Hyperbolic Manifolds and Discrete Groups: Lectures on Thurston's Hyperbolization

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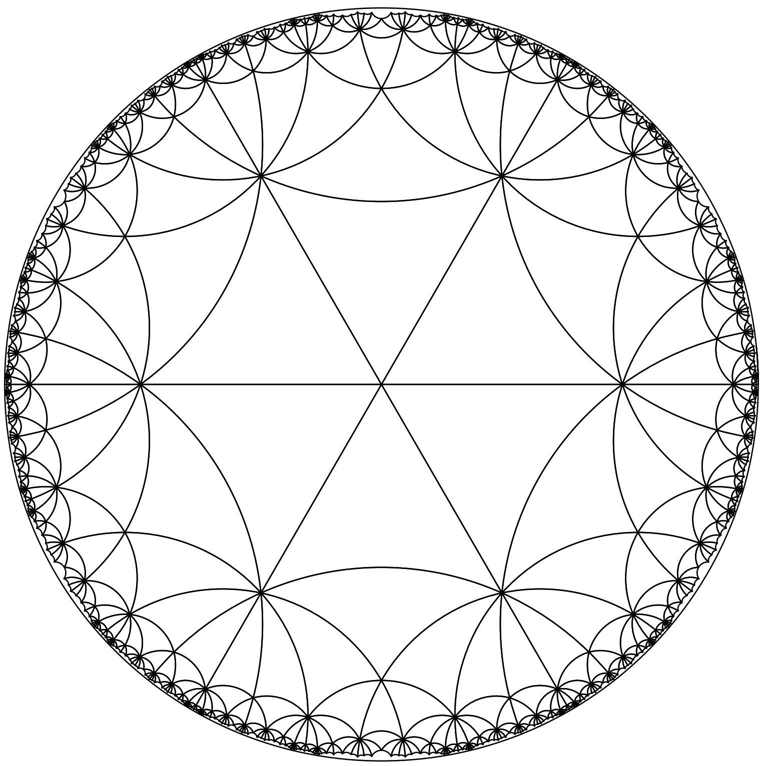

Preface

This book is based upon the course "Hyperbolic Manifolds and Discrete Groups" that I was teaching at the University of Utah during the academic year 1993-94. The main goal of the book is to present a proof of

Thurston's Hyperbolization Theorem. ("The Big Monster".) *Suppose that M is a compact atoroidal Haken 3-manifold, which has zero Euler characteristic. Then the interior of M admits a complete hyperbolic metric of finite volume.*

This theorem establishes a strong link between the geometry and topology of 3-manifolds and the algebra of discrete subgroups of $\mathrm{Isom}(\mathbb{H}^3)$. It completely changed the landscape of 3-dimensional topology and theory of Kleinian groups. This theorem allowed to prove things that were beyond the reach of the standard 3-manifold technique like Smith's Conjecture, residual finiteness of the fundamental groups of Haken manifolds, etc. In this book I present a complete proof of the Hyperbolization Theorem in the "generic case". Initially I was planning to include the detailed proof in the remaining case of manifolds fibered over $\mathbb{S}^1$ as well. However since Otal's book [Ota96] (which treats the fiber bundle case) became available, I will give in this book only a sketch of the proof in the fibered case.

The proof of the Hyperbolization Theorem is by induction on the steps of the *Haken decomposition* of M along incompressible surfaces. The *members of the Haken hierarchy* are manifolds $M = N_0, N_1, N_2, N_3, ..., N_h$ where each N_i is obtained by splitting N_{i-1} along a superincompressible surface and N_h is a disjoint union of 3-balls. There are two cases:

(a) The "generic case" when the decomposition of M starts with an incompressible surface S which is not a *virtual fiber* and thus the manifold N_1 is not an interval bundle (or a disjoint union of two interval bundles) over a surface.

(b) The "exceptional case" when N_1 is an interval bundle over a surface (or a disjoint union of two such bundles). The most important example is when M fibers over the circle with the fiber S, i.e. M is the mapping torus of a homeomorphism $\tau : S \to S$.

Below is a sketch of the proof in the case (a) under the assumption that M has empty boundary.

The first step of induction: each component of N_h (which is a closed 3-ball) admits a hyperbolic structure (just take a ball in the hyperbolic 3-space). We skip for a moment all the intermediate steps of the induction and consider the "last step of induction" (the *final gluing*) which turns out to be the heart of the proof.

The final gluing. Assume that M is a closed atoroidal orientable Haken manifold and $S \subset M$ is an incompressible surface which separates M into compact components M_1, M_2, each of which admits a hyperbolic metric and is not homotopy-equivalent to a surface (thus $N_1 = M_1 \sqcup M_2$). Let $S_i := \partial M_i$ and let τ denote the gluing mapping. We would like to find hyperbolic metrics g_1, g_2 on M_1, M_2 so that:

The gluing map τ is homotopic to an isometry $f : Nbd(S_1) \to Nbd(S_2)$ between product neighborhoods of the surfaces S_1, S_2 which sends S_1 to the surface in $\partial Nbd(S_2)$ which is different from S_2.

Once this is done, we can glue the manifolds (M_1, g_1), (M_2, g_2) via the isometry f and get a hyperbolic structure on the manifold M.

Of course, there will be an obstruction to the isometric gluing (i.e. to the existence of the metrics g_1, g_2 as above), the goal is to show that this obstruction is an incompressible torus $T \subset M$, which thus corresponds to collections of disjoint cylinders in M_1 and M_2. Therefore if (say) the manifold M_1 is *acylindrical* then the isometric gluing is unobstructed no matter what τ is (since there are no incompressible tori). Instead of the hyperbolic metrics we will try to find their *holonomy representations* $\rho_i : \pi_1(M_i) \to PSL(2, \mathbb{C})$. The homomorphisms ρ_1 and ρ_2 have to be chosen to form a commutative diagram:

$$\begin{tikzcd} & \pi_1(M_1) \arrow[dr] & \\ \pi_1(S) \arrow[ur] \arrow[dr] & & PSL(2,\mathbb{C}) \\ & \pi_1(M_2) \arrow[ur] & \end{tikzcd}$$

Once such ρ_i's are found, they induce a homomorphism $\rho : \pi_1(M) \to PSL(2, \mathbb{C})$ (part (1) of the proof). To conclude that ρ is the holonomy of a hyperbolic structure on M (i.e. that one can glue the corresponding metrics along neighborhoods of S_1, S_2) one has to prove that ρ_1, ρ_2 satisfy some further conditions: a combination theorem of Maskit (part (2) of the proof). Roughly speaking, these conditions require that the hyperbolic manifolds with boundary $(M_1, g_1), (M_2, g_2)$ embed isometrically as deformation retracts in compact hyperbolic manifolds $(M_1', g_1'), (M_2', g_2')$ which have convex boundary.

