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FLAT CONFORMAL STRUCTURES ON 3-MANIFOLDS (SURVEY)

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0. By a flat conformal structure (FCS) on n-manifold M we shall mean an equivalence class of atlases $\mathcal{K} = \{ (U_j , \varphi_j)$, $\varphi_j : U_j \subset M \to S^n \}$ with conformal transition maps $\varphi_j \circ \varphi_k^{-1}$. If the dimension $n \geq 3$ then conformal maps are (in fact) mobius; hence flat conformal structures are to be rather called mobius structures. Flat conformal structures 1-1 correspond to conformal classes of conformally- euclidean metrics on M (more on this subject see in [Kul], [Ko], [Ku 1]). So we can speak about "topological" and "Riemannian" nature of FCS. For example, manifolds of constant curvature are conformally flat. Among 8 three-dimensional geometries we have 5 conformally-euclidean: H^3, S^3, E^3, $H^2 \times R$, $S^2 \times R$. The 3 other geometries Nil, Sol, $SL_2(R)$ are not conformally euclidean (see [Sc 1] for details).

0.1. Hierarchy of structures. The best class of FCS consists of uniformizable ones. They arise in the following way:

Let Γ be a discrete group of mobius transformations which acts freely and discontinuously on a domain $\Omega \subset S^n$, then the standard FCS on Ω canonically projects to the uniformizable structure K_Γ on the factor manifold $\Omega / \Gamma = M$.

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Example 1. Let F be a fuchsian torsion free cocompact subgroup of $\mathrm{Isom}(H^2)$. Consider the Poincare extension $\widetilde{F}$ of the group F to $\mathrm{Isom}_+(H^4) = \mathrm{Mob}(S^3)$ - the full group of orientation preserving mobius transformations of S^3. The discontinuity domain Ω of $\widetilde{F} = G$ is the complement of the unit circle. The factor manifold $M = \Omega / G$ is homeomorphic to $\Sigma_g \times S^1$; where Σ_g is a compact surface of genus g. The structure K_G is in fact a $H^2 \times R$ -structure.

The second class: almost uniformizable structures.

Let (M, K) be a manifold with uniformizable FCS K. Let $p_0 \colon M_0 \to M$ be a finite-sheeted covering and $p_0^{-1}(K) = K_0$ be the preimage of structure. Next let $p \colon (M_0, K_0) \to (M_1, K_1)$ be a conformal finite-sheeted covering. Then the structure K_1 is said to be almost uniformizable.

Example 2. Consider uniformizable FCS K_G from Example 1. Let $p_1 \colon S^1 \to S^1$ be the n-sheeted covering; $p \colon S^1 \times \Sigma \to S^1 \times \Sigma$ be the product-map $p_1 \times \mathbf{id}$. Then the structure $p^{-1}(K_G)$ is almost uniformizable but not uniformizable.

The third is the class of structures with nonsurjective development maps (in [Ka 1] these structures are called relatively complete since their development maps are coverings onto image). The development map is a conformal map $dev \colon (\tilde{M}, \tilde{K}) \to (S^n, can)$, inducing the holonomy representation

$hol \colon \pi_1(M) \to \mathrm{Mob}(S^n)$, $hol(g) \circ dev = dev \circ g$ for every $g \in \pi_1(M)$.

Here $(\tilde{M}, \tilde{K})$ is the universal covering over (M, K).

Example 3. Let Γ be a fuchsian torsion-free subgroup of $\mathrm{Isom}(H^2)$, $h \in R \setminus \pi Q$. Consider the following action of

$\Gamma \times Z$ on $H^2 \times R$:

$\gamma(z, t) = (\gamma(z), t)$, $z \in H^2$, $t \in R$, $\gamma \in \Gamma$

$Z = \langle \theta \rangle$, $\theta(z, t) = (z, t + h)$

Then the natural FCS on the factor manifold $H^2 \times R / \Gamma \times Z$ is relatively complete but not almost uniformizable.

However, theorems of [Kam], [KuP 1], [Ka 1], [G K] state that any relatively complete FCS on closed manifold is almost uniformizable unless it is a $H^2 \times R$ -structure of type given by Example 3.

