Consequently, \( \dim [R(d_{\Gamma})/R(d_{\nu})] < \infty \). By Theorem 1, the operator \( d_{\nu} \) is compactly solvable. By Lemma 3, the operator \( d_{\Gamma} \) is compactly solvable. The theorem is proved.

Note that in [1] there have been constructed for every \( k \neq 0, n - 1 \) and \( p = q = 2 \) examples of operators \( d_{\Gamma} \) which are not normally or compactly solvable.

LITERATURE CITED


CONFORMALLY FLAT STRUCTURES ON 3-MANIFOLDS: EXISTENCE PROBLEM. I*

M. È. Kapovich

INTRODUCTION

A conformally flat structure on a manifold \( M \) (of dimension \( n \geqslant 3 \)) is a maximal atlas \( \mathcal{K} = \{(\Omega, \psi_i) \mid \psi_i: U_i \rightarrow \mathbb{R}^n, i \in I \} \), in which the transition maps are conformal (i.e., \( \psi_i \circ \psi_j^{-1} \) is a restriction of a Möbius automorphism of \( \mathbb{R}^n \)). There is also another, classical definition of conformally flat structure (CFS), as the class of conformally equivalent conformally Euclidean metrics on \( M \) [i.e., metrics locally expressible as \( \rho(x)|dx|^2 \), where \( \rho(x) \) is a smooth positive function]. That these definitions are equivalent was proved in [1, 2]. It is well known that metrics of constant sectional curvature are conformally Euclidean (see [3]). Yet another characterization of CFS makes use of Kleinian groups: if a Kleinian group \( \Gamma \) is free and acts discontinuously on a domain \( \Omega \) (for the detailed definitions see below, Sec. 1), then the quotient manifold \( M = \Omega/\Gamma \) admits a natural CFS \( \mathcal{K}_\Gamma \) for which the cover \( p: \Omega \rightarrow M \) is a conformal map. Such structures are said to be uniformizable, and \( \Gamma \) is a uniformizing group.

The particular interest in conformally flat structures on 3-manifolds is due largely to the fact that five of the eight homogeneous Riemann spaces in three dimensions are conformally Euclidean: \( S^3, E^3, H^3, S^2 \times R, H^2 \times R \) (see [4]). The following theorem of Thurston is well known [5, 6]:

THEOREM H. Let \( M \) be a closed atoroidal Haken manifold. Then there exists a hyperbolic structure (i.e., a metric of sectional curvature \(-1\)) on \( M \).

Thus manifolds of this class admit CFSs. On the other hand, it follows from results of Goldman [7] that if \( M \) is a closed 3-manifold whose fundamental group is solvable but not a finite extension of an Abelian group (i.e., \( M \) is either a Sol- or a Nil-manifold; see [4]), then \( M \) does not admit a CFS.

Our aim is to prove the following theorem, according to which there exist CFSs on a broader (than atoroidal) class of Haken manifolds.

THEOREM C. Let \( M \) be a closed Haken 3-manifold with unsolvable fundamental group, such that \( M \), when obtained by gluing hyperbolic and Seifert pieces together along tori, does not contain combinations of hyperbolic manifolds with hyperbolic or Euclidean manifolds (in the sense of [4]). Then there exists a finite-sheeted cover \( M_0 \) over \( M \) which admits a uniformizable CFS.

* Dedicated to Yuri Grigor'evich Reshetnyak on his sixtieth birthday.

The proof will be divided into three steps. In this paper we carry out the first two; the third will be the subject of a forthcoming paper. In Sec. 1 we introduce the necessary definitions. In Sec. 2 we prove

**Theorem A.** Let \( S(g, e) \) be a fiber space over a closed orientable surface \( S_2 \) of genus \( g \), with fiber \( S^1 \) and Euler number \( e > 0 \) such that \( e \leq (g - 1)/11 \). Then the space of \( S(g, e) \) admits a uniformizable CFS.

**Corollary.** If \( M \) is a Seifert fiber space and \( \pi_1(M) \) is unsolvable, then the conclusion of Theorem C is true for this manifold.

As an application we shall construct an example of a discrete uniformly quasiconformal group \( \Gamma \) which is not topologically conjugate to any subgroup of the Möbius group (Corollary 3).

In Sec. 3 we prove

**Theorem B.** Let \( M \) be a closed manifold obtained by gluing Seifert fiber spaces \( Z_1, \ldots, Z_s \) together along boundary tori (i.e., \( M \) is a "graph-manifold" in Waldhausen's sense), such that \( \pi_1(M) \) is unsolvable. Then the conclusion of Theorem C is true for \( M \).

In our forthcoming paper we shall prove Theorem C in the general case — when the manifold is obtained by gluing together both Seifert and hyperbolic components. The main idea of the proof of Theorem C is to deform the CFSs on finite-sheeted covers over the hyperbolic and Seifert components glued together to get \( M \), in such a way that the gluing operation can be done conformally.

We recall that by a result of Kulkarni [2], if \( M_1 \) and \( M_2 \) are conformally flat manifolds, there exists a CFS on their connected sum. In view of Theorem C and Kulkarni's theorem, the following conjecture is plausible.

**Conjecture.** Let \( M \) be a closed 3-manifold satisfying Thurston's geometrization conjecture. Conjecture (see [4, 6]), i.e., obtained from manifolds admitting a geometric structure by gluing together along tori and connected sum operations. Assume further that the decomposition of \( M \) as a connected sum of primitive manifolds does not involve terms with Sol- or Nil-structure. Then \( M \) has a finite-sheeted cover that admits a CFS.

1. Definitions and Notation

1.1. Let \( \mathcal{M} \) be the group of all orientation-preserving Möbius automorphisms of the \( n \)-dimensional sphere \( S^n = \mathbb{R}^n = \mathbb{R}^n \cup \{\infty\} \). If \( \gamma \in \mathcal{M} \), we let \( \text{Fix}(\gamma) \) denote the set \( \{x \in S^n : \gamma(x) = x\} \). The region of discontinuity of a group \( \Gamma \subset \mathcal{M} \) is the set \( \mathbb{R}(\Gamma) \) of all points \( x \in S^n \), having a neighborhood \( U(x) \) such that the intersection \( U(x) \cap \gamma U(x) \) is empty for all but a finite number of \( \gamma \in \Gamma \). A connected component \( R_0 \subset \mathbb{R}(\Gamma) \), which is invariant under \( \Gamma \) is called an invariant component of \( \Gamma \). The group \( \Gamma \) acts freely on \( R_0 \) is the stabilizer \( \Gamma_x \) of every point \( x \in R_0 \) is trivial. Thus, \( \Gamma \) acts freely on an invariant component \( R_0 \subset \mathbb{R}(\Gamma) \) if and only if the natural projection \( q : R_0 \to R_0/\Gamma \) is a cover. A group \( \Gamma \subset \mathcal{M} \), with a nonempty set \( \mathbb{R}(\Gamma) \) is called a Kleinian group, and \( L(\Gamma) = S^n \setminus \mathbb{R}(\Gamma) \) is known as the limit set of \( \Gamma \).

