All Irrational Extended Rotation Algebras are AF

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Rotation Algebras

Let $\theta \in [0,1]$. Recall

$$A_\theta = \mathrm{C}^*\{u, v : \; uv = e^{2\pi i\theta} vu, \; u, v \text{ unitaries}\}$$

If $\theta \in \mathbb{Q}$, then A_θ is a homogeneous C*-algebra with spectrum $\mathbb{T}^2$.
If $\theta \in [0,1] \setminus \mathbb{Q}$, then

- A_θ is simple, has a unique trace, and
- A_θ is the A$\mathbb{T}$-algebra (an inductive limit of $\mathrm{C}(\mathbb{T}) \otimes F$, where F is a finite dimensional C*-algebra) with

$$(\mathrm{K}_0(A_\theta), \mathrm{K}_0^+(A_\theta), [1_{A_\theta}]_0) = (\mathbb{Z} + \mathbb{Z}\theta, (\mathbb{Z} + \mathbb{Z}\theta) \cap \mathbb{R}^+, 1)$$

  and $\mathrm{K}_1(A_\theta) = \mathbb{Z} \oplus \mathbb{Z}$.

(A_θ) form a continuous field.

AF-embedding of A_θ

Let B_θ be the AF-algebra with

$$(\mathrm{K}_0(B_\theta), \mathrm{K}_0^+(B_\theta), [1_{B_\theta}]_0) = (\mathbb{Z} + \mathbb{Z}\theta, (\mathbb{Z} + \mathbb{Z}\theta) \cap \mathbb{R}^+, 1).$$

Then

Theorem (Pimsner-Voiculescu)

There is a unital embedding of A_θ into B_θ.

Remark

Since both A_θ and B_θ are $\mathrm{A}\mathbb{T}$-algebras, by the classification theorem of $\mathrm{A}\mathbb{T}$-algebras, there always exist unital embeddings of A_θ to B_θ. Moreover, all such embeddings are approximately unitarily equivalent.

Extended Rotation Algebras

Let θ be irrational. Consider a representation of A_θ, and consider the spectral projections

$$p = E_u([0,\theta)) \quad \text{and} \quad q = E_v([0,\theta)).$$

Then the extended rotation algebra is

$$\mathcal{B}_\theta := \mathrm{C}^*\{u, v, p, q\}.$$

Generators and Relations

Let

$$\mathrm{C}(\Omega_u) := \mathrm{C}^*\{\mathrm{C}(\mathbb{T}) \cup \{\sigma^{-k}(p) : \ k \in \mathbb{Z}\}\}$$

and

$$\mathrm{C}(\Omega_v) := \mathrm{C}^*\{\mathrm{C}(\mathbb{T}) \cup \{\sigma^{k}(q) : \ k \in \mathbb{Z}\}\},$$

where σ is the rotation by angle θ. Then $\mathcal{B}_\theta$ is the universal C*-algebra generated by $\mathrm{C}(\Omega_u)$ and $\mathrm{C}(\Omega_v)$ with respect to the relations

1. $uv = e^{2\pi i\theta} vu$,
2. $u\sigma^k(q)u^* = \sigma^{k+1}(q)$ for any $k \in \mathbb{Z}$, and
3. $v\sigma^{-k}(p)v^* = \sigma^{-k-1}(p)$ for any $k \in \mathbb{Z}$

That is,

$$\mathcal{B}_\theta = (\mathrm{C}(\Omega_u) \rtimes_\sigma \mathbb{Z}) *_{A_\theta} (\mathrm{C}(\Omega_v) \rtimes_\sigma \mathbb{Z}).$$

Or

$$\mathcal{B}_\theta = (\mathrm{C}(\Omega_u) \rtimes_\sigma \mathbb{Z}) *_{\mathrm{C}(\mathbb{T})} (\mathrm{C}([0,1]));$$

or, $\mathcal{B}_\theta$ is the universal C*-algebra generated by unitaries u and v together with positive elements h_u and h_v satisfying the following relations:

1. $uv = e^{2\pi i\theta} vu$,
2. $\|h_u\| = \|h_v\| = 1$,
3. $u = e^{2\pi i h_u}$, and
4. $v = e^{2\pi i h_v}$.