The worst class of FCS consists of those structures whose development maps are surjective. Examples of such kind may be constructed via "bending" or "grafting" [Go 3]. The structures of this class are very irregular and mysterious. To my best knowledge it is unknown whether there is a compact flat conformal manifold (M, K) such that M admits no relatively complete FCS. The most general fact about FCS whose development map can be surjective is Decomposition Theorem of R.Kulkarni and U.Pinkall [KuP 2].

So we have the hierarchy of strong inclusions:

Uniformizable Structures ⊂ Almost Uniformizable FCS ⊂ Relatively Complete FCS ⊂ FCS.

0.2. In this survey we shall discuss the following questions concerning FCS :

1) Existence problem;

2) Problem of realization of automorphisms;

3) Global properties of deformation spaces of FCS.

The subjects that are concerned with FCS but missed in this paper:

(i) Relations between "Riemannian" and "topological" nature of FCS (see [S Y], [KuP 2]);

(ii) twistor constructions [Le], [A G], [B O];

(iii) description problem, i.e. description of development maps of FCS under some conditions on the holonomy (or fundamental) groups (see for example [Go 3], [Ka 4], [M], [F]);

(iv) infinitesimal and local deformations of FCS (see [A T], [G G], [La], [L], [M J], [Ka 5], [Ka 7]);

(v) Chern-Simons functional and η-invariant [CS], [APS].

Reader is also referred to the beautiful surveys [Go 4], [Kul 2], [Y], [M].

1. Existence problem.

First let's reformulate two well-known problems of 3-dimensional topology in terms of FCS.

1.1. Poincare Conjecture.

Σ is homotopy 3-sphere $\Rightarrow$ Σ admits FCS. (1.1)

The equivalence of (1.1) to the classical Poincare Conjecture is evident: development map $d : \Sigma \longrightarrow S^3$ is a local homeomorphism, hence it is a homeomorphism [Ku 1].

1.2. Generalized Smith Conjecture.

M is orientable manifold covered by 3-sphere $\Rightarrow$ M admits FCS.

The equivalence of (1.2) to the Smith Conjecture about free finite group action on 3-sphere follows from the following:

any compact subgroup of SO(4, 1) is conjugate to a subgroup of SO(4).

1.3. In fact the deepest existence theorem for FCS is Thurston's hyperbolization theorem:

Theorem 1. Let M be a compact Haken 3-manifold which doesn't contain incompressible tori. Then M admits a

Riemannian metric of the curvature (-1).

1.4. Kulkarni's existence theorem [Kul].

Theorem 2. Let M_1 and M_2 be manifolds which admit FCS. Then their connected sum $M_1 \# M_2$ also admits FCS.

Really this theorem is a generalization of Klein's Combination to FCS.

1.5. Goldman's nonexistence theorem [Go 1].

Theorem 3. If M is closed **Sol**- or **Nil**-manifold then M does not admit any FCS. Moreover, if flat conformal manifold (M, K) has almost solvable holonomy group then $\pi_1(M)$ is almost abelian.

In particular, nontrivial S^1- bundles over $S^1 \times S^1$ does not admit any FCS.

1.6. The following theorem is an answer to some question of [Go 1] given independently by several people [G L T], [Ku 3], [Ka 2], [Ka 3]

Theorem 4. Let $M = S(g, e)$ be the total space of S^1-bundle (with Euler number $e \in Z$) over surface Σ of genus g. Then M admits uniformizable FCS under condition :

$o < |e| \leq |\chi(\Sigma)|/22$ [Ka 2], [Ka 3]

or more weakly, $o < |e| \leq |\chi(\Sigma)|/3$ [Ku 3].

Conjecture [GLT]. The inequality $|e| \leq |\chi(\Sigma)|$ is the necessary condition for existence of uniformizable FCS on $S(g, e)$.

For Seifert manifolds which are not circle bundles we have the following existence theorem of F.Luo [Lu]:

Theorem 5. Let $p/q \in Q \cap]o, 1[\setminus \{1/2\}$. Then for every rational $p'/q' > p/q$ sufficiently close to p/q and every positive integer g there is an uniformizable FCS on some Seifert manifold $M = M(g, p/q, p'/q')$ such that:

(i) the base-orbifold of M has genus g and 2 cone points of orders $1/p$, $1/q'$;

(ii) the Euler number of M is equal to $p/q - p'/q'$.

