An example of 2-dimensional hyperbolic group which can't act on 2-dimensional negatively curved complexes

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It is a long standing open problem wether or not any word hyperbolic group admits a discrete faithful cocompact isometric action on a space of negative curvature. The goal of this note is to show that the answer is negative if one restricts to the class of groups of isometries of 2-dimensional $CAT(0)$-complexes. Namely, we will prove the following:

Theorem 0.1 *There exists a word-hyperbolic group G which acts discretely and effectively by isometries with compact quotient on a contractible 2-dimensional complex P of non-positive curvature so that G doesn't admit such action on any negatively curved 2-dimensional polyhedron.*

By negatively (or nonpositively) curved space we mean $CAT(-1)$ (or $CAT(0)$) space in the sense of comparison theorems. We shall need a definition of the angle in $CAT(0)$-space X which we take from [1]. Suppose $c(t), c'(t)$ are geodesics emanating from a point $x \in X$. Let

$$f(t) = \arccos(\frac{2}{t^2}(d(c(t), c'(t))^2 - 1)$$

Then the angle between c, c' at x is defined to be

$$\lim_{t \to 0} f(t)$$

This definition coincides with the usual one in the case of Riemann manifolds. Angle comparison theorem states that for $CAT(-k)$ space angles of geodesic triangles are not greater than angles of comparison triangles in Riemannian space of constant curvature $-k$.

Definition 0.2 *Let X be a topological space and $G \subset Homeo(X)$ is a subgroup. Then G is said to act discretely (= properly discontinuously) on X if for any compact $K \subset X$ there are only finitely many elements $g \in G$ such that $gK \cap K \neq \emptyset$.*

It is known that if X is a proper metric space (i.e. metric balls are compact) then $G \subset Isom(X)$ is discrete iff the G–orbit of any point $x \in X$ is discrete in the sense that the map $G \to Gx \subset X$ is proper, where G has discrete topology.

Definition 0.3 *Suppose that G is a discrete group of isometries of a geodesic length space X. An ideal point $z \in \partial_\infty X$ is called a "point of approximation" if for any geodesic ray l emanating from z there exists an infinite sequence $g_n \in G$ and a compact $K \subset X$ such that $g_n(l) \cap K \neq \emptyset$ for all n.*

Lemma 0.4 *Suppose that $G \subset Isom(X)$ is a discrete group of isometries where X is a space as above and G acts with compact quotient. Then any point $z \in \partial_\infty X$ is a point of approximation for the action of G.*

Proof: Take any geodesic ray l emanating from z. The projection of l to X/G is recurrent. Therefore if F is a (compact) fundamental domain for the action of G on X then infinitely many translates of l intersect F. $\square$

Lemma 0.5 *Suppose that G, X are as above, X is a $CAT(0)$-space and $H \subset G$ is a finite subgroup such that the normalizer of H in G is finite. Then the fixed-point set $F = Fix_H$ of H in X is compact.*

Proof: It is well-known that F is convex. Suppose that F is unbounded and $z \in \partial_\infty F$. Take any geodesic ray $l \subset F$ emanating from z. According to Lemma 0.4 there exists a compact K, an infinite sequence $g_n \in G$ such that $g_n(l) \cap K \neq \emptyset$ for all g_n. Therefore either we have an infinite sequence of distinct finite subgroups $H_n = g_n H g_n^{-1}$ whose fixed-point sets intersect K (which means that G doesn't act discretely) or the sequence H_n contains only a finite number of distinct members. This implies that the normalizer of H in G is infinite. $\square$

Remark 0.6 *Actually one can prove that if X is $CAT(0)$, G is cocompact group of isometries, H is a finite subgroup in G then the subgroup*

$$Stab(Fix_H) = \{g \in G : g(Fix_H) = Fix_H\}$$

acts on Fix_H with compact quotient.