Remark

The order-4 automorphism of A_θ

$$u \mapsto v^*, \quad v \mapsto u$$

and the flip automorphism

$$u \mapsto u^* \quad v \mapsto v^*$$

can be lifted to automorphisms of $\mathcal{B}_\theta$.

Theorem (Elliott-N)

For any irrational θ, the C-algebra $\mathcal{B}_\theta$ is simple amenable unital, has a unique trace,*

$$(K_0(\mathcal{B}_\theta), K_0^+(\mathcal{B}_\theta), [1_{\mathcal{B}_\theta}]_0) = (\mathbb{Z} + \mathbb{Z}\theta, (\mathbb{Z} + \mathbb{Z}\theta) \cap \mathbb{R}^+, 1),$$

and $K_1(\mathcal{B}_\theta) = 0$.

Theorem (Elliott-N)

With a careful construction of rational extended rotation algebras, $(\mathcal{B}_\theta)$ form a upper-semicontinuous field of C-algebras.*

Theorem (Elliott-N)

For all irrational θ, the C-algebra $\mathcal{B}_\theta$ is AF.*

Proof of the main theorem

Show that the C*-algebras $\mathcal{B}_\theta$ and B_θ are covered by a recent classification theorem. Since they have the same invariants, they must be isomorphic.

TAF-algebras

Definition (Lin)

A unital simple C*-algebra A is said to be a tracially approximately finite dimensional algebra (TAF-algebra) if for any finite subset $\mathcal{F} \subset A$, any $\varepsilon > 0$, and nonzero $a \in A^+$, there exist a nonzero projection $p \in A$ and a sub-C*-algebra $F \cong \bigoplus_{i=1}^{m} \mathrm{M}_{r(i)}$ such that

1. $\|[x, \ p]\| \leq \varepsilon$ for any $x \in \mathcal{F}$,
2. for any $x \in \mathcal{F}$, there is $x' \in F$ such that $\|pxp - x'\| \leq \varepsilon$, and
3. $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{aAa}$.

$$a \approx_\varepsilon \begin{pmatrix} \left[(1-p)a(1-p)\right] & \\ & \begin{bmatrix} & & \\ & pap & \\ & & \end{bmatrix} \in_\varepsilon F \end{pmatrix} \in A.$$

Classification of TAF algebras

Theorem (Lin)

Let A and B be two simple amenable separable TAF-algebras satisfying the UCT. Then $A \cong B$ if and only if

$$(K_0(A), K_0^+(A), [1_A]_0, K_1(A)) \cong (K_0(B), K_0^+(B), [1_B]_0, K_1(B)).$$

Remark

The class of TAF algebras classified in the theorem above is exactly the class of simple unital real rank zero AH algebras with slow dimension growth. In particular, it contains all simple unital real rank zero $A\mathbb{T}$ algebras and all simple AF algebras.

Jiang-Su algebra $\mathcal{Z}$

Definition

The *Jiang-Su algebra* $\mathcal{Z}$ is the unique unital simple inductive limit of

$$\mathcal{Z}_{pq} := \{f : [0,1] \to \mathrm{M}_p(\mathbb{C}) \otimes \mathrm{M}_q(\mathbb{C}); \begin{array}{l} f(0) \in \mathrm{M}_p(\mathbb{C}) \otimes 1_{\mathrm{M}_q(\mathbb{C})} \\ f(1) \in 1_{\mathrm{M}_p(\mathbb{C})} \otimes \mathrm{M}_q(\mathbb{C}) \end{array} \}$$

with a unique tracial state, where p and q are coprime natural numbers.