Remark. Pick arbitrary integer $n \geq 7$ which isn't divided by 6. Let $S^2(p, q, r)$ be the 2-dimensional orbifold of genus 0 with 3 cone points of orders p, q, r. Then it can be shown that all Seifert manifolds with the base-orbifolds $S^2(2, 3, n)$, $S^2(3, 3, 4)$ do not admit uniformizable FCS.

1.7. The following existence theorem is an attempt to generalize Thurston's hyperbolization theorem to the wider class of Haken manifolds [Ka 3], [Ka 2]

Theorem 6. Let M be a Haken manifold which is not Sol- or Nil-manifold. Suppose that M is obtained by gluing geometric components in such way that there are no pasting such as :

hyperbolic - hyperbolic

and

hyperbolic - euclidean .

Then some finite-sheeted covering M_0 of M admits an uniformizable FCS.

Remark. In fact Theorem 4 is a special case of Theorem 6 and it was a starting point for its proof. In final section of [G L T] there is also an attempt to use Theorem 4 for construction of FCS on other Haken manifolds. Unfortunately their construction of conformal assembling of (M_1, K_1) ,,, (M_n, K_n) leads to the disjoint union of these flat conformal manifolds only.

Below we present two examples which clarify ideas of Theorem's 6 proof.

Example 4.

Let $Z_j = \Sigma_j \times S^1$, $j = 1, 2$, where Σ_j is the surface of genus

$g_j \neq 0$ and has connected boundary. The decomposition of Z_j into the direct product introduces in $\pi_1(\partial Z_j)$ the natural meridian-longitude basis. Suppose that the manifold M is obtained by gluing Z_j via a homeomorphism $f : \partial Z_1 \to \partial Z_2$ which is defined (in the natural bases) by a matrix $A \in GL(Z, 2)$ with $a_{21}=1$. If the numbers g_k are sufficiently large with respect to $|a_{jj}|$ then there exist the groups $H(g_1, |a_{22}|)$, $H(g_2, |a_{11}|)$ uniformizing the manifolds $S(g_1, |a_{22}|)$, $S(g_2, |a_{11}|)$ (Theorem 4). Next we dispose the constructed groups in S^3 in such way that the complements of their fundamental domains (that look like twisted unknotted solid tori) define a link of index 1 in S^3. It is not hard to see that the group $G = H(g_1, |a_{22}|) * H(g_2, |a_{11}|)$ uniformizes the manifold M. However it is impossible to avoid the condition $|a_{21}|=1$ (for the circumscribed construction of the group G). Proving theorem 6 we find a finite-sheeted covering over M such that the corresponding coefficients a_{21} are equal to 1.

Example 5.

Let G_1 be a torsion-free discrete subgroup of $PSL(2, C)$, $p: H^3 \to H^3/G_1 = M_1$ be the universal covering, the manifold M_1 is compact and contains a simple closed geodesic γ. Suppose that some component $\tilde{\gamma} \subset p^{-1}(\gamma)$ has the hyperbolic stabilizer $\langle g \rangle$ in G_1 (i.e. $Tr(g) \in R_+$). Then γ has an open ε-neighborhood $U_\varepsilon(\gamma)$ which is homeomorphic to the solid torus. It isn't hard to notice that the manifold $M_1^* = M_1 \setminus cl(U_\varepsilon(\gamma))$ is hyperbolic [Koj]. We shall denote by C the euclidean circle that contains the arc $\tilde{\gamma}$.

Let $\Gamma \subset Isom(H^2)$ be a free discrete group of rank $2r$ such that H^2/Γ is the surface with infinite area and one ideal boundary component. Let $\Sigma_c \subset H^2/\Gamma$ be the Nielsen's core (i.e the minimal compact convex subsurface homotopy equivalent to H^2/Γ). Assume that

(1.7) $\text{length}(\gamma) = \text{length}(\partial\Sigma_c)$; $\arccos(1/\cosh(\delta)) = \arcsin(1/\cosh(\varepsilon))$

and the δ-neighborhood $U_\delta(\partial\Sigma_c)$ of $\partial\Sigma_c$ is homeomorphic to the annulus. Put $\Sigma = \Sigma_c \setminus U_\delta(\partial\Sigma_c)$, G_2 be the extension of Γ to S^3. Without loss of generality we can suppose that:

(1) the circle C is invariant under G_2,

(2) $\langle g \rangle \subset [G_2, G_2]$ corresponds to $\pi_1(\partial\Sigma_c) \subset \pi_1(\Sigma_c)$.

