Hyperbolic Manifolds and Discrete Groups

Lectures on Thurston's Hyperbolization

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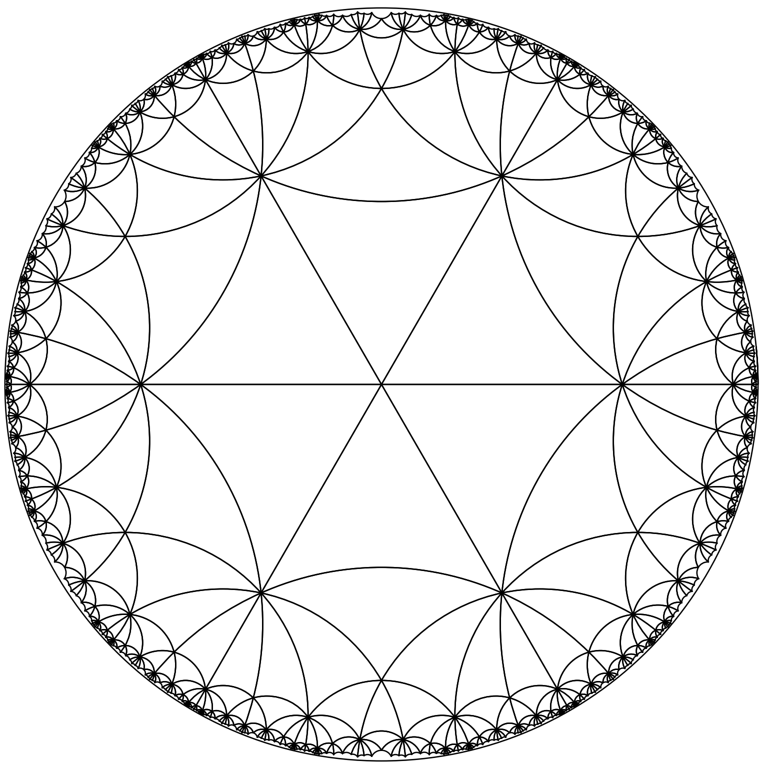

Preface

This book is based upon the course "Hyperbolic Manifolds and Discrete Groups" that I was teaching at the University of Utah during the academic year 1993-94. The main goal of the book is to present a proof of

Thurston's Hyperbolization Theorem. ("The Big Monster".) *Suppose that M is a compact atoroidal Haken 3-manifold, which has zero Euler characteristic. Then the interior of M admits a complete hyperbolic metric of finite volume.*

This theorem establishes a strong link between the geometry and topology of 3-manifolds and the algebra of discrete subgroups of $\mathrm{Isom}(\mathbb{H}^3)$. It completely changed the landscape of 3-dimensional topology and theory of Kleinian groups. This theorem allowed to prove things that were beyond the reach of the standard 3-manifold technique like Smith's Conjecture, residual finiteness of the fundamental groups of Haken manifolds, etc. In this book I present a complete proof of the Hyperbolization Theorem in the "generic case". Initially I was planning to include the detailed proof in the remaining case of manifolds fibered over $\mathbb{S}^1$ as well. However since Otal's book [Ota96] (which treats the fiber bundle case) became available, I will give in this book only a sketch of the proof in the fibered case.

The proof of the Hyperbolization Theorem is by induction on the steps of the *Haken decomposition* of M along incompressible surfaces. The *members of the Haken hierarchy* are manifolds $M = N_0, N_1, N_2, N_3, ..., N_h$ where each N_i is obtained by splitting N_{i-1} along a superincompressible surface and N_h is a disjoint union of 3-balls. There are two cases:

(a) The "generic case" when the decomposition of M starts with an incompressible surface S which is not a *virtual fiber* and thus the manifold N_1 is not an interval bundle (or a disjoint union of two interval bundles) over a surface.

(b) The "exceptional case" when N_1 is an interval bundle over a surface (or a disjoint union of two such bundles). The most important example is when M fibers over the circle with the fiber S, i.e. M is the mapping torus of a homeomorphism $\tau : S \to S$.

Below is a sketch of the proof in the case (a) under the assumption that M has empty boundary.

The first step of induction: each component of N_h (which is a closed 3-ball) admits a hyperbolic structure (just take a ball in the hyperbolic 3-space). We skip for a moment all the intermediate steps of the induction and consider the "last step of induction" (the *final gluing*) which turns out to be the heart of the proof.

The final gluing. Assume that M is a closed atoroidal orientable Haken manifold and $S \subset M$ is an incompressible surface which separates M into compact components M_1, M_2, each of which admits a hyperbolic metric and is not homotopy-equivalent to a surface (thus $N_1 = M_1 \sqcup M_2$). Let $S_i := \partial M_i$ and let τ denote the gluing mapping. We would like to find hyperbolic metrics g_1, g_2 on M_1, M_2 so that:

The gluing map τ is homotopic to an isometry $f : Nbd(S_1) \to Nbd(S_2)$ between product neighborhoods of the surfaces S_1, S_2 which sends S_1 to the surface in $\partial Nbd(S_2)$ which is different from S_2.

Once this is done, we can glue the manifolds (M_1, g_1), (M_2, g_2) via the isometry f and get a hyperbolic structure on the manifold M.

Of course, there will be an obstruction to the isometric gluing (i.e. to the existence of the metrics g_1, g_2 as above), the goal is to show that this obstruction is an incompressible torus $T \subset M$, which thus corresponds to collections of disjoint cylinders in M_1 and M_2. Therefore if (say) the manifold M_1 is *acylindrical* then the isometric gluing is unobstructed no matter what τ is (since there are no incompressible tori). Instead of the hyperbolic metrics we will try to find their *holonomy representations* $\rho_i : \pi_1(M_i) \to PSL(2, \mathbb{C})$. The homomorphisms ρ_1 and ρ_2 have to be chosen to form a commutative diagram:

$$
\begin{tikzcd}
 & \pi_1(M_1) \arrow[dr] & \\
\pi_1(S) \arrow[ur] \arrow[dr] & & PSL(2,\mathbb{C}) \\
 & \pi_1(M_2) \arrow[ur] &
\end{tikzcd}
$$

Once such ρ_i's are found, they induce a homomorphism $\rho : \pi_1(M) \to PSL(2, \mathbb{C})$ (part (1) of the proof). To conclude that ρ is the holonomy of a hyperbolic structure on M (i.e. that one can glue the corresponding metrics along neighborhoods of S_1, S_2) one has to prove that ρ_1, ρ_2 satisfy some further conditions: a combination theorem of Maskit (part (2) of the proof). Roughly speaking, these conditions require that the hyperbolic manifolds with boundary $(M_1, g_1), (M_2, g_2)$ embed isometrically as deformation retracts in compact hyperbolic manifolds $(M_1', g_1'), (M_2', g_2')$ which have convex boundary.

We now give more details. Let $G_i = \pi_1(M_i)$, $i = 1, 2$, $G := (G_1, G_2)$. Thurston's idea is to reduce the problem of finding the metrics g_1, g_2 to the *fixed-point problem* for a certain map σ of the Teichmüller space $\mathcal{T}(G) = \mathcal{T}(G_1) \times \mathcal{T}(G_2)$ and then to prove existence of a fixed point. As in many fixed point theorems, one tries to find this fixed point as the limit of a sequence of iterations $\sigma^n(X) = [\rho_n] \in \mathcal{T}(G)$. The spaces $\mathcal{T}(G_1), \mathcal{T}(G_2)$ are spaces of hyperbolic structures with convex boundary on the manifolds M_1, M_2: they are complete locally compact metric spaces. The mapping σ is a contraction:

$$d(\sigma(p), \sigma(q)) < d(p, q), \text{ unless } p = q.$$

The key part of the proof of the existence of the fixed point is the *Bounded Image Theorem*: it establishes relative compactness of the sequence $[\rho_n]$

in the Teichmüller space $\mathcal{T}(G)$. Algebraically, Teichmüller spaces $\mathcal{T}(G_i)$ correspond to equivalence classes of representations $G_i \to PSL(2, \mathbb{C})$ which are induced by quasiconformal homeomorphisms of the 2-sphere. The proof of precompactness breaks in two parts: (1) the proof of existence of a pair of limiting representation $G_i \to PSL(2, \mathbb{C})$, (2) the proof of the fact that these representations are induced by quasiconformal homeomorphisms.

Thurston's idea of the proof of (1) was based on a detailed study of the geometry of pleated surfaces in hyperbolic manifolds, most of it was presented by Thurston in the paper [Thu86a] and in the unpublished preprints [Thu87a], [Thu87b]. Instead of this approach we shall use a combination of geometry and combinatorics: the theory of group actions on trees. The *tree-theoretic approach* to proving precompactness of sequences of group representations was first developed by Culler, Morgan and Shalen in the papers [CS83], [MS84], [Mor86], [MS88a, MS88b]. The idea is to show that:

(i) Each "divergent" sequence of representations of G_i corresponds to an action of G_i on a tree T_i so that G_i does not fix a point in T_i (the "geometric part").

(ii) The action $G_i \curvearrowright T_i$ has to have a global fixed point (the "combinatorial part"). [1]

The "geometric" part of the proof in [CS83], [MS84], [Mor86] was actually algebro-geometric; the geometric approach presented in the book is a version of the geometric approaches of Bestvina [Bes88], Paulin [Pau89] and Chiswell [Chi91]. The intuitive idea is that the ideal triangles in the hyperbolic space "approximately look like" an infinite *tripod*, i.e. the union of three rays with the common origin. If one multiplies the hyperbolic metrics by a very large constant then the "approximation" gets better. In the limit we get a tree.

The "combinatorial part" of Morgan-Shalen's proof [MS88a, MS88b] was actually topological: it was based on analysis of measured laminations in 3-manifolds. It is replaced in this book by more combinatorial *Rips' Theory of group actions on trees*; our discussion follows the paper of Bestvina and Feighn [BF95]. As an alternative to this part of the proof the reader can use either the paper of Paulin [Pau97], which is essentially another version of the Rips' Theory (although many arguments are quite different from the Rips' ideas) or the original papers of Morgan and Shalen. The proof of Skora's theorem (needed in the proof of part (1)) which we present in the book is again an application of the Rips' Theory, our discussion mainly follows Bestvina's paper [Bes97]. Very briefly, using the Rips Theory we transform the action of G_i on T_i to an action of G_i on a *simplicial tree* where each edge stabilizer is cyclic and fixes a point in T_i. Such action corresponds to a decomposition of G_i as an amalgamated free product (or an HNN-extension) over a cyclic subgroup. This gives rise to an *essential cylinder* in M_i. If M_i is acylindrical we get a contradiction. If M_i is not acylindrical then M_i splits along essential cylinders into submanifolds Z_{i_j} (so called JSJ decomposition) and by applying the Rips Theory to each

[1] What was actually proven in [MS88b] is that certain subgroups of the 3-manifold group have global fixed points. To prove that the whole group fixes a point one has to apply a corollary of Skora's theorem [Sko96].

group $\pi_1(Z_{i_j})$ we conclude that it fixes a point in T_i. This is the most difficult part of the proof. On the other hand, the actions of G_1 and G_2 on the trees T_1 and T_2 are related: an element of $\pi_1(S_1)$ fixes a point in T_1 if and only if the corresponding (under the gluing map τ) element of $\pi_1(S_2)$ fixes a point in T_2. Skora's theorem allows to "collect" all the elements of $\pi_1(S_i)$ which fix points in T_i into subsurfaces S_i' so that the gluing map τ carries S_1' to S_2'. If $S_i' \neq S_i$ then we get an incompressible torus in the manifold M by gluing cylinders in M_i whose boundaries are contained in S_i'. Thus both groups $\pi_1(S_i), i = 1,2$, fix points in T_i. Given this one verifies that both groups G_i fix points in T_i's which is a contradiction: this means that both sequences $[\rho_n : G_i \to PSL(2,\mathbb{C})]$, $i = 1,2$, are relatively compact in

$$Hom(G_i, PSL(2,\mathbb{C}))/PSL(2,\mathbb{C}).$$

This concludes the part (1) of the proof.

The proof of the part (2) of the bounded image theorem presented in the book is probably similar to the one that Thurston had in mind, although the details of this part of the proof were not discussed in Thurston's preprints and the corresponding part of Morgan's outline is somewhat sketchy:

(α) First one has to show that the representation (or rather a pair of representations) arising as the limit of ρ_n in the part (1) does not have *accidental parabolic elements*, i.e. non-parabolic elements of G_i which are mapped to parabolic elements.

(β) Next, one has to show that the limit group is *geometrically finite*.

The proof of (α) is based on Sullivan's cusps finiteness theorem and the theory of algebraic/geometric convergence of sequences of representations (developed by Jorgensen, Thurston and others). Briefly, each accidental parabolic element corresponds to an essential annulus in M_i, such annuli are glued by τ to an incompressible torus in M which is a contradiction.

The part (β) is based on the theory of ends of hyperbolic 3-manifolds (developed by Thurston, Bonahon and others) and, again, algebraic/geometric convergence.

This concludes our sketch of the proof of the *Final Gluing* Theorem. Recall that in our discussion we jumped over all the intermediate steps of induction. The general step of induction is reduced to the *final gluing* via Thurston's "orbifold-trick" outlined by Morgan in [Mor84]. The point of the "trick" is that each member N_i of the Haken hierarchy of M is obtained from N_{i+1} by gluing certain subsurfaces F_{i+1} of ∂N_{i+1}. If $F_{i+1} = \partial N_{i+1}$ then we are in the last step of the induction. Otherwise we have to do something about $\partial N_{i+1} - F_{i+1}$. Thurston's idea is to make $\partial N_{i+1} - F_{i+1}$ "disappear" by putting a hyperbolic *locally reflective orbifold structure* O_{i+1} on this part of the boundary. In this book we construct O_{i+1} by using a small deformation of the existing hyperbolic structure on N_{i+1} and applying Brooks' theorem [Bro86]. Along the way we present a proof of an important theorem of R. Brooks [Bro86] which states that arbitrarily near a geometrically finite subgroup G of $\mathrm{Isom}(\mathbb{H}^3)$ one can always find an isomorphic geometrically finite group G' which embeds in a discrete subgroup $\Gamma \subset \mathrm{Isom}(\mathbb{H}^3)$ such that $\mathbb{H}^3/\Gamma$ has finite volume.

We discuss the orbifold theory in details in Chapters 6, 19. At this moment just note that orbifolds are analogues of manifolds, locally they have the structure of quotients of $\mathbb{R}^n$ by finite isometry groups. Below is the most important for us example of a *locally reflective orbifold.* If M is a smooth manifold and Φ is a group generated by reflections and acting properly discontinuously (but not freely!) on M then M/Φ has a natural orbifold structure. The *underlying space* of this orbifold is the topological quotient $X = M/\Phi$. Roughly speaking, the orbifold structure on X is given by the *singular locus* Σ, which is a stratified subset of X that consists of projections of points in M with nontrivial stabilizers. Each point $x \in \Sigma$ is assigned a subgroup $\Phi_x \subset \Phi$ which stabilizes a point in M which projects to x (defined up to conjugation). Topologically, X is a manifold with boundary, $\partial X = \Sigma$. The top-dimensional strata in Σ are codimension 1 submanifolds in ∂X called *mirrors.* If M is a hyperbolic manifold and Φ acts by isometries then we get a hyperbolic metric on X (so that X has convex boundary) and each stratum of Σ is totally-geodesic. Such orbifold is *right-angled* if all the dihedral angles between faces are $\pi/2$. If M was a manifold with boundary then the orbifold O has boundary as well, ∂O is the projection of the boundary of M. The projection $M \to O$ is called a *manifold cover* of O. The *fundamental group* of O consists of all diffeomorphisms of the universal cover of M which project to the elements of Φ.

By applying Brooks' theorem to the hyperbolic manifold N_{i+1} we get an orbifold O_{i+1} whose underlying space is N_{i+1} and the boundary is F_{i+1}, the rest of ∂N_{i+1} becomes a part of the singular locus of O_{i+1}. To get the hyperbolic orbifold O_{i+1} we deform the given hyperbolic structure on N_{i+1} to a new convex hyperbolic structure so that the part of the boundary corresponding to $\partial N_{i+1} - F_{i+1}$ is the union of totally geodesic mirrors which meet at the right angles. These mirrors (and angles) determine the *locally reflective* orbifold structure O_{i+1}. The hyperbolic metric on O_{i+1} is essentially the *deformed* hyperbolic metric on N_{i+1}.

