# A homology theory for Smale spaces

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## Hyperbolicity

An invertible linear map $T : \mathbb{R}^d \to \mathbb{R}^d$ is *hyperbolic* if $\mathbb{R}^d = E^s \oplus E^u$, $T$-invariant, $C > 0, 0 < \lambda < 1$,

$$\|T^n v\| \leq C\lambda^n \|v\|, \quad n \geq 1 \quad v \in E^s,$$
$$\|T^{-n} v\| \leq C\lambda^n \|v\|, \quad n \geq 1 \quad v \in E^u,$$

Same definition replacing $\mathbb{R}^d$ by a vector bundle (over compact space).

$M$ compact manifold, $\varphi : M \to M$ diffeomorphism is *Anosov* if $D\varphi : TM \to TM$ is hyperbolic.

Smale: $M, \varphi$ *Axiom A*: replace $TM$ above by $TM|_{NW(\varphi)} = E^s \oplus E^u$, where $NW(\varphi)$ is the set of non-wandering points. But $NW(\varphi)$ is usually a fractal, not a submanifold.

**Smale spaces** (D. Ruelle)

$(X, d)$ compact metric space,

$\varphi : X \to X$ homeomorphism $0 < \lambda < 1$,

For $x$ in $X$ and $\epsilon > 0$ and small, there is a local stable set $X^s(x, \epsilon)$ and a local unstable set $X^u(x, \epsilon)$:

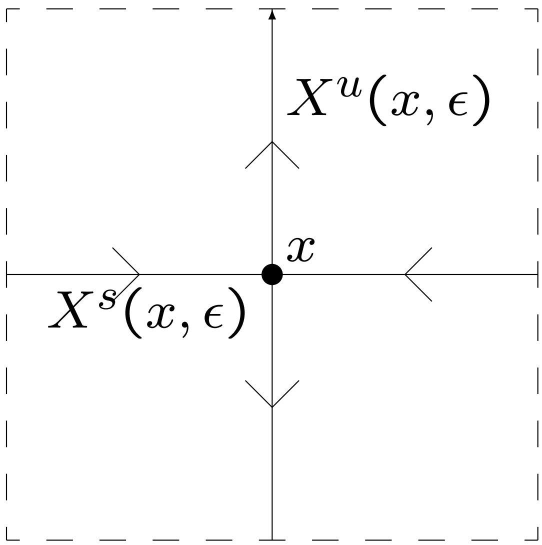
1. $X^s(x, \epsilon) \times X^u(x, \epsilon)$ is homeomorphic to a neighbourhood of $x$,

2. $\varphi$-invariance,

3.

$$\begin{aligned} d(\varphi(y), \varphi(z)) &\leq \lambda d(y, z), \quad y, z \in X^s(x, \epsilon), \\ d(\varphi^{-1}(y), \varphi^{-1}(z)) &\leq \lambda d(y, z), \quad y, z \in X^u(x, \epsilon), \end{aligned}$$

That is, we have a local picture:




Global stable and unstable sets:

$$
\begin{aligned}
X^s(x) &= \{y \mid \lim_{n\to+\infty} d(\varphi^n(x),\varphi^n(y)) = 0\} \\
X^u(x) &= \{y \mid \lim_{n\to+\infty} d(\varphi^{-n}(x),\varphi^{-n}(y)) = 0\}
\end{aligned}
$$

These are equivalence relations.

$X^s(x,\epsilon) \subset X^s(x)$, $X^u(x,\epsilon) \subset X^u(x)$.

