ON THE ABSENCE OF THE AHLFORS AND SULLIVAN
THEOREMS FOR KLEINIAN GROUPS IN HIGHER DIMENSIONS

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Introduction

One of the fundamental results of the theory of discontinuous groups of fractional linear transformations acting on the complex plane $\mathbb{C}$ is the following finiteness theorem of L. Ahlfors [1, 2]:

Let $G$ be a finitely generated nonelementary discrete subgroup of $\text{PSL}(2, \mathbb{C})$ acting freely on a region of discontinuity $\Omega(G)$. Then the quotient surface $\Omega(G)/G$ consists of a finite number of Riemann surfaces $S_1, \ldots, S_n$ of finite hyperbolic area. In particular the groups $\pi_1(S_i)$ are finitely generated and the homotopy type of the surfaces $S_i$ is finite ($i = 1, \ldots, n$).

Subsequently D. Sullivan [3] strengthened Ahlfors' finiteness theorem by showing that any finitely generated discrete group $G \subset \text{PSL}(2, \mathbb{C})$ has at most a finite number of cusps (i.e., conjugacy classes of maximal parabolic subgroups).

Ahlfors [4] and Ohtake [5] attempted to develop analytic methods of studying the problem of finiteness of multidimensional Kleinian groups. However, the results obtained do not give any information about either the topology of the quotient spaces of Kleinian groups or the number of cusps.

In the present article we shall show that even a weakened version of Ahlfors' finiteness theorem fails in dimension 3 and also construct a counterexample to the analog of Sullivan's finiteness theorem in higher dimensions.

**Theorem 1.** There exists a finitely generated torsion-free function group $F \subset \text{Möb}(S^3)$ with invariant component $\Omega \subset \Omega(F)$ such that the fundamental group $\pi_1(\Omega/F)$ is infinitely generated. Moreover the group $F$ itself is infinitely defined.

**Theorem 2.** There exists a finitely generated Kleinian group $F' \subset \text{Möb}(S^3)$ such that

a) $F'$ contains an infinite number of cusps (of rank 1),

b) if $F''$ is a conformal extension of the group $F'$ to $S^n$, then rank$(H_{n-1}(\Omega(F'')/F'', \mathbb{Z})) = \infty$. Thus the manifold $\Omega(F'')/F''$ has an infinite homotopy type.

1. Preliminary information

Let $\text{Möb}(\mathbb{R}^n) \approx \text{Isom}(\mathbb{H}^{n+1})$ be the group of conformal automorphisms of the $n$-dimensional sphere $S^n = \mathbb{H}^{n+1} \setminus \{\infty\}$, where $\mathbb{H}^{n+1} = \{(x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$ is hyperbolic space.

A subgroup $G \subset \text{Möb}(S^n)$ is called Kleinian if the action of $G$ is discontinuous at some point $x \in S^n$, i.e., there exists a neighborhood $U(x)$ such that $g(U(x)) \cap U(x) \neq \emptyset$ for only a finite number of elements $g \in G$. The set of points where $G$ acts discontinuously is called the discontinuity set $\Omega(G)$ and its complement $\Lambda(G) = S^n \setminus \Omega(G)$ the limit set of the group $G$.

A Kleinian group $G$ is called a function group if there exists a connected component $\Omega \subset \Omega(G)$ that is invariant with respect to $G$. If $G$ acts freely on $\Omega$, then the quotient space $M(G) = \Omega/G$ is an $n$-dimensional manifold. We shall denote by $\mathcal{P}(G)$ the isometric fundamental region for $G$ [6] and by $I(g)$ the isometric sphere $g \in \text{Möb}(S^n)$.

In what follows we shall assume (if not otherwise specified) that all manifolds are three-dimensional and piecewise linear. Standard reductions by the theory of Kleinian groups and three-dimensional topology can be found in [2] and [6]–[8]. If $S \subset \mathbb{R}^3$ is a two-sphere, we shall denote by $\text{ext}(S)$ and $\text{int}(S)$ the components of $\mathbb{R}^3 \setminus S$ such that $\infty \in \text{ext}(S)$. The symbol $\text{cl}(\ )$ denotes the closure of a set.
2. Outline of the proof of Theorem 1

Consider the configuration consisting of four mutually tangent Euclidean spheres \( \Sigma_i \subset \mathbb{R}^3 \) (Fig. 1). Each sphere \( \Sigma_i \) is obtained from its neighbor by a reflection \( \tau_j \) in the plane \( \Pi_j \), \( (i = 1, \ldots, 4), \( j = 1, 2) \).

We shall construct discontinuous groups \( \Gamma_i \subset \text{Mob}(\mathbb{R}^3) \) such that the groups \( \Gamma_i \) are isomorphic to the fundamental group of a bundle over a circle with a “surface” as fiber. The group \( \Gamma_1 \) leaves invariant the outside of the sphere \( \Sigma_i \), \( i = 1, \ldots, 4 \). Using Maskit’s combination method we shall show that both groups \( G_1 = (\Gamma_1, \Gamma_2) \) and \( G_2 = (\Gamma_3, \Gamma_4) \) are Kleinian and also isomorphic to the fundamental group of a bundle over a circle (cf. Lemma 3). Let \( F_i \) be normal subgroups in \( G_i \) corresponding to surface subgroups in \( G_i \) (\( i = 1, 2 \)). The proof of Theorem 1 concludes with Lemma 5, in which we establish that the group \( \langle F_1, F_2 \rangle = F \) is the one sought. In particular \( F \) is a normal subgroup of the geometrically finite function group \( G = (G_1, G_2) \).

The proof of Lemma 5 is based on the following reasoning. Using the involution \( \tau_2 \) we represent the manifold \( M(F) = \Omega / F \) in the form of a doubling of some manifold \( M(G) \). There exists an infinite regular covering \( M(G) \rightarrow M(F) \subset M(G) \) induced by the covering \( M(F) \rightarrow M(G) \). The manifold \( M(G) \) is not a bundle over a circle, since \( \partial M(G) \) contains a surface of genus 2. It follows from this that the group \( \pi_1(M(G)) \) cannot be finitely generated \([7]\). It then follows immediately that the group \( \pi_1(M(F)) \) is also infinitely generated.

3. Outline of the proof of Theorem 2

Consider the configuration of four spheres \( \Sigma_1, \Theta_2, \Theta_3, \Sigma_4 \) shown in Fig. 2. We construct groups \( \Gamma'_1, \Gamma'_2, \Gamma'_3, \Gamma'_4 \) conjugate in \( \text{Mob}(S^3) \) to the groups \( \Gamma_i \) of the preceding section such that their limit sets are respectively the spheres \( \Sigma_1, \Theta_2, \Theta_3, \) and \( \Sigma_4 \). The groups \( \Gamma'_3 \) and \( \Gamma'_4 \) are obtained from \( \Gamma'_2 \) and \( \Gamma'_1 \) by conjugation using a symmetry \( \tau'_2 \) with respect to the plane \( L_2 \).

