A von Neumann approach to quantum metrics

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What is a quantum metric?

(In our sense)

Let $\mathcal{M} \subseteq \mathcal{L}(\mathcal{H})$ be a von Neumann algebra. A quantum metric on $\mathcal{M}$ is a weak-* algebra filtration of $\mathcal{L}(\mathcal{H})$ given by $d : \mathcal{L}(\mathcal{H}) \to [0, \infty]$, such that the 0-term is $\mathcal{M}'$. More precisely:

- Each term $\mathcal{V}_t \stackrel{\text{def}}{=} d^{-1}([0, t])$ is an operator system in $\mathcal{L}(\mathcal{H})$ which is closed in the weak-* topology. *I.e.*, d is lower semicontinuous and

$$d(a^*) = d(\lambda a) = d(a) \qquad d(a + b) \leq \max(d(a), d(b)).$$

- Also $\mathcal{V}_s\mathcal{V}_t \subseteq \mathcal{V}_{s+t}$. *I.e.*, $d(ab) \leq d(a) + d(b)$.
- Also $\mathcal{V}_0 = \mathcal{M}'$.

Behold! (But note other definitions, in particular Rieffel's.)

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Independence of $\mathcal{H}$

Doesn't a filtration of $\mathcal{L}(\mathcal{H}) \supseteq \mathcal{M}$ depend on $\mathcal{H}$?

Theorem

No.

To understand this, consider more generally operator spaces

$$\mathcal{V} \subseteq \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2).$$

which are bimodules of $\mathcal{M}_1' \subseteq \mathcal{L}(H_1)$ and $\mathcal{M}_2' \subseteq \mathcal{L}(H_2)$. These are quantum relations between $\mathcal{M}_1$ and $\mathcal{M}_2$. The lattice of quantum relations is stable with respect to $\otimes\mathcal{L}(\mathcal{H})$, and tensoring also preserves *, and composition of relations. This move unites all faithful representations of $\mathcal{M}_1$ and $\mathcal{M}_2$.

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First motivation: Classical metrics

If (X, d) is a metric space, let $\mathcal{M} = \ell^\infty(X)$ and $\mathcal{H} = \ell^2(X)$.

Theorem

Quantum metrics on $\mathcal{M}$ are equivalent to classical metrics on X.

We define a filtration of $\mathcal{L}(\mathcal{H})$ by letting $d(e_{x,y}) = d(x, y)$ for elementary matrices. In general $d(a)$ is the displacement of a, the supremal “distance” of its “motion” as a superposition of $x \mapsto y$.

This construction is reversible, because all bimodules of $\mathcal{M}' = \mathcal{M}$ are spanned by elementary matrices:

$$\begin{pmatrix} 0 & * & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix}.$$

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Classical metrics, continued

In the case $\mathcal{M} = \ell^2(X)$, our quantum axioms correspond well with the axioms of a classical metric space:

$$\begin{aligned} d(x,y) = 0 \text{ iff } x = y \quad &\iff \quad \mathcal{M}' = \mathcal{V}_0 \\ d(x,y) = d(y,x) \quad &\iff \quad \mathcal{V}_t = \mathcal{V}_t^* \\ d(x,z) \le d(x,y) + d(y,z) \quad &\iff \quad \mathcal{V}_s \mathcal{V}_t \subseteq \mathcal{V}_{s+t}. \end{aligned}$$

The only discrepancy is that we allow infinite displacements. This is because the displacement is a supremum, for example:

$$a \in \mathcal{L}(\ell^2(\mathbb{Z})) \qquad a[f](n) = f(n^3) \qquad d(a) = \infty.$$

Besides, we can allow $d(x,y) = \infty$ classically.

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Second motivation: The double commutant theorem

Another alignment: Passing to a subalgebra $\mathcal{N} \subseteq \mathcal{M}$ generalizes the quotient operation $X \twoheadrightarrow X/\sim$ induced by a pseudometric.