Then the group G generated by G_1, G_2 uniformizes a manifold M which is obtained by gluing M_1^* and $\Sigma \times S^1$ along the boundary tori. The condition (1.7) guarantees that ∂M_1^* and $\partial(\Sigma \times S^1)$ are Mobius-equivalent

However only few sewings may be realized in such way and the hyperbolicity of g is very restrictive condition. That is why we have to waive of utilizing groups G_2 with invariant circles. Instead of them we use discrete groups arising after some tiring operations over $H(g, e)$ that have been constructed in Theorem 4.

Conjecture [Ka 2]. Let M^3 be a connected sum of Haken manifolds, where summunds do not admit **Sol-** or **Nil-** structure. Then on some finite-sheeted covering of M there is an uniformizable FCS.

1.8. Birational uniformization.

The class of Haken manifolds treated by Theorem 6 is sufficiently wide. However according to Theorem 3 we have no hope to construct FCS on any Haken 3-manifold (even up to a finite-sheeted covering)

Hence let's remind that the group $\mathbf{Mob}(S^2) = \mathbf{PSL}(2, \mathbf{C})$ is contained in the pseudogroup $\mathbf{Conf}(S^2)$ consisting of conformal injections defined on subdomains of S^2. In R^3 there is one analogy of this inclusion. Let $Bir(\bar{R}^3)$ be the pseudogroup of birational transformations $f : Dom(f) \subseteq \bar{R}^3 \rightarrow \bar{R}^3$ (i.e. $f^{\pm 1} = (f_1^{\pm}, f_2^{\pm}, f_3^{\pm})$ such that

$f_j^{\pm} : Dom(f^{\pm 1}) \rightarrow R$ are rational functions, $(j = 1, 2, 3)$.

As it was proved in [AK], [BM] every closed 3-manifold M admits a real algebraic structure $\mathcal{A}$ such that $(M, \mathcal{A})$ is birationally equivalent to S^3; moreover, the exceptional set for the birational equivalence is a smooth (typically disjoint) curve. It implies (as A.Tyurin explained to me) that any closed 3-manifold admits a smooth atlas with birational transition maps.

<u>Definition</u>. 3-manifold M admits a birational uniformization if there exist a simply connected domain $\Omega \subseteq \bar{R}^3$ and a group $\Gamma \subset Bir(\bar{R}^3)$ which acts freely and properly discontinuously on Ω, so that Ω / Γ is homeomorphic to M.

<u>Question</u> (J.Hempel [He, *Ch. 15*]).

Does any closed 3-manifold admit birational uniformization ?

<u>Remark.</u> Any 3-manifold modelled on some 3-dimensional geometry except of $SL_2(R)$ admits the natural birational uniformization.

<u>Theorem</u> <u>7</u>. Any connected sum of Haken manifolds admits a birational uniformization.

This theorem can be proved by analog of Maskit Combination process for discrete groups of birational transformations.

<u>1.9. Flat conformal structures on open 3-manifolds</u>.

As it was remarked by Whitehead [W] any open orientable 3-manifold admits an immersion into E^3 and, hence, admits FCS. Thus the only sensible problem is existence of uniformizable FCS on these manifolds. Here we discuss only one aspect of this question : can manifolds of infinite homotopy type admit uniformizable mobius structures with finitely generated holonomy group ?

For the case of Riemannian surfaces the negative answer is given by Ahlfors' finiteness theorem (see [A], [Kr], [KS]) :

Let G be a discrete non-elementary finitely generated subgroup of PSL(2, C) acting freely on the domain of discontinuity $\Omega(G)$; then the factor space $\Omega(G)/G$ consists of a finite number of Riemannian surfaces $S_1, \dots, S_n$ each having a finite hyperbolic area. In particular, the group $\pi_1(S_j)$ is finitely generated $(j = 1, \dots, n)$.

In the joint paper of author and L.Potyagailo [K P 1] the affirmative answer for dimension 3 was given :

Theorem 8. There exists a finitely generated, discrete group $F \subset \mathrm{Mob}(S^3)$ without torsion, with invariant component Ω of the domain of discontinuity such that the group $\pi_1(\Omega / F)$ is not finitely generated.

The manifold $M = \Omega / F$ is homeomorphic to double (along part of boundary) of cube with infinitely many handles.