We now give more details. Let $G_i = \pi_1(M_i)$, $i = 1, 2$, $G := (G_1, G_2)$. Thurston's idea is to reduce the problem of finding the metrics g_1, g_2 to the *fixed-point problem* for a certain map σ of the Teichmüller space $\mathcal{T}(G) = \mathcal{T}(G_1) \times \mathcal{T}(G_2)$ and then to prove existence of a fixed point. As in many fixed point theorems, one tries to find this fixed point as the limit of a sequence of iterations $\sigma^n(X) = [\rho_n] \in \mathcal{T}(G)$. The spaces $\mathcal{T}(G_1), \mathcal{T}(G_2)$ are spaces of hyperbolic structures with convex boundary on the manifolds M_1, M_2: they are complete locally compact metric spaces. The mapping σ is a contraction:

$$d(\sigma(p), \sigma(q)) < d(p, q), \text{ unless } p = q.$$

The key part of the proof of the existence of the fixed point is the *Bounded Image Theorem*: it establishes relative compactness of the sequence $[\rho_n]$ in the Teichmüller space $\mathcal{T}(G)$. Algebraically, Teichmüller spaces $\mathcal{T}(G_i)$ correspond to equivalence classes of representations $G_i \to PSL(2, \mathbb{C})$ which are induced by quasiconformal homeomorphisms of the 2-sphere. The proof of precompactness breaks in two parts: (1) the proof of existence of a pair of limiting representation $G_i \to PSL(2, \mathbb{C})$, (2) the proof of the fact that these representations are induced by quasiconformal homeomorphisms.

Thurston's idea of the proof of (1) was based on a detailed study of the geometry of pleated surfaces in hyperbolic manifolds, most of it was presented by Thurston in the paper [Thu86a] and in the unpublished preprints [Thu87a], [Thu87b]. Instead of this approach we shall use a combination of geometry and combinatorics: the theory of group actions on trees. The *tree-theoretic approach* to proving precompactness of sequences of group representations was first developed by Culler, Morgan and Shalen in the papers [CS83], [MS84], [Mor86], [MS88a, MS88b]. The idea is to show that:

(i) Each "divergent" sequence of representations of G_i corresponds to an action of G_i on a tree T_i so that G_i does not fix a point in T_i (the "geometric part").

(ii) The action $G_i \curvearrowright T_i$ has to have a global fixed point (the "combinatorial part"). [1]

The "geometric" part of the proof in [CS83], [MS84], [Mor86] was actually algebro-geometric; the geometric approach presented in the book is a version of the geometric approaches of Bestvina [Bes88], Paulin [Pau89] and Chiswell [Chi91]. The intuitive idea is that the ideal triangles in the hyperbolic space "approximately look like" an infinite *tripod*, i.e. the union of three rays with the common origin. If one multiplies the hyperbolic metrics by a very large constant then the "approximation" gets better. In the limit we get a tree.

The "combinatorial part" of Morgan-Shalen's proof [MS88a, MS88b] was actually topological: it was based on analysis of measured laminations in 3-manifolds. It is replaced in this book by more combinatorial *Rips' Theory of group actions on trees*; our discussion follows the paper of Bestvina and Feighn [BF95]. As an alternative to this part of the proof the reader can use either the paper of Paulin [Pau97], which is essentially another version of the Rips' Theory (although many arguments are quite different from the Rips' ideas) or the original papers of Morgan and Shalen. The proof of Skora's theorem (needed in the proof of part (1)) which we present in the book is again an application of the Rips' Theory, our discussion mainly follows Bestvina's paper [Bes97]. Very briefly, using the Rips Theory we transform the action of G_i on T_i to an action of G_i on a *simplicial tree* where each edge stabilizer is cyclic and fixes a point in T_i. Such action corresponds to a decomposition of G_i as an amalgamated free product (or an HNN-extension) over a cyclic subgroup. This gives rise to an *essential cylinder* in M_i. If M_i is acylindrical we get a contradiction. If M_i is not acylindrical then M_i splits along essential cylinders into submanifolds Z_{i_j} (so called JSJ decomposition) and by applying the Rips Theory to each group $\pi_1(Z_{i_j})$ we conclude that it fixes a point in T_i. This is the most difficult part of the proof. On the other hand, the actions of G_1 and G_2 on the trees T_1 and T_2 are related: an element of $\pi_1(S_1)$ fixes a point in T_1 if and only if the corresponding (under the gluing map τ) element of $\pi_1(S_2)$ fixes a point in T_2. Skora's theorem allows to "collect" all the elements of $\pi_1(S_i)$ which fix points in T_i into subsurfaces S_i' so that the gluing map τ carries S_1' to S_2'. If $S_i' \neq S_i$ then we get an incompressible torus in the manifold M by gluing cylinders in M_i whose boundaries are contained in S_i'. Thus both groups $\pi_1(S_i), i = 1, 2$, fix points in T_i. Given this one verifies that both groups G_i fix points in T_i's which is a contradiction: this means that both sequences $[\rho_n : G_i \to PSL(2,\mathbb{C})]$, $i = 1, 2$, are relatively compact in

$$Hom(G_i, PSL(2,\mathbb{C}))/PSL(2,\mathbb{C}).$$

This concludes the part (1) of the proof.

The proof of the part (2) of the bounded image theorem presented in the book is probably similar to the one that Thurston had in mind, although the details of this part of the proof were not discussed in Thurston's preprints and the corresponding part of Morgan's outline is somewhat sketchy:

(α) First one has to show that the representation (or rather a pair of representations) arising as the limit of ρ_n in the part (1) does not have *accidental parabolic elements*, i.e. non-parabolic elements of G_i which are mapped to parabolic elements.

(β) Next, one has to show that the limit group is *geometrically finite.*

The proof of (α) is based on Sullivan's cusps finiteness theorem and the theory of algebraic/geometric convergence of sequences of representations (developed by Jorgensen, Thurston and others). Briefly, each accidental parabolic element corresponds to an essential annulus in M_i, such annuli are glued by τ to an incompressible torus in M which is a contradiction.

The part (β) is based on the theory of ends of hyperbolic 3-manifolds (developed by Thurston, Bonahon and others) and, again, algebraic/geometric convergence.

[1] What was actually proven in [MS88b] is that certain subgroups of the 3-manifold group have global fixed points. To prove that the whole group fixes a point one has to apply a corollary of Skora's theorem [Sko96].