Then we glue the components of F_{i+1} together and get an orbifold O_i without boundary, the underlying topological space of O_i is N_i. The orbifold O_i is also locally reflective and it admits a finite manifold cover $\tilde{O}_i \to O_i$ so that $\tilde{O}_i$ is an atoroidal Haken 3-manifold which has a hyperbolic metric of finite volume. The hyperbolic structure exists on $\tilde{O}_i$ by the *Final Gluing Theorem* applied to manifolds: $\tilde{O}_i$ is obtained by gluing hyperbolic 3-manifolds $\tilde{O}_{i+1}$ along disjoint closed incompressible surfaces, where the hyperbolic 3-manifolds $\tilde{O}_{i+1}$ are finite covers of the orbifolds O_{i+1} and the hyperbolic metric on $\tilde{O}_{i+1}$ is lifted from O_{i+1}. The orbifold O_i is the quotient of the manifold $\tilde{O}_i$ by a finite group of diffeomorphisms Φ_i (actually isomorphic to $\mathbb{Z}_2^k$, where $k = 2$ since we will use *bipolar* orbifolds). The group Φ_i does not act isometrically, however by applying Mostow rigidity theorem we conclude that the action $\Phi_i \curvearrowright \tilde{O}_i$ is homotopic to an isometric action $\Psi_i \curvearrowright \tilde{O}_i$ of the same group $\mathbb{Z}_2^k$. The orbifolds $O_i = \tilde{O}_i/\Phi_i$ and $\tilde{O}_i/\Psi_i$ have isomorphic *fundamental groups.* To show that O_i and $\tilde{O}_i/\Psi_i$ are homeomorphic we use a generalization (Theorem 6.33) of Johannson-Waldhausen homeomorphism theorem to a certain class of 3-dimensional orbifolds: this theorem was missing in Morgan's outline. This gives us a hyperbolic structure on O_i; thus we also get a convex hyperbolic structure on the underlying topological space N_i of O_i, which is the required intermediate step of induction i+1→i.

We note here that Thurston's approach (as outlined by Morgan) to the "orbifold-trick" was somewhat different: it was based on Thurston's generalization of Andreev's theorem [Thu81, Chapter 13]. Yet another approach was carried out by Paulin and Otal in the unpublished preprints [Pau92] and [OP96], and published by Otal in [Ota98].

(b) We now discuss the "exceptional case" assuming that M is the mapping torus of a homeomorphism $\tau : S \to S$ and $\partial M = \emptyset$. The assumption that the mapping torus of τ is atoroidal is easily seen to be equivalent to the assumption that τ is *aperiodic*, i.e. for any homotopically nontrivial loop $\gamma \subset S$ and for any $m > 0$ the loops $\gamma, \tau^m(\gamma)$ are not freely homotopic in S.

The proof of existence of a hyperbolic structure on M is different from the *generic* case. In this case one can still try to find a hyperbolic metric on $S \times [0,1]$ so that τ is an isometry between neighborhoods of $S \times \{0\}, S \times \{1\}$ and define a sequence of iterations $[\rho_n] = \sigma^n(X) \in \mathcal{T}(G)$. However it turns out that this sequence is unbounded and the map σ of the Teichmüller space has no fixed point. The proof of the existence of a hyperbolic structure on M breaks in two parts:

Part 1. *The Double Limit Theorem.* One proves that the sequence $\{\rho_n\}$ converges (up to a subsequence) to a representation $\rho_\infty : G = \pi_1(S) \to PSL(2, \mathbb{C})$. This part of the proof was discussed by Thurston in his unpublished preprint [Thu87a]. A different proof was given by Otal in [Ota96], who used group actions on trees. In Chapter 18 I will outline Otal's proof; I omitted only one–but central–ingredient: estimates on the length function. I decided to include this outline since it illustrates the power of the theory of group actions on trees.

Part 2. The representation ρ_∞ is injective and its image is a discrete subgroup according to a theorem of Chuckrow. Let $\Gamma_\infty := \rho_\infty(G)$. The automorphism $\tau_* : G \to G$ induced by τ corresponds to a quasiconformal homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of the extended complex plane. Then one has to show that the limit set of Γ_∞ is $\mathbb{S}^2$. This was proven by McMullen in [McM96] and by Otal in [Ota96]. Once this is done, Sullivan's Rigidity Theorem implies that f is conformal. Since $fgf^{-1} = \tau_*(g)$ for each $g \in \Gamma_\infty$ (we identify G with Γ_∞ using ρ_∞), it follows that the group $\langle f, \Gamma_\infty \rangle$ generated by f and Γ_∞ is isomorphic to $\pi_1(M)$. Such group is discrete (since Γ_∞ is). Hence the hyperbolic manifold $\mathbb{H}^3/\langle f, \Gamma_\infty \rangle$ is homeomorphic to M (by a theorem of Stallings).

In Chapter 18 I also explain how to reduce the proof of the hyperbolization theorem in the exceptional case (b) to the generic case (a) provided that $\partial M \neq \emptyset$. The key part of this reduction is Thurston's theorem 2.1 which allows to construct (under an appropriate assumption on the 1-st Betti number of M) an incompressible surface S in M which is not a fiber in a fibration over the circle. The assumption $\partial M \neq \emptyset$ is used to construct a finite covering $M' \to M$ such that $H_2(M', \partial M') > 1$. It is conjectured that such covering exists even if M has empty boundary.

Historical Remarks.[2] Even the idea of something like Thurston's Hyperbolization Theorem was quite astounding in the 1970's when Thurston had first announced his theorem. Nevertheless, it had some historic precursors:

(a) *Andreev's Theorem* [And70], [And71b], where Andreev proves an analogue of the hyperbolization theorem in the case of reflection orbifolds whose underlying space is the closed 3-ball.

(b) *Marden's paper* [Mar74], where in section 11.3 Marden suggests to use Haken hierarchy to analyze Kleinian groups and in section 13.2 (Question 2) asks for necessary and sufficient conditions for a 3-manifold to be hyperbolic and notes that it is necessary to assume that the manifold is irreducible and its fundamental group has trivial center.

(c) *Riley's work* [Ril75], where Riley conjectures the hyperbolization theorem for knot complements in $\mathbb{S}^3$ and gives some "experimental" evidence towards this conjecture.

Thurston's hyperbolization theorem gradually became accepted (at least among experts) as a mathematical fact through the 1980's, however Thurston never wrote a complete proof[3] of his theorem. In [Thu94] Thurston describes his reasons for not writing a proof. Thurston wrote a general introduction to the proof and to the program of geometrization of 3-manifolds in [Thu82]. An outline of Thurston's proof was presented by J. Morgan in [Mor84]. Thurston's lecture notes [Thu81] develop a part of the technique required in the proof (ends of hyperbolic manifolds, strong convergence, generalized Andreev's theorem, etc.). In the mid-1980's Thurston wrote several papers and preprints [Thu86a, Thu87a, Thu87b] in which he fills in some major pieces of the proof of the part (1) of the Bounded Image Theorem and of the fiber bundle case of the Hyperbolization Theorem that are missing in Morgan's outline [Mor84]. In the same time, an alternative proof of most of the part (1) of the Bounded Image Theorem was published by J. Morgan and P. Shalen [MS84, MS88a, MS88b]. Morgan and Shalen used the theory of group actions on trees: the missing ingredient for completion of the part (1) of the proof was *Skora's duality theorem.* Shalen in [Sha87] stated several questions (most importantly, Question D) concerning generalization of the work of Morgan and Shalen; as we will see, one needs the affirmative answer to a *relative version* of Shalen's Question D to replace [MS88b] as a part of the proof of the hyperbolization theorem. Skora's theorem which we mentioned above was also conjectured by Shalen in [Sha87]. R. Skora [Sko96] proved his theorem in 1990, but it remained unpublished for 6 years. A combinatorial replacement to [MS88b] was provided meanwhile by E. Rips in the early 1990's. Rips' work was motivated in part by Shalen's questions in [Sha87] and in part by the work of G. Makanin [Mak82] and A. Razborov [Raz84] on solution of equations in groups. Rips' work still remains unpublished, however several written accounts became available thanks to the work of M. Bestvina and M. Feighn

[2]These and other historical remarks scattered throughout the book are not meant to present a *complete* version of history of Thurston's Hyperbolization Theorem and of various ingredients of its proof.

[3]One may argue here about the definition of the word *proof.* The definition which I use requires a proof to be a publicly available written document.

[BF95] and D. Gaboriau, G. Levitt and F. Paulin (see [Pau97]). On the author's request Bestvina and Feighn also proved in [BF95] a relative version of the Rips' theorem which is required in this book.

On the other hand, works of F. Bonahon [Bon86], R. Canary and Y. Minsky [Can96], [Min94a], [CM96] and K. Ohshika [Ohs92, Ohs98] considerably clarified and generalized Thurston's theory of ends of hyperbolic manifolds and strong convergence of sequences of representations (which is required for the part (2) of the Bounded Image Theorem). In particular, Bonahon in [Bon86] streamlined a major part of the proof of the part (2β) of the Bounded Image Theorem by proving Thurston's conjecture on *tameness of ends* of hyperbolic 3-manifolds with freely indecomposable fundamental groups. Originally, Thurston was proving this conjecture for discrete groups which are limits of geometrically finite groups (the only case required for the proof of the part (2)) for which he greatly extended Jorgensen's theory of algebraic/geometric convergence. This is partially presented in Thurston's lecture notes [Thu81]. Bonahon's theorem made the corresponding part of the proof of (2β) somewhat obsolete, however it did not completely eliminate the theory of algebraic/geometric limits from the proof.

Completely different (and much shorter) *analytical* proof of the Bounded Image Theorem was given by C. McMullen in 1989, [McM89]. McMullen's proof was based on estimates of the norm of the derivative of the map σ. A somewhat different version of McMullen's proof was given in 1996 by D. Barrett and J. Diller in [BD96].

Thurston's "orbifold–trick" for the reduction of the *general step* of induction in the proof of the hyperbolization theorem to the *last step* was modified considerably by F. Paulin (following a suggestion of F. Bonahon to use right-angled orbifolds) in his 1992 preprint [Pau92]. Unaware of Paulin's preprint I decided to use the right angled orbifolds but for a reason different from Paulin's. In this book the right angles come naturally from the proof of the Brooks' theorem, it is also somewhat easier to prove the homeomorphism theorem for orbifolds assuming that all the angles are $\pi/2$. In [Pau92, OP96, Ota98] the right angles seem to appear because in this case it is easy to find manifold covers of such orbifolds and using right angled orbifolds one can avoid Thurston's generalization of Andreev's theorem and apply Andreev's theorem directly.

A complete proof of the hyperbolization theorem in the case (b) (manifolds fibered over the circle) was published by J.-P. Otal [Ota96]. Otal's proof was quite different from the one proposed by Thurston; one of the key ingredients in Otal's proof is the above-mentioned theorem of R. Skora which establishes duality between small surface group actions on trees and measured geodesic laminations. Skora's theorem could be considered as a deep generalization of the Poincaré duality for closed hyperbolic surfaces S in the following sense. Homology classes (represented by real linear combinations of simple loops) are dual to the cohomology classes (represented by homomorphisms from the fundamental group to $\mathbb{R}$). The space of measured geodesic laminations on S is a certain completion of the collection of simple closed geodesics on S (weighted by positive real numbers). The space of small $\pi_1(S)$ actions on trees T could be identified with a certain space of maps $\pi_1(S) \to \mathbb{R}$, each given by the translation length function $g \mapsto \ell_T(g)$.

Earlier, the case (b) of the hyperbolization theorem was discussed by D. Sullivan [Sul81c] and by Thurston [Thu87a].

The present book assembles various pieces of the proof of the Hyperbolization Theorem in the generic case following Morgan's outline [Mor84] of the original (geometric) approach of Thurston, with tree-theoretic replacements of Thurston's convergence theorems in the part (1). The way this is done in this book is by no means original and was probably known to many of those familiar with Thurston's "blueprint" [Mor84] and the main "building blocks" of the proof that became available by the early 1990's. Otal's paper [Ota98] also treats the generic case of the hyperbolization theorem, where the hyperbolic manifold is assumed to have empty boundary. Otal's approach is analytical; it is based on McMullen's proof of the Bounded Image Theorem.

Summary.

This book contains essentially no new results, I preferred to use whatever I could find in the existing literature. I give sketches of the proofs (or just the references) mostly in the cases when I was comfortable with the proofs that are already published. I tried to give proofs if they seemed to me more transparent than those in the standard references, if they are unpublished, if they are not *very* long but are central for our discussion (like Theorems 2.1 and 14.24, etc.). There are only three theorems in this book that seem to be relatively new: I include them since the proofs are not difficult and they provide good illustration to the subject. These are:

(i) Theorem 8.44 on smoothness of the representation varieties of finitely generated Kleinian groups. (ii) Theorem 19.6, which states that every finitely generated Kleinian group G in $PSL(2, \mathbb{C})$ is isomorphic to a geometrically finite group (this is more or less a straightforward application of Thurston's Hyperbolization Theorem, the only problem is torsion in G). (iii) An analogue of the Johannson–Waldhausen homeomorphism theorem for a class of 3-dimensional orbifolds, Theorem 6.33. I could not find any reasonably general homeomorphism theorem for orbifolds in the existing literature; the papers of Takeuchi [Tak88, Tak91] fall somewhat short of what we need for the Hyperbolization Theorem. I found it easier to prove Theorem 6.33 directly than to derive it from Takeuchi's papers.

Otherwise, most of the work done in this book is to assemble the results that are already known in one or another form. There are several deep theorems about 3-manifolds and discrete groups that we will use but their proofs are complicated enough to force me to omit even sketches of the proofs: Haken hierarchy theorem, Waldhausen's homeomorphism theorem for 3-manifolds, Sullivan's Rigidity Theorem and Bonahon's theorem about ends of hyperbolic manifolds. The experts will notice that most of the material of the book is an "introduction" to one or another subject. The actual proof starts in Chapter 15, thus the reader familiar with the preliminary material can start reading the proof of the Hyperbolization Theorem from this chapter. Below is the brief description of the material of each chapter.

Chapter 1: Three-dimensional Topology. We discuss here some basic facts about 3-dimensional manifolds, like the Sphere and Loop theorems, the Dehn Lemma, incompressible surfaces, Haken hierarchy, Seifert manifolds, JSJ decomposition, etc. There are several good books on this subject, so I skipped most of the proofs.

Chapter 2: Thurston Norm. We prove here Thurston's existence theorem for incompressible surfaces which are not fibers in a fibration over $\mathbb{S}^1$. Our discussion mainly follows Thurston's paper [Thu85]. We will use material of this chapter in the chapter 18 to reduce (under extra assumptions) the case (b) of the hyperbolization theorem to the generic case (a). Thurston's paper [Thu85] was probably motivated by such reduction.

Chapter 3: Geometry of the Hyperbolic Space. This is perhaps the most eclectic chapter of the book: I include here basic material on the geometry of spaces of nonpositive and negative curvature as well as the geometry of the hyperbolic space $\mathbb{H}^n$ itself. We introduce here quasi-isometries and quasi-geodesics and prove *stability theorem* for quasi-geodesics in the hyperbolic space: any quasi-geodesic is within bounded distance from a geodesic. This is used (following Mostow) to establish a relation between quasi-isometries of $\mathbb{H}^n$ and quasiconformal mappings of the sphere $\mathbb{S}^{n-1}$. This is an important part of the proof of Mostow rigidity theorem.

Chapter 4: Kleinian Groups. This chapter contains mostly *pre-Thurston* material: results which in some form were known before Thurston came to the field. We cover several very important *pre-Thurston* theorems on the structure of Kleinian groups: Ahlfors finiteness theorem and its companions (Sullivan finiteness theorem, Scott compact core theorem etc.), Klein and Maskit Combination Theorems, Kazhdan-Margulis-Zassenhaus Theorem and characterization of geometrically finite Kleinian groups.

Chapter 5: Teichmüller Theory of Riemann Surfaces. Since there are several excellent books on the Teichmüller Theory, I have tried to give only a bare minimum of the material. We discuss basic properties of quasiconformal mappings of the complex plane, define the Teichmüller spaces, the mapping class group and metrics on the Teichmüller spaces. We also consider here finite subgroups of the mapping class group.

Chapter 6: Introduction to the Orbifold Theory. We start with the basic definitions and examples of orbifolds. Then we introduce a special class of 3-dimensional orbifolds: *all right orbifolds of zero Euler characteristic.* This class is a generalization of the class of atoroidal Haken manifolds of zero Euler characteristic. For this class of orbifolds we prove that an isomorphism of the fundamental groups is always induced by a homeomorphism.

Chapter 7: Complex Projective Structures. These structures are special coordinate coverings of Riemann surfaces where the transition maps belong to $PSL(2, \mathbb{C})$. They provide a useful and important generalization of the Kleinian groups. I include this chapter mostly because I like the subject, however we will use the Holonomy Theorem as a technical tool in the discussion of *strong convergence.*

Chapter 8: Sociology of Kleinian Groups. This chapter is about the "collective behavior" of Kleinian groups in *families* (deformations, algebraic and geometric convergence) and *pairs* (rigidity theorems and realization of isomorphisms by quasiconformal mappings). We discuss various results on algebraic, geometric and strong convergence of sequences of Kleinian groups. We prove the Mostow Rigidity Theorem so that the proof gives us extra information on non-smoothness of quasiconformal mappings conjugating Kleinian groups. (This is used in proving the Hyperbolization Theorem.) In this chapter we prove smoothness of character varieties of Kleinian groups and Bers' theorem about isomorphism between the Teichmüller space of a Kleinian group and the Teichmüller space of the associated Riemann surface. We then prove the Ahlfors' finiteness theorem. I also discuss Douady–Earle *barycentric* extension of homeomorphisms of the unit circle and explain how to justify Poincare's continuity method for proving the uniformization theorem in the case of punctured spheres.