If \( \Gamma \) is a Kleinian group, \( R_0 \) an invariant component of \( \Gamma \) on which the group acts freely, then a set \( \Phi_0 \subset R_0 \) is called a fundamental region for the action of \( \Gamma \) on \( R_0 \) if (a) \( \text{cl} \Phi_0 = \text{cl} \text{int} \Phi_0 \), (b) \( \bigcup_{\gamma \in \Gamma} \gamma \Phi_0 = R_0 \), (c) \( \gamma \Phi_0 \cap \Phi_0 = \emptyset \) for all \( \gamma \in \Gamma \setminus \{1\} \), (d) the family \( \gamma \Phi_0 \) is locally finite. The details may be found, e.g., in [8, 9]. Thus, a manifold \( M(\Gamma) = R_0/\Gamma \) uniformizable by \( \Gamma \) is obtained from \( \text{cl} \Phi_0 \) by identifying boundary points that are equivalent relative to \( \Gamma \) (i.e., \( x \) and \( \gamma x, \gamma \in \Gamma \)).

1.2. A 3-manifold \( M \) is said to be irreducible if any polyhedral sphere embedded in \( M \) bounds a ball. An irreducible 3-manifold \( M \) is called a Haken manifold if it admits an embedding \( i : S \to M \) of a closed surface, neither \( S^2 \) nor \( \mathbb{R}P^2 \), such that the induced map \( i_* : \pi_1(S) \to \pi_1(M) \).

Remark. Throughout this paper we shall be concerned only with orientable 3-manifolds.

Thurston's hyperbolization theorem [5, 6] states that if \( M \) is Haken, \( \partial M \) is the union of finitely many tori \( T_1 \cup \ldots \cup T_s \) and \( M \) is atoroidal (i.e., for any subgroup \( Z + Z \subset \pi_1(M) \) there
exists a conjugate subgroup \( A \subset \pi_i(T) \) for some \( i \), then there exists a complete metric of constant negative curvature on \( \text{int} M \). A manifold satisfying this condition is said to be hyperbolic.

The main definitions and facts from the theory of orbifolds may be found, e.g., in [4], and the definition of compact three-dimensional Seifert fiber spaces in [4, 10, 11]. We mention only that (if \( \lvert \pi_1(M) \rvert = \infty \)) a manifold \( M \) is a Seifert fiber space if and only if it has a finite-sheeted cover which is an ordinary fiber space over an orientable surface (possibly with boundary) with fiber \( S^1 \). In addition, the fundamental group of a Seifert fiber space over an orbifold \( O \) can be embedded in a short exact sequence \( 1 \to \mathbb{Z} \to \pi_1(M) \to \pi_1(O) \to 1 \), where \( \mathbb{Z} \subset \pi_1(M) \) is generated by a regular fiber of the Seifert fiber space (for short, we shall call this a fiber).

1.3. We shall also need the following geometric description of a fiber space \( S(g, e) \) with fiber \( S^1 \), base space \( S_g \) and Euler number \( e \in \mathbb{Z} \).

Let \( \Sigma_g = S_g \setminus \text{int} B^2 \), where \( B^2 \) is a closed disk, \( x \in \partial B^2, \Sigma_x \times S^1, t = (x) \times S^1 \subset \Sigma_x, \beta = \partial B^2 \times \{q\} \), where \( q \in S^1, T = \partial B^2 \times S^1 \) is the boundary of the manifold \( \Sigma_x \). Let \( T = B^2 \times S^1 \) be a solid torus, \( t = (x) \times S^1 \subset \partial T, \kappa = \partial B^2 \times \{q\} \subset \partial T \). The corresponding elements of \( \pi_1(T) \) and \( \pi_1(\partial T) \) are again denoted by \( t, \beta, \tau, \kappa \). Glue \( T \) to \( \partial \Sigma \) so that the loop \( t \) is glued to \( \tau \) and the loop \( \beta \) to the loop \( \kappa \). The manifold thus obtained is precisely \( S(g, e) \) (clearly, only \( |e| \) is of topological significance).

1.4. Let \( M \) be a 3-manifold. We shall say that \( M \) admits a geometric structure (is geometrical) if it has the form \( X/\Gamma \), where \( X \) is one of the eight three-dimensional homogeneous Riemannian spaces (see [4]): \( E^3, S^3, H^3, H^3 \times \mathbb{R}, S^2 \times \mathbb{R}, SL_2(\mathbb{R}), \text{Sol}, \text{Nil} \), and \( \Gamma \) is a discrete subgroup of the group of isometries of \( H^3 \times \mathbb{R} \). It is readily verified that a generator \( t \) of \( \Gamma \) generating a normal cyclic subgroup may be chosen as follows: \( t(z, \varphi) = (z, \varphi + 2\pi) \), where \( z \) is a coordinate on \( H^3 \), and \( \varphi \) a coordinate on \( \mathbb{R} \). Then \( H^3 \times \mathbb{R}/\langle t \rangle \) is isometric to \( X = \mathbb{R}^3 \setminus \{(x_1, x_2, x_3) : x_1 = 0\} \), where we have introduced the metric \( ds^2 = dz^2 + \rho^2 dx_2^2 + dx_3^2 \) and the group \( \Gamma = \Gamma/\langle t \rangle \) acts freely on \( X \) as a discrete group of isometries. Clearly \( F \subset \mathcal{M}_3 \) is the required group uniformizing \( M \).

At the same time, an invariant Riemannian metric on the group \( SL_2(\mathbb{R}) \) is not conformally Euclidean, and so this kind of argument collapses entirely in the attempt to define a CFS on an \( SL_2(\mathbb{R}) \)-manifold.

2. CONFORMALLY FLAT STRUCTURES ON SEIFERT FIBER SPACES

2.1. We first observe that if \( M \) is a Seifert fiber space over a hyperbolic base with Euler number zero, there exists a Kleinian group \( F \) uniformizing \( M \). Indeed, a Seifert fiber space satisfying this condition admits an \( H^2 \times \mathbb{R} \)-structure (see [4]), i.e., it has the form \( H^2 \times \mathbb{R}/\Gamma \), where \( \Gamma \) is a subgroup of the group of isometries of \( H^2 \times \mathbb{R} \). It is readily verified that a generator \( t \) of \( \Gamma \) generating a normal cyclic subgroup may be chosen as follows: \( t(z, \varphi) = (z, \varphi + 2\pi) \), where \( z \) is a coordinate on \( H^3 \), and \( \varphi \) a coordinate on \( \mathbb{R} \). Then \( H^2 \times \mathbb{R}/\langle t \rangle \) is isometric to \( X = \mathbb{R}^3 \setminus \{(x_1, x_2, x_3) : x_1 = 0\} \), where we have introduced the metric \( ds^2 = dz^2 + \rho^2 dx_2^2 + dx_3^2 \) and the group \( F = \Gamma/\langle t \rangle \) acts freely on \( X \) as a discrete group of isometries. Clearly \( F \subset \mathcal{M}_3 \) is the required group uniformizing \( M \).

At the same time, an invariant Riemannian metric on the group \( SL_2(\mathbb{R}) \) is not conformally Euclidean, and so this kind of argument collapses entirely in the attempt to define a CFS on an \( SL_2(\mathbb{R}) \)-manifold.

2.2. Proof of Theorem A. Our main goal will be to construct a Kleinian group \( H = H(g, i) \) such that \( R(H)/H = M(H) \) is homeomorphic to \( S(g, i) \), where \( g = 12 \) [and in that case \( H = \pi_1(S_g) \)]. The fundamental polyhedron \( \Phi \) of \( H \) is homeomorphic to a solid torus and satisfies the following conditions.

(a) The faces of the polyhedron, \( Q_1, R_1, Q_1, R_1, ..., Q_g, R_g, Q_g, R_g \), lie on Euclidean spheres in \( \mathbb{R}^3 \) and are homeomorphic to annuli. Two adjacent faces (i.e., appearing successively in the above chain, and also \( R_g \) and \( Q_1 \)) intersect in a circle; faces which are not adjacent do not intersect (Fig. 1).