- If $\mathcal{M} = \ell^\infty(X)$, then every subalgebra is $\mathcal{N} = \ell^\infty(X/\sim)$.
- If d is a pseudometric on X, we can define a filtration on $\mathcal{L}(\ell^2(X))$ as usual. Then $\mathcal{V}_0 = \mathcal{N}'$, where $\mathcal{N} = \ell^\infty(X/\sim)$ and $x \sim y$ when $d(x, y) = 0$.
- In general a quantum pseudometric on $\mathcal{M}$ is one with $\mathcal{V}_0 \supseteq \mathcal{M}'$ and is viewed as a metric on $\mathcal{N} = \mathcal{V}_0' \subseteq \mathcal{M}$.
- A quantum metric on any $\mathcal{M} \subseteq \mathcal{L}(\mathcal{H})$ is, first, a quantum pseudometric on $\mathcal{L}(\mathcal{H})$.

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Third motivation: Quantum error correction

(My motivation)

In quantum computation, there is an essential theory of quantum error correction.

- Take $\mathcal{M} = \mathcal{L}(\mathcal{H})$, with $\dim \mathcal{H} < \infty$. Usually $\mathcal{L}(\mathcal{H}) = M_2(\mathbb{C})^{\otimes n}$, meaning "a register with n qubits".
- The errors (to be corrected or detected) are an operator system $\mathcal{E} \subseteq \mathcal{L}(\mathcal{H})$. An $\mathcal{E}$-detecting code is a subspace $\mathcal{H}_C \subseteq \mathcal{H}$ such that $pap = \epsilon(a)p$ for $a \in \mathcal{E}$. Here p is projection onto $\mathcal{H}_C$ and $\epsilon : \mathcal{E} \to \mathbb{C}$ is a slope.
- We generalize this to all $\mathcal{M} \subseteq \mathcal{L}(\mathcal{H})$ by letting $p \in \mathcal{M}$ be a self-adjoint idempotent and $pap = \epsilon(a)p$, with $\epsilon : \mathcal{E} \to \mathcal{M}'$.
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Quantum error correction (continued)

Given a quantum code $p \in \mathcal{M} \subseteq \mathcal{L}(\mathcal{H})$ with $pap = \epsilon(a)p$ for $a \in \mathcal{E}$, we could take $\mathcal{E} = \mathcal{V}_t$ for a quantum metric on $\mathcal{M}$. Then p is a quantum minimum distance code.

- An example: There is a trivial quantum metric $d : M_2(\mathbb{C}) \to \{0, 1\}$ with $d(A) = 0$ iff $A \propto I$. Then $(M_2(\mathbb{C}), d)^{\otimes n}$ is quantum Hamming space. (Despite the use of quantum Hamming space from the beginning, a general definition of quantum metrics was overlooked!)
- More examples: Every quantum metric on a qubit $M_2(\mathbb{C})$ is given by $d(X)$, $d(Y)$, and $d(Z)$, up to conjugation, where X, Y, and Z are the Pauli spin matrices. Also $d(X) \leq d(Y) + d(Z)$, etc. A metric qubit is like a classical triangle.

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Fourth motivation: Measurable metric spaces

(Nik's motivation)

Weaver [J. Funct. An., 1996] defined measurable metric spaces. Given an abstract σ-algebra Σ, one defines a distance function $d(a, b)$ for booleans $a, b \in \Sigma$. The distance function satisfies some delicate axioms. Measurable metric spaces have various favorable properties.

Happily, the axioms can be made less delicate. A measurable metric space is exactly a quantum metric on $\mathcal{M} = L^\infty(\Sigma)$, if this is a von Neumann algebra.

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Quantum graphs, posets, uniform spaces, ...

Suppose that $\mathcal{M} \subseteq \mathcal{L}(\mathcal{H})$ and $\mathcal{E} \subseteq \mathcal{L}(\mathcal{H})$ is an $\mathcal{M}'$-bimodule, a quantum relation.

- If $\mathcal{E}$ is an operator system, then $(\mathcal{M}, \mathcal{E})$ is a quantum graph.
- If $\mathcal{E}$ is an algebra with $\mathcal{E} \cap \mathcal{E}^* = \mathcal{M}'$, then $(\mathcal{M}, \mathcal{E})$ is a quantum poset.
- If $\{\mathcal{E}\}$ is a family of quantum graphs satisfying the Weil-Bourbaki axioms, then each $\mathcal{E}$ is a quantum entourage and $(\mathcal{M}, \{\mathcal{E}\})$ is a quantum uniform space.

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Quantum graph theory

Theorem

If a graph Γ has valence v and n vertices, then the independence number $\alpha(\Gamma) \geq \lceil n/(v+1) \rceil$.