Chapter 9: Ultralimits of Metric Spaces. This chapter is almost entirely taken from my joint paper with B. Leeb [KL95]. Ultralimits were introduced to the field by two logicians: L. Van den Dries and A. Wilkie, who used them to give an alternative proof of Gromov's theorem on groups of polynomial growth [VW84]. It appears that I. Chiswell [Chi91] was the first to realize that ultralimits provide a convenient formalism for construction of group actions on trees from divergent sequences of group representations into the isometry groups of (Gromov) hyperbolic spaces. To define ultralimits one needs the Axiom of Choice. If you do not believe in this axiom, you can use for instance [Mor86], [Bes88] or [Pau88] as an alternative.

Chapter 10: Introduction to Group Actions on Trees. Here we show how to use actions of groups on trees to compactify representation varieties. I also give proofs of various elementary facts about group actions on trees.

Chapter 11: Laminations, Foliations and Trees. Most of the material in this chapter was introduced by Thurston (part of it was implicit in the earlier works of Dehn and Nielsen). This material is important for understanding of Thurston's and Otal's approaches to hyperbolization of manifolds fibered over $\mathbb{S}^1$. The reader who is not interested in the hyperbolization of fibrations can skip most of this chapter, except for the formulation of Skora's Duality Theorem (Theorem 11.31). We discuss the relation between several essentially equivalent concepts: measured foliations, measured geodesic laminations, train tracks and small surface group actions on trees. We prove Thurston's characterization theorem for pseudo-Anosov homeomorphisms of surfaces and describe Thurston's compactification of the Teichmüller space by the space of projective classes of measured laminations using the approach of Morgan and Shalen.

Chapter 12: The Rips' Theory. The deepest part of the original Thurston's proof of the Hyperbolization Theorem was certain compactness theorem for sequences of representations. The most efficient (at the present time) way to prove such compactness results is via the *Rips Theory.* In this chapter we also use the Rips' Theory to prove Skora's Duality Theorem.

Here is a brief outline of this chapter. The action of a finitely presented group G on a tree T corresponds to a foliation $\mathcal{F}$ on a 2-dimensional complex Y. The complex Y consists of foliated Euclidean rectangles, called *bands* which are attached to a graph along the edges (called *bases*) transversal to the foliation. $\pi_1(Y)$ is thus free and to get a 2-complex X with the fundamental group G we add a finite number of 2-cells to Y (which correspond to the relations in the fundamental group). The complex Y breakes into a union of subcomplexes of *simplicial* and *pure* type. Roughly speaking, simplicial components correspond to simplicial subtrees in T invariant under appropriate subgroups of G; each leaf of the foliation in the simplicial part of $(Y, \mathcal{F})$ is compact. Each leaf of a pure component of $(Y, \mathcal{F})$ is dense in this component. The *Rips Machine* transforms each *pure component* to a foliated complex of one of the three "model" types: *surface type*, *axial type*, and *thin type* (the 2-cells of X are transformed as well). Each component of the *surface type* is essentially a surface with boundary which is given a measured foliation. The fundamental group of each component of the *axial type* preserves a geodesic in the tree T. Components $C_i \subset Y$ of *thin type* are more difficult to describe. I do it here assuming that G acts freely on T. Briefly, the Machine transforms each *band* in C_i into a union of bands which are arbitrarily thin and long. These bands have the remarkable property that some of them meet the 2-cells of $X - Y$ only along *bases.* If we cut the (transformed) complex X across any such thin band we get a decomposition of G as a free product. One then has to check that this decomposition is nontrivial. In general, even if the action $G \curvearrowright T$ is not free each thin component causes a nontrivial splitting of G over a subgroup stabilizing an arc in T. If the action of G on T is *small*, i.e. arc stabilizers are virtually nilpotent, then presence of any *pure component* in X corresponds to a decomposition of G over a virtually solvable subgroup. The same happens if we have simplicial components. Such decomposition is impossible for instance in the case of the fundamental group of a compact acylindrical atoroidal Haken 3-manifold M with incompressible boundary, thus $G = \pi_1(M)$ would have to fix a point in T. If M is not acylindrical then we get a relative version of the fixed point theorem: the manifold M splits along essential cylinders and Moebius bands so that the fundamental group of each component Z of the decomposition either fixes a point in T or Z is an interval bundle over a surface. These absolute and relative fixed point theorem were originally proven by Morgan and Shalen in [MS88a, MS88b].

Chapter 13: Brooks' Theorem and Circle Packings. The central part of this chapter is a theorem of R. Brooks about deforming geometrically finite subgroups of $PSL(2, \mathbb{C})$ to subgroups of lattices in $\mathrm{Isom}(\mathbb{H}^3)$. A version of this theorem is used to prove existence of geometrically finite hyperbolic structures on orbifolds of *finite type* which admit finite circle packings.

Chapter 14: Pleated Surfaces and Ends of Hyperbolic Manifolds. Thurston invented this area as a technical tool for proving the Hyperbolization Theorem. Instead of *pleated surfaces*, which were introduced by Thurston, we will use their combinatorial counterparts: *singular pleated surfaces*, which are much easier to handle (I believe that they were invented by Bonahon). The central result of this chapter is Thurston's Covering Theorem (Theorem 14.24): if $p : N \to M$ is a locally isometric covering between complete hyperbolic 3-manifolds and E is a *geometrically tame* end of N, then the restriction of p to E has finite multiplicity. This theorem is absolutely critical for proving part (2) of the Bounded Image Theorem. It allows to show that certain subgroups of the fundamental groups of hyperbolic 3-manifolds are geometrically finite.

Chapter 15: Outline of the Proof of the Hyperbolization Theorem. This and the following two chapters mostly follow Morgan's paper [Mor84], I use the results about group actions on trees as a substitute of the compactness results of Thurston. We break the proof in several steps and in the following chapters we fill in the details. There are two different cases: the "generic" Case (a) and the "exceptional" Case (b) of manifolds fibered over $\mathbb{S}^1$. Case (a) is discussed in chapters 16 and 17. Case (b) is discussed in chapter 18.

Chapter 16: Reduction to the Bounded Image Theorem. This is a slight modification of a part of Morgan's paper [Mor84]. I have changed several details which I did not like or did not understand. The existence theorem for a hyperbolic metric is reduced to a certain Fixed-Point Theorem (similarly to the fact that the fixed-point set of a map $\sigma : \mathcal{T}(G) \to \mathcal{T}(G)$ is the intersection of the graph of σ with the diagonal). Standard proofs of such Fixed-Point Theorems require strict contraction property for σ. For σ strict contraction fails, however we check here that σ is a contraction.

Chapter 17: The Bounded Image Theorem. This is the central part of the proof of the Hyperbolization Theorem: the sequence of iterations $\sigma^j(X)$ is relatively compact.

Chapter 18: Hyperbolization of Fibrations. In this chapter we discuss the exceptional case of the Hyperbolization Theorem (case (b)). Our discussion essentially follows Otal's book [Ota96]. In the same chapter I also give a short alternative proof of the Hyperbolization Theorem for manifolds fibered over the circle assuming that the manifold either has nonempty boundary or admits a finite covering with the 1-st Betti number ≥ 2 (conjecturally such covering always exists). The proof is by reduction of the case (b) to the generic case (a) using the results of Chapter 2.

Chapter 19: The Orbifold Trick. I explain how to glue the *all right* orbifolds of zero Euler characteristic from the orbifolds of *finite type*. The latter are similar to the class of acylindrical, atoroidal Haken 3-manifolds. Given a geometrically finite hyperbolic structure on orbifolds of finite type we will construct hyperbolic structures of finite volume on *all right* orbifolds of zero Euler characteristic. We then use Brooks' Theorem to finish the induction argument in the proof of the hyperbolization theorem.

Chapter 20: Beyond the Hyperbolization Theorem. The hyperbolization theorem is a part of Thurston's program for "geometrizing" 3-manifolds. In this chapter I describe Thurston's Geometrization Conjecture (which, among other things, implies Poincare Conjecture), collect various conjectures related to the Geometrization Conjecture and discuss several possible approaches to this conjecture. I also discuss subjects related to Thurston's hyperbolization theorem: higher-dimensional negatively curved manifolds, general geometric structures on 3-manifolds, and hyperbolic groups. The choice of subjects was mostly motivated by my personal mathematical taste and knowledge; the most serious omissions perhaps are the dynamics of holomorphic functions in the complex plane and the theory of circular packings.

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Michael Kapovich
Bonn, Germany,
April 2000.

Contents

Chapter 1

Three-dimensional Topology

1.1. Basic definitions and facts

By a *closed* manifold we shall mean a compact manifold without boundary. We will use the notation I for the closed interval $[0, 1]$. A compact surface Σ is called *pants* (or a *pair of pants*) if it is homeomorphic to the sphere with three holes. Suppose that S is a closed oriented surface of genus g. Remove from S interiors of disjoint closed disks $D_1, ..., D_n$. We say that g is the genus of the resulting surface. The *solid torus* is the product $D^2 \times \mathbb{S}^1$, where D^2 is the closed 2-disk. *The handlebody* of the genus g is the 3-manifold which is obtained by attaching g solid tori along disjoint boundary disks to the 3-ball. Each handlebody is bounded by the surface of genus g.

Recall that every closed 3-manifold has zero Euler characteristic; thus for every compact 3-manifold M we have: $\chi(M) = \chi(\partial M)/2$. In particular, the projective plane cannot be the boundary of a compact 3-manifold. A 3-manifold is called *aspherical* if it is connected and has contractible universal cover. Suppose that M is a compact aspherical 3-manifold, then either ∂M is a single sphere and M is contractible or each boundary component of M has nonpositive Euler characteristic (for otherwise the universal cover $\tilde{M}$ of M would have at least two boundary spheres which implies that $H_2(\tilde{M}) \neq 0$). Thus, if M is compact, aspherical and $\chi(M) = 0$ then either M is contractible or each component of ∂M has zero Euler characteristic, i.e. is the torus or the Klein bottle.

Remark 1.1. If M is a compact contractible 3-manifold then 3-dimensional Poincaré conjecture predicts that M is homeomorphic to the 3-ball.

Throughout this book all manifolds, maps, etc., will be considered either in PL or in the smooth category. This is not a very strong restriction because of

Theorem 1.2. *(E. Moise [Moi77].) Each topological 3-manifold admits a smooth (and PL) structure. This structure is unique.*

However, this choice will allow us to avoid the discussion of "wild" embeddings of 1- and 2-dimensional manifolds into 3-manifolds.

Suppose that M is a 3-dimensional PL manifold and S, F are 2-dimensional PL submanifolds in M. Then S is said to be *transversal* to F if each point of intersection $F \cap S$ locally looks like:

- either the intersection of two distinct coordinates planes in 3-space;
- or (if say F is a surface with boundary) like the intersection of the coordinate xy-plane in $\mathbb{R}^3$ with the coordinate yz-half-plane.

An embedding $\iota : S \to M$ of a surface into 3-manifold is called *proper*, if $\iota(\partial S) \subset \partial M$ and $\iota(S)$ is transversal to ∂M.

A 3-manifold M is called *prime* if it is not a nontrivial connected sum.

Theorem 1.3. *(H. Kneser, J. Milnor, see [Hem76].) Each compact 3-manifold M admits a connected–sum decomposition into prime components. This decomposition is unique under the assumption that M is orientable. (Otherwise we have to assume that there are no $\mathbb{S}^2 \times \mathbb{S}^1$ factors in the decomposition.)*

Remark 1.4. If N is nonorientable then $N \# \mathbb{R}P^3 \cong N \# (\mathbb{S}^2 \times \mathbb{S}^1)$.

Theorem 1.5. *(The "Dehn Lemma", see [Hem76].) Suppose that M is a 3-manifold and $f : D^2 \to M$ is a map so that the restriction of f to a some neighborhood A of ∂D^2 is an embedding and $f^{-1}f(A) = A$. Then $f|_{\partial D^2}$ extends to an embedding $g : D^2 \to M$.*

The usual application of Dehn Lemma is the following: suppose that γ is a simple loop in ∂M which is homotopically trivial in M. Then γ bounds an embedded disk in M.

Theorem 1.6. *(The Loop Theorem, see [Hem76].) Let M be a 3-manifold, F be a connected 2-submanifold of ∂M. If $\pi_1(F) \to \pi_1(M)$ is not injective then there is a proper embedding*

$$g : (D^2, \partial D^2) \to (M, F)$$

such that $g(\partial D^2)$ is homotopically nontrivial in F.

Theorem 1.7. *(The Sphere Theorem, see [Hem76].) Let M be an orientable 3-manifold such that $\pi_2(M) \neq 0$. Then there is an embedding $g : \mathbb{S}^2 \to M$ which represents a nontrivial element of $\pi_2(M)$.*

Note that this theorem is wrong without the assumption that M is orientable: if M is nonorientable one also has to consider homotopically nontrivial embedded projective planes in M.

The first correct proof of the above three theorems (which form a basis for the modern 3-dimensional topology) was given by C. Papakyriakopoulos in [Pap57]. It concluded a long sequence of erroneous proofs of Dehn's lemma originated in Dehn's paper [Deh10].

A 3-manifold M is called *irreducible* if it is prime, contains no 2-sided projective planes and is different from $\mathbb{S}^1 \times \mathbb{S}^2$. Considering irreducible manifolds will allow us to avoid the Poincaré Conjecture.

Assumption. *From now on we shall deal only with connected irreducible manifolds.*

Theorem 1.8. *(The Annulus Theorem, see [CF76], [Sco80].) Suppose that M is an orientable 3-manifold with boundary, γ_1, γ_2 are two homotopically nontrivial (in M) loops on ∂M which are homotopic in M but not in ∂M. Then there is an embedded annulus $(A, \partial A) \subset (M, \partial M)$ so that the boundary components of A are not isotopic on ∂M.*

Suppose that S is a compact surface. A map $f : (S, \partial S) \to (M, \partial M)$ is called *essential* if:

- it cannot be homotoped (relatively to ∂S) to a map $f' : S \to \partial M$;
- the induced homomorphism $\pi_1(f) : \pi_1(S) \to \pi_1(M)$ is injective.

A properly embedded annulus $\iota : A \hookrightarrow M$ is called an *essential annulus* if ι is an essential map. Similarly we define essential Moebius bands in M.

Exercise 1.9. *Consider the manifold $N = T^2 \times [-1, 1]$, let $\tau : T^2 \to T^2$ be the involution so that T^2/τ is the Klein bottle. Extend τ to $\widehat{\tau} : (x, t) \to (\tau(x), -t)$. Take the manifold $M := N/\widehat{\tau}$, it has only one boundary component (the torus T^2). The manifold M contains essential annuli as well as essential Moebius bands. Find all of them (up to isotopy).*

1.2. Incompressible surfaces

A π_1-injective surface in a 3-manifold M is an embedded surface $S \subset M$ whose embedding induces a monomorphism of fundamental groups. In the "first approximation", incompressible surfaces in M are the π_1-injective surfaces $S \subset M$ such that $\chi(S) \leq 0$. Actually, more restrictions on the surface S are need; below we give a complete definition.

Let M be a 3-manifold (possibly with boundary ∂M). Let $(F, \partial F) \subset (M, \partial M)$ be a surface which is either properly embedded or is contained in ∂M. A *compressing disk* for F is an embedding $(D^2, \partial D^2) \to (M, F)$ such that: $F \cap D^2 = \partial D^2$ is a (homotopically) nontrivial loop on F (Figure 1.1).

A *boundary compressing disk* for F is a triple $(D^2, A, B) \subset (M, \partial M, F)$ such that:

(i) $A \cup B = \partial D^2$,

(ii) $D^2 \cap F = B$,

(iii) D^2 is not isotopic (*rel* ∂D^2) to an embedding which lies in $\partial M \cup F$ and that meets each $\partial M, F$ in a disk.

Consider a surface $(F, \partial F) \subset (M, \partial M)$ which is either properly embedded or is contained in the boundary of M.

Definition 1.10. A surface $(F, \partial F) \subset (M, \partial M)$ as above is called an **incompressible surface** if it satisfies the following conditions:

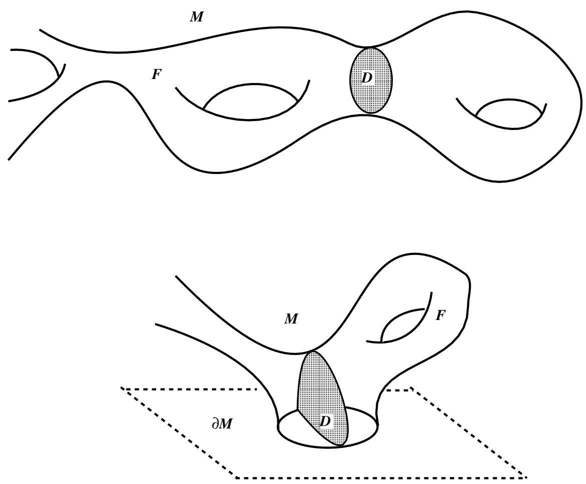

Figure 1.1: *Compressing disk and boundary compressing disk.*

1. F is 2-sided.
2. (i) Either $F = D^2$ and ∂F is a homotopically nontrivial loop on ∂M,

   (ii) or for each component $F_i \subset F$, $\chi(F_i) \le 0$ and F has no compressing disks and boundary compressing disks.

Note that F is not required to be connected. Our assumptions exclude spheres and projective planes from being incompressible surfaces.