The group \( \Gamma'_1 \) contains a parabolic element \( \beta_2 \) such that the isometric spheres \( I(\beta_2) \) and \( I(\beta_2^{-1}) \) are tangent to \( L_2 \). The point of tangency \( x = I(\beta_2) \cap L_2 \) is a fixed point for the parabolic transformation \( u = \tau'_2 \beta_2^{-1} \tau'_2 \beta_2 \). We shall show that the point \( x \) is cusped for the Kleinian group \( G' = (\Gamma'_1, \Gamma'_2, \Gamma'_3, \Gamma'_4) \).

The group \( G' \) contains a normal free subgroup \( F' \supseteq u \) of finite rank, \( G'/F' \simeq \mathbb{Z} \). The action of \( G'/F' \) on \( F' \) by conjugates is induced by a homeomorphism \( \theta \) of a compact surface \( \mathcal{F} \), \( \pi_1(\mathcal{F}) \simeq F' \). The system of three loops \( \alpha = \alpha_1 \cup \alpha_2 \cup \alpha_3 \) is invariant with respect to \( \theta \), gives a reduction of \( \theta \) to irreducible homeomorphisms of infinite order \([9]\). Here \( [u] \) is the image of \( u \) under the isomorphism \( j^{-1} : \mathcal{F}' \rightarrow \pi_1(\mathcal{F}') \); \( [u] \) is not conjugate to any element of \( \pi_1(\mathcal{F}) \) corresponding to components of \( \partial \mathcal{F} \cup \alpha \) (since \( [u] \) is a maximal parabolic subgroup of \( G' \)). It follows from this that \( \theta^m([u]) \) and \( \theta^m([u]) \) are nonconjugate elements of \( \pi_1(\mathcal{F}) \) for all \( m \neq n \in \mathbb{Z} \). Thus the group \( F' \) contains an infinite number of conjugacy classes of maximal parabolic subgroups: \( \{ j(\theta^m([u])), m \in \mathbb{Z} \} \). Property (b) of the group \( F' \) follows from the fact that the point \( x \) is cusped in the group \( F' \).
4. Construction of the group $F$

Let $M$ be an open manifold homeomorphic to the complement of a linkage of Borromean rings. It is known that $M$ is a total bundle space over a circle with a “surface” for fiber \[10\]. In addition $M$ admits a complete hyperbolic structure of finite volume, i.e., $M = H^3/\Gamma$, $\Gamma \subset \text{Isom}(H^3)$ \[8\].

Definition. A group $K$ with a subgroup $S$ is called \textit{S-finitely approximable} if for any element $k \in K \setminus S$ there exists a subgroup $K_1 \subset K$ of finite index that contains $S$ but $k \notin K_1$.

4.1. Lemma 1. The group $\Gamma$ is S-finitely approximable for any geometrically finite subgroup $S \subset \Gamma$.

Proof. Consider the regular ideal octahedron $P \subset H^3$ whose dihedral angles all equal $\pi/2$ \[8\]. Let $Q$ be the group of reflections in the faces of $P$, and let $Q_1$ be a finite extension of it using four automorphisms of order 3. Then $Q_1$ contains $\Gamma$ as a subgroup of finite index \[8\]. The assertion of the lemma follows from \[11\] and the commensurability of the groups $\Gamma$ and $Q$. Lemma 1 is now proved.

4.2. We denote by $B$ the outside of the unit sphere $\Sigma \subset \mathbb{R}^3$ with center at the origin. We shall regard $B$ as a model of the hyperbolic space $H^3$. Further let $H_i$ be certain nonconjugate maximal parabolic subgroups of $\Gamma$ and $\Lambda(H_i) = \{p_i\}$, $i = 1, 2$. We shall assume that the points $p_i$ have coordinates $(0,0,0)$ and $(0,0,1)$. Let $\Pi_i$ be a Euclidean plane tangent to $\Sigma$ at the point $p_i$ (cf. Fig. 1) and $\Pi_i^\perp$ the component of $R^3 \setminus \Pi_i$ that does not intersect $\Sigma (i = 1, 2)$. We set $\overline{\Pi}_i = \Pi_i \cup \{\infty\}$.

In the next lemma we shall show that for some subgroup of finite index $\Gamma \subset \Gamma$ and planes $\Pi_i$ the hypotheses of Maskit’s combination theorem are fulfilled. Consider a certain neighborhood of $\Pi_i$, and let the sphere $W_i$ be tangent to $\Sigma$ at the point $p_i$, so that $W_i \subset \text{ext} B$, $V_i = \text{ext} W_i$, $\text{cl} \Pi_i^\perp \setminus \{p_i\} \subset V_i$, $i = 1, 2$.

4.3. Lemma 2. There exists a subgroup of finite index $\Gamma_0$ in the group $\Gamma$ such that the following conditions hold:

(a) the group $\Gamma_0$ contains a normal subgroup $F \subset \Gamma$ for which $\Gamma_0 = \langle F, t_i \rangle$, $t_i \in H_i \cap \Gamma$, $i = 1, 2$;

(b) the group $\Gamma_0$ has a fundamental set $P \subset B$ such that $P \cap V_i$ is a fundamental set for the action of the group $H_i \cap \Gamma = \widetilde{H}_i$ on $V_i$, $(i = 1, 2)$.

Proof. We denote by $I(g)$ the isometric sphere of the element $g \in \Gamma$. Then there exists at most a finite number of elements $h_k \in H_i$ such that $I(h_k) \cap (\Pi_j \cup (S^3 \setminus P(H_j))) = \emptyset$, $i \neq j$, $i,j = \{1,2\}$, $0 \leq k \leq N$. Using the finite approximability of the group $\Gamma$ \[12\], we choose a subgroup of finite index $\Gamma^* \subset \Gamma$ for which $h_k \notin \Gamma^*$, $0 \leq k \leq N$. We set $\widetilde{H}_i = \Gamma^* \cap H_i$, $i = 1, 2$.

(a) Let $\Phi$ be a normal subgroup of $\Gamma$ corresponding to a fiber of $M$. Then $F^* = \Phi \cap \Gamma^*$ is a normal subgroup of $\Gamma^*$ and $\Gamma^* = \langle F^*, \ell \rangle$. The action of the element $\ell$ on $F^*$ by conjugation is induced by the action of some homeomorphism $\lambda$ of a compact surface $S$ for which $\pi_1(S) \cong F^*$. Let $\gamma_i \in \partial S$ be oriented boundary curves whose homotopy classes $[\gamma_i]$ correspond to elements $\beta_i \in H_i^* \cap F^*$, $i = 1, 2$. Without loss of generality we may assume that $\lambda(\partial S) = \partial S$; therefore there exists a number $n \in \mathbb{Z} \setminus \{0\}$ such that $\lambda^n(\gamma_i) = \gamma_i$, $i = 1, 2$. We denote by $M_0$ the manifold obtained from $S \times [0,1]$ by identifying the points $(x,0)$ and $(\lambda^n(x),1)$, $x \in S$. The manifold $M_0$ is a bundle over a circle and a typical fiber $S^*$ of this bundle is the image of the surface $S \times \{0\}$ under the quotient mapping. Then the intersection of $S^*$ with the component of $\partial M_0$ on which the image $\gamma_i$ lies consists of only the curve $\gamma_i$, $i = 1, 2$. Hence it easily follows that there exist elements $t_i \in \widetilde{H}_i = H_i^* \cap (\Gamma_0 = \langle F^*, t_i \rangle)$ such that $\langle F^*, t_i \rangle = \Gamma_0$. Obviously $|\Gamma : \Gamma_0| < \infty$.