**Example 1**

The linear map $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is hyperbolic. Let $\gamma > 1$ be the Golden mean,

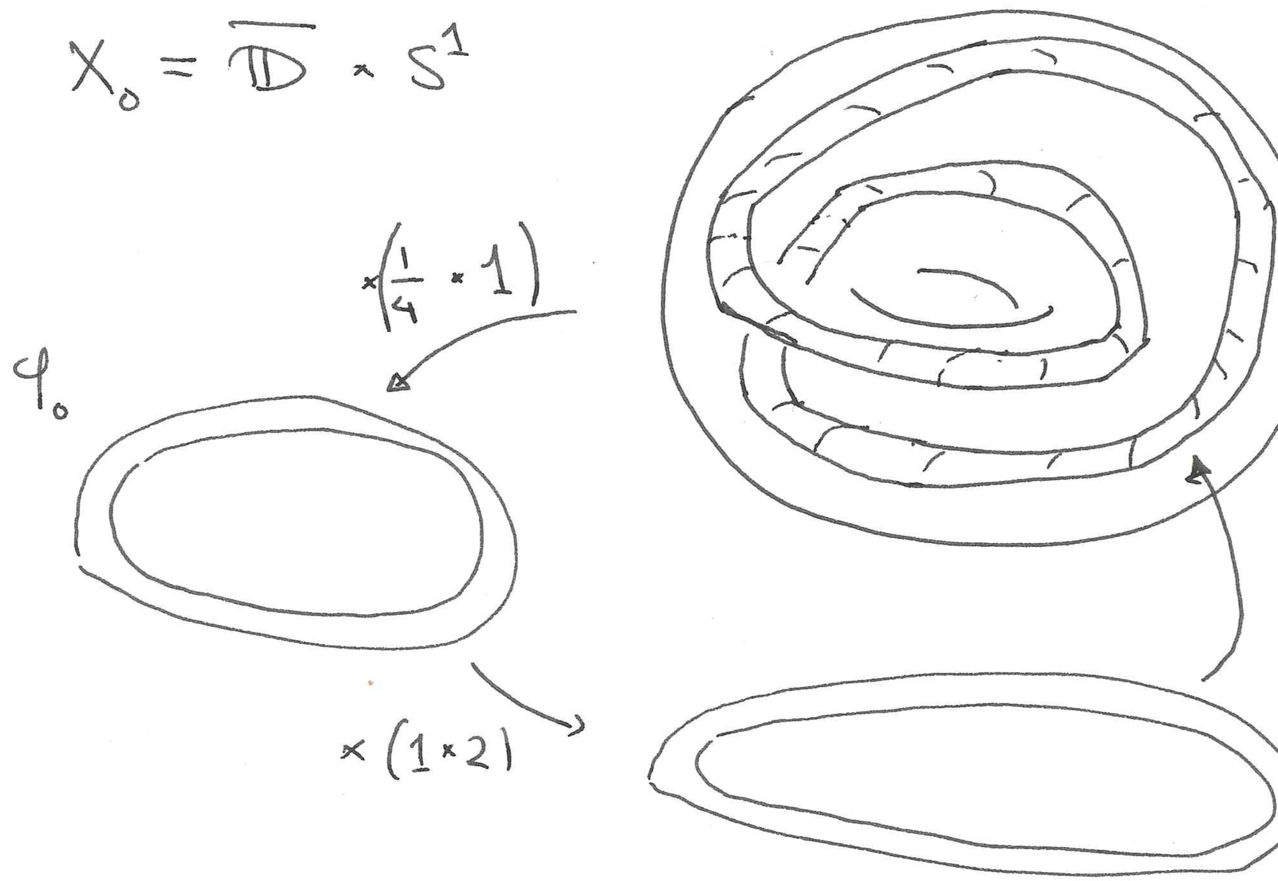
$$\begin{aligned} (\gamma, 1)A &= \gamma(\gamma, 1) \\ (-1, \gamma)A &= -\gamma^{-1}(-1, \gamma) \end{aligned}$$

As $det(A) = -1$, it induces a homeomorphism of $\mathbb{R}^2/\mathbb{Z}^2$ which is Anosov.

$X^s$ and $X^u$ are Kronecker foliations with lines of slope $-\gamma^{-1}$ and $\gamma$.