Exercise 1.11. *(a) Let K be a nontrivial knot in $\mathbb{S}^3$ and F be the boundary of its regular neighborhood. Then F is incompressible in $\mathbb{S}^3 - K$.*

(b) If F is any connected incompressible surface in M then F is π_1-injective.

(c) Let F be a surface of nonpositive Euler characteristic, M be the total space of a nontrivial I-bundle over F. Then the inclusion $F \subset int(M)$ given by a section of this bundle determines a π_1-injective incompressible surface which is not incompressible. The reason is that F is not 1-sided in M.

(d) Let $\tau : T^2 \to T^2$ be an orientation-reversing involution which does not have a fixed point; thus the quotient $F = T^2/\tau$ is the Klein bottle. Then there is a simple (homotopically nontrivial) loop $\gamma \subset T^2$ such that $\tau(\gamma) = \gamma$ and τ reverses orientation on γ. Let $N = (T^2 \times [-1,1])/\widehat{\tau}$, where $\widehat{\tau}$ acts on $T^2 \times [-1,1]$ by $\widehat{\tau}(x,t) = (\tau(x), -t)$. Now glue the solid torus $L = D^2 \times \mathbb{S}^1$ to the boundary of N so that γ bounds a disk in L. Call the resulting manifold M. Then the natural inclusion $F \to M$ satisfies the assumption (2) of the Definition 1.10 but $F \subset M$ is not incompressible (because (1) fails). Show that the manifold M contains no incompressible surfaces.

To prove Thurston's Hyperbolization Theorem we shall need a somewhat restrictive definition than incompressible surface. A *compressing annulus* for $(F, \partial F) \subset (M, \partial M)$ is an embedded annulus

$$i : (A = I \times \mathbb{S}^1, \{0\} \times \mathbb{S}^1, \{1\} \times \mathbb{S}^1) \hookrightarrow (M, F, \partial M)$$

such that:

(i) $i_* : \pi_1(A) \to \pi_1(M)$ is an embedding;

(ii) $A \cap F = \{0\} \times \mathbb{S}^1 = \gamma$;

(iii) γ is not isotopic to ∂F in F.

Example 1.12. Take $M = T^2 \times I$ and let $F = T^2 \times \{0.5\}$. Then F admits a compressing annulus.

Definition 1.13. (Superincompressible surface.) Suppose that M is a 3-manifold, $F \subset M$ is a properly embedded surface. We say that F is **superincompressible** if it is incompressible and has no compressing annuli.

Example 1.14. Each incompressible surface in a closed 3-manifold is superincompressible.

Definition 1.15. A compact irreducible manifold is called **Haken** if it contains a superincompressible surface.

Definition 1.16. (Haken hierarchy.) Start with a connected surface S_1 in $M = M_1$ which is superincompressible. Split M along S_1. As the result we obtain a manifold M_2 each of whose components is either Haken or a 3-ball. Then in each Haken component of M_2 again take a connected surface S_2 which is superincompressible. Continue this process until all components of M_h are homeomorphic to the 3-ball. The collection of manifolds $M_1, M_2, ..., M_h$ together with the gluing data is called a **Haken hierarchy**.

Theorem 1.17. *(W. Haken [Hak61]) For each Haken manifold M the above decomposition process eventually terminates and M admits a finite Haken hierarchy.*

We also refer the reader to [Wal68], [Hem76], [Jac81] for the proofs.

As an example of Haken hierarchy consider a compact 3-manifold M which fibers over the circle. Cut M along the fiber S to obtain $M_2 = S \times [0,1]$. Then cut M_2 along the "vertical cylinders" of the form $\gamma \times [0,1]$ ($\gamma \subset S$) to the get the union of $P_j \times [0,1]$, where P_j are pairs of pants. Finally, in each $P = P_j$ choose two disjoint arcs a, b which connect different boundary components and split $P \times [0,1]$ along $a \times [0,1]$ and $b \times [0,1]$. The result is the disjoint union of balls.

Definition 1.18. Compact manifold M whose boundary is a (possibly empty) collection of surfaces of nonpositive Euler characteristic is called **atoroidal** if:

(i) each rank 2 Abelian subgroup of $\pi_1(M)$ is conjugate into the image of the fundamental group of some boundary component;

(ii) M is different from an I-bundle over the torus or Klein bottle and from a disc bundle over $\mathbb{S}^1$.

Definition 1.19. A compact manifold M whose boundary is a (possibly empty) collection of surfaces of nonpositive Euler characteristic, is called **topologically atoroidal** if:

(i) each incompressible torus and Klein bottle in M is isotopic to a boundary component of M;

(ii) M is different from an I-bundle over the torus or Klein bottle and from a disc bundle over $\mathbb{S}^1$.

Example 1.20. Take $M=$ the pair of pants $\times\mathbb{S}^1$. Then M is not atoroidal but is topologically atoroidal. Let K be the trefoil knot. Then $\mathbb{S}^3 - Nbd(K)$ is not atoroidal but is topologically atoroidal. With exception of so called **Seifert manifolds** that we shall discuss below, being an atoroidal manifold is equivalent to being topologically atoroidal.

Definition 1.21. A compact 3-manifold M is called **acylindrical** if it does not contain essential annuli. Manifold M is **weakly acylindrical** if for each properly embedded annulus $A \subset M$ the pair of loops ∂A bounds an annulus in ∂M.

Theorem 1.8 implies that M is acylindrical iff $\pi_1(M)$ does not split into a nontrivial amalgamated free product and HNN-extension with amalgamation over $\mathbb{Z}$. Note that atoroidal manifold is acylindrical if and only if it is weakly acylindrical.

Recall that a group is called *almost* (or *virtually*) Abelian (almost *nilpotent*, etc.) if it contains an Abelian (nilpotent, etc.) subgroup of finite index.

Theorem 1.22. *If M is compact atoroidal and $H=\{M_j\}_{j\in J}$ is a Haken hierarchy for M, then each manifold M_j is either atoroidal or has almost Abelian fundamental group.*

Proof: According to Seifert– Van Kampen Theorem the inclusions $M_{i+1}\to M_i$ induce monomorphisms of the fundamental groups. Thus, if there is a subgroup $A=\mathbb{Z}^2\subset\pi_1(M_{i+1})$, then A is embedded in $\pi_1(M_i)$ as well. We assume that A is not conjugate into the fundamental group of any boundary component of M_{i+1}. Suppose that A is conjugate in $\pi_1(M_i)$ into the fundamental group of one of the boundary components T_j (T_j must be a torus or a Klein bottle). It follows from the algebraic structure of amalgamated free product (or HNN extension) that T_j must be a boundary component of M_{i+1}. This proves that A is conjugate (in $\pi_1(M_{i+1})$) into $\pi_1(T_j)$. $\square$

1.3. Existence of incompressible surfaces

Theorem 1.23. *Suppose that M is compact, irreducible and*

$$0\neq z\in H_2(M,\partial M;\mathbb{Z}).$$

Then M contains an orientable (possibly disconnected) superincompressible surface S representing the class z. Moreover, if z is represented by an embedded subsurface Σ so that each component of Σ is homologically nontrivial, then $\chi(S)\geq\chi(\Sigma)$.

The proof is based on cut and paste operations. See [Jac81], [Hem76] and Chapter 2.

Corollary 1.24. *Suppose that a 3-manifold M is compact orientable, irreducible, distinct from B^3 and has nonempty boundary. Then M contains a properly embedded superincompressible surface.*

Proof: Note that $\chi(\partial M) \leq 0$, thus $H_1(M, \mathbb{Z}) \neq 0$. By duality we conclude that the group $H_2(M, \partial M; \mathbb{Z})$ is nontrivial. $\square$

Corollary 1.25. *Suppose that $0 \neq z \in H_2(M, \partial M; \mathbb{Z})$ is an element which is different from a multiple of the homology class of the fiber in any fibration of M over $\mathbb{S}^1$. Then any connected component S_j of the surface S representing z (as in Theorem 1.23) is not a fiber in a fibration of M over $\mathbb{S}^1$.*

Proof: The components S_j are disjoint. Suppose that $S_j \subset S$ is a fiber in a fibration over $\mathbb{S}^1$. Split M along S_j, the result is $S_j \times I$ which contains the rest of the surface S. Therefore $[S] = q[S_j]$, which contradicts our assumptions. $\square$

A stronger existence theorem for incompressible surfaces is implicit in the paper of Freedman, Hass and Scott [FHS83] and was later explicitly used by Bonahon [Bon86] and in the work of Canary and Minsky [CM96]. We first formulate the absolute version of this theorem.

Theorem 1.26. *Suppose that (M, ds^2) is a Riemannian 3-manifold, $S \subset M$ is a closed superincompressible surface (which is not necessarily connected), $f : S \to M$ is a map homotopic to the inclusion $\iota : S \hookrightarrow M$. Then in the 2-neighborhood U of $f(S)$ there is an embedded superincompressible surface Σ which is homotopic to S.*

Proof: I first give an informal proof suggested by Bonahon. Recall that the main result of [FHS83] states that a *least area* map $g : S \to M$ homotopic to ι is either

1. an embedding or,
2. a 2-fold covering onto a 1-sided surface.

Note that in the second case we can always approximate g by a homeomorphism g_ϵ onto the boundary of a regular neighborhood of $g(S)$. To guarantee existence of a least area map as well as the fact that it lies in U, we change the Riemannian metric on M as follows. Keep the metric in the 0.5-neighborhood V of $f(S)$ as it was before, outside of the 1-neighborhood W of $f(S)$ change the metric ds^2 to $\rho(x)ds^2$, so that:

- The injectivity radius at each point $x \in (M - W, \rho \cdot ds^2)$ is at least $Area(f(S))$.
- $Vol(B_r(x)) \gg Area(f(S))$ for any $x \in M - V$.

Then, consider a sequence of maps $f_n, n = 1, 2, \ldots$ in the homotopy class of $f = f_1$, which is area-minimizing; this sequence has to stay inside U. This guarantees both existence of a limiting area-minimizing map f_∞ and the fact that the image of $f_\infty : S \to \Sigma \subset M$ is contained in U.

In order to make these arguments formal one has to overcome some technical difficulties. Fortunately there is a relatively easy way out [CM96].

Namely, Jaco and Rubinstein [JR88, JR89] developed a theory of combinatorial least area surfaces generalizing the classical theory, which is easy to handle. Choose a smooth triangulation T of M. Instead of minimizing area of surfaces in M Jaco and Rubinstein minimize a quantity $w(f)$ for *normal* simplicial maps $f : S \to M$. All we need to know about the definition of normality is that it includes transversality to the 1-dimensional skeleton $T^{(1)}$ of T. The exact definition of $w(f)$ is not essential here, what is important is that $w(f)$ is at least the number of points in $f^{-1}(T^{(1)})$. As in [FHS83] it turns out that w-minimizing map $g : S \to M$ homotopic to an embedding is an embedding itself. Now, instead of expanding the metric on M outside of W, we just take a sufficiently deep barycentric subdivision of T in $M - W$ and keep the old triangulation in V. The new triangulation T' of M determines a combinatorial distance between its 3-simplices (the distance between any adjacent simplices is equal to 1). For each $u \in U$ take a top-dimensional simplex $\Delta \in T'$ containing u and let $B(u, w(f))$ be the union of 3-simplices of T' within the combinatorial distance at most $w(f)$ from Δ. We require each $B(u, w(f))$ to be contractible. Now the same arguments as in the case of area-minimizing surfaces imply that image of any w-minimizing π_1-injective map $g : S \to M$ is contained in W. $\Box$

The situation in the case of surfaces S with boundary is slightly more subtle.

Theorem 1.27. *Suppose that (M, ds^2) is a Riemannian 3-manifold, $S \subset M$ is a superincompressible surface, $f : (S, \partial S) \to (M, \partial M)$ is a map homotopic to the inclusion $\iota : (S, \partial S) \hookrightarrow (M, \partial M)$. Assume that f is an embedding near ∂S. Then in the 2-neighborhood U of $f(S)$ there is an embedded superincompressible surface Σ which is homotopic to S.*

Proof: There is a relative version of the theorem of Freedman, Hass and Scott above (see [FHS83, Section 7] and [JR88]). Consider the *relative* minimization problem, i.e. the area-minimization problem (or, in the combinatorial approach, the w-minimization problem) for maps $g : S \to M$ which are equal to f on ∂S and homotopic to f relatively to ∂S. Again there exists an area–minimizing surface which is either embedded or is a 2-fold covering of an embedded 1-sided surface. Then we repeat the arguments in the proof of the previous theorem. $\Box$

1.4. Homeomorphisms of Haken manifolds

Theorem 1.28. *(F. Waldhausen [Wal68], see also [Hem76].) Suppose that M_1 is a closed orientable Haken manifold, M_2 is another irreducible 3-manifold which is homotopy-equivalent to M_1. Then any isomorphism of the fundamental groups $\phi : \pi_1(M_1) \to \pi_1(M_2)$ is induced by a homeomorphism.*

An earlier version of this theorem was proven by J. Stallings [Sta62] in the case of 3-manifolds fibered over the circle.

Theorem 1.29. *(Waldhausen's Homeomorphism Theorem, see [Wal68], [Hem76].) Let M_1 be an orientable Haken manifold, M_2 be another compact irreducible 3-manifold and*

$$f : (M_1, \partial M_1) \to (M_2, \partial M_2)$$

be a homotopy-equivalence of pairs. Then f is homotopic to a homeomorphism.

Suppose that M is a compact Haken 3-manifold with boundary. Following [Hem76] we define the *peripheral structure* of $\pi_1(M)$ as follows. For each boundary component $S_j \subset \partial M$ $(j = 1, ..., k)$ fix the homomorphism

$$\theta_{M,j} : \pi_1(S_j) \to \pi_1(M)$$

induced by the inclusion. Then the collection

$$\mathcal{P}(\pi_1(M)) = \{(\pi_1(S_j), \theta_{M,j}), j = 1, ..., k\}$$

is called the *peripheral structure* of $\pi_1(M)$. A subgroup $G \subset \pi_1(M)$ is called *peripheral* if there exists a boundary surface S_j so that G is conjugate to a subgroup of $\theta_{M,j}(\pi_1(S_j))$ in $\pi_1(M)$.

Suppose that M, N are Haken manifolds with the peripheral structures

$$\mathcal{P}(\pi_1(M)) = \{\theta_{M,j} : \pi_1(S_j) \to \pi_1(M), j = 1, ..., k\}$$

$$\mathcal{P}(\pi_1(N)) = \{\theta_{N,j} : \pi_1(T_j) \to \pi_1(N), j = 1, ..., k\}.$$

Define an isomorphism of the peripheral structures $\mathcal{P}(\pi_1(M)) \to \mathcal{P}(\pi_1(N))$ as a collection of isomorphisms $\psi : \pi_1(M) \to \pi_1(N)$, $\phi_j : \pi_1(S_j) \to \pi_1(T_j)$ so that

$$\psi \circ \theta_{M,j} = \theta_{N,j} \circ \phi_j, \quad j = 1, ..., k.$$

Theorem 1.30. *(See [Wal68], [Hem76].) Suppose that M, N are Haken manifolds as above, $\psi : \pi_1(M) \to \pi_1(N)$ is an isomorphism and*

$$h_j : S_j \to T_j, \quad j = 1, ..., k$$

is a collection of homeomorphisms which induce an isomorphism of the peripheral structures of $\pi_1(M)$ and $\pi_1(N)$. Then this collection of homeomorphisms can be extended to a homeomorphism $M \to N$.

Theorem 1.31. *(Johannson Homeomorphism Theorem [Joh79].) Suppose that M_1 is Haken and acylindrical. Then for every irreducible compact 3-manifold M_2, each homotopy–equivalence $h : M_1 \to M_2$ is homotopic to a homeomorphism.*

Below we give two simple examples which show that the assumptions of Theorems 1.29 and 1.31 are necessary.

(1) Let P be the pair of pants and T^* be the torus with a hole. Then manifolds $P \times \mathbb{S}^1$ and $T^* \times \mathbb{S}^1$ are homotopy-equivalent but not homeomorphic.

(2) Consider four copies S_1, S_2, S_3, S_4 of the torus with a hole. On the boundary of a solid torus $\widehat{T}$ draw four disjoint parallel loops which are (homotopically) nontrivial in $\widehat{T}$. These loops separate the boundary of $\widehat{T}$ into four annuli A_j. Now glue the manifolds $N_j = S_j \times I$ along the boundary annuli $\partial S_j \times I \cong A_j$ to $\widehat{T}$. The result is a compact Haken 3-manifold M. There is an obvious automorphism φ of the fundamental group $\pi_1(M)$ which preserves $\pi_1(N_1), \pi_1(N_3)$ but interchanges $\pi_1(N_2), \pi_1(N_4)$. Clearly φ is not induced by a self-homeomorphism of the manifold M.

1.5. Pared 3-manifolds

A *pared manifold* is a pair (M, P) such that:

- M is a compact orientable irreducible 3-manifold;
- $P \subset \partial M$ is a disjoint union of incompressible tori and annuli P_j;
- no two components of P are isotopic in ∂M;
- every Abelian noncyclic subgroup of $\pi_1(M)$ is conjugate to $\pi_1(P_j)$ for some j;
- there are no essential cylinders $(A, \partial A) \hookrightarrow (M, P)$.