The example above was elaborated by author [Ka 6], [K P 2] as follows:

Theorem 9. There exists a finitely generated free Kleinian group $K_3 \subset \mathrm{Mob}(S^3)$ such as

(a) The number of conjugacy classes of maximal parabolic subgroups of K_3 is infinite ;

(b) If $K_n \subset \mathrm{Mob}(S^n)$ is the conformal extension of K_3 to S^n $(n \geq 3)$, then

$$\mathrm{rank}(H_{n-1}(\Omega(K_n)/K_n, Q)) = \infty .$$

Thus the manifold $M(K_n) = \Omega(K_n) / K_n$ has infinite homotopy type.

The manifold $\Omega(K_3) / K_3$ is homeomorphic to an open handlebody X_r from which some ∞-component link L is removed. Each component of the link is an unknot presenting free generator of $\pi_1(X_r)$.

This theorem also demonstrates failure of Sullivan's cusp's finiteness theorem [Su] for Kleinian groups in higher dimensions.

The group K_3 is interesting also from algebraic point of view: small deformations of it produce a family of finitely generated discrete subgroups of SO(4 , 1) each possessing infinitely many conjugacy classes of finite order elements (see also [FM]).

Theorem 1Ø. For each $q \in \mathbb{Z}$ there exist :

1) an integer r = (rank of free group $F_r = \langle x_1, \dots, x_r \rangle$) ,

2) automorphism $\varphi : F_r \to F_r$ such that the group

$\Gamma(r, q) = \langle x_1, \dots, x_r : (\varphi^n(x_1))^q = 1, n = 1, 2, \dots \rangle$

has infinitely many conjugacy classes of finite order elements $[y_n = \varphi^n(x_1)]$ and $\Gamma(r, q)$ admits discrete faithful representation $\rho : \Gamma(r, q) \longrightarrow \mathrm{Mob}(S^3)$.

Remark. According to Selberg's lemma [Se] $\rho\Gamma(r, q)$ has a torsion-free finite index subgroup $\Gamma_o(r, q)$ (as finitely generated linear group). The group $\Gamma_o(r, q)$ can't be the fundamental group of compact aspherical manifold (see [FM]). Underlying space of the orbifold $\Omega(r, q) / \rho\Gamma(r, q)$ is homeomorphic to X_r, where the singular set is L (see above). Generalizations and discussions of Theorems 8, 9, 10 see in [P1], [P2], [Bo M].

2. Realization of automorphisms.

2.1. As a consequence of Thurston's geometrization theorem for orbifolds [KOS] and results of [MS] we have that:

for any 3-manifold M possessing one of 8 geometries (X, G) and for any finite group $F \subset \mathcal{D}i\ell\ell_+(M)$ which acts nonfreely, there exists a (X, G)-structure on M which is F-invariant. Moreover, if (X, G) is not H^3 nor S^3, then the restriction to orientation preserving and nonfree actions may be dropped [Sc 2]. Conjecturally these restrictions can be omitted for any geometry.

In contrast to these results we have the following situation for flat conformal structures.

2.2. Nonfree actions

Theorem 11 [Ka 2]. There exists a manifold $M = S(g, e)$ (with $0 < e \le (2g-2)/3$) and finite group $F \subset \mathcal{D}i\ell\ell_+(M)$ which acts nonfreely on M, such that F does not preserve any FCS on M.

2.3. Free actions

Let $\mathcal{O}$ be the orbifold supported by $S^1 \times [0, 1]$ and possessing one singular cone point of order 2. Then we choose a Seifert fibration $N \to \mathcal{O}$ over $\mathcal{O}$, such that $H = \pi_1(N) \simeq \langle a, b, c, t : c^2 = t, abc = 1, [a, t] = [b, t] = 1 \rangle$. The manifold N has two boundary tori T_1 and T_2; let $i_k : T_k \to N$ be the inclusions, $k = 1, 2$. The fundamental groups of T_1 and T_2 are generated by $\{a_1, t_1\}$ and $\{b_2, t_2\}$, where $i_{1*}(a_1, t_1) = (a, t)$ and $i_{2*}(b_2, t_2) = (b, t)$. There exists an orientation reversing homeomorphism $f: T_1 \to T_2$ such that $f_*(a_1) = t_2$, $f_*(t_1) = b_2$. Let M be the manifold $N / (x \equiv f(x))$.