## Example 2: $2^\infty$-solenoid

$X_0 = \overline{\mathbb{D}} \times S^1$




$$X = \bigcap_{n \geq 0} \varphi_0^n(X_0), \quad \varphi = \varphi_0|X$$

$X^s((x,y),\varepsilon) \cong \overline{\mathbb{D}} \times \{y\} \cap X$ Cantor

$X^u((x,y),\varepsilon) \cong \{x\} \times (y-\varepsilon, y+\varepsilon)$

**Example 3: Shifts of finite type (SFTs)**

Let $G = (G^0, G^1, i, t)$ be a finite directed graph. Then we have the shift space and shift map:

$$\begin{aligned}\Sigma_G &= \{(e^k)_{k=-\infty}^{\infty} \mid e^k \in G^1, \\ &\quad\; i(e^{k+1}) = t(e^k), \text{ for all } n\} \\ \sigma(e)^k &= e^{k+1}, \text{ "left shift"}\end{aligned}$$

The local product structure is given by

$$\Sigma^s(e,1) = \{(\ldots, *, *, *, e^0, e^1, e^2, \ldots)\}$$
$$\Sigma^u(e,1) = \{(\ldots, e^{-2}, e^{-1}, e^0, *, *, *, \ldots)\}$$

## Example 4

Let $m < n$ be relatively prime, and

$$X = \mathbb{Q}_m \times \mathbb{R} \times \mathbb{Q}_n / \mathbb{Z}[1/mn]$$

and

$$\varphi(a, r, b) = \left(\frac{n}{m}a, \frac{n}{m}r, \frac{n}{m}b\right).$$

The $\mathbb{Q}_m \times \mathbb{R}$ coordinates are expanding while the $\mathbb{Q}_n$ coordinate is contracting.

Smales spaces have a large supply of periodic points and it is interesting to count them.

Adjacency matrix of $G$: $G^0 = \{1, 2, \ldots, N\}$, $A_G$ is $N \times N$ with

$$(A_G)_{i,j} = \#\text{edges from } i \text{ to } j$$

**Theorem 1.** *Let $A_G$ be the adjancency matrix of the graph $G$. For any $p \geq 1$, we have*

$$\#\{e \in \Sigma_G \mid \sigma^p(e) = e\} = Tr(A_G^p).$$

This is reminiscent of the Lefschetz fixed-point formula for smooth maps of compact manifolds.

**Question 2.** *Is the right hand side actually the result of $\sigma$ acting on some homology theory of $(\Sigma_G, \sigma)$?*

Positive answers by Bowen-Franks and Krieger.

**Krieger's invariants for SFT's**

W. Krieger defined invariants, which we denote by $D^s(\Sigma_G, \sigma), D^u(\Sigma_G, \sigma)$, for shifts of finite type by considering stable and unstable equivalence as groupoids and taking its groupoid $C^*$-algebra:

$$K_0(C^*(X^s)), K_0(C^*(X^s))$$

In this case, these are both AF-algebras and

$$D^s(\Sigma_G, \sigma) = \lim \mathbb{Z}^N \xrightarrow{A_G} \mathbb{Z}^N \xrightarrow{A_G} \cdots$$

(For the unstable, replace $A_G$ with $A_G^T$.) Each comes with a canonical automorphism.

Returning to Smale spaces . . .

## Bowen's Theorem

**Theorem 3** (Bowen)**.** *For a non-wandering Smale space, $(X, \varphi)$, there exists a SFT $(\Sigma, \sigma)$ and*

$$\pi : (\Sigma, \sigma) \to (X, \varphi),$$

*with $\pi \circ \sigma = \varphi \circ \pi$, continuous, surjective and finite-to-one.*

First, this means that SFT's have a special place among Smale spaces. Secondly, one can try to understand $(X, \varphi)$ by investigating $(\Sigma, \sigma)$. For example, they will have the same entropy. Of course, $(\Sigma, \sigma)$ is not unique.

A. Manning used Bowen's Theorem to provide a formula counting the number of periodic points for $(X, \varphi)$.

For $N \geq 0$, define

$$\begin{aligned}\Sigma_N(\pi) &= \{(e_0, e_1, \ldots, e_N) \mid \\ &\quad \pi(e_n) = \pi(e_0), \\ &\quad 0 \leq n \leq N\}.\end{aligned}$$

For all $N \geq 0$, $(\Sigma_N(\pi), \sigma)$ is also a shift of finite type. Observe that $S_{N+1}$ acts on $\Sigma_N(\pi)$.