The submanifold P is called the *designated parabolic locus* of M: its components will correspond to the cusps in a complete hyperbolic metric that we will construct on $M - P$ in the process of proving Thurston's Hyperbolization Theorem. We let $\partial_0 M := \partial M - P$. Any pared manifold (M, P) satisfies the following:

Theorem 1.32. *Suppose that $\pi_1(M)$ splits as a nontrivial free product $A * B$, so that the fundamental group of each component P_j is conjugate either into A or into B. Then M contains a compressing disk $(D, \partial D) \subset (M, \partial_0 M)$.*

Theorem 1.33. *Suppose that $\pi_1(M)$ splits as a nontrivial amalgamated free product $A *_C B$ or HNN extension $A *_C$ with infinite cyclic amalgamated subgroup C, so that the fundamental group of each component P_j is conjugate either into A or into B. Then M contains an essential annulus (or a Moebius band) $(A, \partial A) \subset (M, \partial_0 M)$.*

We will discuss proofs of these theorems in §10.2. For the future purposes I will make the following convention for Haken hierarchies of pared manifolds. If (M, P) is pared, $S \subset M$ is a superincompressible surface, then we choose S within its isotopy class so that:

- If a component α of ∂S is isotopic in ∂M to a boundary component of ∂P, then $\alpha \subset P$.
- S minimizes the number of components of $\partial S \cap P$ in the isotopy class of S.

1.6. Seifert manifolds

This is a very important class of 3-manifolds, so we will give several equivalent definitions.

Definition 1.34. A **Seifert manifold** is a compact 3-manifold which admits a foliation by circles.

Definition 1.35. In the class of compact prime 3-manifolds with infinite fundamental groups, **Seifert manifolds** M are those whose fundamental groups have infinite normal cyclic subgroups. Their fundamental groups fit into short exact sequence:

$$1 \to \mathbb{Z} \to \pi_1(M) \to F \to 1, \tag{1.1}$$

where F is a discrete group of isometries of $\mathbb{S}^2$ or $\mathbb{E}^2$ or $\mathbb{H}^2$.

Definition 1.36. In the class of compact 3-manifolds with infinite fundamental group, **Seifert manifolds** are those which are finitely covered by circle bundles over surfaces.

Example 1.37. The sphere $\mathbb{S}^3$ is a Seifert manifold (the $\mathbb{S}^1$-fibration is the Hopf fibration). If S is a compact surface then $S \times \mathbb{S}^1$ is a Seifert manifold. The total space of the unit tangent bundle of a closed Riemannian surface is a Seifert manifold.

Theorem 1.38. *In the class of compact prime 3-manifolds with infinite fundamental groups all three definitions of Seifert manifolds are equivalent.*

The proof is a combination of [Eps72], [Hem76], [Sco80], [Gab92], [CJ94]. Moreover,

Theorem 1.39. *Suppose that M is a compact irreducible 3-manifold whose fundamental group contains an infinite cyclic normal subgroup. Then M is Seifert.*

The proof of this theorem is a culmination of work of many people: if M is Haken then this is a result of F. Waldhausen (see [Hem76]). If the quotient of the fundamental group of M by a normal infinite cyclic subgroup is isomorphic to a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^2)$ or $\mathrm{Isom}(\mathbb{E}^2)$ this theorem was proven by P. Scott [Sco83b]. Then Theorem 1.39 was reduced by G. Mess [Mes90] (see also [Bow99]) to the problem of proving that each discrete *uniform convergence* subgroup of $Homeo(\mathbb{S}^1)$ is isomorphic to a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^2)$. The latter was proven independently by D. Gabai [Gab92] and by A. Casson and D. Jungreis [CJ94] using earlier results of P. Tukia [Tuk88].

Thurston realized that each compact Seifert manifold M is *geometric*, i.e. there exists a unimodular simply-connected homogeneous Riemannian manifold X and a discrete group $\Gamma \subset \mathrm{Isom}(X)$ acting freely on X so that $int(M) = X/\Gamma$. Note that in general we cannot assume that X/Γ has finite volume (consider for instance the solid torus or an interval bundle over a torus). We refer the reader to Chapter 20 and to [Sco83a, Thu97a] for the discussion of this result and the description of the possible spaces X.

1.7. The decomposition theorem

Theorem 1.40. *(The Torus Theorem.) Suppose that M is an (irreducible) 3-manifold, $\mathbb{Z}^2 \subset \pi_1(M)$ is a* **nonperipheral** *subgroup. Then either M contains an embedded incompressible torus (or Klein bottle) which is not boundary-parallel or M is a Seifert manifold.*

This theorem was originally proven for Haken manifolds (see [Fe76a, Fe76b], [JS79], [Joh79]), later P. Scott [Sco80] reduced this theorem to proving Theorem 1.39.

Theorem 1.41. *(The JSJ Decomposition Theorem, W. Jaco, P. Shalen, K. Johannson.) Suppose that M is a Haken 3-manifold whose boundary has zero Euler characteristic. Then there is a collection C of disjoint embedded incompressible tori and Klein bottles in M such that $M - C$ consists only of Seifert manifolds and atoroidal manifolds. Minimal (with respect to the inclusion) collection C_{min} is unique up to isotopy. The decomposition of M into components $M - C_{min}$ is called the* **Jaco-Shalen-Johannson** *decomposition of M (the JSJ decomposition).*

Below we define a modification of the JSJ decomposition, called *geometric decomposition*. Cut the above manifold M open along a disjoint union K of π_1-injective tori and Klein bottles, *we no longer assume that they are 2-sided*. Thus, if F is one of these tori and Klein bottles, if we cut M open along F, we get a manifold N with a boundary surface $\tilde{F}$ and an involution $\tau : \tilde{F} \to \tilde{F}$ so that $N/\tau = M$. We assume that each component of the decomposition $M - int(Nbd(K))$ is either atoroidal or Seifert or is so called Sol-manifold. The latter either fibers over the circle with toral fiber or is finitely covered by the total space of such fibration. Finally, we assume that this decomposition is chosen to have the least number of components in K and $M - int(Nbd(K))$. The geometric decomposition is obtained from the JSJ decomposition $M - C$ as follows. If M is a Sol-manifold, do not split it at all. Otherwise, suppose that a component M_i of the JSJ decomposition is a nontrivial interval bundle over torus or Klein bottle. Then M_i contains a 1-sided π_1-injective subsurface $F \subset int(M_i)$ (the image has of a section of the fibration $M_i \to F$). The boundary of M_i is a connected surface $\tilde{F}$; this surface is a 2-fold cover of F, $F = \tilde{F}/\tau$. Let M_j be a component of $M - int(Nbd(C))$ which is adjacent to M_i along $\tilde{F}$. Then eliminate M_i from the list of components of the JSJ decomposition of M; we recover M by gluing M_j to itself via τ. In other words, we adjoin M_i to M_j and cut the result open along F; thus we reduced the number of components of the decomposition by 1. Continue this process until we eliminate all interval bundles among the components of JSJ decomposition. Call the result the *geometric decomposition* of M. For example, suppose that M_1 is an atoroidal 3-manifold so that $\chi(\partial M) = 0$ and M_1 has a single boundary component which is the torus T^2. Let M_2 be the total space of an interval bundle over the Klein bottle so that ∂M_2 is the torus. Let M be obtained by gluing M_1 and M_2 along T^2. Then the JSJ decomposition of M consists of two components, M_1, M_2. On the other hand, the geometric decomposition of M has only one component, namely M_1.

Now we can formulate the central theorem of this book:

Theorem 1.42. *(Thurston's Hyperbolization Theorem.) Suppose that M is a compact Haken atoroidal 3-manifold, whose boundary has zero Euler characteristic. Then the interior of M admits a complete hyperbolic metric of finite volume.*

By abusing notation we shall say that M is *hyperbolic* if $int(M) = \mathbb{H}^3/\Gamma$, where Γ is a discrete torsion-free group of isometries of $\mathbb{H}^3$ so that $vol(\mathbb{H}^3/\Gamma) < \infty$. More generally we have:

Theorem 1.43. *Suppose that (M, P) is a compact atoroidal Haken pared 3-manifold. Then M admits a complete geometrically finite hyperbolic structure with the parabolic locus P. (I will give necessary details in the Definition 4.77.)*

Uniqueness of the hyperbolic structure follows from

Theorem 1.44. *(Mostow Rigidity Theorem [Mos73].) Suppose that two complete hyperbolic manifolds of finite volume M, N have isomorphic fundamental groups and dimension $n > 2$. Then any isomorphism of their fundamental groups can be realized by an isometry $M \to N$.*

I will prove Mostow Rigidity Theorem in §8.5. The proof of Theorem 1.42 that we shall discuss in this book, is based on a wide range of mathematical technique:

(a) discrete groups of isometries of hyperbolic space (Kleinian groups),
(b) 3-dimensional topology,
(c) Teichmüller theory,
(d) pleated surfaces,
(e) actions of groups on trees.

Theorem 1.42 has a number of implications, I formulate here only one of them:

Corollary 1.45. *(W. Thurston.) Let M be a Haken 3-manifold. Then M admits a decomposition along a collection of disjoint incompressible tori (and Klein bottles) into components M_j each of which is* **geometric***.*

Proof: If M is atoroidal then M is either Seifert or admits a geometrically finite hyperbolic structure, in the both cases M is geometric itself (recall that our definition of *geometric structure* does not require finiteness of volume). Otherwise apply the torus theorem to split M inductively into atoroidal components along disjoint incompressible tori and Klein bottles. □

1.8. Characteristic submanifolds

In this section we discuss how a compact Haken manifold can fail to be acylindrical. Consider a compact 3-manifold pair (N, R), where $R \subset \partial N$ is an incompressible surface. There exists a 3-manifold subpair $(X, S) \subset (N, R)$, where $S \subset \partial X$, X is possibly disconnected codimension zero submanifold in N, which is called a *characteristic subpair* of (N, R) and which satisfies the following properties:

1. Each component of (X, S) is either a pair (I–bundle, ∂I–subbundle) over a surface or is a solid torus fibered by circles, where S is the union of fibered annuli.
2. $\partial X \cap \partial N = S$.
3. The components of $\partial_1(X) = \partial X - S$ are essential annuli.
4. Any essential incompressible annulus (and Moebius band) $(A, \partial A) \to (N, R)$ is homotopic (as a map of pairs) into (X, R).
5. The characteristic subpair is unique up to isotopy.

The submanifold X is called the *characteristic submanifold* in N. Theory of characteristic subpairs was developed in [JS79], [Joh79], [Jac81].

Suppose that $P = \partial N - R$ and (N, P) is a pared manifold. Then the characteristic subpair $(X, S) \subset (N, R)$ is called the *characteristic pair* of the pared manifold (N, P). The decomposition of (N, P) into the union of components of X and $N - int(X)$ is called the *canonical* JSJ (Jaco-Shalen-Johannson) decomposition of (N, P). Components N_j of $N - int(X)$ are:

- *Relatively acylindrical* in the sense that any essential annulus (and Moebius band) in N_j must intersect $P \cup (N_j \cap X)$.
- *Have relatively incompressible boundary* in the sense that any compressing disk for ∂N_j must intersect $P \cup (N_j \cap X)$.

Far-reaching generalization of the canonical JSJ decomposition is developed in [RS94], [RS97], [DS99], [DS], [FP97].

Example 1.46. *Let N_j, $j = 1, 2$ be a pair of acylindrical manifolds with boundary surfaces of the genus 2. Choose a pair of simple homotopically nontrivial loops $\gamma_j \subset \partial N_j$ and a compact oriented surface H with two boundary components. Glue H to $N_1 \cup N_2$ so that ∂H is glued to the union $\gamma_1 \cup \gamma_2$. Then take an I-bundle $I(H) = I \times H$ over H and use it to "thicken" the surface H in the complex $N_1 \cup H \cup N_2$. The result is a 3-manifold N. Then the pair $(I(H), \{\pm 1\} \times H)$ is the characteristic subpair in $(N, \partial N)$.*

Example 1.47. *In this example the characteristic subpair contains a foliated solid torus. Take two copies of the manifold $I \times F$ where F is a closed orientable surface of the genus 2, choose a separating simple homotopically nontrivial loop γ in $F \times \{1\}$, let $\mathcal{N}(\gamma)$ be a regular neighborhood of γ on $F \times \{1\}$. Let $V \subset U$ be regular neighborhoods of γ in $I \times F$ so that boundaries of U and V are disjoint. Glue two copies of $I \times F$ along $\mathcal{N}(\gamma)$. The result is a manifold L which admits a rotation of order 2 around γ so that the corresponding cyclic group $\mathbb{Z}_2$ acts freely on γ. Now take the quotient of L by $\mathbb{Z}_2$. The result is a manifold N. The characteristic submanifold in N consists of two components which are: a copy of $I \times F - int(U)$ and the projection of V to N.*

We now return to the discussion of characteristic subpairs. Let (X, S) be the characteristic subpair in (N, R). Choose a component (X_j, S_j) of (X, S) which is a solid torus. Then $\partial X_j - S_j$ is a union of essential annuli in N. For each j thicken these annuli and consider them as I-bundles. Add to them all components of (X, S) which are I-bundles. The result is a submanifold $W \subset N$ (which is an I-bundle). W is called the *window* in N. The frontier $\partial_1 W = cl(\partial W - \partial N)$ is the union of essential annuli. Then the property (4) of the characteristic subpair implies that for each component L of $N - intW$:

(a) either L is the solid torus (which appears as above),

(b) or all essential annuli in $(L, \partial N \cap L)$ are parallel in L to essential annuli on the boundary of the window.

Exercise 1.48. *Let S be a compact orientable surface of genus 1 with two holes, $f : \gamma_1 \to \gamma_2$ is a degree 2 map between the boundary circles of S, Z is the CW-complex obtained from gluing S to itself via f. Show that $\pi_1(Z)$ is not a 3-manifold group. Construct a finite covering $Z' \to Z$ so that $\pi_1(Z')$ is a 3-manifold group.*

1.9. 3-manifolds fibered over $\mathbb{S}^1$

Suppose that (M, P) is a pared orientable 3-manifold, where $P = \partial M$ is a collection of incompressible boundary tori. Let $S \subset M$ be a connected superincompressible surface. Define (M', P') as a compact pared manifold obtained by splitting M along S. Note that the manifold $M' - P'$ has two boundary components which are homeomorphic copies of S. We assume that at least one component M_1' of M' is an interval bundle over a surface Σ_1 so that $P_1' := P' \cap M_1$ is the interval bundle over ∂S_1. We will abbreviate this by saying: (M_1, P_1') is an interval bundle over a surface. There are two cases that may occur.

Case 1. M' is connected. Since $\partial M' - P'$ has two components we conclude that $M_1 = M'$ is the trivial interval bundle $M' = \Sigma_1 \times I$. Thus the manifold M is a surface bundle over $\mathbb{S}^1$ with the fiber Σ_1.

Case 2. M' consists of two components M_1', M_2'. Each of these manifolds contains exactly one component of $\partial M_j' - P'$. If the manifold M_1' is a nontrivial interval bundle then the base of the bundle has to be nonorientable since the manifold M is orientable.

Subcase 2a. The manifold M_2' is also a nontrivial interval bundle over a surface Σ_2. Note that the base of this fibration is uniquely determined by the topology of $\partial M_2' - P'$ and hence Σ_2 is homeomorphic to Σ_1, which is a nonorientable surface.

Proposition 1.49. *Under the above conditions there exists a 2-fold covering $N \to M$ so that:*

- *N is fibered over $\mathbb{S}^1$.*
- *The lift of Σ_1 to N is a fiber of this fibration.*

- *This lift is connected and is isotopic to a component of the lift of S.*

Proof: Let $\Sigma \cong \Sigma_1 \cong \Sigma_2$, then $S \to \Sigma$ is the 2-fold orientation covering over Σ. Now construct the covering $N \to M$ as follows. We assume that Σ_1, Σ_2 are embedded in M as zero sections of the corresponding interval bundles M_1', M_2'. The union $\Sigma_1 \cup \Sigma_2 = Z$ determines a nontrivial class $[Z]$ in $H_2(M, \partial M; \mathbb{Z}/2)$ (by the Mayer-Vietoris sequence). Note that the homology class of S in $H_2(M, \partial M; \mathbb{Z}/2)$ is trivial. By duality $[Z]$ defines a nontrivial homomorphism $H_1(M, \mathbb{Z}/2) \to \mathbb{Z}/2$. Thus we get an epimorphism

$$\psi : \pi_1(M) \to H_1(M, \mathbb{Z}/2) \to \mathbb{Z}/2$$

whose kernel defines the covering $N \to M$. Since $\pi_1(S) \subset \pi_1(\Sigma_1)$ lies in the kernel of ψ we conclude that the lift $\tilde{\Sigma}_1$ of Σ_1 to N is a single orientable surface homeomorphic to S. The same is true for Σ_2. The complement $M - Z$ is a trivial interval bundle $S \times I$ over S, its lift to N consists of two homeomorphic components bounded by $\tilde{\Sigma}_1 \cup \tilde{\Sigma}_2$. Thus the whole manifold N is a surface bundle over $\mathbb{S}^1$. $\square$

Definition 1.50. In the circumstances of Proposition 1.49, the surface $S \subset M$ is called a **virtual fiber** in a fibration over $\mathbb{S}^1$.