(b) We have already shown that $cl(S^3 \setminus P(H_j)) \subset P(\widetilde{H}_{ij})$, $i \neq j$, $i,j = \{1,2\}$. Therefore by Klein’s combination theorem the set $P(H_1) \cap P(H_2)$ is a fundamental region for a group of Schottky type $\widetilde{H} = \langle \widetilde{H}_1, \widetilde{H}_2 \rangle \cong \widetilde{H}_1 \ast \widetilde{H}_2$ \[6\]. Thus the set $R = P(\widetilde{H}_1) \cap P(\widetilde{H}_2) \cap cl(V_1 \cup V_2)$ cannot have equivalent points with respect to the action of the group $\widetilde{H}$. The closure of the set $T = R \cap (V_1 \cup V_2)$ is compact in $B$ and therefore there exists at most a finite number of elements $g_m \in \Gamma_0$ such that $g_m(T) \cap T \neq \emptyset$, $m = 1, \ldots, K$. The group $\widetilde{H}$ is geometrically finite \[6\] and by Lemma 1 $\Gamma_0$ is an $\widetilde{H}$-finitely approximable group. Consequently there exists a subgroup of finite index $\Gamma_0 \subset \Gamma_0$, in which $\widetilde{H}$ is contained, but which contains none of the elements $g_m$. Obviously for all $g \in \Gamma_0$ we have $g(R) \cap R = \emptyset$ and $R$ is a fundamental set for the action of the group $\Gamma$ in the orbit $\Gamma(cl(V_1 \cup V_2))$. It is also clear that $\Gamma$ satisfies condition (a) in the statement of this lemma.
We choose an arbitrary fundamental set $A$ for the action of the group $\Gamma$ in $B \setminus \Gamma(\text{cl}(V_1 \cup V_2))$. The set $P = R \cup A$ will be fundamental in $B$, and condition (b) will hold for it. Lemma 2 is now proved.

4.4. Let $\tau_i$ be reflection in the plane $\Pi_i$ ($i = 1, 2$). We introduce the following notation: $\Gamma_1 = \Gamma$, $\Gamma_2 = \tau_1 \Gamma_1 \tau_2$, $G_1 = \langle \Gamma_1, \Gamma_2 \rangle$, $G_2 = \tau_2 \Gamma_1 \tau_2$, $G = \langle G_1, G_2 \rangle$. In what follows we shall write $H_1$ and $H_2$ instead of $\Gamma_1$ and $\Gamma_2$.

**Lemma 3.** The group $G_1$ is discontinuous and contains a finitely generated normal subgroup $F_1$ such that $G_1/F_1 \simeq \mathbb{Z}$, $G_1 = \langle F_1, t \rangle$, $t \in H_2$.

**Proof.** Consider the fundamental set $P_1 = P$ of the group $\Gamma_1$, which was constructed in Lemma 2. The group $H_1 = \Gamma_1 \cap \Gamma_2$ stabilizes the plane $\Pi_1$ in the groups $\Gamma_1$ and $\Gamma_2$. By assertion (b) of Lemma 2 and the maximality of the parabolic subgroup $H_1 \subset \Gamma_1$ the region $\text{cl}\Pi_1$ is precisely invariant with respect to $H_1$ in the group $\Gamma_1$. Similarly the region $\tau_1(\text{cl}\Pi_1)$ is precisely invariant with respect to $H_1$ in the group $\Gamma_2$. Thus all the hypotheses of the first combination theorem of Maskit [13] are satisfied (the multidimensional variant of the combination theorem can be found, for example, in [14]). Consequently the group $G_1$ is discontinuous and isomorphic to $\Gamma_1 \ast H_1 \Gamma_2$, and the set $R_1 = \tau_1(P_1)$ is fundamental for the action of $G_1$ on the invariant component $\Omega_1 \subset \Omega(G_1)$ containing the point $\infty$.

Suppose further that $F_1 = \langle F_1, t \rangle$, and $F_1$ is normal in $G_1$ and finitely generated. Then it remains only to remark that $t_1 \in \Gamma_1 = \langle F_1, t \rangle$, and so $G_1 = \langle F_1, t \rangle$, $t = t_2 \in H_2$. Lemma 3 is now proved.

**Proposition 1** (cf. also [15]). The manifold $M(G_1) = \Omega_1/G_1$ is homeomorphic to the interior of a bundle over a circle whose fiber is a compact surface with $\pi_1(\Omega_1) = \{1\}$.

**Proof.** It follows from the geometrical decomposition of the group $G_1 = \Gamma_1 \ast H_1 \Gamma_2$ that $M(G_1)$ is obtained by gluing together two manifolds $M_1$ and $M_2$, where $M_1 = M(\Gamma_1 \setminus (\Pi_1/H_1))$ and $M_2 = M(\Gamma_2 \setminus (\tau_2 \Pi_1/H_1))$. We further have $\Pi_1/H_1 \cong \tau_1 \Pi_1/H_1 \cong S^1 \times S^1 \times (0, 1)$. Consequently each of the manifolds $M_i$ is homeomorphic to a bundle over a circle and the interior of $M_i$ is a finite-sheeted covering of the original manifold $M$.

The gluing homeomorphism $f : \partial M_1 \to \partial M_2$ preserves the bundle structure, since it is covered by the identity homeomorphism $\tilde{f} : \tilde{\Pi}_1 \to \tilde{\Pi}_1$. It follows from the Seifert-van Kampen theorem that $\pi_1(M(G_1)) \simeq \Gamma_1 \ast H_1 \Gamma_2 \cong G_1$. The group $G_1 \subset \text{M"ob}(S^3)$ is a Hopf group [12], and therefore $\pi_1(\Omega_1) = \{1\}$. Each of the manifolds $M_i$ admits a compactification (by adjoining tori); therefore $M(G_1)$ is also compactifiable. Proposition 1 is now proved.

4.5. We set $F = \langle F_1, F_2 \rangle$, where $F_2 = \tau_2 F_1 \tau_2$, $G = \langle G_1, G_2 \rangle$.

**Lemma 4.** The following assertions hold:

(A) The group $G$ is the result of Maskit combination of the groups $G_1$ and $G_2$.

(B) The group $G$ is discontinuous and possesses an invariant component $\Omega \subset \Omega(G)$ containing the point $\infty$.

(C) The finitely generated group $G$ is normal in $G$.

(D) The manifold $M(G) = \Omega/G$ is homeomorphic to the interior of a compact manifold.