Theorem 0.7 *There exists a word-hyperbolic group G which acts discretely and effectively by isometries with compact quotient on a 2-dimensional polyhedron P of non-positive curvature so that G doesn't admit such action on a negatively curved 2-dimensional polyhedron.*

Proof: This theorem is an application of a construction of a hyperbolic group due to W.Ballman and M.Brin [2]. Note that W.Ballman and S.Buyalo [3] proved that for any finite-index subgroup $G' < G$ there is is only one G'-invariant $CAT(0)$-metric on P. Our theorem was motivated by this result and a question of S.Gersten.

First I describe the properties of the construction [2]. There exists a 2-dimensional $CAT(0)$ polyhedron P with the following properties:

(a) all faces of P are flat regular hexagons,

(b) links L_x of all vertices of P are tetrahedrons (complete graphs on 4 vertices);

(c) for each vertices $x, y \in P$ and isometries $\phi : L_x \to L_y$ there exists (a unique) global isometry g of P which sends x to y and induces the map ϕ between links L_x, L_y.

These properties imply that the group $G(P) = Isom(P)$ acts on P with compact quotient which is isometric to a Euclidean triangle $\Delta(\pi/2, \pi/3, \pi/6)$. The properties (a-c) determine the space P almost uniquely. Namely, there are exactly two polyhedra P, P' which satisfy these properties. However the corresponding groups $G(P), G(P')$ are quite different: one is hyperbolic, another contains Z^2. Moreover, $G(P')$ is a nonuniform lattice in $\mathbb{H}^3$.

To distinguish these polyhedra (and the corresponding groups) we introduce the notion of "twist" around the face F_0 of P. Take the union of all faces of P adjacent to F_0 along edges of F_0. For each edge we get exactly 2 faces, therefore we have the union of 12 faces F_j. Now delete from

$$St(F) = \cup_{j=1}^{12} F_j$$

the edges which belong to F_0. There are exactly two possible cases:

(d) $St(F_0) - F_0$ is connected (the polyhedron P has a "twist");

(d') $St(F_0) - F_0$ consists of two connected components (the polyhedron P has no "twist").

We require P to satisfy the condition (d).

Lemma 0.8 *Suppose that P satisfies conditions (a-d). Then the space X is Gromov-hyperbolic.*

Proof: The space P is $CAT(0)$ and has cocompact group of isometries. Therefore hyperbolicity of P is equivalent to the absence of 2-flats: this was proven for nonpositively curved simply-connected Riemannian manifolds by P.Eberlein [5] and the general case was treated by M.Bridson [4]. Suppose that P contains a flat L. Then L must contain a face F_0 of P. By examining how L can extend from F_0 we conclude that L must contain all faces in $St(F_0)$. This contradicts the assumption that L is totally-geodesic. $\square$

Remark 0.9 *If P' satisfies conditions (a- d') then the arguments above will imply just that the flat L contains all faces in a component of $St(F_0) - F_0$ which is perfectly legal for a flat. Moreover one can see that extending F this way we produce a periodic flat in P'.*

Our goal is to show that the group G can't act on a negatively curved 2-complexes (as we shall see we will prove a stronger statement). Denote by A_j the centers of faces F_j and by C_i the vertices of F_0. Let $\tau_0 \in G$ be the reflection in the hexagon F_0, θ_j be the rotation of order 12 around A_j (so $\theta_0^6 = \tau_0$). Finally we let H_j be stabilizers of the vertices C_j. These groups are isomorphic to the permutation group S_4. Note that the normalizer of each group H_j is equal to H_j (for example since their fixed-point sets are points C_j). This implies that H_j are maximal finite subgroups in G. The normalizer of $< \theta_0 >$ in G is finite (this is Z_2-extension of $< \theta_0 >$).