Subcase 2b. The manifold M_2' is not an interval bundle over a surface. In this case as in the proof of Proposition 1.49 the surface Σ_1 defines a cycle Z in $H_2(M, P; \mathbb{Z}/2)$ which by duality corresponds to an epimorphism

$$\psi : \pi_1(M) \to H_1(M, \mathbb{Z}/2) \to \mathbb{Z}/2$$

whose kernel defines a covering $N \to M$. The surface Σ_1 lifts to this covering as an orientable surface $F \cong S$. By the assumption the complement $N - F$ is not an interval bundle over S. It proves the following:

Proposition 1.51. *Under assumptions of the Subcase 2b, the manifold (M, P) admits a 2-fold covering (N, Q) which contains a nonseparating superincompressible surface F so that $(N - F, Q - F)$ is not an interval bundle over a surface.*

Let S be a compact surface (maybe with boundary). A homeomorphism $\tau : S \to S$ is called *aperiodic* if for each nontrivial nonperipheral element $\gamma \in \pi_1(S)$ the following holds:

for every integer $n \neq 0$ the elements $\tau_*^n(\gamma)$ and γ are not conjugate in $\pi_1(S)$.

Suppose now that S has negative Euler characteristic. The fundamental group of the mapping torus M_τ (of τ) splits as

$$1 \to \pi_1(S) \to \pi_1(M_\tau) \to \mathbb{Z} \to 1$$

where a generator t of $\mathbb{Z}$ acts on $\pi_1(S)$ as τ_* (up to composition with an inner automorphism).

It is an easy exercise to show that τ is aperiodic if and only if the mapping torus M_τ is an atoroidal 3-manifold. For instance, if M_τ is not atoroidal, then there is a subgroup $\mathbb{Z} \times \mathbb{Z}$ in it which does not come from one of the boundary tori. Hence there has to be an element $\gamma \in \pi_1(S)$ which is not peripheral so that $t^m \gamma = \gamma t^m$, i.e. $\tau_*^m(\gamma) = \gamma$.

1.10. Baumslag-Solitar relations

Suppose that N is a Haken manifold, $\Sigma', \Sigma'' \subset \partial N$ are disjoint incompressible subsurfaces, $\gamma' \in \pi_1(\Sigma')$ and $\gamma, \gamma'' \in \pi_1(\Sigma'')$ are nontrivial elements. Let $\alpha' \in \pi_1(\Sigma'), \alpha'' \in \pi_1(\Sigma'')$ denote the primitive roots of γ', γ''. Assume that $\gamma, \gamma', \gamma''$ are conjugate in $\pi_1(N)$. Let $\Sigma := \Sigma' \cup \Sigma''$.

Lemma 1.52. *Under the above conditions $g\alpha' g^{-1} = \alpha''$ and γ, γ'' are conjugate in $\pi_1(\Sigma'')$.*

Proof: Take the characteristic submanifold $(X, S) \subset (N, \Sigma)$. Then there exists a component $(X_j, S_j) \subset N$ so that $\gamma, \gamma', \gamma''$ are represented by loops ℓ, ℓ', ℓ'' in $\partial X_j \cap \Sigma$, the loops ℓ, ℓ', ℓ'' are homotopic in X_j. There are two cases: either (N_j, S_j) is an interval bundle over a surface Σ_j or X_j is a solid torus. In the both cases the assertions of lemma are obvious. $\square$

Suppose that a, b are elements of a group G. These elements satisfy *Baumslag–Solitar relation* if

$$a^p = ba^q b^{-1} \quad \text{for some} \quad p, q \neq 0. \tag{1.2}$$

Note that if $p = q$, then the relation reduces to the "torus" relation

$$cb = bc \quad \text{where} \quad c := a^p.$$

If $p = -q$ then we get the "Klein bottle" relation:

$$cb = bc^{-1} \quad \text{where} \quad c := a^p.$$

If $|p| \neq |q|$ and $a \neq 1$, then we say that the Baumslag-Solitar relation (1.2) is *nonelementary.* The group $BS_{p,q}$ given by the presentation

$$\langle a, b | a^p = ba^q b^{-1} \rangle$$

is called a *Baumslag–Solitar* group.

Exercise 1.53. *Show that $BS_{2,3}$ contains $\mathbb{Z}^2$.*

Theorem 1.54. *(W. Jaco, P. Shalen, [JS79].) Suppose that G is the fundamental group of a Haken[1] 3-manifold M, then there are no nontrivial elements $a, b \in G$ which satisfy a nonelementary Baumslag-Solitar relation.*

Proof: The proof is done by induction on the length of Haken hierarchy of M. If M is a ball there is nothing to prove. Assume that $G = \pi_1(M)$ does have a nonelementary Baumslag-Solitar relation.

Lemma 1.55. *Suppose that $G = X *_W Y$ (amalgamated free product) or $G = X*_W$ (HNN extension). Then for every pair of elements $a, b \in G$ which satisfy a nonelementary Baumslag-Solitar relation, the element a is conjugate either into X or into Y.*

[1] P. Shalen proved that the Haken assumption is actually unnecessary, [Sha].

Proof: The best proof of this lemma that I know is via actions of groups on trees. So, skip it or read first the material of Chapter 10. The amalgamated free product as well as the decomposition as HNN-extension corresponds to an effective action of G on a simplicial tree T so that the quotient T/G is either a segment $[pq]$ (in the case of amalgamated free product) or is a circle with one vertex p (in the case of HNN-extension). Stabilizers (in G) of the lifts of p, q into T are subgroups conjugate either to X or to Y. Elements of G which are not conjugate into X, Y act freely on T (they are called *hyperbolic elements*). Hyperbolic elements have the property: each hyperbolic element g has an unique invariant geodesic $A_g \subset T$, where g acts as a translation. The translation length $\ell_T(g)$ is the distance $d(x, gx)$, $x \in A_g$. This distance is an invariant of the conjugacy class of g. If the Baumslag-Solitar relation (1.2) is satisfied, the elements a^p, ba^qb^{-1}, a^q have the same translation length. If $\ell_T(a) > 0$ then $|p| = |q|$, i.e. the Baumslag-Solitar relation is elementary. Otherwise, a has a fixed point in T (the element a is *elliptic*), i.e. it is conjugate either into X or into Y. $\square$

So, let $S \subset M$ be an incompressible surface properly embedded into M, the manifold N is obtained by splitting M along S. Let $S', S'' \subset \partial N$ be incompressible surfaces corresponding to S, $\tau : S' \to S''$ be the gluing map such that $N/\tau = M$. Let $\tau_* : \pi_1(S') \to \pi_1(S'')$ be the isomorphism induced by τ. There are two possible cases: either N is connected or it consists of two components. Let me consider the former case (the case of two components is similar), then G is an HNN extension of $\pi_1(N)$:

$$G = \langle \pi_1(N), t | tgt^{-1} = \tau_*(g), \quad g \in \pi_1(S') \rangle.$$

By the induction hypothesis, $\pi_1(N)$ has no nonelementary Baumslag-Solitar relations. According to Lemma 1.55, the only way $G = \pi_1(M)$ can satisfy a nonelementary Baumslag-Solitar relation is that there are nontrivial elements $a^p, a^q \in \pi_1(N)$ which are conjugate in $\pi_1(M)$. After choosing a, b within their $\pi_1(N)$-conjugacy classes, we can assume that the relation is:

$$t^s a^q t^{-s} = a^p.$$

Let me consider the case $s = 1$. Then the conjugation between the elements a^p, a^q is given by a map $\alpha : A = \mathbb{S}^1 \times I \to M$ so that:

- $\alpha_0 = a^p, \alpha_1 = a^q$;
- $\alpha(\bullet, 0.5) : \mathbb{S}^1 \to S$.

Thus $\alpha(\bullet, 0.5)$ gives us two nontrivial loops γ', γ'' on S', S''. Now the assertion of Theorem follows from Lemma 1.52. $\square$

Theorem 1.56. *(B. Evans, W. Jaco, [EJ73].) Suppose that G is a nilpotent*[2] *subgroup of $\Gamma = \pi_1(M)$, where M is a Haken 3-manifold. Then G is finitely generated.*

[2] We refer the reader unfamiliar with nilpotent groups to §4.1 for the definition of n-step nilpotent groups.

Proof: We first exclude the case of G isomorphic to the additive group of rational numbers (cf. [EM72]). If A is a group, $a \in A - \{1\}$, then we say that the *order of the maximal root* of a in A is $\sup\{r \in \mathbb{Z}_+ : a = b^r, b \in A\} \in \mathbb{Z}_+ \cup \{\infty\}$.

The proof is induction on the length of Haken hierarchy of M similarly to the previous theorem. If M is a ball there is nothing to prove. Suppose that M is obtained by gluing M_1, M_2 (which might be equal) along an incompressible boundary subsurface, so that M_1, M_2 satisfy the conclusion of theorem. The corresponding decomposition of $\pi_1(M)$ as amalgamated free product or HNN-extension gives rise to an action of $\pi_1(M)$ on a simplicial tree T. The group G acts on the tree T, thus G consists of elliptic automorphisms of T if and only if a single nontrivial element $\gamma \in G$ is elliptic. If some (any) element γ of G is hyperbolic then $\ell_T(\gamma) \geq 1$ which implies that $\ell_T(\gamma) \geq 1$. This means that one can extract only finitely many roots from γ and hence G is finitely generated in this case.

So we assume that G consists of elliptic elements. If $g \in G - \{1\}$ fixes only one vertex in T then this vertex is also fixed by the whole group G, which implies that G is conjugate either to $\pi_1(M_1)$ or to $\pi_1(M_2)$ and we are done by induction. Let $\gamma \in G$ be a nontrivial elliptic element, then γ fixes an edge e of T, i.e. it is a peripheral element of $\pi_1(M_j)$ (up to conjugation). Lemma 1.52 implies that for each edge $f \in T$ and its stabilizer $\Gamma_f \subset \Gamma$, the order of the maximal root of γ in Γ_f is the same as the order r_e of the maximal root of γ in Γ_e. Since each element of $G \cong \mathbb{Q}$ fixes an edge in T, there exists $\alpha \in G$ such that $\gamma = \alpha^r, r > r_e$ and $\alpha \in \Gamma_f$ for some edge f. Contradiction.

Now consider the general case. We proved that G contains no infinitely divisible elements. If G is a torsion-free abelian group without infinitely divisible elements and G is not finitely generated then G contains direct products $\mathbb{Z}^m$ for arbitrarily large m. Hence, for $m \geq 4$ the covering $\tilde{M} \to M$ corresponding to $\mathbb{Z}^m$ has $H_4(\tilde{M}) \neq 0$, which is impossible for a 3-manifold.

Suppose that G is non-abelian. Briefly, the cohomological dimension of G in this case has to be 3 in which case G is a finite index subgroup of $\pi_1(M)$; this implies that G is finitely generated. We give now more details without assuming that the reader is familiar with the properties of cohomological dimension of groups. If G is a nonabelian torsion-free nilpotent group without infinitely divisible elements then G contains a non-abelian 2-step nilpotent subgroup N which fits into a short exact sequence

$$1 \to \mathbb{Z}^a \to N \to \mathbb{Z}^b \to 1.$$

Let $m := a + b$ and let $K(N, 1)$ be an Eilenberg-MacLane space of N; $K(N, 1)$ can be chosen a closed m-manifold which fibers over T^b with the fiber T^a. Note that $H^m(K(N, 1), \mathbb{Z}) \neq 0$ which (as in the abelian case) implies that $m \leq 3$. If $m = 3$ then we have a covering $\tilde{M} \to M$ so that $\pi_1(\tilde{M}) = N$. The manifold $\tilde{M}$ is homotopy-equivalent to $K(N, 1)$, hence $H^3(\tilde{M}) = \mathbb{Z}$; this implies that $\tilde{M}$ is compact and the covering $\tilde{M} \to M$ is finite. Thus N, and therefore G, is a finite index subgroup in $\pi_1(M)$ and G is finitely generated. It remains to analyze the case $a + b = 2$. Since N is nonabelian it follows that $a = b = 1$. However, since N is nilpotent, $\mathbb{Z}$ is a central subgroup in N which implies that N is abelian. Contradiction. $\square$

Chapter 2

Thurston Norm

In this chapter we prove the following theorem:

Theorem 2.1. *(W. Thurston [Thu85].) Suppose that M is a compact atoroidal orientable 3-manifold such that rank $H_2(M, \partial M; \mathbb{Z}) \geq 2$. Then M contains an embedded superincompressible surface which is not a fiber in a fibration of M over $\mathbb{S}^1$ and which represents a nontrivial element of $H_2(M, \partial M; \mathbb{Z})$.*

Proof of this theorem will be finished in §2.3. Our proof is essentially the same as Thurston's.

2.1. Norms defined over $\mathbb{Z}$

Consider the space $\mathbb{R}^n$ with the lattice $\mathbb{Z}^n$ embedded in the standard way. Suppose that $p : \mathbb{Z}^n \to \mathbb{Z}$ is a non-negative function which satisfies the following axioms:

1. p is linear along rays, i.e. $p(m \cdot z) = |m|p(z)$ for each $z \in \mathbb{Z}^n$, $m \in \mathbb{Z}$.
2. p is convex over $\mathbb{Z}$, i.e. for each $k, m \in \mathbb{Z}$ and $z, w \in \mathbb{Z}^n$ we have:
$$p(k \cdot z + m \cdot w) \leq |k|p(z) + |m|p(w).$$
3. p is nondegenerate, i.e. $p(z) = 0$ iff $z = 0$.

Function p satisfying these properties is called a *norm defined over* $\mathbb{Z}$.

Theorem 2.2. *Each norm defined over $\mathbb{Z}$ extends to a (usual) norm defined on the whole space $\mathbb{R}^n$.*

Proof: Suppose that $\alpha \in \mathbb{Q}^n - \{0\}$. Then there exists $m \in \mathbb{Z}$ such that $m \cdot \alpha \in \mathbb{Z}^n$ and we let $p(\alpha) = p(m \cdot \alpha)/m$. This extension of the function p to $\mathbb{Q}^n$ is well-defined since p is linear on rays in $\mathbb{Z}^n$. Clearly the function $p : \mathbb{Q}^n \to \mathbb{Q}$ is linear on rays and $\mathbb{Q}$-convex in the sense that $p(k\alpha + m\beta) \leq |k|p(\alpha) + |m|p(\beta)$ for all $k, m \in \mathbb{Q}$ and $\alpha, \beta \in \mathbb{Q}^n$.

Lemma 2.3. *Sublevel sets of the function $p : \mathbb{Q}^n \to \mathbb{Q}$ are bounded.*

Proof: Each sublevel set $\{p(\alpha) \leq c\}$ is $\mathbb{Q}$-convex, thus if this set is unbounded, it must contain a complete $\mathbb{Q}$-ray $\{q \cdot \alpha : q \in \mathbb{Q}_+\}$ for some $\alpha \neq 0$. However $p(k\alpha) = kp(\alpha) \to \infty$ as $k \to \infty$. Contradiction. $\square$

Therefore p is uniformly continuous and hence admits a continuous extension to a convex function $p : \mathbb{R}^n \to \mathbb{R}_+$. To prove that p is a norm we have to check that that $p(\alpha) = 0$ if and only if $\alpha = 0$.

Lemma 2.4. $p(\alpha) = 0$ *if and only if* $\alpha = 0$.

Proof: Suppose $p(\alpha) = 0$ and $\alpha \neq 0$. Then $p(t\alpha) = 0$ for all $t \in \mathbb{R}$. Let $\epsilon = 1/2$, then $p(z) \geq 2\epsilon$ for all $z \in \mathbb{Z}^n - \{0\}$. The sublevel set $C(\epsilon) = \{p(\sigma) < \epsilon : \sigma \in \mathbb{R}^n\}$ is an open convex cone since it contains the line $\mathbb{R} \cdot \alpha$ and p is a continuous convex function. Thus $C(\epsilon)$ must contain at least one integral point $z \in \mathbb{Z}^n - \{0\}$. Contradiction. $\square$

This finishes the proof of Theorem 2.2. $\square$

We retain the name *norm defined over* $\mathbb{Z}$ for the extension of p to $\mathbb{R}^n$. Let

$$B^{(p)}(r) = \{v \in \mathbb{R}^n : p(v) \leq r\}$$

denote the ball of radius r centered at zero, with respect to the norm p.

Theorem 2.5. *Any norm $p : \mathbb{R}^n \to \mathbb{R}$ defined over $\mathbb{Z}$ has the following property. Suppose that z is an element of $\mathbb{Z}^n$. Then there exists a linear function l such that:*

- $l(z/p(z)) = 1$;
- $l : \mathbb{Z}^n \to \mathbb{Z}$, *i.e. the function l is defined over $\mathbb{Z}$;*
- *the operator norm of l with respect to p is equal to* 1, *i.e.* $|l(\alpha)| \leq 1$ *for all* $\alpha \in B^{(p)}(1)$.