This concludes our sketch of the proof of the *Final Gluing* Theorem. Recall that in our discussion we jumped over all the intermediate steps of induction. The general step of induction is reduced to the *final gluing* via Thurston's "orbifold-trick" outlined by Morgan in [Mor84]. The point of the "trick" is that each member N_i of the Haken hierarchy of M is obtained from N_{i+1} by gluing certain subsurfaces F_{i+1} of ∂N_{i+1}. If $F_{i+1} = \partial N_{i+1}$ then we are in the last step of the induction. Otherwise we have to do something about $\partial N_{i+1} - F_{i+1}$. Thurston's idea is to make $\partial N_{i+1} - F_{i+1}$ "disappear" by putting a hyperbolic *locally reflective orbifold structure* O_{i+1} on this part of the boundary. In this book we construct O_{i+1} by using a small deformation of the existing hyperbolic structure on N_{i+1} and applying Brooks' theorem [Bro86]. Along the way we present a proof of an important theorem of R. Brooks [Bro86] which states that arbitrarily near a geometrically finite subgroup G of $\mathrm{Isom}(\mathbb{H}^3)$ one can always find an isomorphic geometrically finite group G' which embeds in a discrete subgroup $\Gamma \subset \mathrm{Isom}(\mathbb{H}^3)$ such that $\mathbb{H}^3/\Gamma$ has finite volume.

We discuss the orbifold theory in details in Chapters 6, 19. At this moment just note that orbifolds are analogues of manifolds, locally they have the structure of quotients of $\mathbb{R}^n$ by finite isometry groups. Below is the most important for us example of a *locally reflective orbifold.* If M is a smooth manifold and Φ is a group generated by reflections and acting properly discontinuously (but not freely!) on M then M/Φ has a natural orbifold structure. The *underlying space* of this orbifold is the topological quotient $X = M/\Phi$. Roughly speaking, the orbifold structure on X is given by the *singular locus* Σ, which is a stratified subset of X that consists of projections of points in M with nontrivial stabilizers. Each point $x \in \Sigma$ is assigned a subgroup $\Phi_x \subset \Phi$ which stabilizes a point in M which projects to x (defined up to conjugation). Topologically, X is a manifold with boundary, $\partial X = \Sigma$. The top-dimensional strata in Σ are codimension 1 submanifolds in ∂X called *mirrors.* If M is a hyperbolic manifold and Φ acts by isometries then we get a hyperbolic metric on X (so that X has convex boundary) and each stratum of Σ is totally-geodesic. Such orbifold is *right-angled* if all the dihedral angles between faces are $\pi/2$. If M was a manifold with boundary then the orbifold O has boundary as well, ∂O is the projection of the boundary of M. The projection $M \to O$ is called a *manifold cover* of O. The *fundamental group* of O consists of all diffeomorphisms of the universal cover of M which project to the elements of Φ.

By applying Brooks' theorem to the hyperbolic manifold N_{i+1} we get an orbifold O_{i+1} whose underlying space is N_{i+1} and the boundary is F_{i+1}, the rest of ∂N_{i+1} becomes a part of the singular locus of O_{i+1}. To get the hyperbolic orbifold O_{i+1} we deform the given hyperbolic structure on N_{i+1} to a new convex hyperbolic structure so that the part of the boundary corresponding to $\partial N_{i+1} - F_{i+1}$ is the union of totally geodesic mirrors which meet at the right angles. These mirrors (and angles) determine the *locally reflective* orbifold structure O_{i+1}. The hyperbolic metric on O_{i+1} is essentially the *deformed* hyperbolic metric on N_{i+1}.

Then we glue the components of F_{i+1} together and get an orbifold O_i without boundary, the underlying topological space of O_i is N_i. The orbifold O_i is also locally reflective and it admits a finite manifold cover $\tilde{O}_i \to O_i$ so that $\tilde{O}_i$ is an atoroidal Haken 3-manifold which has a hyperbolic metric of finite volume. The hyperbolic structure exists on $\tilde{O}_i$ by the *Final Gluing Theorem* applied to manifolds: $\tilde{O}_i$ is obtained by gluing hyperbolic 3-manifolds $\tilde{O}_{i+1}$ along disjoint closed incompressible surfaces, where the hyperbolic 3-manifolds $\tilde{O}_{i+1}$ are finite covers of the orbifolds O_{i+1} and the hyperbolic metric on $\tilde{O}_{i+1}$ is lifted from O_{i+1}. The orbifold O_i is the quotient of the manifold $\tilde{O}_i$ by a finite group of diffeomorphisms Φ_i (actually isomorphic to $\mathbb{Z}_2^k$, where $k = 2$ since we will use *bipolar* orbifolds). The group Φ_i does not act isometrically, however by applying Mostow rigidity theorem we conclude that the action $\Phi_i \curvearrowright \tilde{O}_i$ is homotopic to an isometric action $\Psi_i \curvearrowright \tilde{O}_i$ of the same group $\mathbb{Z}_2^k$. The orbifolds $O_i = \tilde{O}_i/\Phi_i$ and $\tilde{O}_i/\Psi_i$ have isomorphic *fundamental groups.* To show that O_i and $\tilde{O}_i/\Psi_i$ are homeomorphic we use a generalization (Theorem 6.33) of Johannson-Waldhausen homeomorphism theorem to a certain class of 3-dimensional orbifolds:

this theorem was missing in Morgan's outline. This gives us a hyperbolic structure on O_i; thus we also get a convex hyperbolic structure on the underlying topological space N_i of O_i, which is the required intermediate step of induction i+1→i.

We note here that Thurston's approach (as outlined by Morgan) to the "orbifold-trick" was somewhat different: it was based on Thurston's generalization of Andreev's theorem [Thu81, Chapter 13]. Yet another approach was carried out by Paulin and Otal in the unpublished preprints [Pau92] and [OP96], and published by Otal in [Ota98].

(b) We now discuss the "exceptional case" assuming that M is the mapping torus of a homeomorphism $\tau : S \to S$ and $\partial M = \emptyset$. The assumption that the mapping torus of τ is atoroidal is easily seen to be equivalent to the assumption that τ is *aperiodic*, i.e. for any homotopically nontrivial loop $\gamma \subset S$ and for any $m > 0$ the loops $\gamma, \tau^m(\gamma)$ are not freely homotopic in S.