Remark

$$(\mathrm{K}_0(\mathcal{Z}), \mathrm{K}_0(\mathcal{Z}), [1_{\mathcal{Z}}]_0) \cong (\mathbb{Z}, \mathbb{Z}^+, 1), \quad \mathrm{K}_1(\mathcal{Z}) = \{0\}, \quad \mathrm{T}(\mathcal{Z}) = \{\mathrm{pt}\}.$$

For well-behaved C*-algebra A (e.g., all simple amenable TAF algebras), one has

$$A \cong A \otimes \mathcal{Z}.$$

Classification of $\mathcal{Z}$-stable Rationally TAF algebras

Definition

A C*-algebra A is called rationally TAF if $A \otimes UHF$ is TAF.

Remark

All TAF algebras are rationally TAF. $\mathcal{Z}$ is rationally TAF.

Theorem (Winter, Lin-N)

Let A and B be two amenable simple rationally TAF algebras satisfying the UCT. Suppose that A and B are $\mathcal{Z}$-stable. Then $A \cong B$ if and only if

$$(K_0(A), K_0^+(A), [1_A]_0, K_1(A)) \cong (K_0(B), K_0^+(B), [1_B]_0, K_1(B)).$$

The AF algebra B_θ is $\mathcal{Z}$-stable, and thus is covered by this classification theorem. How about $\mathcal{B}_\theta$?

A result of Matui and Sato

Theorem (Matui-Sato)

Let A be a unital simple separable amenable C-algebra which is quasi-diagonal and has a unique trace. Then $A \otimes \mathrm{UHF}$ is TAF.*

Lemma

The C-algebra $\mathcal{B}_\theta$ quasi-diagonal. Hence, by Matui-Sato, $\mathcal{B}_\theta$ is rationally TAF.*

$\mathcal{Z}$-stability of $\mathcal{B}_\theta$

Another result of Matui and Sato.

Theorem

Let A be a unital separable simple amenable C-algebra. Suppose that A has finitely many extreme traces. If A has strict comparison for positive elements, then $A \otimes \mathcal{Z} \cong A$.*

Definition

Let $a \in A^+$ and $\tau \in \mathrm{T}(A)$. Define $d_\tau(a) = \lim_{n\to\infty}(\tau(a^{\frac{1}{n}}))$.
The C*-algebra A is said to have strict comparison for positive elements if

$$d_\tau(a) < d_\tau(b), \quad \forall \tau \in \mathrm{T}(A)$$

implies

$$a \preceq b,$$

i.e., there are $x_n \in A$, $n = 1, 2, ...$, in A such that $x_n^* b x_n \to a$.

Large subalgbera

Definition (Phillips)

Let A be a unital sub-C*-algebra of B. Then A is said to be large in B if for any $a \in B$, any $b \in B^+ \setminus \{0\}$, and any $\varepsilon > 0$, there exist $c \in A$ and $g \in B$ such that

1. $\|a - (c + g)\| < \varepsilon$, and
2. $g^* g \precsim b$ in B.

Example

Let σ be a minimal homeomorphism on X, and let $x \in X$. Then

$$A_x := \mathrm{C}^*\{f, gu : \ f, g \in \mathrm{C}(X), \ g(x) = 0\} \subseteq \mathrm{C}(X) \rtimes_\sigma \mathbb{Z}.$$

Theorem

If A is large in B, and if A has strict comparison for positive elements, then B has strict comparison for positive elements.

Theorem

The rotation algebra A_θ is large in $\mathcal{B}_\theta$.

Corollary

The C-algebra $\mathcal{B}_\theta$ has strict comparison for positive elements. Since it is also amenable and has a unique trace, it is $\mathcal{Z}$-stable.*

It also can be shown that the C*-algebra $\mathcal{B}_\theta$ satisfies the UCT. Therefore

Theorem

The C-algebra $\mathcal{B}_\theta$ is a simple amenable $\mathcal{Z}$-stable rationally TAF C*-algebra satisfying the UCT. Hence it is isomorphic to B_θ.*

Remark

Consider the C*-algebra generated by A_θ together with arbitrary spectral projections of u and v. It is always an AF algebra.