Proof: We consider the sequence of *integral balls* of the radius r in $\mathbb{R}^n$: $B^{(p)}(r) \cap \mathbb{Z}^n$, where $r \in \mathbb{Z}_+ \cdot p(z)$. Then

$$\bigcup_{r \in \mathbb{Z}^+} \frac{1}{r} \cdot (B^{(p)}(r) \cap \mathbb{Z}^n) = B^{(p)}(1) \cap \mathbb{Q}^n.$$

Note that the integer vector $\frac{r}{p(z)} \cdot z$ belongs to the boundary of $B^{(p)}(r)$ for each r as above. The convex hulls $C(r)$ of $B^{(p)}(r) \cap \mathbb{Z}^n$ are finite polyhedra whose faces are defined over $\mathbb{Z}$ in the sense that each top-dimensional face is contained in zero level set of a linear function with integer coefficients. Thus for each point $\frac{r}{p(z)} \cdot z$ there is linear function l_r with integer coefficients such that $l_r(\frac{r}{p(z)} \cdot z) = r$ and the level set $\{l_r(w) = r\}$ is disjoint from the interior of $C(r)$. Hence $l_r(z/p(z)) = 1$ and the level set $\{l_r(v) = 1\}$ is disjoint from the interior of $\frac{1}{r}C(r)$. However the convex sets $\frac{1}{r}C(r)$ exhaust the unit ball $B^{(p)}(1)$. Thus the operator norms (with respect to p) of the linear functions l_r are convergent to 1 as $r \to \infty$. This means that the sequence of linear functions l_r with integer coefficients is subconvergent as $r \to \infty$ to a linear function l whose norm is equal to 1. Clearly l has integer coefficients and $l(z/p(z)) = 1$. $\square$

The ratios $z/p(z), z \in \mathbb{Z}^n$, are dense in the unit sphere $\partial B^{(p)}(1)$. Thus the unit ball $B^{(p)}(1)$ is the intersection of the sublevel sets $\{l \leq 1\}$, where the linear functions l are as in Theorem 2.5. However the number of such linear functions l is finite. We conclude that $B^{(p)}(1)$ is a polyhedron with finitely many faces which are given by linear equations with integer coefficients.

Corollary 2.6. *Suppose that v is a vertex of the polyhedron $B^{(p)}(1)$. Then there exists an element $z \in \mathbb{Z}^n$ such that $v = z/p(z)$.*

2.2. Variation of fiber-bundle structure

Suppose that M is a compact 3-manifold (smoothly) fibered over $\mathbb{S}^1$, let $\mathcal{F}$ denote the fibration and F_t denote the fiber of $\mathcal{F}$ over $t \in \mathbb{S}^1$. The fibers F_t are transversal to the boundary of M, we assume that M is given a Riemannian metric such that F_t are orthogonal to ∂M. Tangent planes to the fibers F_t define a plane subbundle $P \subset T(M)$. Let L denote the unit vector field on M which is orthogonal to P. Along each boundary curve ∂F_t we choose the unit tangent field X on F_t which is orthogonal to ∂M. Thus we get a section X of $P|_{\partial M}$. The pair (P, X) determines an element of $H_2(M, \partial M)^* = H^1(M)$ as follows. Suppose that $h : (\Sigma, \partial\Sigma) \to (M, \partial M)$ is a proper smooth map from a surface Σ which is a representative of a relative cycle $\zeta \in Z_2(M, \partial M)$. Then $h^*(P)$ is a 2-dimensional vector bundle over Σ with the prescribed section $h^*(X)$ over $\partial\Sigma$. There is a well-defined obstruction (the relative Euler number) $e(h^*(P), X)) \in \mathbb{Z}$ to the extension of $h^*(X)$ to a nonzero section of the bundle $h^*(P)$. Thus we define $\tau \in H_2(M, \partial M)^*$ as $\tau([\zeta]) = e(h^*(P), X)$. It is easy to see that τ is well-defined. Obviously for the relative class $[\zeta] \in H_2(M, \partial M)$ which is represented by a fiber F_t we get: $\tau([\zeta]) = \chi(F_t)$.

There is a closed nondegenerate integer 1-form θ on M whose kernel is the tangent subbundle of $\mathcal{F}$. Namely, if $f : M \to \mathbb{S}^1$ is the fibration, then θ is the pull-back under f of the angle form dt from $\mathbb{S}^1$. The converse to this is true as well:

Theorem 2.7. *(D. Tishler, [Tis70].) Suppose that ω is a closed nondegenerate 1-form on M which has integer periods (i.e. it determines an element of $H^1(M, \mathbb{Z})$). Assume that the restriction of ω to ∂M is nondegenerate. Then there exists a fibration $\mathcal{G}$ of M over $\mathbb{S}^1$ such that fibers of $\mathcal{G}$ are tangent to the kernel-distribution of ω.*

Sketch of the proof: Choose a base-point $p \in M$ and consider the indefinite integral

$$f(q) = \int_p^q \omega \in \mathbb{R}/\mathbb{Z}.$$

The function $f(q)$ is well-defined, smooth and local calculation shows that f has maximal rank at each point. Thus f is a fibration. $\square$

Now suppose that ω is a closed rational nondegenerate 1-form on M which is sufficiently close to θ, namely we assume that the kernel-distribution of ω is transversal to the vector-field L. Since ω is rational, the kernel-distribution Q of ω is tangent to a fibration $\mathcal{G}$ of M by surfaces $G_t, t \in \mathbb{S}^1$. The fibers of $\mathcal{G}$ determine a relative class $[\xi] \in H_2(M, \partial M)$.

Lemma 2.8. $\tau([\xi]) = \chi(G_t)$.

Proof: Since both $\mathcal{F}$ and $\mathcal{G}$ are transversal to the vector field L, we conclude that for each fiber $G = G_t$ of $\mathcal{G}$:

$$P|_G \cong T(M)|_G / Span(L) \cong Q|_G.$$

Thus the bundles $P|_G$ and $Q|_G$ are isomorphic and moreover the isomorphism λ between them carries the section X (of $P|_{\partial G}$) to the tangent vector field $\lambda(X)$ on G which is normal to ∂G. Therefore the obstruction $\chi(G)$ to the extension of $\lambda(X)$ to a nonzero field on G is the same as the relative Euler number $\tau([\xi]) = e(P|_G, X)$. $\square$

2.3. Application to incompressible surfaces

We consider a compact irreducible atoroidal orientable manifold M with (possibly empty) incompressible boundary of zero Euler characteristic. Suppose that $\xi \in H_2(M, \partial M; \mathbb{Z})$ is a relative homology class. Define *Thurston's norm* $x(\xi)$ as

$$x(\xi) = ||\xi|| := \min\{|\chi(S)| : (S, \partial S) \subset (M, \partial M) \text{ is an embedded surface representing the class } \xi\}.$$

Since the manifold M is atoroidal, $x(\xi) \neq 0$ for each $\xi \neq 0$. There is a generalization of Thurston's norm which is defined by taking minimum over immersed surfaces representing the given homology class. It turns out that two norms coincide, see [Gab83], [Per93].

Lemma 2.9. *Every element $\xi \in H_2(M, \partial M; \mathbb{Z})$ is represented by an embedded oriented surface S. If $\xi/n \in H_2(M, \partial M; \mathbb{Z})$ for some $n > 0$, then any surface S representing ξ is the union of n disjoint subsurfaces S_j each representing ξ/n.*

Remark 2.10. The surfaces S_j could be disconnected.

Proof: Recall that (by duality) each element $\zeta \in H_2(M, \partial M; \mathbb{Z})$ determines an element $\zeta^* \in H^1(M; \mathbb{Z}) \cong H^1(\pi_1(M), \mathbb{Z})$, the latter is a homomorphism $\zeta^* : \pi_1(M) \to \mathbb{Z}$. The homomorphism ζ^* is induced by a PL map $f : M \to \mathbb{S}^1$. If c is a regular value of f then the surface $F = f^{-1}(c)$ is embedded, has canonical orientation and represents the class ζ. This proves the first assertion of Lemma (cf. Lemma 1.23).

Suppose that $\zeta \in H_2(M, \partial M; \mathbb{Z})$ is represented by an embedded relative 2-cycle F. The image of the 1-st homology group of F in $H_1(M, \mathbb{Z})$ lies in the kernel of ζ^*, hence the restriction of f to F is homotopic to a constant and we can choose f within its homotopy class to be constant on F. We can also assume that f has regular value at $c = f(F)$. The inverse image $f^{-1}(c)$ can be larger then the surface F. However, since $f^{-1}(c)$ is homologous to F, we conclude that $F_0 = f^{-1}(c) - F$ is homologically trivial.

If two maps $f, g : M \to \mathbb{S}^1$ correspond to the same homology class ζ, then f is homotopic to g (since they induce the same homomorphism of $\pi_1(M)$). Now let us use this to prove the second assertion of Lemma. Suppose that $\alpha = \xi/n \in H_2(M, \partial M; \mathbb{Z})$, $f_\alpha, f_\xi : M \to \mathbb{S}^1$ are the corresponding maps and the subsurface $S \subset f_\xi^{-1}(c)$ represents the homology class ξ. Let $\lambda : \mathbb{S}^1 \to \mathbb{S}^1$ be the n-fold covering, the inverse image $\lambda^{-1}(c)$ is a set $\{b_1, ..., b_n\}$. Hence the function $\lambda \circ f_\alpha$ corresponds to the class $n\alpha = \xi$; this function is homotopic to f_ξ. Thus f_ξ lifts to a function $\tilde{f}_\xi : M \to \mathbb{S}^1$ via the covering λ. Therefore the surface

$$S \cup S_0 = f_\xi^{-1}(c) = \bigcup_{i=1}^{n} \tilde{f}_\xi^{-1}(b_i)$$

is a disjoint union of n subsurfaces $S_1, ..., S_n$. Recall that the relative homology class $[S_0]$ is trivial. Since the function $\tilde{f}_\xi$ is homotopic to f_α, we conclude that each of these subsurfaces represents the homology class α. $\square$

From now on we shall assume that all components of surfaces representing nontrivial homology classes in $H_2(M, \partial M; \mathbb{Z})$ have negative Euler characteristic. (We can make this assumption since ∂M is incompressible and M is atoroidal and aspherical.)

Theorem 2.11. *Suppose that M is a 3-manifold as above. Then the function*

$$x : H_2(M, \partial M; \mathbb{Z}) \to \mathbb{Z}$$

is a norm defined over $\mathbb{Z}$.

Proof: According to Lemma 2.9 we have:

$$x(n\xi) \geq |n| x(\xi) \text{ for all } \ n \in \mathbb{Z}.$$

The opposite inequality is obvious. Thus $x(n\xi) = |n| x(\xi)$ for all $n \in \mathbb{Z}$, i.e. x is linear on rays.

Lemma 2.12. *If $\alpha, \beta \in H_2(M, \partial M; \mathbb{Z})$ then $x(\alpha + \beta) \leq x(\alpha) + x(\beta)$.*

Proof: Represent the classes α, β by PL-embedded oriented surfaces A, B. We can assume that these surfaces intersect transversally: their intersection is a 1-dimensional submanifold Γ. By Theorem 1.23 we can assume that each surface A, B is incompressible. Consider components of $A - \Gamma$ and $B - \Gamma$. If a component of, say, $A - \Gamma$ is a disk D_A bounded by a loop

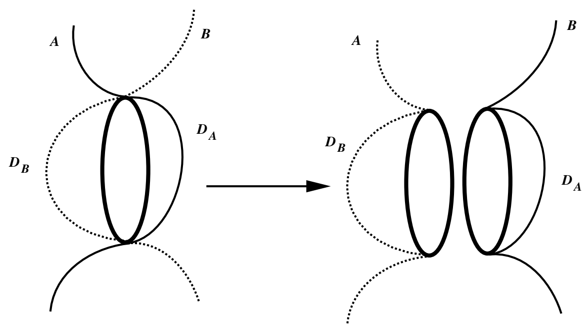

Figure 2.1: *Trading disks and pushing surfaces apart.*

$\gamma \subset A$ then γ bounds a disk D_B on B and we can "trade" these disks: take $A := A - D_A \cup D_B$, $B := B - D_B \cup D_A$ (see Figure 2.1).

By pushing the new surfaces apart near γ we eliminate the intersection along γ. This procedure preserves the homology classes of A and B (since M is aspherical and the sphere $D_A \cup D_B$ is contractible in M) and preserves the Euler characteristics of the surfaces. Thus we may assume that each component of $A - \Gamma$ and $B - \Gamma$ is different from the disk. The union of surfaces $A \cup B$ with the natural orientation of simplices on A, B represents the class $\alpha + \beta$. Now we remove Γ from both A and B and glue the components of the complement as follows. Suppose that $C_A \subset A - \Gamma$ is adjacent to a loop $\gamma \subset \Gamma$. There are exactly two components of $B - \Gamma \cap Nbd(\gamma)$ which are adjacent to γ. We glue C_A to those which induces orientation on γ opposite to the orientation induced by C_A and do the same with component $C_B \subset B - \Gamma$ (exactly the same way as we did with disks D_A, D_B above, see Figure 2.1). We repeat this procedure for all components of $A \cup B - \Gamma$. The resulting singular 2-cycle $R(A \cup B)$ has non-transversal self-intersections along Γ. Clearly $R(A \cup B)$ has the same homology class as $\alpha + \beta$. Thus we push the self-intersections apart and the result is an embedded oriented surface $A \oplus B$ which represents the homology class $\alpha + \beta$.

Direct calculation shows that $\chi(A) + \chi(B) = \chi(A \oplus B)$. Also, no component of $A \oplus B$ is a sphere. This proves Lemma. $\square$

The above lemma and linearity of the function x on rays imply that x is convex. Finally, the norm x is nondegenerate since there are no elements of $H_2(M, \partial M; \mathbb{Z}) - \{0\}$ which are represented by tori. $\square$

Theorem 2.2 implies that the norm x (defined over $\mathbb{Z}$) extends from $H_2(M, \partial M; \mathbb{Z})$ to $H_2(M, \partial M; \mathbb{R})$.

Theorem 2.13. *Suppose that* $\dim(H_2(M, \partial M; \mathbb{R})) \geq 2$. *Let* $\zeta \in H_2(M, \partial M; \mathbb{Z})$ *be such that* $v = \zeta / x(\zeta)$ *is a vertex of the polyhedron* $B^{(x)}(1)$. *Then* ζ **cannot** *be represented by a fiber in a fibration of* M *over* $\mathbb{S}^1$.

Proof: Suppose that ζ is a class in $H_2(M, \partial M; \mathbb{Z})$ represented by a fibration of M over $\mathbb{S}^1$. Recall that such fibration determines a nonzero element $\tau \in H_2(M, \partial M)^*$. We proved that there exists a positive number ϵ such that the linear function τ satisfies the following properties:

- The linear function τ is integer, i.e. $\tau : H_2(M, \partial M; \mathbb{Z}) \to \mathbb{Z}$.
- For all integer classes $\xi \in H_2(M, \partial M; \mathbb{Z})$ such that
$$\|\xi/x(\xi) - v\| \leq \epsilon$$
we have: $x(\xi) = -\tau(\xi)$.

Rational classes are dense in the cone

$$C_\epsilon(v) = \{\alpha \in H_2(M, \partial M; \mathbb{R}) : \|\alpha/x(\alpha) - v\| \leq \epsilon\}.$$

Thus we conclude that in this cone the integer linear function $-\tau$ coincides with Thurston's norm x. Therefore the unit vector v belongs to the interior of a top-dimensional face of the unit sphere $\partial B^{(x)}(1)$, which locally (near v) is given by the equation $\tau(\beta) = -1$. This contradicts our assumptions. □

Now we can prove the main theorem of this chapter (Theorem 2.1).