**Theorem 4** (Manning)**.** *For a non-wandering Smale space $(X, \varphi)$, $(\Sigma, \sigma)$ as above and $p \geq 1$, we have*

$$\begin{aligned}&\#\{x \in X \mid \varphi^p(x) = x\} \\ = \;& \textstyle\sum_N (-1)^N Tr(\sigma_*^p : D^s(\Sigma_N(\pi))^{alt} \\ &\to D^s(\Sigma_N(\pi))^{alt}).\end{aligned}$$

**Question 5** (Bowen)**.** *Is there a homology theory for Smale spaces $H_*(X, \varphi)$ which provides a Lefschetz formula, counting the periodic points?*

In fact, the groups $D^s(\Sigma_N(\pi))^{alt}$ appear to be giving a chain complex.

Idea: for $0 \leq n \leq N$, let $\delta_n : \Sigma_N(\pi) \to \Sigma_{N-1}(\pi)$ be the map which deletes entry $n$.

Let $(\delta_n)_* : D^s(\Sigma_N(\pi))^{alt} \to D^s(\Sigma_{N-1}(\pi))^{alt}$ be the induced map and $\partial = \sum_{n=0}^{N}(-1)^n(\delta_n)_*$ to make a chain complex.

This is wrong: a map

$$\rho : (\Sigma, \sigma) \to (\Sigma', \sigma)$$

between shifts of finite type does *not* always induce a group homomorphism between Krieger's invariants.

While it is true that $\rho$ will map $R^s(\Sigma)$ to $R^s(\Sigma')$ the functorial properties of the construction of groupoid $C^*$-algebras is subtle.

Let $\pi : (Y, \psi) \to (X, \varphi)$ be a factor map between Smale spaces. For every $y$ in $Y$, we have $\pi(Y^s(y)) \subseteq X^s(\pi(y))$.

**Definition 6.** *$\pi$ is* $s$-bijective *if* $\pi : Y^s(y) \to X^s(\pi(y))$ *is bijective, for all $y$.*

**Theorem 7.** *If $\pi$ is $s$-bijective then $\pi : Y^s(y, \epsilon) \to X^s(\pi(y), \epsilon')$ is a local homeomorphism.*

**Theorem 8.** *Let $\pi : (\Sigma, \sigma) \to (\Sigma', \sigma)$ be a factor map between SFT's.*

*If $\pi$ is $s$-bijective, then there is a map*

$$\pi^s : D^s(\Sigma, \sigma) \to D^s(\Sigma', \sigma).$$

*If $\pi$ is $u$-bijective, then there is a map*

$$\pi^{s*} : D^s(\Sigma', \sigma) \to D^s(\Sigma, \sigma).$$

Bowen's $\pi : (\Sigma, \sigma) \to (X, \varphi)$ is not $s$-bijective or $u$-bijective if $X$ is a torus, for example.

**A better Bowen's Theorem**

Let $(X, \varphi)$ be a Smale space. We look for a Smale space $(Y, \psi)$ and a factor map

$$\pi_s : (Y, \psi) \to (X, \varphi)$$

satisfying:

1. $\pi_s$ is $s$-bijective,

2. $dim(Y^u(y, \epsilon)) = 0$.

That is, $Y^u(y, \epsilon)$ is totally disconnected, while $Y^s(y, \epsilon)$ is homeomorphic to $X^s(\pi_s(y), \epsilon)$.

This is a "one-coordinate" version of Bowen's Theorem.

Similarly, we look for a Smale space $(Z, \zeta)$ and a factor map $\pi_u : (Z, \zeta) \to (X, \varphi)$ satisfying $dim(Z^s(z, \epsilon)) = 0$, and $\pi_u$ is $u$-bijective.

We call $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ a *$s/u$-bijective pair* for $(X, \varphi)$.