**Proof.** (A) Let $H_3 = \tau_1 H_2 \tau_1$ and $H = \langle H_2, H_3 \rangle$. By Lemma 3 the group $G_1$ acts discontinuously on $\Omega_1 \supset \infty$ and has a fundamental set $P_1 = P \cap \tau_1 P$. It follows from Lemma 2 that $R_1 \cap \text{cl}\Pi_2$ is a fundamental set for the action of the group $H$ on $\text{cl}\Pi_2$. Moreover in the neighborhood $V = V_2 \cap \tau_1 V_2$ of the set $\Pi_2 \setminus \Lambda(H)$ we have $R_1 \cap V = P(H_2) \cap P(H_3) \cap V$ (Lemma 2) and the open surface $\Pi_2 \setminus \Lambda(H)$ is precisely invariant with respect to $H$ in the group $G_1$. Consequently there exists a neighborhood $N$ of the surface $\Pi_2 \setminus \Lambda(H) \subset \Omega(G_1)$ such that $N \subset \Omega(G_1)$ and $N$ is precisely invariant with respect to $H$ in $G_1$. To verify assertion (A) it now remains to prove the following result.

**Proposition 2.** The sphere $\Pi_2$ is precisely invariant under the action of the group $H \subset G_1$.

**Proof.** Assume that there exists an element $g \in G_1 \setminus H$ such that $g(\Pi_2) \cap \Pi_2 = \{x\} \subset \Lambda(H)$. The group $H$ of Schottky type is geometrically finite [6, 16], and so the following alternative holds [17]: either (1) $x$ is an approximation point for the group $H$, or (2) $x$ is a fixed point of a parabolic element $\gamma \in H$.
In the first case there exists a sequence \( h_n \in H \) such that \( \lim_{n \to \infty} h_n(x) = x_0 \in \Pi_2 \) and \( x_0 \neq y_0 = \lim_{n \to \infty} h_n(z) \) for any point \( z \in \text{cl} \Pi_2 \setminus \{x\} \), and \( y_0 \in \Pi_2 \). It follows from this that the sequence of spheres \( h_n g(\Pi_2) \) converges to \( \overline{\Pi}_2 \subset S^3 \). Consequently \( h_n g(N) \cap N \neq \emptyset \) for large values of \( n \). The latter is impossible by the precise invariance of \( N \) with respect to \( H \) in the group \( G_1 \).

In the second case there exist elements \( h \) and \( h' \) such that \( hgh'({p_2, p_3}) = {p_2, p_3} \), where \( p_3 = \tau_1(p_2) \). From the maximality and nonconjugacy of the parabolic subgroups \( H_2, H_3 \subset G_1 \) it follows that \( g \in H \) and this is impossible. Proposition 2 is now proved.

(B) Assertion (B) follows immediately from (A) and Maskit's combination theorem.

(C) We shall verify that the inclusion \( g_1F_2g_1^{-1} \subset F = \langle F_1, F_2 \rangle \) holds for any \( g_1 \in G_1 \). The element \( g_1 \) has the form \( f_1t^n \), where \( f_1 \in F_1, t \in H_2 \subset G_2 \cap G_1, G_2 = \langle F_2, t \rangle \) (cf. Lemma 3). Thus \( g_1F_2g_1^{-1} = f_1t^nF_2t^{-n}f_1^{-1} = f_1F_2f_1^{-1} \subset F \). Analogously \( g_2F_2g_2^{-1} \subset F \) for any \( g_2 \in G_2 \). It follows from this that \( F \) is normal in \( G \) and assertion (C) is thus verified.

(D) As we have already seen, both manifolds \( M(G_1) \) and \( M(G_2) \) admit a natural compactification by the adjunction of cusped tori. Consequently the manifolds \( M^-(G_1) = M(G_1) \setminus (\Pi_2^-/H) \) and \( M^-(G_2) = M(G_2) \setminus (\tau_2(\Pi_2^-))/H \) also admit a compactification. Therefore the manifold \( M(G) \) obtained by gluing together \( M^-(G_1) \) and \( M^-(G_2) \) along the compact boundary surface \( S_2 = (\overline{\Pi}_2 \setminus \Lambda(H))/H \) is also compactifiable. Lemma 4 is now proved.

By assertions (B) and (C) of Lemma 4 the groups \( G \) and \( F \) have a common invariant component \( \Omega \ni \infty \). We set \( M(F) = \Omega/F \).

4.6. Lemma 5. The group \( \pi_1(M(F)) \) is not finitely generated.

Proof. Step 1. We begin by verifying that the orbits of \( G_1(\Pi_2^-) \) and \( F_1(\Pi_2^-) \) coincide. Indeed \( G_1 = \langle F_1, t \rangle, t \in H_2, t(\Pi_2^-) = \Pi_2^- \). Hence \( G_1 \Pi_2^- = F_1 \Pi_2^- \). We shall further show that \( G = \langle F, t \rangle \). For any element \( g \in G \) the decomposition \( g = g_1g_2 \cdots g_n (g_i \in G_1 \cup G_2) \) holds and from the equality \( g_i = f_it^{m_i} \) (\( f_i \in F_1 \cup F_2, t \in H \setminus F \)) we obtain \( g = ft^m, f \in F, m \in \mathbb{Z} \). In analogy with Lemma 3 the subgroup \( F \) is normal in \( G \).

Remark. We are not asserting here that \( G/F \simeq \mathbb{Z} \). This will follow from the reasoning below.

Step 2. By the construction we have \( \tau_2G\tau_2 = G \). Therefore using the covering \( \pi : \Omega \to \Omega/G = M(G) \) the involution \( \tau_2 \) projects to an involution \( \tilde{\tau}_2 : M(G) \to M(G) \). Obviously the surface \( S_2 = \pi(\overline{\Pi}_2 \setminus \Lambda(H)) \) is the fixed set for this involution. Similarly the involution \( \tau_2 \) projects to an involution \( \tilde{\tau}_2 : \Omega/F \to \Omega/F = M(F) \). Thus we have the commutative diagram

\[
\begin{array}{ccc}
\Omega & \overset{q}{\longrightarrow} & M(F) \\
\downarrow \tau_2 & & \downarrow \tau_2 \\
\Omega & \overset{q}{\longrightarrow} & M(F) \\
& & \downarrow \tilde{\tau}_2 \\
& & M(G)
\end{array}
\]

where \( p = r \circ q \) and \( r \) is a regular covering with the group of covering transformations \( G/F \). The surface \( \tilde{S} = r^{-1}(\tilde{S}) = q(\overline{\Pi}_2 \setminus \Lambda(H)) \) is connected (Step 1) and coincides with the fixed set of the involution \( \tilde{\tau}_2 \).

Step 3. Since the group \( G \) is the result of the Maskit combination of the groups \( G_1 \) and \( G_2 \), the region \( \Omega_1 \setminus G_1(\Pi_2^-) \) is contained in \( \Omega \) and \( p(\Omega_1 \setminus G_1(\Pi_2^-)) \) is the closure of one of the components of \( M(G) \setminus S_2 \). We denote this closure by \( M^-(G) \) and use \( M^-(F) \) to denote the preimage \( r^{-1}(M^-(G)) \). On the other hand, \( M^-(F) \) and \( M^-(G) \) are homeomorphic to \( M(F_1) \setminus (\Pi_2^- / H \cap F) \) and \( M(G_1) \setminus (\Pi_2^- / H) \) respectively. Thus the covering \( r : M^-(F) \to M^-(G) \) is the restriction of the infinite cyclic covering \( M(F_1) \to M(G_1) \).