The proof of existence of a hyperbolic structure on M is different from the *generic* case. In this case one can still try to find a hyperbolic metric on $S \times [0,1]$ so that τ is an isometry between neighborhoods of $S \times \{0\}, S \times \{1\}$ and define a sequence of iterations $[\rho_n] = \sigma^n(X) \in \mathcal{T}(G)$. However it turns out that this sequence is unbounded and the map σ of the Teichmüller space has no fixed point. The proof of the existence of a hyperbolic structure on M breaks in two parts:

Part 1. *The Double Limit Theorem.* One proves that the sequence $\{\rho_n\}$ converges (up to a subsequence) to a representation $\rho_\infty : G = \pi_1(S) \to PSL(2, \mathbb{C})$. This part of the proof was discussed by Thurston in his unpublished preprint [Thu87a]. A different proof was given by Otal in [Ota96], who used group actions on trees. In Chapter 18 I will outline Otal's proof; I omitted only one–but central–ingredient: estimates on the length function. I decided to include this outline since it illustrates the power of the theory of group actions on trees.

Part 2. The representation ρ_∞ is injective and its image is a discrete subgroup according to a theorem of Chuckrow. Let $\Gamma_\infty := \rho_\infty(G)$. The automorphism $\tau_* : G \to G$ induced by τ corresponds to a quasiconformal homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of the extended complex plane. Then one has to show that the limit set of Γ_∞ is $\mathbb{S}^2$. This was proven by McMullen in [McM96] and by Otal in [Ota96]. Once this is done, Sullivan's Rigidity Theorem implies that f is conformal. Since $fgf^{-1} = \tau_*(g)$ for each $g \in \Gamma_\infty$ (we identify G with Γ_∞ using ρ_∞), it follows that the group $\langle f, \Gamma_\infty \rangle$ generated by f and Γ_∞ is isomorphic to $\pi_1(M)$. Such group is discrete (since Γ_∞ is). Hence the hyperbolic manifold $\mathbb{H}^3 / \langle f, \Gamma_\infty \rangle$ is homeomorphic to M (by a theorem of Stallings).

In Chapter 18 I also explain how to reduce the proof of the hyperbolization theorem in the exceptional case (b) to the generic case (a) provided that $\partial M \neq \emptyset$. The key part of this reduction is Thurston's theorem 2.1 which allows to construct (under an appropriate assumption on the 1-st Betti number of M) an incompressible surface S in M which is not a fiber in a fibration over the circle. The assumption $\partial M \neq \emptyset$ is used to construct a finite covering $M' \to M$ such that $H_2(M', \partial M') > 1$. It is conjectured that such covering exists even if M has empty boundary.

Historical Remarks.[2] Even the idea of something like Thurston's Hyperbolization Theorem was quite astounding in the 1970's when Thurston had first announced his theorem. Nevertheless, it had some historic precursors:

(a) *Andreev's Theorem* [And70], [And71b], where Andreev proves an analogue of the hyperbolization theorem in the case of reflection orbifolds whose underlying space is the closed 3-ball.

(b) *Marden's paper* [Mar74], where in section 11.3 Marden suggests to use Haken hierarchy to analyze Kleinian groups and in section 13.2 (Question 2) asks for necessary and sufficient conditions for a 3-manifold to be hyperbolic and notes that it is necessary to assume that the manifold is irreducible and its fundamental group has trivial center.

[2]These and other historical remarks scattered throughout the book are not meant to present a *complete* version of history of Thurston's Hyperbolization Theorem and of various ingredients of its proof.

(c) *Riley's work* [Ril75], where Riley conjectures the hyperbolization theorem for knot complements in $\mathbb{S}^3$ and gives some "experimental" evidence towards this conjecture.

Thurston's hyperbolization theorem gradually became accepted (at least among experts) as a mathematical fact through the 1980's, however Thurston never wrote a complete proof[3] of his theorem. In [Thu94] Thurston describes his reasons for not writing a proof. Thurston wrote a general introduction to the proof and to the program of geometrization of 3-manifolds in [Thu82]. An outline of Thurston's proof was presented by J. Morgan in [Mor84]. Thurston's lecture notes [Thu81] develop a part of the technique required in the proof (ends of hyperbolic manifolds, strong convergence, generalized Andreev's theorem, etc.). In the mid-1980's Thurston wrote several papers and preprints [Thu86a, Thu87a, Thu87b] in which he fills in some major pieces of the proof of the part (1) of the Bounded Image Theorem and of the fiber bundle case of the Hyperbolization Theorem that are missing in Morgan's outline [Mor84]. In the same time, an alternative proof of most of the part (1) of the Bounded Image Theorem was published by J. Morgan and P. Shalen [MS84, MS88a, MS88b]. Morgan and Shalen used the theory of group actions on trees: the missing ingredient for completion of the part (1) of the proof was *Skora's duality theorem*. Shalen in [Sha87] stated several questions (most importantly, Question D) concerning generalization of the work of Morgan and Shalen; as we will see, one needs the affirmative answer to a *relative version* of Shalen's Question D to replace [MS88b] as a part of the proof of the hyperbolization theorem. Skora's theorem which we mentioned above was also conjectured by Shalen in [Sha87]. R. Skora [Sko96] proved his theorem in 1990, but it remained unpublished for 6 years. A combinatorial replacement to [MS88b] was provided meanwhile by E. Rips in the early 1990's. Rips' work was motivated in part by Shalen's questions in [Sha87] and in part by the work of G. Makanin [Mak82] and A. Razborov [Raz84] on solution of equations in groups. Rips' work still remains unpublished, however several written accounts became available thanks to the work of M. Bestvina and M. Feighn [BF95] and D. Gaboriau, G. Levitt and F. Paulin (see [Pau97]). On the author's request Bestvina and Feighn also proved in [BF95] a relative version of the Rips' theorem which is required in this book.

On the other hand, works of F. Bonahon [Bon86], R. Canary and Y. Minsky [Can96], [Min94a], [CM96] and K. Ohshika [Ohs92, Ohs98] considerably clarified and generalized Thurston's theory of ends of hyperbolic manifolds and strong convergence of sequences of representations (which is required for the part (2) of the Bounded Image Theorem). In particular, Bonahon in [Bon86] streamlined a major part of the proof of the part (2β) of the Bounded Image Theorem by proving Thurston's conjecture on *tameness of ends* of hyperbolic 3-manifolds with freely indecomposable fundamental groups. Originally, Thurston was proving this conjecture for discrete groups which are limits of geometrically finite groups (the only case required for the proof of the part (2)) for which he greatly extended Jorgensen's theory of algebraic/geometric convergence. This is partially presented in Thurston's lecture notes [Thu81]. Bonahon's theorem made the corresponding part of the proof of (2β) somewhat obsolete, however it did not completely eliminate the theory of algebraic/geometric limits from the proof.

Completely different (and much shorter) *analytical* proof of the Bounded Image Theorem was given by C. McMullen in 1989, [McM89]. McMullen's proof was based on estimates of the norm of the derivative of the map σ. A somewhat different version of McMullen's proof was given in 1996 by D. Barrett and J. Diller in [BD96].