We can define the valence of a quantum graph $(\mathcal{M}, \mathcal{E})$ as $v = r - 1$, where r is the rank of $\mathcal{E}$ as a left $\mathcal{M}'$-module.

Theorem (Knill-Laflamme-Viola [1999])

If $\Gamma = (M_n(\mathbb{C}), \mathcal{E})$ *is a finite, purely quantum graph, then*

$$\alpha(\Gamma) \geq \left\lceil \left\lceil \frac{n}{v} \right\rceil \frac{1}{v+1} \right\rceil.$$

The quantum independence number is exactly the maximal $\mathcal{M}'$-rank of a quantum code. The proof uses Tverberg's theorem from convex geometry.

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Quantum graphs (continued)

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If Γ is a graph with valence v, then the chromatic number $\chi(\Gamma) \le v + 1$.

But here there is a surprise...

Theorem (Steven Lu)

There exists a quantum graph $\Gamma = (M_n(\mathbb{C}), \mathcal{E})$ with valence one, and with

$$\chi(\Gamma) \ge \lceil \log_2(n) \rceil.$$

Here a quantum coloring is a homomorphism $f : \ell^2(S) \to \mathcal{M}$, where S is a set of colors and each $f(s)$ is a quantum code.

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Topologies from metrics

- Rieffel begins with a C^*-algebra $\mathcal{A}$ and considers compatible quantum metrics in his sense. This is fine, but backwards relative to undergraduate analysis.
- Traditionally, a set X is a canvas, a metric $d(x, y)$ is a painting on the canvas, and the topology $\mathcal{T}$ induced by d is the painting's impression.
- A quantum topology on $\mathcal{M}$ is a weakly dense C^*-subalgebra $\mathcal{A} \subseteq \mathcal{M}$, an algebra of bounded, "continuous" elements. If $\mathcal{M} = \ell^\infty(X)$, then $\mathcal{A} \subseteq \mathcal{M}$ comes from a topology on X with a compactification.
- If $\mathcal{M} = \ell^\infty(X)$ and X has a metric d, then there a C^*-algebra $\mathcal{A}$ of bounded, uniformly continuous functions. We want a quantum version of this.

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The continuous C^* algebra

Given $\mathcal{M} \subseteq \mathcal{L}(\mathcal{H})$ and $\{\mathcal{V}_t\}$, an element $a \in \mathcal{L}(\mathcal{H})$ is commutation uniform if for every $\epsilon > 0$, there exists a $\delta > 0$, such that

$$x \in \mathcal{V}_\delta \quad \Longrightarrow \quad ||[x, a]|| < \epsilon ||x||.$$

Let $\mathcal{A}$ be the set of commutation-uniform a.

Theorem

$\mathcal{A}$ is a C^*-subalgebra of $\mathcal{M}$ and is weak- dense.*

The construction generalizes to quantum uniformities.

Example: Rieffel tori.

One can also make a Lipschitz algebra by taking $\delta \propto \epsilon$.

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Tropicalization

- The tropicalization of the semiring $\mathbb{R}_+$ is the limit

$$\log(\exp(ta) + \exp(tb))/t \longrightarrow \max(a, b)$$
$$\log(\exp(ta)\exp(tb))/t^2 \longrightarrow a + b$$

  as $t \to \infty$. This limit is recently important in algebraic geometry...

- And was always important in error correction. Minimum distance is the tropicalization of error likelihood:

$$[\exp(-t||x||^2) \lesssim \exp(-t)] \longrightarrow [||x|| > 1]\,.$$

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Rigidity

- Von Neumann quantum metrics are "rigid" while Rieffel metrics are "soft". Our definition is "tropical".
- Our definition is clearly correct for quantum error correction.
- Rieffel obtains Gromov-Hausdorff convergence

$$[\mathrm{SO}(3) \to M_{2\ell+1}(\mathbb{C})] \longrightarrow C(S^2).$$

  We do not know whether our quantum metrics can do this.
- Unf. we have two definitions of Gromov-Hausdorff limits.
- A natural map like this also seems possible:

$$\{\text{Von Neumann quantum metrics}\} \longrightarrow \{\text{Rieffel quantum metrics}\}.$$

  What is the quality of this correspondence?

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A factory for C^*-algebras

- The single most important use of classical metrics is to construct topological spaces.
- Von Neumann quantum metrics are a way to construct C^*-algebras from Von Neumann algebras. Is this a useful source of C^*-algebras?