The faces of \( \Phi \) are identified by Möbius transformations \( A_1: Q_1 \to Q_1', B_1: R_1 \to R_1', ..., A_g: Q_g \to Q_g', B_g: R_g \to R_g' \), which generate the group \( H \).

Let \( x_0 \in Q_1 \cap R_1, x_1 = B_1^{-1} \cdot A_1^{-1} \cdot B_1 \cdot A_1(x_0) = [A_1, B_1](x_0) \in Q_1 \cap R_1 \) and so on, \( x_n = [A_1, B_1] \cdot ... \cdot [A_1, B_1](x_0) \in R_1 \cap Q_1 \).

(b) We stipulate that \( x_g = x_g \). If in addition the sum of the dihedral angles of \( \Phi \) is \( 2\pi \), then \( \Phi \) is a fundamental region for the group \( H = \langle A_1, B_1, ..., A_g, B_g \rangle \times ... \times [A_1, B_1] = 1 \).

In order to see this, it suffices to extend \( \Phi \) into the hyperbolic space \( H^3 \) (every sphere can be extended to a geodesic hypersurface) and to apply the arguments of [13].
Let $\alpha_1$ be a simple curve on $Q_1$ connecting $x_2$ and $A_1^{-1}B_1A_1(x_2)$. Let $\gamma_1 = R_1$ be a curve connecting $A_1(x_0)$ and $x_1$. $\alpha'_1 = A_1(\alpha_1)$, $\gamma'_1 = B_1(\gamma_1)$. Similar constructions yield curves $\alpha_2, \alpha'_2, \ldots, \gamma_g, \gamma'_g$ (see Fig. 1). Thanks to condition (b), the union of these curves is a simple closed curve on $\partial \Phi$, which we denote by $\eta$. Assume that the following condition holds:

(c) The linking number of $\eta$ and the axis of the solid torus $S^3 \setminus \Phi$ is $|e| = 1$.

It is easy to see that condition (c) is equivalent to the following: $\eta$ is homotopic on $\partial \Phi$ to a loop $t + \kappa$, where $t = Q_1 \cap R_2$, and the class $[\kappa]$ generates the kernel of the homomorphism $n_1(\partial \Phi) \to n_1(\Phi)$ (the loop $\kappa$ is homotopic in $S^3 \setminus \Phi$ to the axis of the solid torus).

2.3. We claim that if conditions (a)-(c) are fulfilled, then $H$ uniformizes $S(g, 1)$ (the fiber space over $S_g$ with fiber $S^1$ and Euler number 1). Let $T' \subset \Phi$ be a torus parallel to $\partial \Phi$ and $\Gamma$ a component of $\partial \Phi \setminus T'$, lying between $\partial \Phi$ and $T'$. The manifold $M(H) = R(H)/H$ is homeomorphic to $\Phi$, provided that points of the boundary equivalent relative to $H$ are identified. Let $q: \Phi \to M(H)$ be the natural projection, $M = q(\Phi)$, $\beta = q(\beta')$, where $\beta' < T'$ is a loop parallel in $\partial \Phi$ to $\eta$. Then the manifold $M(H)$ is obtained by gluing together $M$ (which is homeomorphic to $S_g \times S^1$) and $T = q(\Phi \setminus T')$. But this is precisely the construction of Sec. 1.3 for the case $|e| = 1$.

2.4. We now proceed to the construction of $\Phi$. Note that on the twice twisted tape $L_1$ (Fig. 2) the linking number of the central line $\sigma$ and the curve $\eta$ is 1. In the same figure we also see an equivalent tape $L_2$ in which the folded-over sections have been "separated." Our problem will be to "pave" $L_2$ with spheres in such a way that conditions (a)-(c) of 2.2 will be satisfied.

Dividing $L_2$ into two parts: $L_2^1$, lying in the horizontal plane $\Pi'$, and $L_2^2$ in which the central line $\sigma$ lies in the vertical plane $\Pi''$. Let $l = \Pi' \cap \Pi''$ and let $\Lambda' \subset \Pi'$ be the axis of symmetry of $L_2$, $O = l \cap \Lambda'$. We shall treat $l$ and $\Lambda'$ as coordinate axes in $\Pi''$ (Fig. 3).

Let $O_2$ and $O_2'$ be the points with coordinates $(0, 1)$ and $(2, 1)$, respectively, and $l_1 \subset \Pi'$ the straight line through $O_1$ and $O_2$. Let $\alpha = \pi/8$, $\varepsilon = \pi/24$, and let $C_1$ be the point with coordinates $(1, 1 - \tan(\alpha/2))$. Define $Q_1$ (the same letter will denote the sphere and the face of the polyhedron $\Phi$ on it) to be the sphere with center $C_1$ and radius $r = \tan(\alpha/2)/\cos(\varepsilon/2)$. The spheres $R_1^1, Q_1^1, R_1^2, Q_2$ are obtained from $Q_1$ by rotations about $O_2$ through angles $\alpha, 2\alpha, 3\alpha, 4\alpha$. Similarly, the spheres $R_2^1, Q_1^2, R_2^2, Q_1^3$ are obtained by rotating the same sphere about $O_1$ through the same angles (see Fig. 3). It is readily seen that the angles between adjacent spheres are $\varepsilon$, and the centers of $R_2$ and $Q_2$ lie on the axis $\varepsilon$. We have thus constructed the required "paving" of $L_2^1$. Let $J_1$ be inversion with respect to $Q_1$ and $O_2$ symmetry with respect to the plane orthogonal to $\Pi'$ and passing through $O_1$ and the center of the sphere $R_1^1$; define $A_1 = \sigma \cdot J_1$. Similarly, we let $I_1$ be inversion with respect to $R_1^1$ and $\sigma_1$ symmetry with respect to the plane orthogonal to $\Pi_1$ and passing through $O_1$ and the center of $Q_1^1, B_1 = \sigma_1 \cdot I_1$. It is easy to see that $A_1(Q_1^1) = Q_1^1, B_1(R_1^1) = R_1^1, A_1(Q_1^1 \cap R_1^1) = R_1^1 \cap Q_1^1$ and so on.

We now turn to the plane $\Pi''$. Let $\Lambda'' = \Pi''$ be the straight line orthogonal to $l$ and passing through $O$. Introduce a coordinate system $(l, O, \Lambda'')$ on $\Pi''$ (see Fig. 3). Let $O_3 = (2, 1), O_4 = (1, 0)$ be points on $\Pi''$. The spheres $R_3^2, Q_2, R_2^2, \ldots, R_5, Q_5$ are obtained from $Q_2$ by rotation about $O_3$ through angles $\alpha, 2\alpha, 3\alpha, \ldots, 11\alpha, 12\alpha$. All these spheres are orthogonal to
Finally, the spheres $R^0_1, Q^0_2$ and $R^0_3$ are obtained from $Q_2$ by rotation about $O_1$ through angles $\alpha, 2\alpha, 3\alpha$. The center of $R^0_3$ is on the line $l$. The system of spheres $Q_5, R^1_5, \ldots, Q_9, R^1_9$ is obtained by symmetry about the axis $l^1$ from the already constructed family of spheres. The angle between any two adjacent spheres is $\epsilon$. The exterior of the spheres $Q_1, \ldots, R_{12}$ is the required polyhedron $\Phi$. Indeed, the sum of its dihedral angles is $48\epsilon = 2\pi$.