Thurston's "orbifold–trick" for the reduction of the *general step* of induction in the proof of the hyperbolization theorem to the *last step* was modified considerably by F. Paulin (following a suggestion of F. Bonahon to use right-angled orbifolds) in his 1992 preprint [Pau92]. Unaware

[3]One may argue here about the definition of the word *proof*. The definition which I use requires a proof to be a publicly available written document.

of Paulin's preprint I decided to use the right angled orbifolds but for a reason different from Paulin's. In this book the right angles come naturally from the proof of the Brooks' theorem, it is also somewhat easier to prove the homeomorphism theorem for orbifolds assuming that all the angles are $\pi/2$. In [Pau92, OP96, Ota98] the right angles seem to appear because in this case it is easy to find manifold covers of such orbifolds and using right angled orbifolds one can avoid Thurston's generalization of Andreev's theorem and apply Andreev's theorem directly.

A complete proof of the hyperbolization theorem in the case (b) (manifolds fibered over the circle) was published by J.-P. Otal [Ota96]. Otal's proof was quite different from the one proposed by Thurston; one of the key ingredients in Otal's proof is the above-mentioned theorem of R. Skora which establishes duality between small surface group actions on trees and measured geodesic laminations. Skora's theorem could be considered as a deep generalization of the Poincaré duality for closed hyperbolic surfaces S in the following sense. Homology classes (represented by real linear combinations of simple loops) are dual to the cohomology classes (represented by homomorphisms from the fundamental group to $\mathbb{R}$). The space of measured geodesic laminations on S is a certain completion of the collection of simple closed geodesics on S (weighted by positive real numbers). The space of small $\pi_1(S)$ actions on trees T could be identified with a certain space of maps $\pi_1(S) \to \mathbb{R}$, each given by the translation length function $g \mapsto \ell_T(g)$.

Earlier, the case (b) of the hyperbolization theorem was discussed by D. Sullivan [Sul81c] and by Thurston [Thu87a].

The present book assembles various pieces of the proof of the Hyperbolization Theorem in the generic case following Morgan's outline [Mor84] of the original (geometric) approach of Thurston, with tree-theoretic replacements of Thurston's convergence theorems in the part (1). The way this is done in this book is by no means original and was probably known to many of those familiar with Thurston's "blueprint" [Mor84] and the main "building blocks" of the proof that became available by the early 1990's. Otal's paper [Ota98] also treats the generic case of the hyperbolization theorem, where the hyperbolic manifold is assumed to have empty boundary. Otal's approach is analytical; it is based on McMullen's proof of the Bounded Image Theorem.

Summary.

This book contains essentially no new results, I preferred to use whatever I could find in the existing literature. I give sketches of the proofs (or just the references) mostly in the cases when I was comfortable with the proofs that are already published. I tried to give proofs if they seemed to me more transparent than those in the standard references, if they are unpublished, if they are not *very* long but are central for our discussion (like Theorems 2.1 and 14.24, etc.). There are only three theorems in this book that seem to be relatively new: I include them since the proofs are not difficult and they provide good illustration to the subject. These are:

(i) Theorem 8.44 on smoothness of the representation varieties of finitely generated Kleinian groups. (ii) Theorem 19.6, which states that every finitely generated Kleinian group G in $PSL(2,\mathbb{C})$ is isomorphic to a geometrically finite group (this is more or less a straightforward application of Thurston's Hyperbolization Theorem, the only problem is torsion in G). (iii) An analogue of the Johannson–Waldhausen homeomorphism theorem for a class of 3-dimensional orbifolds, Theorem 6.33. I could not find any reasonably general homeomorphism theorem for orbifolds in the existing literature; the papers of Takeuchi [Tak88, Tak91] fall somewhat short of what we need for the Hyperbolization Theorem. I found it easier to prove Theorem 6.33 directly than to derive it from Takeuchi's papers.

Otherwise, most of the work done in this book is to assemble the results that are already known in one or another form. There are several deep theorems about 3-manifolds and discrete groups that we will use but their proofs are complicated enough to force me to omit even sketches of the proofs: Haken hierarchy theorem, Waldhausen's homeomorphism theorem for 3-manifolds, Sullivan's Rigidity Theorem and Bonahon's theorem about ends of hyperbolic manifolds. The experts will notice that most of the material of the book is an "introduction" to one or another subject. The actual proof starts in Chapter 15, thus the reader familiar with the preliminary material can start reading the proof of the Hyperbolization Theorem from this chapter. Below is the brief description of the material of each chapter.

Chapter 1: Three-dimensional Topology. We discuss here some basic facts about 3-dimensional manifolds, like the Sphere and Loop theorems, the Dehn Lemma, incompressible surfaces, Haken hierarchy, Seifert manifolds, JSJ decomposition, etc. There are several good books on this subject, so I skipped most of the proofs.

Chapter 2: Thurston Norm. We prove here Thurston's existence theorem for incompressible surfaces which are not fibers in a fibration over $\mathbb{S}^1$. Our discussion mainly follows Thurston's paper [Thu85]. We will use material of this chapter in the chapter 18 to reduce (under extra assumptions) the case (b) of the hyperbolization theorem to the generic case (a). Thurston's paper [Thu85] was probably motivated by such reduction.

Chapter 3: Geometry of the Hyperbolic Space. This is perhaps the most eclectic chapter of the book: I include here basic material on the geometry of spaces of nonpositive and negative curvature as well as the geometry of the hyperbolic space $\mathbb{H}^n$ itself. We introduce here quasi-isometries and quasi-geodesics and prove *stability theorem* for quasi-geodesics in the hyperbolic space: any quasi-geodesic is within bounded distance from a geodesic. This is used (following Mostow) to establish a relation between quasi-isometries of $\mathbb{H}^n$ and quasiconformal mappings of the sphere $\mathbb{S}^{n-1}$. This is an important part of the proof of Mostow rigidity theorem.

Chapter 4: Kleinian Groups. This chapter contains mostly *pre-Thurston* material: results which in some form were known before Thurston came to the field. We cover several very important *pre-Thurston* theorems on the structure of Kleinian groups: Ahlfors finiteness theorem and its companions (Sullivan finiteness theorem, Scott compact core theorem etc.), Klein and Maskit Combination Theorems, Kazhdan-Margulis-Zassenhaus Theorem and characterization of geometrically finite Kleinian groups.

Chapter 5: Teichmüller Theory of Riemann Surfaces. Since there are several excellent books on the Teichmüller Theory, I have tried to give only a bare minimum of the material. We discuss basic properties of quasiconformal mappings of the complex plane, define the Teichmüller spaces, the mapping class group and metrics on the Teichmüller spaces. We also consider here finite subgroups of the mapping class group.