Suppose that $\rho : G \to Isom(X)$ is a discrete, faithful representation where X is a $CAT(0)$ 2-complex so that $X/\rho(G)$ is compact. Each group $H_j^* = \rho(H_j)$ has nonempty fixed point set $Fix(H_j^*)$ which must be a bounded convex set (according to Lemma 0.5). The fixed-point sets

$$Fix(H_j^*), Fix(< \rho\theta_0 >)$$

are disjoint (otherwise H_j are not maximal finite in G). Therefore we can choose for each j points

$$C_j^* \in Fix(H_j^*), A_0^* \in Fix(< \theta_0^* >)$$

to realize the minimal distance between these compact sets. These points may be nonunique, however sets of minimizing points are convex in which case we choose centers of masses of these convex subsets. It follows that our choice is invariant under the action of G^* and moreover θ_0^* acts on $\{C_1^*, ..., C_6^*\}$ as element of order 6.

Consider the triangle $\Delta_j = A_0^* C_{j-1}^* C_j^* \subset X$. By minimality of A_0^* the edges of Δ emanating from this point intersect only at A_0^*. Now let's prove the same for C_j^*. We denote by $[C_j^*, Q_j^*]$ the intersection $[C_j^*, C_{j+1}^*] \cap [C_j^*, A_0^*]$. Let R be rotation in H_j which sends C_{j-1} to C_{j+1}. Then Q_j^* is fixed under $\rho(R)$ and under $\rho(\tau_0)$ (the last element fixes all the segments $[C_k^*, C_{k+1}^*]$. However the group H_j is generated by R and τ_0. This means that $Q_j^* \in Fix(H_j^*)$. By minimality of C_j^* we conclude that $Q_j^* = C_j^*$.

Minimality and injectivity of ρ also implies that $\Delta_j \cap \Delta_{j+1} = [A_0^*, C_j^*]$.

Denote by $6(r)$ a regular 1-dimensional hexagon with sides equal to r. Choose $r = \angle C_{j-1}^* A_0^* C_j^*$, then there exists a local isometry $i : 6(r) \to L_{A_0^*}$ where $L_{A_0^*}$ is the link of A_0^* in X; values of i on the vertices of $6(r)$ are given by the segments $[C_j^*, A_0^*]$. Since X is a $CAT(0)$ space the length of $i(6(r))$ is at least 2π, therefore $r \geq \pi/3$. We conclude that $\beta = \angle C_j^* \ C_{j-1}^* A_0^* \leq \pi/3$ and equality iff the triangle Δ is flat. However we assume that X is negatively curved, therefore $\beta < \pi/3$. Hence $\angle C_{j+1}^* C_j^* C_{j-1}^* \leq 2\beta < 2\pi/3$. Now we use invariance to conclude that the link $L_{C_j^*}$ contains a (locally) embedded circuit of the length strictly less than $3\angle C_{j+1}^* C_j^* C_{j-1}^* \leq 6\pi/3 = 2\pi$. This contradicts the assumption that X is $CAT(0)$.

□

Remark 0.10 *One can relax the assumption that X is negatively curved and assume just that X is $CAT(0)$. The conclusion in this case would be that there exists a positive number s and an equivariant local isometry $f : (P, s \cdot d) \to X$. Namely, one subdivides faces of P into equilateral triangles. On 1-skeleton of this triangulation we get a local similarity from P to X by sending points C_j, A_k to C_j^*, A_k^*. (The corresponding triangles in X are equilateral flat triangles). Then extend this to P by combing from C_j. This* *map is actually a totally-geodesic isometric embedding: check it locally and use $CAT(0)$–property.*

References

[1] A.Alexandrov , V.Berestovskii, I.Nikolaev, *Generalized Riemannian Spaces,* Russian Math. Surveys, 41:3 (1986) 1– 54.

[2] W.Ballman, M.Brin, *Polygonal Complexes and Combinatorial Group Theory,* Geom. Dedicata (1994).

[3] W.Ballman, S.Buyalo, *Nonpositively curved metrics on 2-polyhedra*, Preprint, 1993.

[4] M.Bridson, *On the existence of flat planes in spaces of nonpositive curvature,* Preprint, 1992.

[5] P.Eberlein, *Geodesic flow on certain manifolds without conjugate points*, Transaction of AMS (1972) 151– 170.