Let $x_0 \in Q \cap l_1$ be the point nearest $O_2$. It is readily seen that $[A_2, B_2] \cdots [A_1, B_1](x_0) = x_0$, and the curve $\eta$ and $\delta \Phi$ constructed as in Sec. 2.2 has linking number 1 with the axis of the solid torus $R^3 \Phi$. We have thus constructed the required group $H = H(12, 1)$ uniformizing $S(12, 1)$.

2.5. We now show that for any $g$ and $e$ [such that $1 \leq |e| \leq (g - 1)/14$] there exists a Kleinian group $H(g, e)$ uniformizing $S(g, e)$. Let $H$ be a subgroup of $H(12, 1)$ of index $j$. It follows at once from Lemma 3.5 of [4] and the Riemann-Hurwitz formula that $H = H(11j + 1, j)$. If $H(12, 1) = H + h_1H + \ldots + h_jH$ is the coset decomposition of this group, then the fundamental polyhedron $\Psi$ of $H$ is the union $\Phi \cup h_1(\Phi) \cup \ldots \cup h_j(\Phi)$. The elements $h_1, \ldots, h_j$ may be so chosen that $\Psi$ is homeomorphic to a solid torus. We may assume that the boundary of $\Psi$ contains the piece $h_1(\Phi \cap (Q_1 \cup \ldots \cup R_{12}))$. The transformations $A_{11} = h_1A_1h_1^{-1}, B_{11} = h_1B_1h_1^{-1}, A_{12} = h_1A_{12}h_1^{-1}$ and $B_{12} = h_1B_{12}h_1^{-1}$ of $H$, which identify the faces of this piece, leave invariant a certain circle $C$ [the image under $h_1$ of the circle about $O_1$ of radius $1 - r^2 \sin^2(\epsilon/2)$, in the plane $H^1$]. Let $\Gamma_m$ be a Kleinian group leaving $C$ invariant (as well as the Euclidean disc $D$ spanned by the circle), such that $(D(L(\Gamma_m))/\Gamma_m$ is homeomorphic to a surface of genus $m + 2$ with one boundary component $\Gamma_m = \langle E_{t_1}, D_{t_1}, \ldots, E_{t_{12} - m}, D_{t_{12} - m} \rangle$. Then $\Gamma_m$ can be combined in Maskit's sense (see [14]), also [15, Chap. IV, Sec. 1, p. 169]) with the group $H'$ generated by the elements of $H$ that identify the faces of the polyhedron $\Psi \cap h_1(Q_1 \cup \ldots \cup R_{12})$ (the amalgamated subgroup is $h = \langle A_{12}, B_{12} \rangle$). It is not hard to see that the combined group thus formed, $H^*(m) = H' \ast \Gamma_m$, uniformizes the manifold $S(11j + 1 + m, j)$; hence, setting $m = g - (11j + 1), j = |e|$, we obtain the required group $H(g, e)$, completing the proof of the theorem.

2.6. Let $\tilde{H}(g, e)$ be an extension of $H(g, e)$ to $\tilde{H}^4 = \{(x_1, x_2, x_3, x_4): x_4 \geq 0\} \cup \{\infty\} = H^1 \cup S^3, M(g, e) = \tilde{H}^4 \setminus L(H(g, e)) / H(g, e)$. Note that the manifold $M(g, e)$ is a fiber space over $S^g$ whose fiber is a "closed disk," and the absolute value of its Euler number is $e$. In order to see that $M(g, e)$ is the total space of the fiber, it will suffice to extend the fundamental region $\Phi$ of $H(g, e)$ to a polyhedron $\Phi$ in $H^1$, whose faces are hyperplanes based on corresponding spheres in $S^3$. The natural foliation of $\Phi$ into circles extends to a foliation of $\Phi$ into two-dimensional planes in $H^1$, which in turn extends to a foliation of a $\Phi$ having the local structure of a product. The structure of the foliation is now dropped to $M(g, e)$, which becomes a fiber space over $S^g$ with fiber $D^2$. The Euler class of the resulting fibration is equal in absolute value to $e$; this follows from the fact that $\delta M(g, e) = S(g, e)$ is a fiber space with Euler number $e$.

**Corollary 1:** Let $E \to S_g$ be a fiberation with fiber $R^2$ and Euler number $e = \mathbb{Z}$, such that $|e| \leq \chi(S_g)/22, g \geq 12$ [where $\chi(S_g)$ is the Euler characteristic of $S_g$]. Then there exists a complete metric of constant negative curvature on $E$.

**Remark.** Analogues of Theorem A and Corollary 1—though without explicit estimates of $|e|$—have been proved independently in a preprint of Gromov, Lawson, and Thurston [16].

**Corollary 2.** Any Seifert fiber space with hyperbolic base (see [4]) is almost conformally flat (i.e., it has a finite-sheeted cover by which is a manifold admitting a CFS).

**Proof.** It will suffice to consider the case of a closed Seifert fiber space with Euler number zero. The group $\pi_1(M)$ can be embedded in a short exact sequence $1 \to \mathbb{Z} \to \pi_1(M) \to F \to 1$, where $F$ is isomorphic to a discrete subgroup of $\text{PSL}(2, \mathbb{R})$. Then $F$ contains a subgroup of finite index $F_0$ which is isomorphic to $\pi_1(S_g)$, where the genus of $S_g$ is at least 12. Let $G_0 = \pi^{-1}(F_0)$. Then $G_0$ has a corepresentation $\langle a_1, b_1, \ldots, a_g, b_g, t: [a_1, t] = [b_1, t] = [a_1, b_1] \cdots [a_g, b_g]t^{-e} = 1 \rangle$, where $e \neq 0$. If $\tau = t^0$, then the index of the subgroup $G'_0 = \langle a_1, b_1, \ldots, a_g, b_g, r: [a_1, b_1] \cdots [a_g, b_g]\tau^{-1} = 1 \rangle$ in $\pi_1(M)$ is finite.
The cover constructed on the basis of this subgroup is homeomorphic to \( S(g, 1) \) and is the required conformally flat manifold (since \( g \geq 12 \) and Theorem A is applicable).

**Remark.** The analogous assertion for the case \( g = 1 \), i.e., when the base space is Euclidean, is no longer true [7].

2.7. Later we shall need a certain modification of the groups \( H(g, e) \) constructed in Theorem A. Consider a circle in a plane \( \pi \), say \( O(P, p) \) with center \( P \) and radius \( \rho \); let \( l \) be a straight line in the same plane, whose distance from \( P \) is \( \rho + R \), where \( R > 0 \). Rotating \( O(P, \rho) \) in \( R^3 \) about \( l \), we obtain a torus, denoted by \( T(R, \rho) \); call \( \rho \) the inner and \( R \) the outer radius of the torus.

