left(\frac{1}{d_i} \sum_{v=1}^{d_i} \rho(\varphi(v), \psi(v))^2 \right)^{1/2}.$$

Set

$$h_\Sigma(\rho) = \sup_{\varepsilon > 0} \inf_{F, \delta} \limsup_{i \to \infty} \frac{1}{d_i} \log N_\varepsilon(\mathrm{Map}(\rho, F, \delta, d_i))$$

where $N_\varepsilon(\cdot)$ denotes the max cardinality of a (ρ_2, ε)-separated set.

Definition

$h_\Sigma(X, G)$ is defined to be the common value of $h_\Sigma(\rho)$ over all dynamically generating ρ.

Theorem (variational principle)

Let $G \curvearrowright X$ be an action on a compact metrizable space. Then

$$h_\Sigma(X, G) = \sup_\mu h_{\Sigma,\mu}(X, G)$$

where μ ranges over all invariant Borel probability measures on X.

The sofic topological and measure entropies coincide with their classical counterparts when G is amenable, and so this extends the classical variational principle.

Application to surjunctivity

Gottschalk's surjunctivity problem asks which countable groups G are **surjunctive**, which means that, for each $k \in \mathbb{N}$, every injective G-equivariant continuous map $\{1, \ldots, k\}^G \to \{1, \ldots, k\}^G$ is surjective.

Theorem (Gromov)

Every countable sofic group is surjunctive.

One can give an entropy proof of Gromov's theorem by showing the following.

Theorem (K.-Li)

Let G be a countable sofic group and $k \in \mathbb{N}$. Then with respect to every sofic approximation sequence the shift $G \curvearrowright \{1, \ldots, k\}^G$ has entropy $\log k$ and all proper subshifts have entropy less than $\log k$.

Bernoulli actions

Like in the amenable case:

- the entropy of a Bernoulli action of a sofic group is equal to the Shannon entropy of its base (Bowen, K.-Li)
- Bernoulli actions of nontorsion sofic groups are classified by their entropy (Bowen)

Unlike in the amenable case:

- if G is nonamenable then there are Bernoulli actions of G which factor onto every Bernoulli action of G (Ball)
- if G contains F_2 then any two nontrivial Bernoulli actions of G factor onto one another (Bowen)
- many nonamenable groups, including property (T) groups, have Bernoulli actions with non-Bernoulli factors (Popa)

Bernoulli actions

Definition

An action of a sofic group has **completely positive entropy** if every nontrivial factor has positive entropy with respect to every Σ.

A Bernoulli action $G \curvearrowright (Y, \nu)^G$ of an amenable G has completely positive entropy because all factors are Bernoulli. One can see this more directly as follows.

Bernoulli actions

Let $\mathcal{P}$ be a finite partition of $(Y, \nu)^G$ and $\varepsilon > 0$. Find a partition $\mathcal{Q}$ such that the members of $\mathcal{Q}$ are unions of cylinder sets over a finite set $K \subseteq G$ and $\max(H(\mathcal{P}|\mathcal{Q}), H(\mathcal{Q}|\mathcal{P})) < \varepsilon$. Given a finite set $F \subseteq G$ and an $F' \subseteq F$ for which the translates $s^{-1}K$ for $s \in F'$ are pairwise disjoint and belong to F, we then have

$$\begin{aligned}\frac{1}{|F|}H\bigg(\bigvee_{s\in F} s^{-1}\mathcal{P}\bigg) &\geq \frac{1}{|F|}H\bigg(\bigvee_{s\in F'} s^{-1}\mathcal{P}\bigg) \\ &\geq \frac{1}{|F|}\bigg(H\bigg(\bigvee_{s\in F'} s^{-1}\mathcal{Q}\bigg) - |F'|\varepsilon\bigg) \\ &= \frac{|F'|}{|F|}(H(\mathcal{Q}) - \varepsilon).\end{aligned}$$

If ε is small, as F runs through a Følner sequence this last quantity will be asymptotically bounded below by a positive number.

Bernoulli actions

Theorem (K.)

A Bernoulli action $G \curvearrowright (Y, \nu)^G$ of a sofic group has completely positive entropy.

Bernoulli actions

For a partition $\mathcal{Q}$ consisting of cylinder sets over e:

To show that the entropy is bounded below by $H_\mu(\mathcal{Q})$, enumerate the elements of $\mathcal{Q}$ as $A_1, \dots, A_n$ and think of homomorphisms from the algebra generated by $\mathcal{Q}$ to the algebra of subsets of $\{1, \dots, d_i\}$ as elements of $\{1, \dots, n\}^{d_i}$, which we regard as a probability space under the measure ν^{d_i}. One shows that with high probability an element of this space

(1) is approximately equivariant with distribution like that of $\mathcal{Q}$,

(2) has measure roughly $e^{-d_i H(\mathcal{Q})}$.

For an arbitrary partition:

Relativize the above argument using the positive density of independent translates like in the amenable case but over the sofic approximation space $\{1, \dots, d_i\}$.

Bowen's f-invariant

Let $F_r \curvearrowright (X, \mu)$ be a measure-preserving action of a free group on r generators $s_1, \ldots, s_r$. Write B_n for the set of words in $s_1, \ldots, s_r$ of length at most n. For a finite partition $\mathcal{P}$ of X set

$$F(\mathcal{P}) = (1 - 2r)H(\mathcal{P}) + \sum_{i=1}^{r} H(\mathcal{P} \vee s_i^{-1}\mathcal{P}),$$

$$f(\mathcal{P}) = \inf_{n \in \mathbb{N}} F\Big(\bigvee_{s \in B_n} s^{-1}\mathcal{P}\Big)$$

This last quantity is the same for all generating partitions $\mathcal{P}$, and in the case that there exists a generating partition we define the f-**invariant** of the action to be this common value.

Bowen's f-invariant

Bowen showed that the f-invariant coincides with a version of sofic entropy which is locally computed by **averaging over all sofic approximations** on a finite set instead of using a given sofic approximation.

Corollary

Every nontrivial factor of a Bernoulli action of F_r possessing a finite generating partition has strictly positive f-invariant.