Step 4. As we have already seen in Lemma 4, the manifold \( M^-(G) \) can be compactified to a manifold \( N^-(G) \). The boundary component \( S_2 \subset \partial M^-(G) \) is a compact surface of genus 2 (since \( H \simeq (\mathbb{Z} \oplus \mathbb{Z}) \ast (\mathbb{Z} \oplus \mathbb{Z}) \) acts as a group of Schottky type on the sphere \( \overline{\Pi}_2 \)). Consequently the manifold \( N^-(G) \) cannot be a total bundle space over the circle. Moreover neither of the manifolds \( M^-(G) \) and \( N^-(G) \) contains any fake cells, since they can be covered by a region in \( \mathbb{R}^3 \).

Step 5. The group \( \pi_1(M^-(F)) \) is not finitely generated.
Proof. By Step 3 we have an exact sequence

\[ 1 \to \pi_1(M^-(F)) \to \pi_1(N^-(G)) \to \mathbb{Z} \to 1. \]

Assume that the group \( \pi_1(M^-(F)) \) is finitely generated. The manifold \( M^-(G) \) contains no projective planes in view of the \( \mathbb{P}^2 \)-irreducibility of the manifold \( M(G_1) \). Further \( \pi_1(M^-(F)) \) is not an abelian group, and so it follows from [7, Theorem 11.1] that \( N^-(G) \) is homeomorphic to a bundle over \( S^1 \). The last result contradicts Step 4.

Step 6. It remains for us to verify that \( \pi_1(M(F)) \) is also not a finitely generated group. Let \( \omega : \tilde{M} \to M(F) \) be the universal covering with group of covering transformations \( \pi = \pi_1(M(F)) \). We remark that the manifold \( M^-(F) \) is homomorphic \( M(F)/\tilde{G}_2 \). Consider the lifting \( \tilde{G}_2 : \tilde{M} \to \tilde{M} \) of the involution \( \tilde{G}_2 \). We have \( \tilde{G}_2 \tilde{G}_2 = \pi \) and the group \( \tilde{G} = (\pi_1(M(F)), \tilde{G}_2) \) acts discontinuously on \( \tilde{M} \). We denote the normal subgroup of \( \tilde{G} \) generated by the elements of finite order by \( TORS \).

By Armstrong's theorem [18] the group \( \pi_1(M^-(F)) \) is isomorphic to \( \mathbb{Z}/TORS \) and so \( \tilde{G} \) is infinitely generated. One can easily see that the group \( \pi_1(M(F)) \), being a subgroup of index 2 in the infinitely generated group \( \tilde{G} \), also cannot be finitely generated. Lemma 5 is now proved.

By construction the group \( F = \langle F_1, F_2 \rangle \subset \text{Möb}(S^3) \) is finitely generated and its quotient manifold \( M(F) = \Omega/F \) has an infinitely generated fundamental group. We shall show finally that the group \( F \) is infinitely defined.

The group \( I = H \cap F \) is the stabilizer in \( F \) of the sphere \( \overline{\Pi}_2 \). It follows immediately from the fact that \( G \) is the result of the Maskit combination of the groups \( G_1 \) and \( G_2 \) that \( F \) is also obtained from \( F_1 \) and \( F_2 \) by a Maskit combination. Therefore \( F = F_1 * I F_2 \) is the free product with the combined subgroup \( I \). We remark that the subgroup \( I \) is normal in \( H \) (by the normality of \( F \) in \( G \)) and has infinite index, since \( G/F \cong \mathbb{Z} \) and \( t^n \notin F \) for \( n \in \mathbb{Z} \setminus \{0\} \). It follows easily from this that the group \( I \) is infinitely generated. The fact that the group \( F \) is infinitely defined now follows immediately from the results of [19].

Theorem 1 is now proved.

5. Proof of Theorem 2

Consider the group \( \Gamma'_1 = \Gamma_1 \) constructed in Lemma 2; \( \Gamma_2 = \tau_1 \Gamma_1 \tau_1 \), and in the group \( \Gamma_1 \) there is a parabolic element \( \beta \in F \subset \Gamma_1 \) (cf. the proof of Lemma 2, part (a)). We denote by \( L_2 \) the plane parallel to \( \Pi_2 \) and tangent to the isometric spheres of the elements \( \beta_2 \) and \( \beta_2^{-1} \), \( L_2 \subset \Pi_2 \) (cf. Fig. 2). We set \( \tilde{L}_2 = L_2 \cup \{\infty\} \) and let \( \tilde{L}_2^{-} \) be the component of \( \mathbb{R}^3 \setminus L_2 \) contained in \( \Pi_2 \). Let \( \Theta_2 \) be a sphere tangent to \( \Sigma_1 \) at the point \( p_1 \) so that \( \Theta_2 \subset \text{int} \Sigma_2 \); \( z = L_2 \cap \tilde{I}(\beta_2) \), \( y = L_2 \cap \tilde{I}(\beta_2^{-1}) \), \( z = \Theta_2 \cap L_2 \). Then there exists a unique transformation \( T \in \text{Möb}(S^3) \), that commutes with each element of the group \( H_1 \) and maps the point \( p_3 \) to the point \( z \).

Remark. Passing to a subgroup of finite index \( \langle \tilde{F}, \tilde{t}_2 \rangle \subset \Gamma_1 \) if necessary, we may assume that for any \( h \in H_2 \setminus \{\beta_2, \beta_2^{-1}\} \) the intersection \( I(h) \cap L_2 \) is empty.

5.1. It is easy to see that \( \tau_1 \tilde{T}^{-1}(L_2) \subset \Pi_2^{-} \) and the sphere \( \tau_1 \tilde{T}^{-1}(\tilde{L}_2) \) is tangent to \( \Sigma_1 \) at the point \( p_1 \). It therefore follows from Lemma 2 that \( L_2^{-} \) is precisely invariant with respect to the subgroup \( \Gamma'_2 = T \Gamma_2^{-1} \) in the group \( \Gamma'_2 = T T_2 T^{-1} \). Also by Lemma 2 the hypotheses of Maskit's first combination theorem hold for the groups \( \Gamma'_1 \) and \( \Gamma'_2 \).

We denote by \( G'_1 \) the group \( \langle \Gamma'_1, \Gamma'_2 \rangle \). It follows from Maskit's combination theorem that the group \( G'_1 \) is Kleinian and has an invariant component \( \Omega'_1 \ni \infty \).

Lemma 6. The region \( \Omega'_1 \) is simply connected. The quotient manifold \( \Omega'_1/G'_1 \) is a bundle over the circle formed by gluing together two hyperbolic manifolds that are bundles over \( S^1 \) and homeomorphic to \( (B \setminus \Gamma'_1(\Pi_2))\Gamma'_1 \). The group \( F'_1 = \langle \tilde{F}, T \tau_1 \tilde{F}^{-1} \rangle \) is normal in \( G'_1 \) and corresponds to the fundamental group of a fiber of the manifold \( M(G'_1) \). There exists a fundamental set \( D_1 \) for the action of \( G'_1 \) on \( \Omega'_1 \) such that

1) \( (D_1 \cap \text{cl} L^{-}) \cup \{y\} \) is a fundamental region for the action of the group \( H'_3 \) in \( \text{cl} L_2^{-} \).
2) for some plane $L_1'$ parallel to $L_2$ and lying between $L_2$ and $\Pi_2$ the intersection $L_1' \cap D_1 \cap \tau_1(\Pi_2^-)$ coincides with $L_1' \cap \mathcal{P}((\beta_2)) \cap \tau_1(\Pi_2^-)$.