Note that the exterior of the fundamental polyhedron \( \Phi \) of any group \( H(g, e) \) as constructed in Sec. 2.3 is contained in a ball of radius 4 (centered at \( O \)), and the radius of any sphere (containing a face of \( \Phi \)) is at most \( r = \tan \alpha / \cos \epsilon < 0.2 \). For every natural number \( m \geq 0 \), let us consider the torus \( T = T(10(m + 1), 8) \) with rotation axis \( l \). Within this torus, consider the solid torus \( T_m \) obtained by rotating the disk \( D(Q, 0.5) \) about \( l \) (Fig. 4), where the center \( Q \) of the disk is situated on the perpendicular dropped from \( P \) to \( l \), at a distance 2 from \( P \). Then for given \( m \) and Euler number \( e \) there exists a number \( g_0 = g_0(m, e) \) such that for all \( g \geq g_0 \) there is a Kleinian group \( H_m(g, e) \) [the above-mentioned modification of \( H(g, e) \)] with the following properties:

(a) \( H_m(g, e) \) uniformizes \( S(g, e) \);

(b) \( H_m(g, e) \) has a fundamental polyhedron \( \Phi_m(g, e) \) homeomorphic to a solid torus, whose complement in \( S^3 \) (1) lies in the union of the solid torus \( T_m \) and the ball \( B(P, 8) \) of radius 8 about \( P \), (2) forms a link of index 1 (as the construction of this group is entirely analogous to the construction of Sec. 2.5, we shall not go into details).

2.8. Recall that a group \( \Gamma \) of homeomorphisms of \( S^n \) is said to be (uniformly) quasiconformal if \( \sup \{K(\gamma), \gamma \in \Gamma\} < \infty \), where \( K(\gamma) \) is the quasiconformality coefficient (see, e.g., [17]). Various examples have been constructed [18-20] to refute the conjecture, advanced in [21], that any such group is quasiconformally conjugate to a conformal group. We are going to show how Theorem A can be used to construct an example of a quasiconformally nonstandard (i.e., not conjugate to a topologically conformal) action of the group \( \pi_1(S) \times Z_n \) on the 3-sphere.

Let \( H = H(12, 1) \) be the group constructed in Theorem A, \( \varphi: M(H) \to M(H) \) a diffeomorphism of order \( n \geq 2 \), isotopic to the identity (which exists because Seifert fiber spaces admit an \( S^1 \)-action [10]). Let \( \bar{\varphi} \) denote a lifting of order \( n \) of \( \varphi \) to the region of discontinuity \( R(H) \). Then \( K(\bar{\varphi}) < \infty \), \( \varphi \circ h = h \circ \bar{\varphi} \) for all \( h \in H \), so \( \bar{\varphi} \) extends to a quasiconformal homeomorphism on the whole of \( S^3 \) (see [22-24]).

**Remark.** We have thus proved that \( L(H) \) is an unknotted circle in \( S^3 \) for any group \( H \) that uniformizes a Seifert fiber space over a hyperbolic orbifold [24]. Denote the extension of \( \varphi \) to \( S^3 \) by \( \bar{\varphi} \). Then \( \Gamma = H \times \langle \bar{\varphi} \rangle \cong \pi_1(S) \times Z_n \) is a discrete quasiconformal group. In addition, every element of \( \Gamma \) is quasiconformally conjugate to some Möbius transformation, and \( \Gamma \) itself is isomorphic to a subgroup of \( \mathcal{M} \).

**COROLLARY 3.** The group \( \Gamma \) is not topologically conjugate to any subgroup of \( \mathcal{M} \).
Proof. Suppose that there is such a conjugation \( g \), then the group \( G = g \Gamma g^{-1} \) leaves the Euclidean circle \( \text{Fix}(g/g^{-1}) \) invariant. But the manifold \( \text{M}(g \Gamma g^{-1}) \) is homeomorphic to \( \text{M}(H) \) and has a nontrivial Euler class, which is impossible since there is an \( H^3 \times \mathbb{R} \) structure on \( \text{M}(g \Gamma g^{-1}) \) (cf. Sec. 2.1 in this paper, and also \([4, \text{Sec. 4}]\)).

3. CONFORMAL GLUING OF SEIFERT FIBER SPACES

3.1. Let \( Z_1, \ldots, Z_8 \) be a collection of Seifert fiber spaces and \( M \) an orientable manifold obtained by gluing them together at boundary tori (i.e., \( M \) is a "graph-manifold"). Assume that \( \pi_1(M) \) is not solvable. In this section we shall prove that there exists a finite-sheeted cover \( M_0 \) of \( M \) which admits a uniformizable conformally flat structure.

Before proceeding to the proof, we outline the main idea. Let \( Z_1 = S'_g \times S^1 \), \( Z_2 = S'_g \times S^1 \), where \( S'_g \) is a surface of genus \( g > 0 \) with one boundary component. Splitting \( Z_1 \) into a direct product determines a "natural" basis in \( \pi_1(\partial Z_1) \) (for more details, see Sec. 3.3). Suppose that \( M \) is obtained by gluing \( Z_1 \) and \( Z_2 \) together by means of a homeomorphism \( j: \partial Z_1 \rightarrow \partial Z_2 \), defined relative to the natural bases by a matrix \( A \in GL_+^3(Z) \), where \( a_{21} = 1 \). Take the groups \( H(g_1, a_{22}) \) and \( H(g_2, a_{11}) \), constructed in Theorem A (they exist if \( g_1 \) and \( g_2 \) are sufficiently large), and place them in \( S^3 \) in such a way that the complements of the fundamental polyhedra form a link of index 1. It is not hard to see that the Klein combination \( G = H(g_1, a_{22}) \ast H(g_2, a_{11}) \) of these groups uniformizes \( M \) (note that with this method of constructing the condition \( a_{21} = 1 \) is absolutely unavoidable). Our goal will be to construct a finite-sheeted cover of \( M \) (in Theorem B) obtained by gluing products of surfaces of large genus to a circle, with coefficients \( a_{21} \) equal to unity for all the gluing homeomorphisms.

3.2. Proof of Theorem B. By Theorem A, we may assume without loss of generality that \( M \) is not a Seifert fiber space. Our first task is to construct a cover over \( M \) which, when cut along incompressible tori, will contain as components only trivial Seifert fiber spaces (i.e., products of a surface and a circle). Let \( Z_1 \) be a fiber space over an orbifold \( O_1 \), other than \( S^1 \times [0, 1] \) (we may assume without loss of generality that there are no components \( T^2 \times [0, 1] \) among the \( Z_i \)). To each component \( Z_i \) we glue a disk \( D_i \) with a singular conical point \( g_i \) (with angle \( 2\pi/p \), \( p \) a prime). Denote the resulting orbifold by \( O_i \). It is not hard to see that \( O_i \) is a "good" orbifold (see \([4, \text{Sec. 2}]\)), and therefore there exists an even-sheeted regular cover \( \varphi_i: \mathcal{O}_i \rightarrow O_i \) of the orbifold which is orientable by a surface. Remove the disks \( D_i \) from \( O_i \). The resulting surface \( O_i \) covers our original orbifold \( O_1 \). It is not hard to see that there exist a Seifert fiber space \( W_i \) over \( O_i \), and a cover \( \psi_i: W_i \rightarrow Z_i \), corresponding to a cover \( \varphi_i: O_i \rightarrow O_i \) of the bases and a \( p \)-fold cover of the fiber of \( Z_i \) by the fiber of \( W_i \). Let \( W_i \) be a fiber space over \( O_i \) (cf. [25]). Since \( \varphi_i \neq O_i \), the surface \( O_i \) is orientable and the Seifert fibration \( W_i \rightarrow O_i \) has no singular fibers, it follows that \( W_i \) is homeomorphic to \( O_i \times S^1 \) ([4]). The cover \( \psi_i \) has the property that if \( T_{ij} \) is a component of \( O_i \) and \( \psi_{ij}: T_{ij} \rightarrow T_{ij} \) is the restriction of \( \psi_i \) to a component of \( \psi_i(T_{ij}) \), the the gluing subgroup of \( \psi_{ij} \) is the subgroup \( p(Z + Z) \subset Z + Z \approx \pi_1(T_{ij}) \). Thanks to this property we can glue the manifolds \( W_i \) together to get a cover \( M_1 \) over \( M \) (cf. [25, Proposition 1.1]).

