One-sided shift spaces over infinite alphabets

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Classical Construction

- Let $\mathcal{A}$ be a finite set (called the alphabet or symbol space).
- Give $\mathcal{A}$ the discrete topology, then $\mathcal{A}$ is compact (since $\mathcal{A}$ is finite).
- Consider the set
$$\mathcal{A}^{\mathbb{N}} := \mathcal{A} \times \mathcal{A} \times \dots$$
consisting of all (one-sided) sequences of elements of $\mathcal{A}$.
- $\mathcal{A}^{\mathbb{N}}$ with the product topology is compact (by Tychonoff's theorem).

Classical Construction Con't

- The shift map $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ defined by $\sigma(x_1x_2x_3\ldots) := x_2x_3x_4\ldots$ is continuous.
- The pair $(\mathcal{A}^{\mathbb{N}}, \sigma)$ is called the (one-sided) full shift space.

Definition

The pair $(X, \sigma|_X)$ is a shift space if X is subset of $\mathcal{A}^{\mathbb{N}}$ such that X is closed and $\sigma(X) \subseteq X$.

Since X is a closed subset of a compact space, X is also compact.

Natural Extension

Let $\mathcal{A} = \{a_1, a_2, \dots\}$ be a countably infinite set and give $\mathcal{A}$ the discrete topology.
Consider the space

$$\mathcal{A}^{\mathbb{N}} := \mathcal{A} \times \mathcal{A} \times \dots$$

with the product topology.
The shift map $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ defined by $\sigma(x_1 x_2 x_3 \dots) := x_2 x_3 x_4 \dots$ is continuous.
However, the space $\mathcal{A}^{\mathbb{N}}$ is **NOT** compact (or even locally compact).

Why doesn't it work?

Example

Any open set U in $\mathcal{A}^{\mathbb{N}}$ must contain a basis element of the form

$$Z(x_1 \dots x_m) = \{x_1 \dots x_m z_{m+1} z_{m+2} \dots \in \mathcal{A}^{\mathbb{N}} : \ z_k \in \mathcal{A} \text{ for } k \geq m+1\}.$$

Define $x^n := x_1 \dots x_m a_n a_n a_n \dots$, then $\{x^n\}_{n=1}^{\infty}$ is a sequence in $Z(x_1 \dots x_m)$ without a convergent subsequence.
Hence the closure of U is not (sequentially) compact, therefore $\mathcal{A}^{\mathbb{N}}$ is not locally compact.
If we define a shift space over $\mathcal{A}$ to be a pair $(X, \sigma|_X)$ where X is a closed subset of $\mathcal{A}^{\mathbb{N}}$ with the property that $\sigma(X) \subseteq X$, then the set X will be a closed, but not necessarily compact, subset of $\mathcal{A}^{\mathbb{N}}$.

Benefits

The full shift and all shift spaces are compact!

Our new definition reduces to the classical definition when $\mathcal{A}$ is finite.

New Construction

- Let $\mathcal{A}$ be an infinite alphabet with the discrete topology.
- Let $\mathcal{A}_\infty = \mathcal{A} \cup \{\infty\}$ denote the one-point compactification of $\mathcal{A}$. Since $\mathcal{A}_\infty$ is compact, the product space

  $$X_{\mathcal{A}} := \mathcal{A}_\infty \times \mathcal{A}_\infty \times \dots$$

  is compact.

Note: We do not take $X_{\mathcal{A}}$ as our definition of the full shift, since it includes sequences that contain the symbol ∞, which is not in our original alphabet.

New Construction Con't

We identify elements of $X_{\mathcal{A}}$ with infinite and finite sequences of elements in $\mathcal{A}$.
How do we do it?

- Infinite sequences with no ∞ in them are of no concern.
- For infinite sequences with ∞, we consider the first place that ∞ appears; for example, write $x = x_1 \dots x_n \infty \dots$ with $x_i \neq \infty$ for $1 \leq i \leq n$ and identify x with the finite sequence $x_1 \dots x_n$.
- In this way we define an equivalence relation $\sim$ on $X_{\mathcal{A}}$ such that the quotient space $X_{\mathcal{A}}/\sim$ of all equivalence classes is identified with the collection of all sequences of symbols from $\mathcal{A}$ that are either infinite or finite.
- We let $\Sigma_{\mathcal{A}}$ denote the set of all finite and infinite sequences of elements of $\mathcal{A}$.

Topology on $\Sigma_{\mathcal{A}}$

We use the identification of $\Sigma_{\mathcal{A}}$ with $X_{\mathcal{A}}/\sim$, to give $\Sigma_{\mathcal{A}}$ the quotient topology it inherits from $X_{\mathcal{A}}$.