Proof. All the assertions except the last are proved as in Lemma 2 and Proposition 1. To prove property 2) of the set $D_1$ it suffices to use the remark preceding Lemma 6. For the definition of the set $\mathcal{P}$ see the proof of Lemma 2. Lemma 6 is now proved.

5.2. We introduce the following notation: $J = H_1 \searrow X_2 = L_2^- \cap (L_2 \setminus J(\{x, y, z\}))$, $\tau_2'$ is a symmetry in the plane $L_2$, $X_1 = \tau_2'(X_2)$, $G_2' = \tau_2'G_1'\tau_2'$, $F_2' = \tau_2'F_1'\tau_2'$, $D_2 = \tau_2D_1$.

Direct verification shows that the sets $X_1$ and $X_2$ are interactive (in the sense of Maskit [20, Ch. VII]) for the pair of groups $(G_1', G_2')$, i.e., $J(X_i) = X_i$ and any element $G_i' \setminus J$ maps $X_i$ onto $X_2$. In addition $D_i \cap X_i$ is a fundamental set for the action of the group $J$ in $X_i \cap L_2$, $i = 1, 2$. It follows from this that $g(D_1 \cap X_2) \subset X_2$ for any element $g \in G_1'$ and conversely, for any $g \in G_2'$ the inclusion $g(D_2 \cap X_1) \subset X_1$ holds.

Lemma 7. 1. The groups $G_1'$ and $G_2'$ satisfy the hypotheses of the weak combination theorem of Maskit [20, Ch. VII, Theorem A.15]).

2. The group $G' = (G_1', G_2')$ is isomorphic to $G_1' *_{J} G_2'$.

3. The set $D = (D_1 \cap X_1) \cup (D_2 \cap X_2)$ contains no points that are equivalent with respect to $G'$, and $\text{int } D$ is contained in $\Omega(G')$.

Proof. The first assertion follows from the reasoning preceding the lemma. The second and third follow from Maskit's weak combination theorem. Lemma 7 is now proved.

We denote by $\Omega'$ the component of $\Omega(G')$ containing $\infty$. It is easy to see that $\Omega'$ is invariant with respect to $G'$. Let $\beta_2 = \tau_2'\beta_2\tau_2'$. Obviously the element $u = (\beta_2)^{-1}o\beta_2$ is parabolic, is conjugate to a shift in $\mathbb{R}^3$, and leaves the point $x$ fixed.

5.3. Lemma 8. Let $G^n$ be a conformal extension of the group $G'$ to $\mathbb{R}^n$, $n \geq 3$. Then $x$ is a cusped parabolic point in the group $G^n$. (For the definition of a cusped point cf., for example [21].)

Proof. We begin by constructing a cusped neighborhood for the point $x \in \mathbb{R}^3$. We denote by $\Pi$ the plane passing through the points $0, p_1$, and $p_2$. Let $l$ be the line containing the point $x$ and perpendicular to the plane $\Pi_2$ (Fig. 3). Let $\Delta$ be any closed disk lying in the plane $\Pi$ and tangent to $l$ at the point $x$ such that the diameter of $\Delta$ is less than the radius of $I(\beta_2)$ and $\Delta \cap L_2 = \emptyset$ (cf. Lemma 6).

As a cusped neighborhood $\mathcal{O}$ of the point $x$ we choose the set obtained from $\Delta$ by rotation about the axis $l$. It is obvious that $u(\mathcal{O}) = \emptyset$. We shall show that $\mathcal{O}$ is precisely invariant in $G'$ with respect to $(u)$. By Lemmas 6 and 7 the intersection $\mathcal{O}_- = (\mathcal{O} \setminus \{x\}) \cap \text{cl ext } (I(\beta_2')) \cap \text{cl ext } (I(\beta_2))$ lies in the region $D$ and contains no points that are equivalent with respect to $G'$. Let $w$ be the shift in $\mathbb{R}^3$ along the line $L_2 \cap \Pi$ that takes the point $x$ to the point $y$. Then the set $w(\mathcal{O}_-) \setminus (I(\beta_2^{-1}) \cup I((\beta_2')^{-1}) = \mathcal{O}_-$ is also contained in $D$. It follows from this that $\mathcal{O}_- \cup \beta_2^{-1}(\mathcal{O}_-)$ has no points that are equivalent (with respect to $G'$) and is a
fundamental set for the action of \( \langle u \rangle \) in \( \mathcal{O} \). Thus \( \mathcal{O} \) is a cusped neighborhood of the parabolic point \( x \) in the group \( G' \).

Consider the fractional linear transformation \( \gamma \) that makes the element \( u \) conjugate to the shift \( U : x \mapsto x + e_1 \), where \( e_1 \) is the vector with coordinates \((1,0,0)\). Then \( Q = \mathbb{R}^3 \setminus \gamma(\mathcal{O}) \) is a solid Euclidean cylinder with axis \( \lambda_1 = R \cdot e_1 \) and a certain radius \( r \). In the group \( G' = \gamma G' \gamma^{-1} \) the region \( \mathcal{O}' = \mathbb{R}^3 \setminus Q \) is precisely invariant with respect to \( \langle U \rangle \). Let \( G^n_\bullet \) be a conformal extension of the group \( G_\bullet \) in \( \mathbb{R}^n \) and \( Q^n \) the solid Euclidean cylinder in \( \mathbb{R}^n \) of radius \( 3r \) with axis \( \lambda_1 \). We shall show that the region \( \mathbb{R}^n \setminus Q^n \) is precisely invariant with respect to \( \langle U \rangle \) in the group \( G^n_\bullet \).

We remark that for an arbitrary element \( g \in G_\bullet \) the center of the isometric sphere \( I(g) \) lies inside \( Q \) (otherwise the limit point \( g^{-1}(\infty) \) would lie in the precisely invariant region \( \mathcal{O}_* \)). We shall verify that the radius of \( I(g) \) is less than \( 2r \). Indeed elementary computations show that for radius of the sphere \( I(g) \) larger than \( 2r \) the area of \( I(g) \cap \mathcal{O}_* \) is less than half the area of \( I(g) \). However the radii of \( I(g) \) and \( I(g^{-1}) \) are equal and \( g : I(g) \to I(g^{-1}) \) is a Euclidean isometry. Therefore \( g(I(g) \cap \mathcal{O}_*) \cap \mathcal{O}_* \neq \emptyset \), contradicting the precise invariance of the region \( \mathcal{O}_* \). Thus each sphere \( I(g) \) lies inside a cylinder \( Q' \), from which it follows that the region \( \mathbb{R}^n \setminus Q' \) is precisely invariant. Lemma 8 is now proved.