3.3. As \( \pi_1(M) \) is not solvable, we may assume without loss of generality that \( M \) does not contain components \( T^2 \times [0, 1] \) (since a fiber space over \( S^1 \) with toric fiber can finitely cover only manifolds that admit \( E^3 \), \( \text{Sol} \)- or \( \text{Nil} \)-structure ([4])). All components of the decomposition of \( M_1 \) are products \( S^1 \times O_i \), where \( O_i \) has an even number of boundary components. Fix the orientation on all the \( W_i \)'s so that the homeomorphisms gluing them together to get \( M_1 \) reverse the induced orientation of the boundary (recall that \( M \) is orientable). Let \( \sigma_{ij} \) be a component of \( \partial O_i \) - we shall use the same symbol to denote its natural embedding in \( S^1 \times O_i \). Let \( \tilde{t}_{ij} = S^1 \times \{x_0 \in O_i \} \) denote a representative of the fiber of \( S^1 \times O_i \) on the boundary component \( S^1 \times \sigma_{ij} = \mathcal{T}_{ij} \). Orient all \( \tilde{t}_{ij} \), \( \tilde{t}_{i2} \),..., in the same way and \( \sigma_{ij} \), \( \sigma_{i2} \),... in such a way that the sum of the corresponding elements of \( H_1(W_i, Z) \) vanishes and the orientation of the pairs \( (t_{ij}, \sigma_{ij}) \), \( (t_{i2}, \sigma_{i2}) \),... coincides with the chosen orientation of \( \partial W_i \). The same letters \( t_{ij} \), \( t_{i2} \),... denote basis elements of the groups \( \pi_1(\mathcal{T}_{ij}) = \langle t_{ij} \rangle \oplus \langle \sigma_{ij} \rangle \). From now on we shall call these bases "natural." Let \( W_i \) and \( W_k \) be components of the toric decomposition of \( M_1 \), \( \mathcal{T}_{ij} \subset \partial W_i \subset \partial W_k \) components of the boundary glued together by the homeomorphism \( e = f_{ij} \). Assuming that the manifold thus obtained is not a Seifert fiber space, then \( f_{ij}(t_{ij}) = a_{21} t_{ij} + a_{22} \sigma_{ij} \), \( f_{ij}(\sigma_{ij}) = a_{21} \sigma_{ij} + a_{22} \sigma_{ij} \) (where \( a_{21} \neq 0 \), otherwise the gluing operation produces a Seifert fiber space). We shall call \( A = (a_{22}) = GL^+_{22}(Z) \) the gluing matrix (relative to the natural bases). Let \( \sigma_{ij} = a_{ij} \sigma_i \) and \( \sigma_{ij} = a_{ij} \sigma_k \). Then \( f_{ij}(t_{ij}) = a_{21} t_{ij} + \sigma_{ij} \), \( f_{ij}(\sigma_{ij}) = a_{21} a_{ij} \sigma_i + a_{22} \sigma_k \), therefore \( f_{ij}(t_{ij} \oplus \sigma_{ij}) = (t_{ij} \oplus \sigma_i) \). We thus select loops \( \sigma_{ij} \) on all the tori \( \mathcal{T}_{ij} \), along
which the manifold $M_1$ will be cut. For all surfaces $\mathcal{P}_i$, construct covers $p_i: \mathcal{P}_i \rightarrow \mathcal{P}_i$ such that for each component $a_{ij} \subset \partial \mathcal{P}_i$ the defining subgroup of the corresponding restriction of $p_i$ is the subgroup $\langle a_{ij} \rangle$ (cf. Sec. 3.2). Let $\Pi_i: \bar{W}_i \rightarrow W_i$ be the cover induced by the cover $p_i$ of the base space and the trivial cover of the fiber $S^1$. Lifting the loops $a_{ij}$ and $t_{ij}$ to $\bar{W}_i$ clearly yields natural bases for the components $\Pi_i^{-1}(\mathcal{F}_{ij})$, relative to which the gluing matrix $A = (\overline{a_{ij}})$ has its entry $\overline{a_{ij}}$, equal to 1 (the gluing is carried out by lifting the map $f_{ij}$ to the covering spaces).

3.4. Let $M_2 \rightarrow M_1$ be a finite-sheeted cover, glued together from Seifert fiber spaces $Y_i$ (each of which is homeomorphic to some one of the $\bar{W}_i$'s). Associated with each $Y_i$, which has $r_i$ boundary components, we have a collection of numbers $a_{ij}, j = 1, \ldots, r_i$ -- the elements of the gluing matrix $A(i, j)$ (see Sec. 3.2). Let $e_i = |a_{ij}(i, 1) + \ldots + a_{ij}(i, r_i)|$, and let $g_i$ be the genus of the surface $\mathcal{P}_i$ (the base space of $Y_i$).

Recall that by construction (see Sec. 3.2) the numbers $r_i$ are even for all $i$. Hence each surface $\mathcal{P}_i$ admits a regular cyclic cover $\eta_i: \Sigma_i \rightarrow \mathcal{P}_i$ of arbitrary multiplicity $q_i$, where the number of boundary components of $\Sigma_i$ is, as before, $r_i$. The genus $k_i$ of $\Sigma_i$ is $1 + r_i(q_i - 1)/2 + q_i(g_i - 1)$, and we shall choose the numbers $q_i$ to be the same prime number $q$ (for all $i$). Moreover, we choose $q$ so large that $k_i > g_i(e_i, r_i)$, where $g_i(e, m)$ is the same function as in Sec. 2.7 [the condition $k_i > g_i(e_i, r_i)$ guarantees the existence of the modified group $H_{r_i}(k_i, e_i)$; see Sec. 2.7]. Finally, consider the covers $\xi_i: X_i = S^1 \times \Sigma_i \rightarrow Y_i = S^1 \times \mathcal{P}_i$, where $0: S^1 \rightarrow S^1$ is a $q$-sheeted cover. Then the homeomorphisms by means of which $M_2$ is glued together from the manifolds $Y_i$ lift to homeomorphisms $\gamma_{ij}$ of the boundaries $Y_i$, with the same gluing matrix $A$. The components $X_i$ are now glued together to get a manifold $M_0$ which is a finite-sheeted cover of $M$. Our next goal is to construct a Kleinian group $G$ uniformizing $M_0$.

3.5. Let $G_i$ denote the groups $H_{r_i}(k_i, e_i)$ (see Sec. 3.4). These groups (and their conjugates in $\mathbb{M}_3$) will be combined in the Klein–Maskit sense (see [14, 15]) to construct the required group $G$. We begin the operation with the group $G_1 = G$. The boundary of the fundamental region of $G_1$ is in the interior of the torus $T(10(r_1 + 1), 8)$ (see Sec. 2.7). It is readily seen that, together with $B(P, 8)$, the interior of this torus also contains $r_1$ disjoint balls $B(P_j, 8)$ of the same radius, whose centers $P_j$ lie at the same distance $8 + 10(r_1 + 1)$ from the axis of rotation $l$ as the point $P (j = 1, \ldots, r_1)$.