With this topology the space $\Sigma_{\mathcal{A}}$ is both compact and Hausdorff.

The shift map $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$, which simply removes the first entry from any sequence, is a map on $\Sigma_{\mathcal{A}}$ that is continuous at all points except the empty sequence.

We define the one-sided full shift to be the pair $(\Sigma_{\mathcal{A}}, \sigma)$.

Defining Shift Spaces

Definition

If $\mathcal{A}$ is an alphabet and $X \subseteq \Sigma_{\mathcal{A}}$, we say X has the infinite-extension property if for all $x \in X$ with $l(x) < \infty$, there are infinitely many $a \in \mathcal{A}$ such that $Z(xa) \cap X \neq \emptyset$.

Definition

Let $\mathcal{A}$ be an alphabet, and $(\Sigma_{\mathcal{A}}, \sigma)$ be the full shift over $\mathcal{A}$. A shift space over $\mathcal{A}$ is defined to be a subset $X \subseteq \Sigma_{\mathcal{A}}$ satisfying the following three properties:

(i) *X is a closed subset of $\Sigma_{\mathcal{A}}$.*

(ii) *$\sigma(X) \subseteq X$.*

(iii) *X has the infinite-extension property.*

Defining Shift Spaces

For any shift space X we define $X^{\text{inf}} := X \cap \Sigma_{\mathcal{A}}^{\text{inf}}$ and $X^{\text{fin}} := X \cap \Sigma_{\mathcal{A}}^{\text{fin}}$.

1. All shift spaces are compact since $\Sigma_{\mathcal{A}}$ is compact.
2. $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ restricts to a map $\sigma|_X : X \to X$. Thus we will often attach the map $\sigma|_X$ to X and refer to the pair $(X, \sigma|_X)$ as a shift space. Note that our definition allows the empty set $X = \emptyset$ as a shift space.
3. If $X \neq \emptyset$, then $X^{\text{inf}} \neq \emptyset$, so that nonempty shift spaces will always have sequences of infinite length. Moreover, X^{inf} is dense in X.

Classical Shifts of Finite Type

Definition

Let X be a shift space over a finite alphabet $\mathcal{A}$. Then X is a shift of finite type if $X = X_{\mathcal{F}}$ for a finite set $\mathcal{F}$ of blocks.

For a finite alphabet $\mathcal{A}$, X is a shift of finite type if and only if X is an M-step shift (i.e., $X = X_{\mathcal{F}}$ for a set $\mathcal{F}$ with each block in $\mathcal{F}$ having length $M+1$) if and only if X is an edge shift (i.e., X is the shift space coming from a finite directed graph with no sinks where the edges are used as symbols).

Edge shifts vs. M-step shifts

Proposition

If X_E is an edge shift, then X_E is a 1-step shift.

The converse is false.

Example

Let $\mathcal{A} = \{a_1, a_2, a_3, \dots\}$ be a countably infinite alphabet, and let

$$\mathcal{F} := \{a_i a_j : i \neq 1 \text{ and } i \neq j\}.$$

Then $X_{\mathcal{F}}$ is a 1-step shift, since every forbidden block in $\mathcal{F}$ has length 2. $X_{\mathcal{F}}$ is not an edge shift.

Shifts of Finite Type vs. M-step shifts

Proposition

If X is a shift of finite type, then X is an M-step shift for some $M \in \mathbb{N} \cup \{0\}$.

The converse is false.

Shifts of Finite Type vs. M-step shifts

Example of an edge shift (therefore an M-step shift) that is not a shift of finite type.

Example

Let E be the graph

$$\bullet \xrightarrow{e_1} \bullet \xrightarrow{e_2} \bullet \xrightarrow{e_3} \bullet \xrightarrow{e_4} \cdots$$

and let X_E be the edge shift associated to E. We shall argue that X_E is not a shift of finite type over $\mathcal{A} := E^1$. Suppose $\mathcal{F}$ is a finite subset of $\Sigma_{\mathcal{A}}^{\text{fin}}$. Since $\mathcal{F}$ is a finite collection of finite sequences of edges, there exists $n \in \mathbb{N}$ such that the edge e_n does not appear in any element of $\mathcal{F}$. Thus the infinite sequence $e_n e_n \ldots$ is allowed, and $e_n e_n \ldots \in X_{\mathcal{F}}$. However, $e_n e_n \ldots \notin X_E$, so $X_E \neq X_{\mathcal{F}}$.