**Theorem 9.** *If $(X, \varphi)$ is a non-wandering Smale space, then there exists an $s/u$-bijective pair.*

Consider the fibred product:

$$\Sigma = \{(y, z) \in Y \times Z \mid \pi_s(y) = \pi_u(z)\}$$

with

$$\begin{tikzcd} & \Sigma \arrow[dl, "\rho_u"'] \arrow[dr, "\rho_s"] & \\ Y \arrow[dr, "\pi_s"'] & & Z \arrow[dl, "\pi_u"] \\ & X & \end{tikzcd}$$

$\rho_s(y, z) = z$ is $s$-bijective, $\rho_u(y, z) = y$ is $u$-bijective. Hence, $\Sigma$ is a SFT.

For $L, M \geq 0$, we define

$$\begin{aligned}\Sigma_{L,M}(\pi) &= \{(y_0, \ldots, y_L, z_0, \ldots, z_M) \mid \\ &\quad y_l \in Y, z_m \in Z, \\ &\quad \pi_s(y_l) = \pi_u(z_m)\}.\end{aligned}$$

Each of these is a SFT.

Moreover, the maps

$$\begin{aligned}\delta_{l,} : \ \Sigma_{L,M} &\to \ \Sigma_{L-1,M}, \\ \delta_{,m} : \ \Sigma_{L,M} &\to \ \Sigma_{L,M-1}\end{aligned}$$

which delete $y_l$ and $z_m$ are $s$-bijective and $u$-bijective, respectively.

This is the key point! We have avoided the issue which caused our earlier attempt to get a chain complex to fail.

We get a double complex:

$$\begin{array}{ccccccc}
\uparrow & & \uparrow & & \uparrow & & \\
D^s(\Sigma_{0,2})^{alt} & \longleftarrow & D^s(\Sigma_{1,2})^{alt} & \longleftarrow & D^s(\Sigma_{2,2})^{alt} & \longleftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
D^s(\Sigma_{0,1})^{alt} & \longleftarrow & D^s(\Sigma_{1,1})^{alt} & \longleftarrow & D^s(\Sigma_{2,1})^{alt} & \longleftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
D^s(\Sigma_{0,0})^{alt} & \longleftarrow & D^s(\Sigma_{1,0})^{alt} & \longleftarrow & D^s(\Sigma_{2,0})^{alt} & \longleftarrow &
\end{array}$$

$$\begin{aligned}
\partial^s_N : \quad & \oplus_{L-M=N} D^s(\Sigma_{L,M})^{alt} \\
\to \quad & \oplus_{L-M=N-1} D^s(\Sigma_{L,M})^{alt}
\end{aligned}$$

$$\partial^s_N = \sum_{l=0}^{L} (-1)^l \delta^s_{l,} + \sum_{m=0}^{M+1} (-1)^{m+M} \delta^{s*}_{,m}$$

$$H^s_N(\pi) = \ker(\partial^s_N)/Im(\partial^s_{N+1}).$$

Recall: beginning with $(X, \varphi)$, we select an $s/u$-bijective pair $\pi = (Y, \psi, \pi_s, Z, \zeta\pi_u)$ construct the double complex and compute $H_N^s(\pi)$.

**Theorem 10.** *The groups $H_N^s(\pi)$ do not depend on the choice of $s/u$-bijective pair $\pi$.*

From now on, we write $H_N^s(X, \varphi)$.

**Theorem 11.** *The functor $H_*^s(X, \varphi)$ is covariant for $s$-bijective factor maps, contravariant for $u$-bijective factor maps.*

**Theorem 12.** *The groups $H_N^s(X, \varphi)$ are all finite rank and non-zero for only finitely many $N \in \mathbb{Z}$.*

We can regard $\varphi : (X, \varphi) \to (X, \varphi)$, which is both $s$ and $u$-bijective and so induces an automorphism of the invariants.

