Localization of Matrix Factorizations

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Acknowledgements

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- Physics, i.e. the Anderson model
- Density matrices in quantum chemistry

Wiener's Lemma

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Definition

We denote by $\mathscr{A}(\mathbb{T})$ the Banach algebra of functions with absolutely convergent Fourier series endowed with the norm

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Theorem (Wiener's Lemma, 1932)

If $f \in \mathscr{A}(\mathbb{T})$ and $f(t) \neq 0$ for all $t \in \mathbb{T}$, then $1/f \in \mathscr{A}(\mathbb{T})$.

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- $\{a_k\} \in \ell^1$ means that A_f satisfies some off-diagonal decay condition.

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Theorem (Wiener's Lemma)

The Banach algebra of functions with absolutely convergent Fourier series, $\mathscr{A}(\mathbb{T})$ is inverse closed in the Banach algebra of continuous functions $C(\mathbb{T})$.

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- $\mathscr{C}_v := \left\{\mathbf{M} : \sum_{j\in\mathbb{Z}} \sup_{k\in\mathbb{Z}} |m_{k,k-j}| v(j) < \infty\right\}$.

Some Algebraic Properties

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Definition

Let $\mathscr{A}$ be a Banach algebra of matrices and let $\mathscr{L}$ and $\mathscr{L}_0^* = \mathscr{A} \setminus \mathscr{L}$ be the sub-algebras of lower- and strictly-upper-triangular matrices, respectively. Then, we say that $\mathscr{A}$ is *strongly decomposable* if there exists a bounded projection $\mathcal{P}$ which maps $\mathscr{A}$ onto $\mathscr{L}$ parallel to $\mathscr{L}_0^*$. Let $\mathcal{Q} = I - \mathcal{P}$.

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Definition

An invertible matrix $\mathbf{A} \in \mathscr{A}$ admits a *canonical factorization* in $\mathscr{A}$ if $\mathbf{A} = \mathbf{LU}$ where $\mathbf{L}, \mathbf{L}^{-1} \in \mathscr{L}$ and $\mathbf{U}, \mathbf{U}^{-1} \in \mathscr{L}^*$.

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3. $\mathbf{A} \in \mathscr{L}^* \cap \mathscr{B}_c$ if and only if $f_{\mathbf{A}}$ has a bounded holomorphic extension outside of $\mathbb{D}$ which is continuous in $\mathbb{C} \setminus \mathbb{D}$.

Two useful results

Theorem (Baskakov, Krishtal, 2005)

Let $\mathbf{A} \in \mathscr{L} \cap \mathscr{B}_c$. *Then* $\mathbf{A}^{-1} \in \mathscr{L}$ *if and only if* $f_{\mathbf{A}}(z)$ *is invertible for all* $z \in \overline{\mathbb{D}}$.

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Lemma (Gohberg, Laiterer, 1972)

Let $\mathscr{A} \subset \mathscr{A}_c \subset \mathscr{B}(\ell^2)$ *be a strongly decomposable inverse-closed sub-algebra that satisfies* $\|\mathbf{A}\|_{\mathscr{B}(\ell^2)} \leq C\|\mathbf{A}\|_{\mathscr{A}}$. *Then, if* $\|\mathbf{A} - \mathbf{I}\|_{\mathscr{B}(\ell^2)} < 1$, $\mathbf{A}$ *admits a canonical factorization* $\mathbf{A} = \mathbf{LU}$ *in* $\mathscr{A}$ *such that*

$$\mathbf{L}^{-1} = \mathbf{I} - \mathcal{P}\mathbf{V} + \mathcal{P}[\mathbf{V}\mathcal{P}\mathbf{V}] - \mathcal{P}[\mathbf{V}\mathcal{P}[\mathbf{V}\mathcal{P}\mathbf{V}]] + \dots, \tag{1}$$

$$\mathbf{U}^{-1} = \mathbf{I} - \mathcal{Q}\mathbf{V} + \mathcal{Q}[[\mathcal{Q}\mathbf{V}]\mathbf{V}] - \mathcal{Q}[\mathcal{Q}[[\mathcal{Q}\mathbf{V}]\mathbf{V}]\mathbf{V}] + \dots, \tag{2}$$

where $\mathbf{V} = \mathbf{A} - \mathbf{I}$ *and the series converge in* $\mathscr{A}$.

Main Result

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Theorem (Krishtal, Strohmer, W., 2013)

Let $\mathscr{A} \subset \mathscr{B}_c \subset \mathscr{B}(\ell^2)$ be an strongly decomposable inverse-closed sub-algebra that satisfies

$$\|\mathbf{A}\|_{\mathscr{B}(\ell^2)} \leq C\|\mathbf{A}\|_{\mathscr{A}}.$$

Then, if $\mathbf{A}$ admits a canonical factorization $\mathbf{A} = \mathbf{L}\mathbf{U}$ in $\mathscr{B}_c$, we have $\mathbf{L}, \mathbf{U} \in \mathscr{A}$.

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1. Define the holomorphic extensions

$$f_{\mathbf{L}}(z) = \sum_k z^k \mathbf{L}_k, z \in \mathbb{D} \qquad \text{and} \qquad f_{\mathbf{U}}(z) = \sum_k z^k \mathbf{U}_k, z \in \mathbb{C} \setminus \overline{\mathbb{D}}.$$

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2. Choose $\varepsilon \in (0, 1)$ such that $\|[f_{\mathbf{L}}(\varepsilon)]^{-1} \mathbf{L}\mathbf{U}[f_{\mathbf{U}}(1/\varepsilon)]^{-1} - I\|_{\mathscr{B}(\ell^2)} < 1$.

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3. Then $\mathbf{A}' = [f_{\mathbf{L}}(\varepsilon)]^{-1}\mathbf{L}\mathbf{U}[f_{\mathbf{U}}(1/\varepsilon)]^{-1} = \mathbf{L}'\mathbf{U}'$.

4. So $(\mathbf{L}')^{-1}[f_{\mathbf{L}}(\varepsilon)]^{-1}\mathbf{L} = \mathbf{D} = \mathbf{U}' f_{\mathbf{U}}(1/\varepsilon)\mathbf{U}^{-1}$.

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Corollary

Suppose that $\mathbf{A} \in \mathscr{A}$ *admits a QR factorization* $\mathbf{A} = \mathbf{QR}$*. Then* $\mathbf{Q}, \mathbf{R} \in \mathscr{A}$*.*

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Consider $\mathbf{A}^*\mathbf{A} = \mathbf{R}^*\mathbf{Q}^*\mathbf{QR} = \mathbf{R}^*\mathbf{R}$ and apply the previous corollary.

Next steps

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- Eigenvector localization

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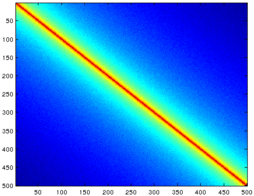
- More general decay patterns

Thanks!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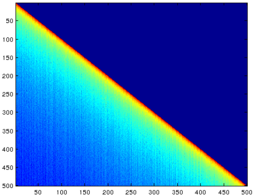
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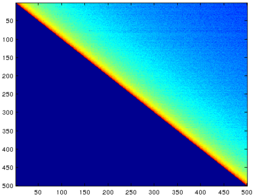
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