Chapter 6: Introduction to the Orbifold Theory. We start with the basic definitions and examples of orbifolds. Then we introduce a special class of 3-dimensional orbifolds: *all right orbifolds of zero Euler characteristic.* This class is a generalization of the class of atoroidal Haken manifolds of zero Euler characteristic. For this class of orbifolds we prove that an isomorphism of the fundamental groups is always induced by a homeomorphism.

Chapter 7: Complex Projective Structures. These structures are special coordinate coverings of Riemann surfaces where the transition maps belong to $PSL(2, \mathbb{C})$. They provide a useful and important generalization of the Kleinian groups. I include this chapter mostly because I like the subject, however we will use the Holonomy Theorem as a technical tool in the discussion of *strong convergence.*

Chapter 8: Sociology of Kleinian Groups. This chapter is about the "collective behavior" of Kleinian groups in *families* (deformations, algebraic and geometric convergence) and *pairs* (rigidity theorems and realization of isomorphisms by quasiconformal mappings). We discuss various results on algebraic, geometric and strong convergence of sequences of Kleinian groups. We prove the Mostow Rigidity Theorem so that the proof gives us extra information on non-smoothness of quasiconformal mappings conjugating Kleinian groups. (This is used in proving the Hyperbolization Theorem.) In this chapter we prove smoothness of character varieties of Kleinian groups and Bers' theorem about isomorphism between the Teichmüller space of a Kleinian group and the Teichmüller space of the associated Riemann surface. We then prove the Ahlfors' finiteness theorem. I also discuss Douady–Earle *barycentric* extension of homeomorphisms of the unit circle and explain how to justify Poincare's continuity method for proving the uniformization theorem in the case of punctured spheres.

Chapter 9: Ultralimits of Metric Spaces. This chapter is almost entirely taken from my joint paper with B. Leeb [KL95]. Ultralimits were introduced to the field by two logicians: L. Van den Dries and A. Wilkie, who used them to give an alternative proof of Gromov's theorem on groups of polynomial growth [VW84]. It appears that I. Chiswell [Chi91] was the first to realize that ultralimits provide a convenient formalism for construction of group actions on trees from divergent sequences of group representations into the isometry groups of (Gromov) hyperbolic spaces. To define ultralimits one needs the Axiom of Choice. If you do not believe in this axiom, you can use for instance [Mor86], [Bes88] or [Pau88] as an alternative.

Chapter 10: Introduction to Group Actions on Trees. Here we show how to use actions of groups on trees to compactify representation varieties. I also give proofs of various elementary facts about group actions on trees.

Chapter 11: Laminations, Foliations and Trees. Most of the material in this chapter was introduced by Thurston (part of it was implicit in the earlier works of Dehn and Nielsen). This material is important for understanding of Thurston's and Otal's approaches to hyperbolization of manifolds fibered over $\mathbb{S}^1$. The reader who is not interested in the hyperbolization of fibrations can skip most of this chapter, except for the formulation of Skora's Duality Theorem (Theorem 11.31). We discuss the relation between several essentially equivalent concepts: measured foliations, measured geodesic laminations, train tracks and small surface group actions on trees. We prove Thurston's characterization theorem for pseudo-Anosov homeomorphisms of surfaces and describe Thurston's compactification of the Teichmüller space by the space of projective classes of measured laminations using the approach of Morgan and Shalen.

Chapter 12: The Rips' Theory. The deepest part of the original Thurston's proof of the Hyperbolization Theorem was certain compactness theorem for sequences of representations. The most efficient (at the present time) way to prove such compactness results is via the *Rips Theory.* In this chapter we also use the Rips' Theory to prove Skora's Duality Theorem.

Here is a brief outline of this chapter. The action of a finitely presented group G on a tree T corresponds to a foliation $\mathcal{F}$ on a 2-dimensional complex Y. The complex Y consists of foliated Euclidean rectangles, called *bands* which are attached to a graph along the edges (called *bases*) transversal to the foliation. $\pi_1(Y)$ is thus free and to get a 2-complex X with the fundamental group G we add a finite number of 2-cells to Y (which correspond to the relations in the fundamental group). The complex Y breakes into a union of subcomplexes of *simplicial* and *pure* type. Roughly speaking, simplicial components correspond to simplicial subtrees in T invariant under appropriate subgroups of G; each leaf of the foliation in the simplicial part of $(Y, \mathcal{F})$ is compact. Each leaf of a pure component of $(Y, \mathcal{F})$ is dense in this component. The *Rips Machine* transforms each *pure component* to a foliated complex of one of the three "model" types: *surface type*, *axial type*, and *thin type* (the 2-cells of X are transformed as well). Each component of the *surface type* is essentially a surface with boundary which is given a measured foliation. The fundamental group of each component of the *axial type* preserves a geodesic in the tree T. Components $C_i \subset Y$ of *thin type* are more difficult to describe. I do it here assuming that G acts freely on T. Briefly, the Machine transforms each *band* in C_i into a union of bands which are arbitrarily thin and long. These bands have the remarkable property that some of them meet the 2-cells of $X - Y$ only along *bases*. If we cut the (transformed) complex X across any such thin band we get a decomposition of G as a free product. One then has to check that this decomposition is nontrivial. In general, even if the action $G \curvearrowright T$ is not free each thin component causes a nontrivial splitting of G over a subgroup stabilizing an arc in T. If the action of G on T is *small*, i.e. arc stabilizers are virtually nilpotent, then presence of any *pure component* in X corresponds to a decomposition of G over a virtually solvable subgroup. The same happens if we have simplicial components. Such decomposition is impossible for instance in the case of the fundamental group of a compact acylindrical atoroidal Haken 3-manifold M with incompressible boundary, thus $G = \pi_1(M)$ would have to fix a point in T. If M is not acylindrical then we get a relative version of the fixed point theorem: the manifold M splits along essential cylinders and Moebius bands so that the fundamental group of each component Z of the decomposition either fixes a point in T or Z is an interval bundle over a surface. These absolute and relative fixed point theorem were originally proven by Morgan and Shalen in [MS88a, MS88b].

Chapter 13: Brooks' Theorem and Circle Packings. The central part of this chapter is a theorem of R. Brooks about deforming geometrically finite subgroups of $PSL(2, \mathbb{C})$ to subgroups of lattices in $\mathrm{Isom}(\mathbb{H}^3)$. A version of this theorem is used to prove existence of geometrically finite hyperbolic structures on orbifolds of *finite type* which admit finite circle packings.