**Remark.** Unfortunately we were unable to use either Lemma 4.15 of [22] or its proof in our reasoning, since there are errors in the proof [22, p. 94].

**Corollary.** The group \( \langle u \rangle \) is a maximal parabolic subgroup of \( G' \).

We shall use the notation \( \mathcal{O}_n^u \) below to denote the cusped neighborhood \( \gamma^{-1}(\mathbb{R}^n \setminus Q^n) \).

5.4. As already noted in Lemma 6, the quotient manifolds \( M(G'_1) \) and \( M(G'_2) \) are bundles over a circle whose fibers \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) correspond to normal subgroups \( F'_1 \subset G'_1 \) and \( F'_2 \subset G'_2 \). Let \( T_i \) be the peripheral tori in \( M(G'_i) \) corresponding to the parabolic subgroup \( J = G'_1 \cap G'_2 \) and \( T_i \) a component of \( M(G'_i) \setminus T_i \) homeomorphic to \((\infty, \infty) \times T^2 \). Consider the manifold \( N \) obtained from \( M(G'_i) \setminus T_i \) by compactification and gluing using a homeomorphism \( h : T_1 \to T_2 \) that induces the identity mapping \( J \to J \).

We set \( \beta_3 = \tau_1 \beta_2 T^{-1}_1 \), \( I = \{ \beta_3 \} = F'_i \cap J \), \( i = 1,2 \). Without loss of generality we may assume that \( h(\mathcal{T}_1 \cap T_1) = \mathcal{T}_2 \cap T_2 \), so that the manifold \( N \) is also homeomorphic to a bundle over a circle whose typical fiber \( \mathcal{T} \) is formed by gluing together the compactifications of the surfaces \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) along \( \mathcal{T}_1 \cap T_1 \). It follows from the Seifert-van Kampen theorem and Lemmas 6 and 7 that there exists an isomorphism \( j : \pi_1(N) \to G'_i \), \( j : \pi_1(\mathcal{T}_i) \to F'_i \), \( i = 1,2 \). Thus the group \( F'_n = \langle F'_1, F'_2 \rangle \) is isomorphic to \( F'_1 * F'_2 \) and normal in \( G'_i \), and the action of the cyclic quotient group \( G'/F'_n \) is induced by some homeomorphism \( \theta : \mathcal{T} \to \mathcal{T} \). The manifold \( N \) is formed by gluing together four copies \( M_i \) of the compactification of the manifold \( M(G'_i) = H^3/G_i \). We denote the boundary tori of the \( M_i \) along which the gluing is done by \( \mathcal{T}_1, \mathcal{T}_2, \) and \( \mathcal{T}_3 \); they correspond to parabolic subgroups of rank 2 in \( G'_i \). Here \( \mathcal{T} \cap \mathcal{T}_i \) consists of a single loop \( \alpha_i \) (Lemma 2). We shall denote the element of \( \pi_1(N) \) corresponding to it by \( [\alpha_i] \).

We remark that for all \( i \) there exists a parabolic subgroup of rank 2 in \( G'_i \) containing \( j([\alpha_i]) \) while for the element \( u \) there is no such subgroup (cf. the corollary to Lemma 8). Therefore \( j([\alpha_i]) \) and \( u \) are not conjugate in the group \( G'_i \) and \( \alpha_i \) are not conjugate in \( G'_i \). We note also that \( \theta(\alpha_i) = \alpha_i, i = 1,2,3 \).

5.5. The manifold \( N \) can be obtained from \( \mathcal{T} \times [0,1] \) by identifying the points \((x,0)\) and \((\theta(x),1)\) for \( x \in \mathcal{T} \). We denote by \( \omega : S^1 \to \mathcal{T} \) the loop corresponding to the element \( u \) under the isomorphism \( j : \pi_1(\mathcal{T}) \to F'_i \).

5.6. Lemma 9. For any \( m, k \in \mathbb{Z}, m \neq k \), the loops \( \theta^k(\omega) \) and \( \theta^m(\omega) \) are not freely homotopic on the surface \( \mathcal{T} \).

**Proof.** Denote by \( \nu \) the loop \( \theta^k(\omega) \). Then \( \theta^m(\omega) = \theta^m(\omega), n = m - k \neq 0 \). Assume that the loops \( \nu \) and \( \theta^n(\nu) \) are freely homotopic on \( \mathcal{T} \) and that \( \mu : S^1 \times [-1,0] \to \mathcal{T} \) is the corresponding homotopy. The manifold \( \tilde{N} = \mathcal{T} \times [0,1]/\theta^n \) is an \( n \)-sheeted regular covering of \( N \). Consider the continuous mapping \( \tilde{\eta} : [-1,1] \times S^1 \to \mathcal{T} \times [0,1] \) such that the restriction of \( \tilde{\eta} \) to \([-1,0] \times S^1 \) coincides with \( \mu \) and the restriction of \( \tilde{\eta} \) to \([0,1] \) is given by the formula \( \tilde{\eta}(t,x) = (t, \theta^n \omega(\mu(x))) \). It is obvious that \( \tilde{\eta} \) projects to a continuous
mapping $\eta : S^1 \times S^1 \to \mathcal{F} \times [0,1]/\theta^m = \tilde{N}$. Passing to a covering of $\tilde{N}$ with defining subgroup $\pi_1(\mathcal{F})$, we see that the nontriviality of the loop $\omega$ implies that the mapping $\eta_* : \pi_1(S^1 \times S^1) \to \tilde{N}$ is injective. Thus $\eta(T^2)$ is an incompressible singular torus in the manifold $\tilde{N}$.

We now lift the tori $\mathcal{T}_i$ to tori $\tilde{\mathcal{T}}_i \subset \tilde{N}$, $i = 1, 2, 3$. Each component of $\tilde{N} \setminus (\tilde{\mathcal{T}}_1 \cup \tilde{\mathcal{T}}_2 \cup \tilde{\mathcal{T}}_3)$ is an $n$-sheeted covering of one of the hyperbolic manifolds $N \setminus (\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3)$, and is therefore itself hyperbolic and atoroidal. Consequently $\mathcal{F} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$ defines a “canonical system of tori” of the manifold $\tilde{N}$ (cf. [23]). Thus the components of a regular neighborhood $\mathcal{X}$ of the submanifold $\partial \tilde{N} \cap \mathcal{F}$ are a complete set of characteristic submanifolds of $\tilde{N}$ [23]. By the results of [23] the continuous mapping $\eta : T^2 \to \tilde{N}$ is homotopic to some mapping $\eta : \mathcal{M} \to \mathcal{X}$. It follows from this that the loop $\tilde{\omega} = \eta|_{\partial \mathcal{M} \times S^1}$ is homotopic to a loop $\hat{\omega}$ of $\mathcal{X}$. Considering the elements $[\tilde{\omega}]$ and $[\hat{\omega}] \in \pi_1(\tilde{N})$ corresponding to $\tilde{\omega}$ and $\hat{\omega}$, we verify that they are conjugate in the group $\pi_1(\tilde{N})$. However $[\omega] = [\tilde{\omega}] \in \pi_1(\mathcal{F} \times \{0\})$ is the fundamental group of the fiber of the bundle $\tilde{N}$ and is normal in $\pi_1(\tilde{N}) \subset \pi(N)$.