Summary

Here's what we know:

$$\{\text{edge shifts}\} \subsetneq \{M\text{-step shifts}\}$$

$$\{\text{shifts of finite type}\} \subsetneq \{M\text{-step shifts}\}$$

$$\{\text{shifts of finite type}\} \neq \{\text{edge shifts}\}.$$

Summary Con't

Here's what we don't know:

- Is every shift of finite type an edge shift? Conjecture: No
- For each $M \in \mathbb{N} \cup \{0\}$ does there exist an $(M+1)$-step shift space that is not conjugate to any M-step shift? Conjecture: Yes

We have proven that

$$\{\text{0-step shifts}\} \subseteq \{\text{1-step shifts}\} \subseteq \{\text{2-step shifts}\} \subseteq \ldots,$$

but we don't know if the containment is proper.

Definition of row-finite

Definition

Let $\mathcal{A}$ be an alphabet, and let $X \subseteq \Sigma_{\mathcal{A}}$ be a shift space over $\mathcal{A}$. We say that X is finite-symbol (or finite) if $B_1(X)$ is finite, and we say X is infinite-symbol (or infinite) otherwise. We say that X is row-finite if for every $a \in \mathcal{A}$, the set $\{b \in \mathcal{A} : ab \in B(X)\}$ is finite.

Proposition

If $\mathcal{A}$ is an infinite alphabet and X is a shift of finite type over $\mathcal{A}$, then X is not row-finite.

$$\{\text{row-finite shifts of finite type over infinite alphabets}\} = \emptyset.$$

Proposition

If $\mathcal{A}$ is an alphabet and X is a 1-step shift space over $\mathcal{A}$ that is row-finite, then X is conjugate to the edge shift of a row-finite graph.

$$\{\text{row-finite edge shifts}\} = \{\text{row-finite } M\text{-step shifts}\}.$$

Theorem (Curtis, Hedlund, Lyndon)

Every shift morphism is equal to a sliding block code.

Outline of proof:

- If $\phi : X \to Y$ is a shift morphism, then the continuity of ϕ and the compactness of X implies that ϕ is uniformly continuous with respect to the standard metric on X giving the topology.
- Any two sequences in X that are close in this metric are equal along some initial segment, and hence one may define a block map $\Phi : B_n(X) \to \mathcal{A}$ and use the fact that ϕ commutes with the shift to show ϕ is the sliding block code coming from Φ.

For shifts over infinite alphabets, this proof does not work.

Sliding Block Codes

Definition

If X and Y are shift spaces over a countable alphabet $\mathcal{A}$, and X is row-finite, we say that a function $\phi : X \to Y$ is a sliding block code if the following two criteria are satisfied:

(a) *If $\{x^n\}_{n=1}^{\infty} \subseteq X$ and $\lim_{n\to\infty} x^n = \vec{0}$, then $\lim_{n\to\infty} \phi(x^n) = \vec{0}$.*

(b) *For each $a \in \mathcal{A}$ there exists a natural number $n(a) \in \mathbb{N}$ and a function $\Phi^a : B_{n(a)}(X) \cap Z(a) \to \mathcal{A}$ such that*

$$\phi(x_1x_2x_3\dots)_i = \Phi^{x_i}(x_i \dots x_{n(x_i)+i-1})$$

for all $i \in \mathbb{N}$ and for all $x_1x_2x_3\dots \in X^{\mathrm{inf}}$.

We say that a sliding block code is bounded if there exists $M \in \mathbb{N}$ such that $n(a) \le M$ for all $a \in \mathcal{A}$, and unbounded otherwise.

Sliding Block Codes

Theorem

Let $\mathcal{A}$ be a countable alphabet, and let X and Y be shift spaces over $\mathcal{A}$. If X is row-finite and $\phi : X \to Y$ is a function, then ϕ is a shift morphism if and only if ϕ is a sliding block code. Moreover, if ϕ is a bounded sliding block code, then ϕ is an M-block code from some $M \in \mathbb{N}$.

Summary of Results

Theorem

Let E and F be countable graphs with no sinks and no sources. If $X_E \cong X_F$, then $\mathcal{G}_E \cong \mathcal{G}_F$, which implies $C^*(E) \cong C^*(F)$.

Theorem

Let E and F be countable graphs with no sinks and no sources. If $X_E \cong X_F$, then $L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$.

Theorem

If E and F are row-finite graphs with no sinks, and if $\psi : X_F \to X_E$ is a bounded conjugacy with bounded inverse, then $C^*(E) \cong C^*(F)$ via an explicit isomorphism.

Thanks for listening

Any Questions?