Chapter 14: Pleated Surfaces and Ends of Hyperbolic Manifolds. Thurston invented this area as a technical tool for proving the Hyperbolization Theorem. Instead of *pleated surfaces*, which were introduced by Thurston, we will use their combinatorial counterparts: *singular pleated surfaces*, which are much easier to handle (I believe that they were invented by Bonahon). The central result of this chapter is Thurston's Covering Theorem (Theorem 14.24): if $p : N \to M$ is a locally isometric covering between complete hyperbolic 3-manifolds and E is a *geometrically tame* end of N, then the restriction of p to E has finite multiplicity. This theorem is absolutely critical for proving part (2) of the Bounded Image Theorem. It allows to show that certain subgroups of the fundamental groups of hyperbolic 3-manifolds are geometrically finite.

Chapter 15: Outline of the Proof of the Hyperbolization Theorem. This and the following two chapters mostly follow Morgan's paper [Mor84], I use the results about group actions on trees as a substitute of the compactness results of Thurston. We break the proof in several steps and in the following chapters we fill in the details. There are two different cases: the "generic" Case (a) and the "exceptional" Case (b) of manifolds fibered over $\mathbb{S}^1$. Case (a) is discussed in chapters 16 and 17. Case (b) is discussed in chapter 18.

Chapter 16: Reduction to the Bounded Image Theorem. This is a slight modification of a part of Morgan's paper [Mor84]. I have changed several details which I did not like or did not understand. The existence theorem for a hyperbolic metric is reduced to a certain Fixed-Point Theorem (similarly to the fact that the fixed-point set of a map $\sigma : \mathcal{T}(G) \to \mathcal{T}(G)$ is the intersection of the graph of σ with the diagonal). Standard proofs of such Fixed-Point Theorems require strict contraction property for σ. For σ strict contraction fails, however we check here that σ is a contraction.

Chapter 17: The Bounded Image Theorem. This is the central part of the proof of the Hyperbolization Theorem: the sequence of iterations $\sigma^j(X)$ is relatively compact.

Chapter 18: Hyperbolization of Fibrations. In this chapter we discuss the exceptional case of the Hyperbolization Theorem (case (b)). Our discussion essentially follows Otal's book [Ota96]. In the same chapter I also give a short alternative proof of the Hyperbolization Theorem for manifolds fibered over the circle assuming that the manifold either has nonempty boundary or admits a finite covering with the 1-st Betti number ≥ 2 (conjecturally such covering always exists). The proof is by reduction of the case (b) to the generic case (a) using the results of Chapter 2.

Chapter 19: The Orbifold Trick. I explain how to glue the *all right* orbifolds of zero Euler characteristic from the orbifolds of *finite type*. The latter are similar to the class of acylindrical, atoroidal Haken 3-manifolds. Given a geometrically finite hyperbolic structure on orbifolds of finite type we will construct hyperbolic structures of finite volume on *all right* orbifolds of zero Euler characteristic. We then use Brooks' Theorem to finish the induction argument in the proof of the hyperbolization theorem.

Chapter 20: Beyond the Hyperbolization Theorem. The hyperbolization theorem is a part of Thurston's program for "geometrizing" 3-manifolds. In this chapter I describe Thurston's Geometrization Conjecture (which, among other things, implies Poincare Conjecture), collect various conjectures related to the Geometrization Conjecture and discuss several possible approaches to this conjecture. I also discuss subjects related to Thurston's hyperbolization theorem: higher-dimensional negatively curved manifolds, general geometric structures on 3-manifolds, and hyperbolic groups. The choice of subjects was mostly motivated by my personal mathematical taste and knowledge; the most serious omissions perhaps are the dynamics of holomorphic functions in the complex plane and the theory of circular packings.

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Contents

1 Three-dimensional Topology **1**
- 1.1 Basic definitions and facts . . . 1
- 1.2 Incompressible surfaces . . . 3
- 1.3 Existence of incompressible surfaces . . . 5
- 1.4 Homeomorphisms of Haken manifolds . . . 7
- 1.5 Pared 3-manifolds . . . 8
- 1.6 Seifert manifolds . . . 9
- 1.7 The decomposition theorem . . . 10
- 1.8 Characteristic submanifolds . . . 11
- 1.9 3-manifolds fibered over $\mathbb{S}^1$ . . . 13
- 1.10 Baumslag-Solitar relations . . . 14

2 Thurston Norm **18**
- 2.1 Norms defined over $\mathbb{Z}$ . . . 18
- 2.2 Variation of fiber-bundle structure . . . 20
- 2.3 Application to incompressible surfaces . . . 21

3 Geometry of the Hyperbolic Space **25**
- 3.1 General definitions and notation . . . 25
- 3.2 $CAT(\lambda)$-spaces . . . 26
- 3.3 Basic properties of the hyperbolic space . . . 28
- 3.4 Models of the hyperbolic space . . . 32
- 3.5 Isometries of the hyperbolic space . . . 33
- 3.6 The convergence property . . . 35
- 3.7 Convex polyhedra in the hyperbolic space . . . 37
- 3.8 Cayley graphs of finitely generated groups . . . 38
- 3.9 Quasi-isometries . . . 39
- 3.10 Quasiconformal mappings . . . 43
- 3.11 Distortion of distance by quasiconformal maps . . . 45
- 3.12 Harmonic functions on the hyperbolic space . . . 45

4 Kleinian Groups **47**
- 4.1 Nilpotent groups . . . 47
- 4.2 Residual finiteness and Selberg lemma . . . 49
- 4.3 Representation varieties . . . 50
- 4.4 Cohomology of groups and sheaves . . . 53
- 4.5 Group cohomology and representation varieties . . . 56

4.6 Basics of discrete groups . . . 59
4.7 Properties of limit sets . . . 61
4.8 Quotient spaces of discrete groups . . . 62
4.9 Conical and parabolic limit points . . . 63
4.10 Fundamental domains . . . 64
4.11 Poincaré theorem on fundamental polyhedra . . . 66
4.12 Kazhdan-Margulis-Zassenhaus theorem . . . 68
4.13 Geometry of Margulis tubes and cusps . . . 71
4.14 Geometrically finite groups . . . 75
4.15 Criteria of geometric finiteness . . . 76
4.16 Kleinian groups and Riemann surfaces . . . 77
4.17 The convex hull and the domain of discontinuity . . . 80
4.18 The combination theorems . . . 83
1 Klein combination . . . 83
2 Maskit combination . . . 85
4.19 Ahlfors finiteness theorem . . . 89
4.20 Extensions of the Ahlfors finiteness theorem . . . 91
4.21 Limit sets of geometrically finite groups . . . 93
4.22 Groups with Kleinian subgroups . . . 94
4.23 Ends of hyperbolic manifolds . . . 94