Consequently $[\tilde{\omega}]$ also lies in $\pi_1(\mathcal{F}) \subset \pi_1(\tilde{N}) \subset \pi(N)$. Thus the loop $\omega \subset \mathcal{F} \subset \tilde{N}$ is freely homotopic to a loop of a regular neighborhood of $\partial \mathcal{M} \cup \alpha_1 \cup \alpha_2 \cup \alpha_3$, which is impossible by Sec. 6.4. This contradiction proves Lemma 9.

5.7. Proof of assertion (a) of Theorem 2. We choose some element representing a generator of the group $G'/F'$, for example $t_2 \in H_2$. By Lemma 9 for any $k \neq m \in \mathbb{Z}$ the elements $t_2^k u t_2^{-k}$ and $t_2^m u t_2^{-m}$ are not conjugate in the group $G'$. However $t_2^m(u)t_2^{-m}$ is a maximal parabolic subgroup of $G'$ (and hence also of $F'$) for $m \in \mathbb{Z}$. Thus the group $F'$, being a free group of finite rank, contains an infinite number of conjugacy classes of the maximal parabolic subgroups $(u_m) = (t_2^m u t_2^{-m})$. This proves assertion (a).

5.8. Proof of assertion (b) of Theorem 2. The point $x$ is a cusped parabolic point of the group $G^n \subset \text{Möb}(S^n)$ for $n \geq 3$. A cusped neighborhood of this point $\mathcal{O}_0^n$ was constructed in the proof of Lemma 8. Since the elements $u_m$ and $u = u_1$ are conjugate in $G'$, the point $t_2^m(x)$ is also cusped and $\mathcal{O}_m^n = t_2^m(\mathcal{O}_0^n)$ is a cusped neighborhood of it (for the group $G'$). We denote by $E(n, m)$ the projection of $\mathcal{O}_m^n$ in the manifold $M(G^n) = \Omega(F^n)/F^n$. The manifold $E(n, m)$ is homeomorphic to $S^{n-2} \times S^1 \times [0, \infty)$, and the closed orientable submanifold $\partial E(n, m)$ in $M(G^n)$ is the boundary of a “parabolic end.” If $m \neq k$, the parabolic ends corresponding to $\partial E(n, m)$ and $\partial E(n, k)$ are distinct and the manifold $M(G^n)$ possesses an infinite number of ends. It is easy to see that the system of cycles $\{\partial E(n, m), m \in \mathbb{Z}\}$ is linearly independent in $H_{n-1}(M(F^n), \mathbb{Z})$. Thus $\text{rank } H_{n-1}(M(F^n), \mathbb{Z}) = \infty$, and Theorem 2 is proved completely.

6. Concluding remarks

6.1. In the theory of discrete subgroups of Lie groups the following theorem of Selberg is well-known [24].

Theorem C. For any finitely generated subgroup $\Gamma$ in the Lie group $G$ the number of $G$-conjugacy classes of elements of $\Gamma$ of finite order is finite.

The following result also holds.

Theorem 3. There exists a sequence of representations $\rho_n : F' \to \text{Möb}(S^3)$ that converges to $\rho_\infty = \text{id}$ and is such that for all $n, m \in \mathbb{Z}$ the order of the element $\rho_n(u_m)$ is finite.

In a subsequent publication we shall show that the elements $\rho_n(u_m)$ and $\rho_n(u_i)$ are not conjugate in the group $\rho_n(F')$ for any $n \in \mathbb{N}$, $m, i \in \mathbb{Z}$, $m \neq i$.

Proof. We denote by $\Sigma_1(s)$ the sheaf of spheres tangent to one another at the point $p_1$, where $\Sigma_1(0) = \Sigma_1$ and $\Sigma_1(1)$ is the sphere whose radius is equal to the distance from the center of $\Sigma_1$ to the plane $L_2$. Let $p_2(s)$ be the point of $\Sigma_1(s)$ closest to the plane $L_2$.

We choose a parabolic transformation $\zeta_s$ that commutes with the group $H_1$ and maps the point $p_2$ to the point $p_2(s)$; $\zeta_s(\Pi) = \Pi$. Consider the parabolic element $\zeta_s^2 = \zeta_s^2 \zeta_s^{-1}$. It is easy to see that the isometric spheres $I(\zeta_s^2)$ and $I(\zeta_s^{-1})$ meet $L_2$ at equal angles $\varphi(s)$, $\varphi(0) = 0$, $\varphi(1) = \pi/2$, $\varphi(s)$ being a continuous function. Let $s(n)$ be a sequence of numbers, $0 \leq s(n) \leq 1$, such that $\varphi(s(n)) = \pi/2n$. Let $\rho_n : \Gamma_1 \to \text{Möb}(S^3)$ be the homomorphism defined by the conjugation $\rho_n(\gamma) = \zeta(s(n)) \gamma \zeta(s(n))$; the restriction of $\rho_n$ to $\Gamma_2$ is the identity. By the equality $G' = \Gamma_1 \ast H_1 \Gamma_2$ and the fact that $\zeta(s(n))$ commutes with the group $H_1$, the mapping $\rho_n : G' \to \text{Möb}(S^3)$ is a homomorphism. We define a mapping $\rho_n : G'_2 \to \text{Möb}(S^3)$ by
the formula $\tau_2' \rho_n(\tau_2^2 g \tau_2^4) \tau_2' = \rho_n(g)$, $g \in G_2'$. It is obvious that the extension of $\rho_n$ to the group $G'$ is a homomorphism and that $\lim_{n \to \infty} \rho_n = \text{id}$.

At the same time $\rho_n(u)$ is an elliptic element of order $n \in \mathbb{Z}$. Since $u = u_0$ and $u_m \in G'$ are conjugate in $G'$, it follows that $\rho_n(u_m)$ is an element of finite order for all $n, m \in \mathbb{Z}$. Thus the sequence of homomorphisms $\rho_n$ is the one sought. Theorem 3 is now proved.

6.2. Numerous variant proofs of the Ahlfors finiteness theorem based on topological and other ideas appeared in the mid-80's [25-27].

The idea that a normal subgroup in a geometrically finite group could be a counterexample to Ahlfors' Theorem in dimension 3 occurred to the second author of this paper while working on [15] (taking account of the Jaco-Hempel theorem [7, Theorem 11.1]). The first example was constructed by the authors in a joint paper [28]. As B. I. Apanasov has pointed out to us, a configuration of spheres similar to [28] covering a "trefoil" knot was used in his Theorem 7.21 of [22]. The group constructed in Theorem 7.21 of [22] was a free, geometrically finite group having a wild knot as limit set (cf. also [20, VIII.F]). In the present article the example of [28] has been significantly simplified and the original configuration of 52 spheres has been replaced by the four spheres $\Sigma_i$, $i = 1, \ldots, 4$ (Theorem 1). Theorems 2 and 3 are due to the second author.

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Literature Cited