Let $\pi_j$ be the plane through $l$ and $P_j$, and $l_i \subset \pi_i$ the straight line parallel to $l$ at a distance 2 from $P_j$. Construct a torus $T(1,1)$ with axis of rotation $l_i$ and take its image under inversion with respect to the sphere of radius 1 about $P_j$ (Fig. 5). Let $T_0(1,1)$ be the image of the resulting torus after dilation with center $P_j$ and coefficient 7.5. We shall call $P_j$ the center of this torus. It is readily verified that $T_0(1,1)$ is contained in the ball $B(P_j, 8)$, and if $F_0(1,1)$ denotes the solid torus bounded by $T_0(1,1)$ and not containing the point $\infty$, then $F_0(1,1)$ and the solid torus $T^*(1,1)$ (see Sec. 2.7 and Fig. 6) form a link in $\mathbb{R}^3$ of index 1.

We now place tori $F_{ij} \cong T^*(1,1)$, as well as $T_0(1,1)$ in the interior of each ball $B(P_j, 8) \subset \text{int}(T_{0}(1,1) = T(10(r_1 + 1), 8))$.

3.6. Suppose the manifold $X_2$ is glued to $X_1$ along several boundary components $\gamma_{ij}: F_{ij} \subset \partial X_1 \rightarrow \partial X_2 \subset \partial X_2, \ldots, \gamma_{2q}: F_{2q} \subset \partial X_1 \rightarrow \partial X_2 \subset \partial X_2$. Working with $X_2$, construct a torus $T(10(r_2 + 1), 8)$, group $G_2 = H_{r_2}(k_2, e_2)$ and system of $q$ tori $F_{2q}$, isometric to $T(1,1)$, situated in balls of radius 8 and forming with $T_{2q}$ a link of index 1 (as done previously inside the torus $T_{(1,1)}$). The remaining $r_2 - q$ disjoint balls inside $T(2)$ will be filled with tori of the form $T(1,1)$ or $T_0(1,1)$ at the end of this subsection. Let $F_{1i}$ and $F_{2i}$ be any two tori in the
interior of $T(1)$ and $T(2)$, respectively. There exists a Möbius transformation $\gamma_{11}^m$: $\text{ext} \mathcal{F}_{11} \rightarrow \text{int} \mathcal{F}_{11}$ [see the definition of $T(1,1)$ and $T(1,1)$]. It is not hard to see that the groups $H_r(k_1, e_1) = G_1^*$ and $G_2^* = \gamma_{11}^m G_1^* \gamma_{11}^m$ form exactly the same "link" as described in Sec. 3.1. The elements $\gamma_{11}^m$ are clearly not uniquely determined. However, if we confine attention to the induced isomorphism $(\gamma_{11}^m)_*: \pi_1(\mathcal{F}_{11}) \rightarrow \pi_1(\mathcal{F}_{11})$, there exist exactly two possible choices for the map $\gamma_{11}^m$ (differing from one another by a Euclidean axial symmetry of $\mathcal{F}_{11}$). We shall see later how to choose $\gamma_{11}^m$.

Let $\gamma_{22}^m$: $\text{ext} \mathcal{F}_{22} \rightarrow \text{int} \mathcal{F}_{12}$, ..., $\gamma_{2q}^m$: $\text{ext} \mathcal{F}_{2q} \rightarrow \text{int} \mathcal{F}_{1q}$ be Möbius transformations. We construct a successive HNN-extension of the group $G_1^* G_2^*$ by the elements $\gamma_{12}^m \gamma_{11}^m \gamma_{12}^m \gamma_{11}^m$. It is easy to see that under these conditions the conditions of Maskit's combination theorem (see [14]) are fulfilled, since the solid tori $\text{int} \mathcal{F}_{1i}$, $\text{int} \mathcal{F}_{2i}$ are strictly invariant (with respect to the identity subgroup).

This process can be continued, considering the Klein-Maskit combinations of the groups $G_i = H_r(k_i, e_i)$ (and their conjugates) in accordance with the way in which $M_0$ is glued together from components $X_i$. When this is done, if manifolds $X_i$ and $X_j$ are to be glued together, we place in each of the unfilled balls of radius 8 in $\text{int} T(i)$, $\text{int} T(j)$ one torus, interlinked with $T_r(i)$ (resp., $T_r(j)$) if the torus placed in $T(i)$ was of type $T_r(i)$, that placed in a ball of $T(j)$ will be of type $T(1,1)$. The group $G$ resulting from this combination procedure is the required group.

3.7. In this section we shall indicate how to choose the Möbius transformations $\gamma_{ij}^{mn}$ and explain why $G$ uniformizes the manifold $M_0$.

We consider the natural orientation of the curve $\eta \subset \partial \Phi$, defined by the ordering $\alpha_1$, $\alpha_1, \gamma_1, \ldots$ (see Fig. 1, Sec. 2.2, and Fig. 2, Sec. 2.4), where $\Phi$ is the fundamental polyhedron of the group $H(g, e)$. The very same orientation can be considered on the loop $\kappa \subset \partial \Phi$, parallel to the axis of the solid torus $S^1 \Phi$ (see Fig. 2). The orientation of the loop $t < \partial \Phi$, $t = Q \cap R_r$ (see Sec. 2.2) is defined by the condition $\eta \sim \kappa < t + \eta$. In a similar manner we orient the loops $\eta_i, \kappa_i, t_i \subset \partial \Phi_r$, where $\Phi_r$ is the fundamental polyhedron of the group $H_r(k_i, e_i)$. The loop $\eta_i$ generates the kernel of the homomorphism $\pi_1(\partial \Phi_r) \rightarrow \pi_1(\Phi_r)$, and the loop $t_i$ the kernel of $\pi_1(\partial \Phi_r) \rightarrow \pi_1(S^1 \Phi_r)$. Let $\mathcal{F}_0 = \text{int} \mathcal{F}_i$, on this torus we then obtain a pair of basis loops $\tau_i, \kappa_i$, parallel in $\Phi_r \cap \mathcal{F}_i$ to $t_i$ and $\kappa_i$, respectively. We now choose the Möbius transformation $\gamma_{ij}^{mn}$: $\text{ext} \mathcal{F}_{ij} \rightarrow \text{int} \mathcal{F}_{mn}$ subject to the condition $\gamma_{ij}^{mn} \gamma_{ij}^{mn} = \gamma_{mn} \gamma_{mn}$, $(\gamma_{ij}^{mn})^* \gamma_{mn} = \gamma_{mn} \gamma_{mn}$.

Now put $\lambda_{ij} = \tilde{a}_{22}(i, j) \tau_{ij} + \kappa_{ij} \in \pi_1(\mathcal{F}_{ij})$ (see Secs. 3.3, 3.4); the same symbol $\lambda_{ij}$ will denote a simple loop on $\mathcal{F}_{ij}$, representing this element of $\pi_1(\mathcal{F}_{ij})$. A direct check now shows that $(\gamma_{ij}^{mn})^* - \lambda_{ij}$, $(\gamma_{ij}^{mn})^* \kappa_{ij} = \tilde{a}_{12}(i, j) \tau_{mn} + \lambda_{mn} \tilde{a}_{22}(i, j)$, where $\tilde{a}_{11}(i, j) = -\tilde{a}_{22}(m, n)$, $\tilde{a}_{12}(i, j) = \tilde{a}_{22}(m, n)$, $\tilde{a}_{11}(i, j) \tilde{a}_{22}(i, j) + 1$.