5 Teichmüller Theory of Riemann Surfaces 97
5.1 Tensor calculus on Riemann surfaces . . . 97
5.2 Properties of quasiconformal maps . . . 99
5.3 Geometry of quadratic differentials . . . 101
5.4 Teichmüller spaces of Riemann surfaces . . . 102
5.5 The Weil-Petersson metric on $\mathcal{T}(S)$. . . . 106
5.6 Torsion in Mod_S . . . 107

6 Introduction to the Orbifold Theory 109
6.1 Definitions and examples . . . 109
6.2 2-dimensional orbifolds . . . 114
6.3 3-dimensional locally reflective orbifolds . . . 117
6.4 Glossary of orbifolds: the good, the bad and 119
6.5 A homeomorphism theorem for 3-orbifolds . . . 125

7 Complex Projective Structures 129
7.1 Basic definitions . . . 129
7.2 Grafting . . . 133

8 Sociology of Kleinian Groups 135
8.1 Algebraic convergence of representations . . . 135
8.2 Geometric convergence . . . 137
8.3 Isomorphisms of geometrically finite groups . . . 140
8.4 Douady-Earle extension . . . 142
8.5 Mostow rigidity theorem . . . 148
8.6 Sullivan rigidity theorem . . . 151
8.7 Bers isomorphism . . . 151
8.8 Smoothness of representation varieties . . . 154

8.9 Applications to quasiconformal stability . . . 160
8.10 Calabi-Weil infinitesimal rigidity theorem . . . 162
8.11 Space of quasifuchsian representations . . . 162
8.12 Distortion of the translation length . . . 163
8.13 Proof of the recurrence theorem . . . 164
8.14 Proof of the Ahlfors finiteness theorem . . . 164
8.15 A generalization of the Bers' isomorphism . . . 168
8.16 Totally degenerate groups . . . 169
8.17 Algebraic topology versus geometric topology . . . 171
8.18 Justification of the Poincaré's continuity method . . . 172

9 Ultralimits of Metric Spaces **175**
9.1 Ultrafilters . . . 175
9.2 Ultralimits of metric spaces . . . 176
9.3 The asymptotic cone of a metric space . . . 179

10 Introduction to Group Actions on Trees **181**
10.1 Basic definitions and properties . . . 181
10.2 Actions on simplicial trees . . . 185
10.3 Limits of isometric actions on $CAT(0)$-spaces . . . 188
10.4 Compactification of character varieties . . . 188
10.5 Proper actions . . . 191

11 Laminations, Foliations and Trees **193**
11.1 The Euclidean motivation . . . 193
11.2 Geodesic currents . . . 195
11.3 Measured foliations on hyperbolic surfaces . . . 196
11.4 Interval exchange transformations . . . 200
11.5 Train tracks . . . 202
11.6 Measured geodesic laminations . . . 203
11.7 Topology on measured laminations . . . 208
11.8 From foliations to laminations . . . 209
11.9 From laminations to foliations . . . 210
11.10 Action of Mod_S on geodesic laminations . . . 211
11.11 The geometric intersection number . . . 212
11.12 From laminations to trees . . . 212
11.13 An application of Skora's theorem . . . 214
11.14 A characterization of aperiodic homeomorphisms . . . 215
11.15 Dynamics of aperiodic homeomorphisms . . . 219
11.16 A compactification of the Teichmüller space . . . 220

12 The Rips' Theory **223**
12.1 Stable trees . . . 224
12.2 Unions of bands and band complexes . . . 225
12.3 Pushing . . . 227
12.4 Transversal measure on union of bands . . . 229
12.5 Dynamical decomposition of unions of bands . . . 230
12.6 Band complexes . . . 231
12.7 Holonomy of vertical paths in X . . . 235

12.8 Kazhdan-Margulis theorem for actions on trees . . . 236
12.9 The moves . . . 239
12.10 Preliminary motions of the Rips machine . . . 244
12.11 The machine . . . 244
12.12 The machine output . . . 250
12.13 Proof of the decomposition theorem . . . 255
12.14 Proof of Skora's duality theorem . . . 256
12.15 Geometric actions on trees . . . 257
12.16 Nongeometric actions on trees . . . 260
12.17 Compactness of representation varieties . . . 262

13 Brooks' Theorem and Circle Packings 266
13.1 Orbifolds and patterns of circles . . . 267
13.2 Brooks' Theorem . . . 270
13.3 A packing invariant of patterns of disks . . . 271
13.4 A packing invariant of Kleinian groups . . . 276
13.5 Proof of the Brooks' theorem . . . 279

14 Pleated Surfaces and Ends of Hyperbolic Manifolds 281
14.1 Singular hyperbolic metrics . . . 281
14.2 Existence theorem for singular pleated maps . . . 283
14.3 Compactness theorem for pleated maps . . . 284
14.4 Geometrically tame ends of hyperbolic manifolds . . . 285
14.5 Ending laminations . . . 288
14.6 Infinite coverings of hyperbolic manifolds . . . 289
14.7 Geometric profile of algebraic convergence . . . 292

15 Outline of the Proof of the Hyperbolization Theorem 295
15.1 Case A: The generic case . . . 297
15.2 Case B: Manifolds fibered over the circle . . . 300

16 Reduction to the Bounded Image Theorem 302
16.1 Step 1. The Maskit combination . . . 302
16.2 Step 2. Formulation of the fixed–point theorem . . . 302
16.3 Step 3. Proof of the contraction theorem . . . 304
16.4 Step 4. End of the proof of the fixed–point theorem . . . 305

17 The Bounded Image Theorem 306
17.1 Limit exists . . . 306
17.2 Geometry of the limit . . . 307
1 Preliminaries and notation . . . 307
2 Idea of the proof . . . 311
3 The limit has no accidental parabolics . . . 312
4 The endgame: the limit is geometrically finite . . . 314

18 Hyperbolization of Fibrations **316**
18.1 Compactness theorem for aperiodic homeomorphisms . . . 316
18.2 The double limit theorem: an outline . . . 317
18.3 Proof of Theorem 15.17 . . . 318
18.4 An alternative approach . . . 319

19 The Orbifold Trick **320**
19.1 Manifold coverings of bipolar orbifolds . . . 320
19.2 Building hyperbolic orbifolds of finite type . . . 321
19.3 Gluing orbifolds of zero Euler characteristic . . . 325
19.4 Hyperbolization of locally reflective orbifolds . . . 327
19.5 Hyperbolic design . . . 327

20 Beyond the Hyperbolization Theorem **331**
20.1 Thurston's geometrization conjecture . . . 331
20.2 Other structures . . . 337
20.3 Higher dimensions . . . 339
20.4 Hyperbolic groups . . . 340