It is easy to see that M obeys the conditions of Theorem 6 (since there are no hyperbolic and euclidean components in the canonical splitting of M). Then a finite-sheeted covering $p: M_0 \to M$ exists, such that M_0 possesses a FCS.

Theorem 12. [Ka 2], [Ka 3]
The manifold M does not admit a FCS.

The proof is based on investigation of representations of $\pi_1(M)$ in $\mathrm{Mob}(S^3)$. It can be shown that every such representation has almost solvable image. Then we can apply Theorem 3.

So the finite group $F = Aut(p : M_o \to M) \subset \mathcal{D}i\ell\ell_+(M_o)$ acts freely on M_o, but there are no FCS which are F-invariant.

3. Deformations of FCS

3.1. Let M be a closed 3-manifold and $C(M)$ be the space of all FCS on M. Roughly speaking [Th], the topology on $C(M)$ is given by small perturbations of charts defining FCS (precise definition see in [CEG], [Go 4], [L]). Another way to introduce this topology is to use C^1-topology on the space of conformally-euclidean metrics on M.

Here we shall discuss only global properties of $C(M)$ and some speculations on this subject.

3.2. Theorem 13 ([Ka 2], [Ka 3], [Ka 5]).

Let $M = S(g, e)$, $o < e \leq |2g - 2|/3$. Then number of connected components of $C(M)$ is not less than

$$[(2g-2)/3e] = \nu(g, e).$$

Below we indicate those FCS which lie in the different components of $C(M)$. Consider the set of manifolds

$$\mathfrak{S} = \{ S(ne, g) : 0 < n \leq \nu(e, g) \}.$$

All manifolds of $\mathfrak{S}$ admit uniformizable FCS K_n, due to Theorem 4. There exists a covering $p: S(g, e) \to S(g, en)$ and hence the structures K_n lift to structures $\tilde{K}_n$ on the manifold $S(g, e)$. Then the holonomy groups of the structures $\tilde{K}_n$ are the groups $H(g, ne)$. The groups $H(g, me)$ and $H(g, ne)$ can not be deformed one to other in the space of all pseudofuchsian groups (if $n \neq m$). Therefore, results of [Ka 1], [Ka 5] imply that the structures $\tilde{K}_n$ and $\tilde{K}_m$ lie in different components $\mathcal{U}_n$ and $\mathcal{U}_m$ of $C(M)$. Another way to see this is to calculate η-invariants associated with conformally-euclidean metrics corresponding to $\tilde{K}_n$.

It is known that for all n the set $\mathcal{U}_n$ entirely consists of almost uniformizable structures [Ka 1], [Ka 5]; and these spaces are manifolds of dimension $1o(2g - 2)$ [Go 2]. Unfortunately we don't know anything more about topology of these spaces. Conjecturally $C(M)$ consists only of almost uniformizable structures and has only finitely many components ($M = S(g, e)$ and $e \neq o$). Last conjecture implies that given g there exists a constant $E(g)$ such that $S(g, e)$ does not admit uniformizable FCS for all $e > E(g)$ (c.f. conjectures of [GLT] and [Ku 3]).

3.3. Here we present some table of speculations about analogy between FCS on closed 3-manifolds and self-dual connections on SU(2)-bundles over simply connected closed 4-manifolds.

Self-dual connections Moduli space $\mathcal{M}$:	Flat Conformal Structures Moduli space $C(M)$:
1. Finite dimensional	Finite dimensional
2. Subset of critical set of Yang-Mills functional **YM** $YM_A = 1/2 \int_M \|F_A\|^2 dvol$	Set of critical points of Chern-Simons functional CS (or η-invariant) $CS_K = \int_M tr(A \wedge F_A - 2/3\, A^3)$, $Tors(A)= 0$
3. **YM** is locally constant	η and CS are locally constant
4. Moduli space admits a natural compactification	$C(M= S^1 \times \Sigma_g)$ has no compactification as a finite CW-complex
5. Taubes' connectedness theorem ($M = S^4$) [Tau 1]	Conjecture: $\eta^{-1}(x)$ is connected for any $x \in \mathbb{R}$
6. Taubes' existence theorem if $-c_2(P) \geq b_-(M)$ [Tau 2]	? (Poincare Conjecture ??)

Unfortunately here there are much more questions than answers on them.

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