On the other hand, we recall that $e_i = |\tilde{a}_{22}(i, 1) + \ldots + \tilde{a}_{22}(i, r_i)|$ (see Sec. 3.4). Therefore, in the manifold

$$X_i = \left( R(G_i) \setminus \bigcup_{\varepsilon \in G_i} g \left( \bigcup_{j=1}^{r_i} \text{int} \mathcal{F}_{ij} \right) \right) / G_i$$

the sum of projections of the loops $\lambda_{ij}$ bounds a surface $\Sigma_i$ (recall that $G_i = H_r(k_i, e_i)$, and $R(G_i)$ is the region of discontinuity of $G_i$). Denoting the projections of $\lambda_{ij}$ in $X_i$ by $\delta_{ij}$ and the projections of $\tau_{ij}$ by $\tau_{ij}$, we see that the pairs $(\delta_{ij}, \tau_{ij})$ are natural bases of $\Sigma_i$, and the gluing matrix of the homeomorphism $\gamma_{ij}^{mn}$, obtained when $\gamma_{ij}^{mn}$ descends to $\delta X_i$ and $\delta X_j$, coincides with $A(i, j)$ (see Secs. 3.3, 3.4). In sum, the manifold $M(F) = R(G)/G$ (obtained from $M(G) = R(G)/G$ by gluing together at boundary points which are equivalent relative to $G_i$ and the elements $\gamma_{ij}^{mn}$) is homeomorphic to $M_0$. Thus $M_0$, which finitely covers $M$, is uniformized by the Kleinian group $G$. This completes the proof of Theorem B.

3.8. As an application of Theorem B, we shall construct an example of a 3-manifold $M$ which does not admit a CFS, but $M$ has a uniformizable finite-sheeted cover.

Let $\mathcal{O}$ be an orbifold whose support is the annulus $S^1 \times [0, 1]$ and whose singular set a conical point with angle $\pi$. Let $N$ be a Seifert fiber space over $\mathcal{O}$ whose fundamental group has the corepresentation $\langle a, b, c, t: e^2 = t, abc = 1, [a, t] = [b, t] = 1 \rangle$. The boundary of $N$ consists
of two toric components whose fundamental groups are generated by the elements $a$ and $t$, $b$ and $t$, respectively. Let $f$ be a homeomorphism mapping one boundary component onto the other, $f_\alpha (a) = t$, $f_\alpha (b) = t$, where $f_\alpha$ is the induced homomorphism of the fundamental groups [the generators of $\pi_1(M)$ can be so chosen that $f$ reserves the induced orientation of the boundary]. Let $M$ denote the manifold obtained by identifying points $x, f(x) = \partial N$.

It is easy to see that $M$ satisfies the assumptions of Theorem B (since the base orbifold $\sigma$ is not Euclidean). Hence there exists a finite-sheeted cover over $M$ that admits a uniformizable conformally flat structure.

**THEOREM D.** There exist no conformally flat structure on $M$.

**Proof.** Let us suppose that there exists a conformally flat structure $K$ on $M$, and let $d_*: \pi_1(M) \to \mathfrak{so}_3$ be the holonomy homomorphism (for the definition see [1, 2, 7]). If $g \equiv \pi_1(M)$, we let $g^\alpha$ denote $d_\alpha(g)$. The fundamental group of $M$ has a corepresentation $(a, b, c, abc = 1, [a, t] = [b, t] = 1, q^{-1}aq = t, q^{-1}bq = b)$. We claim that the group $H = d_\alpha(\pi_1(M))$ must satisfy one of the following conditions: it is conjugate to a subgroup of $\mathfrak{so}(4) \subset \mathfrak{so}_3$, it has two fixed points in $\mathbb{R}^3$, it is Abelian; it is polycyclic of rank $r \leq 3$, it is nilpotent. Since $|\pi_1(M)| = \infty$, the first possibility cannot occur (cf. [26]); that the second case is impossible follows from [24, lemma and Theorem 1]. The group $H$ can be neither nilpotent nor polycyclic of rank $r \leq 3$, in view of results of Kuiper [27] and Goldman [7] (see also [28]), since $\pi_1(M)$ is not Abelian. Thus verification of our claim will complete the proof.

(a) Suppose first that $t^\alpha = 1$. Then $a^\alpha = b^\alpha = 1$, $c^\alpha = 1$, and therefore $H$ is a cyclic group.

(b) Now let $1 \neq t^\alpha$ be an elliptic transformation. Then the elements $a^\alpha, b^\alpha, c^\alpha$ are also elliptic. If $t^\alpha$ has no fixed points in $\mathbb{R}^3$, then its extension to $\mathbb{H}^4$ leaves exactly one point fixed there (denote this point by $q$). Clearly, $q$ is also a fixed point of $a^\alpha, b^\alpha$. Thus the group $d_\alpha(\pi_1(N))$ leaves $q$ fixed. In addition, it follows at once from the condition $(\phi^\alpha)^{-1} a^\alpha \phi^\alpha = t^\alpha$ that $\phi^\alpha(q) = q$. Therefore $H(q) = q$ and $H$ is conjugate to a subgroup of $\mathfrak{so}(4)$.

Suppose now that $t^\alpha$ leaves a circle $l \subset S^3$ fixed point for point. Then the fixed sets of $a^\alpha, b^\alpha$ are circles $l_a, l_b \subset S^3$. If at least one of these circles is $l_a$, then $l_a = l_b = l_\alpha$ and $H$ is Abelian. Note that for any $g \equiv \pi_1(N) \ g^\alpha(l_a) = l_a$. Hence there exists only one possibility in case (b): the pairs $l_a$, $l_b$ and $l_\alpha$, have linking number 1. But then, as is easily seen, $(c^\alpha)^2 = (a^\alpha b^\alpha)^{-2} \neq 1$ and this element cannot have a circle of fixed points $l_\alpha$; consequently, $(c^\alpha)^2 = t^\alpha$, which is false.

(c) Suppose that $t^\alpha$ is a loxodromic element with fixed points $0$ and $\infty \in \mathbb{R}^3$. Then $a^\alpha$ and $b^\alpha$ are also loxodromic transformations and their fixed points are $0$ and $\infty$ (since $[a^\alpha, t^\alpha] = [b^\alpha, t^\alpha] = 1$). Therefore $\phi^\alpha(0) = 0$, $\phi^\alpha(\infty) = \infty$ and the entire group $H$ leaves $0$ and $\infty$ fixed.

(d) The last case: $t^\alpha$ is a parabolic transformation, $t^\alpha(\infty) = (\infty)$. It is readily seen that then the group $d_\alpha(\pi_1(N))$ leaves invariant either a straight line or a plane in $\mathbb{R}^3$. This invariant line (or plane) may be so chosen that it is also invariant to $\phi^\alpha$ [note that $\phi^\alpha(\infty) = \infty$]. It follows at once that $H$ is either polycyclic of rank $r \leq 3$ or nilpotent. This completes the proof.

**COROLLARY.** The manifold $M$ just constructed does not admit a CFS, but it has a uniformizable finite-sheeted cover.

This settles Problem No. 41 in [8].

**Remark.** The author's preprint [29] contains a proof of Theorem A and a sketch of the proof of Theorem B.

In conclusion the author would like to express his profound gratitude to the participants in a seminar led by S. L. Krushkal' for their useful comments, and to S. L. Krushkal' and N. A. Gusevskii for their scientific guidance and constant support. Thanks are also due to W. Goldman, J. Kamishima, R. Kulkarni, N. Kuiper, H. Lawson and many other mathematicians, who kindly sent preprints.

**LITERATURE CITED**