**Theorem 13.** *(Lefschetz Formula) Let $(X, \varphi)$ be any non-wandering Smale space and let $p \geq 1$.*

$$\begin{aligned}\sum_{N \in \mathbb{Z}} (-1)^N \quad Tr[(\varphi^s)^p : \quad & H_N^s(X, \varphi) \otimes \mathbb{Q} \\ \to \quad & H_N^s(X, \varphi) \otimes \mathbb{Q}] \\ \\ = \quad & \#\{x \in X \mid \varphi^p(x) = x\}\end{aligned}$$

**Example 1: Shifts of finite type**

If $(X, \varphi) = (\Sigma, \sigma)$, then $Y = \Sigma = Z$ is an $s/u$-bijective pair.

The double complex $D_a^s$ is:

$$\begin{tikzcd}
0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] & \arrow[l] \\
0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] & \arrow[l] \\
D^s(\Sigma) \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] & \arrow[l]
\end{tikzcd}$$

and $H_0^s(\Sigma, \sigma) = D^s(\Sigma)$ and $H_N^s(\Sigma, \sigma) = 0, N \neq 0$.

**Example 2:** $\mathbf{dim(X^s(x, \epsilon)) = 0}$.

(As an example, the solenoid we saw in example 2.)

We may find a SFT and $s$-bijective map

$$\pi_s : (\Sigma, \sigma) \to (X, \varphi).$$

The $Y = \Sigma, Z = X$ is an $s/u$-bijective pair and the double complex $D^s$ is:

$$\begin{tikzcd}
\phantom{0} & \phantom{0} & \phantom{0} & \\
0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] & \arrow[l] \\
0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] & \arrow[l] \\
D^s(\Sigma_0)^{alt} \arrow[u] & D^s(\Sigma_1)^{alt} \arrow[l] \arrow[u] & D^s(\Sigma_2)^{alt} \arrow[l] \arrow[u] & \arrow[l]
\end{tikzcd}$$

**Example 2':** $(\mathbf{X}, \varphi) = \mathbf{2}^{\infty}$**-solenoid** (Bazett-P.)

An $s/u$-bijective pair is $Y = \{0, 1\}^{\mathbb{Z}}$, the full 2-shift, $Z = X$ and the double complex $D^s$ is

$$
\begin{array}{ccccccc}
\uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathbb{Z}[1/2] & \longleftarrow & \mathbb{Z} & \longleftarrow & 0 & \longleftarrow &
\end{array}
$$

and we get

$$H_0^s(X, \varphi) \cong \mathbb{Z}[1/2], H_1^s(X, \varphi) \cong \mathbb{Z},$$

$H_N^s(\Sigma_G, \sigma) = 0, N \neq 0, 1.$

Generalized 1-solenoids (Williams, Yi, Thomsen): Amini, P, Saeidi Gholikandi.

**Example 4**(N. Burke-P.)

Let $m < n$ be relatively prime, and

$$X = \mathbb{Q}_m \times \mathbb{R} \times \mathbb{Q}_n / \mathbb{Z}[1/mn],$$

and

$$\varphi(a, r, b) = \left( \frac{n}{m} a, \frac{n}{m} r, \frac{n}{m} b \right).$$

$$H_0^s(X, \varphi) \cong \mathbb{Z}[1/n]$$

$$H_1^s(X, \varphi) \cong \mathbb{Z}[1/m]$$

and

$$H_N^s(X, \varphi) = 0, N \neq 0, 1.$$

**Example 3: Our Anosov example** (Bazett-P.):

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$$

The double complex $D^s$ looks like:

$$\begin{tikzcd}
\uparrow & \uparrow & \uparrow & \\
0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] & \arrow[l] \\
\mathbb{Z}^2 \arrow[u] & \mathbb{Z} \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] & \arrow[l] \\
\mathbb{Z}^3 \arrow[u] & \mathbb{Z}^2 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] & \arrow[l]
\end{tikzcd}$$

and

| $N$ | $H^s_N(X, \varphi)$ | $\varphi^s$ |
|---|---|---|
| $-1$ | $\mathbb{Z}$ | $1$ |
| $0$ | $\mathbb{Z}^2$ | $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ |
| $1$ | $\mathbb{Z}$ | $-1.$ |