Proof: Take any vertex v of the polyhedron $B^{(x)}(1)$. According to Corollary 2.6 there exists an integral class $\zeta \in H_2(M, \partial M; \mathbb{Z})$ such that $v = \zeta/x(\zeta)$. On the other hand, by Theorem 2.13, the class ζ is not represented by a fiber in any fibration of M over $\mathbb{S}^1$. The class ζ is nontrivial, thus it can be represented by a superincompressible surface S according to Theorem 1.23. □

Corollary 2.14. *Suppose that M is an orientable atoroidal 3-manifold which fibers over the circle with the fiber Σ. Assume that M has at least two boundary components. Then M contains an embedded superincompressible surface which is not a fiber in a fibration of M over $\mathbb{S}^1$ and which represents a nontrivial element of $H_2(M, \partial M; \mathbb{R})$.*

Proof: It is clear that $\dim H_2(M, \partial M; \mathbb{R}) \geq 1$ since the relative homology class $[\Sigma]$ is nontrivial. By duality $H_2(M, \partial M; \mathbb{R}) \cong H_1(M, \mathbb{R})$. Let T_1, T_2 be distinct boundary tori of M Suppose that $[\gamma] \in H_1(T_1, \mathbb{R}) - \{0\}$ is in the kernel of the homomorphism $H_1(T_1, \mathbb{R}) \to H_1(M, \mathbb{R})$. Then $\gamma = \partial\sigma$ where $[\sigma] \in H_2(M, T_1; \mathbb{R})$. Hence the image of $[\sigma]$ in $H_1(T_2, \mathbb{R})$ is zero. Therefore $[\sigma]$ and $[\Sigma]$ are linearly independent since $[\Sigma]$ has nontrivial image in $H_1(T_j, \mathbb{R})$ for each boundary torus T_j. Thus $\dim H_2(M, \partial M; \mathbb{R}) \geq 2$ in this case. The remaining case is that $H_1(T_1, \mathbb{R})$ injects into $H_1(M, \mathbb{R})$, and $\dim H_1(M, \mathbb{R}) \geq 2$. This again implies that $\dim H_2(M, \partial M; \mathbb{R}) \geq 2$. Thus the assertion of corollary follows from Theorem 2.1. □

Corollary 2.15. *Suppose that M is an orientable atoroidal 3-manifold which fibers over the circle with the fiber Σ. Assume that M has at least one boundary component. Then M admits a finite cover $M' \to M$ so that M' contains an embedded superincompressible surface which is not a fiber in a fibration of M over $\mathbb{S}^1$ and which represents a nontrivial element of $H_2(M, \partial M; \mathbb{R})$.*

Proof: According to the above corollary, it suffices to find a finite covering over M which has at least two boundary components. Let $\Sigma' \to \Sigma$ be the characteristic covering corresponding to the kernel of the homomorphism $\pi_1(\Sigma) \to H_1(\Sigma, \mathbb{Z}_2)$. Then the surface Σ' has at least two boundary components. The manifold M is the mapping torus of a homeomorphism $\tau : \Sigma \to \Sigma$. The characteristic subgroup $\pi_1(\Sigma')$ is invariant under τ_*. There is $n \geq 1$ such that τ^n lifts to a homeomorphism τ' of Σ' which maps each boundary circle to itself. The mapping torus of τ' is the required finite covering M' over M. $\square$

Chapter 3

Geometry of the Hyperbolic Space

3.1. General definitions and notation

If G is a group acting on a set X then for each $g \in G$ we let $Fix_X(g)$ denote the fixed point set for the action of g on X and $Fix_X(G)$ the subset of X fixed by the whole group G (pointwise). Sometimes we shall omit the subscript X in this notation when it is clear what the space X is. If $Y \subset X$ is a subset then we will use the notation $Fix_G(Y)$ to denote the collection of elements $g \in G$ which fix Y pointwise. A sequence x_n in a topological space X is said to be *subconvergent* if it is relatively sequentially compact in X (i.e. each subsequence in $\{x_n\}$ contains a convergent subsequence). We will say that x_n *subconverges* to $x \in X$ if x is the limit of a subsequence in $\{x_n\}$.

Suppose that X is a metric space and A is a subset. We shall use the notation $Nbd_t(A)$ to denote the t-neighborhood of A in X, i.e.

$$Nbd_t(A) = \{x \in X : \exists a \in A \text{ so that } d(x, a) \leq t\}.$$

We let $B_r(x) := Nbd_r(\{x\})$ be the metric r-ball centered at x. Recall that the *Hausdorff distance* between two closed subsets A, B of X is

$$dist_H(A, B) = \inf\{t : A \subset Nbd_t(B), B \subset Nbd_t(A)\}.$$

This distance can be infinite. The *minimal distance* between A, B is

$$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}.$$

We shall assume that geodesics and geodesic rays are parameterized by the unit speed.

Suppose that X is a locally compact metrizable topological space with a countable basis of topology, $C(X)$ is the collection of closed subsets of X. We topologize $C(X)$ as follows. We say that a sequence $C_n \in C(X)$ is convergent to $C \in C(X)$ in the *Chabauty topology* if

- For every point $x \in C$ there exists a sequence $x_n \in C_n$ such that $\lim_{n\to\infty} x_n = x$.
- For every sequence $x_{n_j} \in C_{n_j}$ which is convergent in X we have: $\lim_{j\to\infty} x_{n_j} \in C$.

Alternatively one can describe the Chabauty topology as follows. A sequence $C_n \in C(X)$ is convergent to $C \in C(X)$ in the *Chabauty topology* if for every metric ball $B_r(x_0) \subset X$ we have:

$$\lim_{n\to\infty} dist_H(C_n \cap B_r(x_0), C \cap B_r(x_0)) = 0 \ .$$

Proposition 3.1. *(See [BP92].) $C(X)$ is compact, Hausdorff and metrizable.*

Suppose that M is a complete Riemannian manifold. Then the *injectivity radius* $InRad_x(M)$ of M at a point $x \in M$ is the smallest number r such that M contains a nontrivial geodesic with the both end-points at x and length $2r$. Another way to define it is to say that the exponential map $\exp_x : T_x M \to M$ is injective on the open ball of radius r and is not injective on its closure. The injectivity radius of M is defined as

$$InRad(M) = \inf_{x\in M} InRad_x(M) \ .$$

Definition 3.2. A **geodesic metric space** is a metric space where any two points x, y can be connected by a geodesic path γ (so that the length of γ equals $d(x,y)$). **Proper** metric space is a space where closed metric balls are compact.

We say that a subset $C \subset M$ of a geodesic metric space M is *convex* if any distance-minimizing geodesic $g \subset M$ connecting points of C is contained in C.

3.2. $CAT(\lambda)$-spaces

The term "$CAT(\kappa)$-space" was (I believe) introduced by M. Gromov in his essay [Gro87] and has nothing to do with *cats*: *CAT* stands for Cartan-Alexandrov-Toponogov. I think that historically correct term should be $RAT(\lambda)$: Rauch-Alexandrov-Toponogov. However the name $CAT(.)$ is already widely used and, besides, who likes *rats* anyway...

I give the definition in the case $-\infty \le \kappa \le 0$. A complete simply-connected geodesic metric space X is called $CAT(\kappa)$ if it satisfies the conclusion of Rauch-Toponogov comparison theorem (see [CE75]). Namely, for any geodesic triangle $[ABC]$ in X place a point D in the side $[AC]$. For each λ such that $0 \ge \lambda \ge \kappa$ (or $0 \ge \lambda > \kappa$ if $\kappa = -\infty$) take a simply-connected complete Riemannian 2-manifold $M(\lambda)$ of the constant curvature λ. Draw in $M(\lambda)$ a *comparison triangle* $\bar{A}\bar{B}\bar{C}$ which has the same side-lengths as the original triangle and place a point $\bar{D} \in [\bar{A}\bar{C}]$ within the distance $d(A,D)$ from $\bar{A}$. Then we require $d(B,D) \le d(\bar{B},\bar{D})$. See Figure 3.1.

Consider $\mathbb{R}^2$ with the usual metric and let X be the subset of $\mathbb{R}^2$ which consists of three distinct geodesic rays emanating from the origin. A subset $\Delta \subset X$ (with the induced path-metric) is called a *tripod* if Δ is closed and connected. Thus each tripod is either a point or is isometric to a closed interval (in the first two cases the tripod is called *degenerate*) or is the union of three distinct geodesic segments emanating from the origin.

A *metric tree* is a (metrically complete) nonempty geodesic metric space T where for any geodesic triangle $[abc]$ each edge $[ab]$ is contained in the union of two other edges, $[bc] \cup [ca]$. In other words, each triangle $[abc] \subset T$ is isometric to a *tripod*.

The *center* of the tripod $\Delta = [abc]$ is the point $[ab] \cap [bc] \cap [ca]$, the reader will verify that such point exists and is unique.

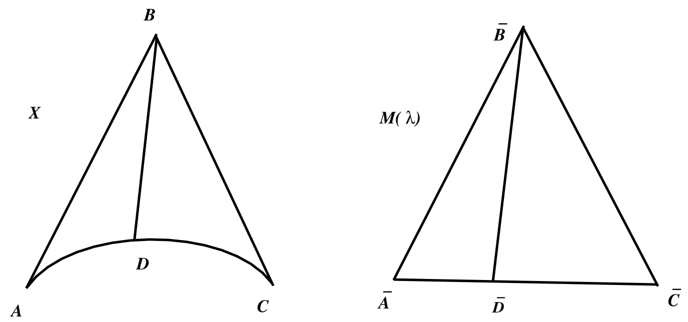

Figure 3.1: *Comparison triangles*

Basic examples of $CAT(\lambda)$-spaces:

(1) Any simply-connected complete Riemannian manifold with sectional curvature $\leq \lambda$. This is a corollary of Rauch and Toponogov comparison theorems (see [CE75]).

(2) Suppose that S is a 2-dimensional combinatorial cell-complex. Assume that d is a geodesic metric on S whose restriction to each cell is a Riemannian metric of the constant curvature λ so that the edges of S are geodesic. Then we can define total angles around vertices of S by adding up angles in the adjacent corners of adjacent cells. These angles determine a metric on each connected component of the link of each vertex of S. We require each embedded topological circle in the link of each vertex to have length at least 2π. Then the universal cover of S is $CAT(\lambda)$. The reader can find a proof of this in [Bal95].

(3) Suppose that X, Y are $CAT(0)$-spaces, then their direct product $X \times Y$ is again $CAT(0)$-space. Recall that the metric on the direct product is defined by the formula:

$$d((x, y), (x', y'))^2 = d(x, x')^2 + d(y, y')^2.$$

(4) Suppose that X is a $CAT(-\infty)$-space. Then X is a metric tree: any geodesic triangle in X degenerates to a tripod.

The distance function on $CAT(0)$-spaces is convex in the following sense: take two geodesics $\gamma_1(t), \gamma_2(t)$ in $CAT(0)$-space which are parameterized by the arc-length. Then the function $f(t) = d(\gamma_1(t), \gamma_2(t))$ is convex.

Exercise 3.3. *Show that metric balls in $CAT(0)$-spaces are strictly convex, i.e. if X is a $CAT(0)$-space and $x, y, z \in X$ such that $d(x,y) = d(x,z) = r$ then for each point $u \in [yz] - \{y, z\}$ we have $d(u,x) < r$.*

Proposition 3.4. *Suppose that X is a $CAT(0)$-space, F is a finite group of isometries of X. Then F has a fixed point in X.*

Proof: Pick a point $x \in X$ and consider the orbit $F(x) = \{x_1, ..., x_n\}$. Define the function

$$r(y) = \max_{j=1}^{n} d(y, x_j).$$

This function is strictly convex, hence if it has a point of minimum (called the *center* of $F(x)$) in X, this minimum is unique. Existence of the minimum follows from completeness of X (see [BGS85]). The center is clearly invariant under the group F. $\square$

The ideal boundary. With each $CAT(0)$-space X one can associate the *ideal boundary.* We first describe this boundary geometrically, assuming that X has *infinitely extendible geodesics*, i.e. if $\gamma : [0, a] \to X$ is a geodesic segment, then γ extends[1] to an infinite geodesic ray $\gamma : [0, \infty) \to X$. Choose a base-point $x_0 \in X$. Then *points at infinity* $\xi \in \partial_\infty(X, x_0)$ are rays $r_\xi : [0, \infty) \to X$ (parameterized as usual by their arc-length) emanating from x_0. We will refer to them as *rays going to the points ξ at infinity* (or *asymptotic to the points ξ at infinity*) and $\partial_\infty(X, x_0)$ as the *ideal boundary* of (X, x_0).

If X is a Riemannian manifold of nonpositive curvature then topology on $\partial_\infty(X, x_0)$ is defined via the angles between corresponding rays: two points at infinity are close if and only if the angle between rays is small. In the general case the topology can be defined as follows. Let ρ_1, ρ_2 are rays emanating from x_0. For $\epsilon > 0$ we say that the corresponding points at infinity ξ_1, ξ_2 are ϵ-close if $d(\rho_1(1/\epsilon), \rho_2(1/\epsilon)) \le \epsilon$. We say that a point $x \in X$ is ϵ-close to $\xi \in \partial_\infty(X, x_0)$ (which is represented by a ray ρ) if $d(x, \rho(\mathbb{R}_+)) \le \epsilon$ and $d(x, x_0) \ge 1/\epsilon$. If X is locally compact, then $\bar{X} = X \cup \partial_\infty X$ is compact.

If we have two distinct base-points x_0, x_1, then we identify the ideal boundaries $\partial_\infty(X, x_0)$, $\partial_\infty(X, x_1)$ as follows. Suppose ρ_0, ρ_1 are geodesic rays emanating from x_0, x_1. Then $\rho_0 \sim \rho_1$ if

$$d(\rho_0(t), \rho_1(t)) \le C \quad \text{for all} \quad t \in [0, \infty)$$

for some constant C which depends only on ρ_0, ρ_1. This gives a natural identification between the ideal boundaries. It is an easy exercise to verify that this identification is a homeomorphism. With this in mind we will drop the base-point label in $\partial_\infty(X, x_0)$ and will use the notation $\partial_\infty X$ instead.

[1] The extension is not assumed to be unique.

Let $\rho : [0, \infty) \to X$ be a geodesic ray which is asymptotic to ξ. The *Busemann function* β_ξ corresponding to ρ is the monotone limit :

$$\beta_\xi(z) := \lim_{t\to\infty} [d(z, \rho(t)) - t]$$

for $z \in X$. It is easy to see that the function β_ξ is a convex function and is well-defined up to an additive constant. For every ray ρ asymptotic to ξ we have:

$$\beta_\xi \circ r(t) = -t + const.$$

Thus we may identify the points in $\partial_\infty X$ with the Busemann functions on X (considered up to additive constants). For each point $x \in X$ consider the distance function

$$d_x : z \mapsto d(z, x).$$

Now choose a base-point $x_0 \in X$ and normalize d_x as:

$$\alpha_x(z) := d_x(z) - d(x_0, x).$$

Then $\alpha_x(x_0) = 0$ and its sublevel sets are metric balls in X. Let $\tilde{C}(X, \mathbb{R})$ be the quotient of the space of continuous functions on X (with the topology of pointwise convergence) by the subspace of constant functions. Then we have a homeomorphic embedding

$$\iota : X \hookrightarrow \tilde{C}(X, \mathbb{R}), x \mapsto \alpha_x \pmod{Const}.$$

The limit of the functions α_x as $x \to \xi \in \partial_\infty X$ is the Busemann function β_ξ (normalized at x_0). Hence the topology on $\bar{X}$ is the topology induced on the closure of $\iota(X)$ in $C(X, \mathbb{R})$. See [BGS85] for more details.

Horoballs in X are the sublevel sets of Busemann functions. Hence each horoball is approximated by a sequence of metric balls in the Chabauty topology.

3.3. Basic properties of the hyperbolic space

The n-dimensional hyperbolic space $\mathbb{H}^n$ is the complete simply-connected n-dimensional Riemannian manifold of the constant sectional curvature -1. I recall the basic properties of this space.

- **The Gauss-Bonnet formula:** If $[abc]$ is a geodesic triangle in $\mathbb{H}^n$ then the sum of angles in $[abc]$ equals $\pi - Area([abc])$.
- **Thin triangles property:** If Δ is any geodesic triangle in $\mathbb{H}^n$, then each side of Δ is contained in the 2-neighborhood of the union of two other sides.
- **The distance function is convex,** i.e. if $\gamma_1(t), \gamma_2(t)$ are geodesics in $\mathbb{H}^n$, then the distance function $d(t) = d(\gamma_1(t), \gamma_2(t))$ is convex.
- **The distance function to convex sets is convex.** In particular, if $C \subset \mathbb{H}^n$ is a convex subset, then for each $r > 0$ the neighborhood $Nbd_r(C)$ is also convex.

- **The space $\mathbb{H}^n$ has no conjugate points:** any two complete distinct geodesics intersect in at most one point.
- **Busemann functions in $\mathbb{H}^n$ are convex.**

Below are several useful formulae of the hyperbolic trigonometry. First consider triangle with the vertices ABC so that the angles at A, B, C are α, β, γ and the sides opposite to A, B, C have the lengths a, b, c respectively.

(A) **Hyperbolic cosine formulae:**

$$\cos(\gamma) = -\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)\cosh(c) \tag{3.1}$$

$$\cosh(c) = \cosh(a)\cosh(b) - \sinh(a)\sinh(b)\cos(\gamma) \tag{3.2}$$

(B) **Hyperbolic sine formula:**

$$\frac{\sinh(a)}{\sin(\alpha)} = \frac{\sinh(b)}{\sin(\beta)} = \frac{\sinh(c)}{\sin(\gamma)} \tag{3.3}$$

(see [Thu97a, page 81]).

(C) Let $[ABCD] \subset \mathbb{H}^2$ be an (embedded) quadrilateral so that the angles at A, B, D are $\pi/2$ and the angle at C is γ. Then

$$\sinh(d(A, B))\sinh(d(A, D)) = \cos(\gamma) \tag{3.4}$$

(see [Bea83, Theorem 7.17.1]).

Lemma 3.5. *Consider a* **twisted** quadrilateral *in $\mathbb{H}^n$: a configuration of four points A, B, C, D in $\mathbb{H}^n$ such that*

$$\angle ABC \geq \pi/2, \angle DAB = \pi/2, \angle ADC \geq \pi/2$$

and $d(D, A) \geq h$. Then $\sinh d(A, B) \leq 1/\sinh h$.

Proof: First, consider a quadrilateral $[A'B'C'D'] \subset \mathbb{H}^2$ so that

$$\angle A'B'C' = \angle A'D'C' = \angle D'A'B' = \pi/2.$$

Then the formula (C) above reads:

$$\sinh(d(A', B'))\sinh(d(A', D')) = \cos(\angle D'C'B'),$$

which implies

$$\sinh(d(A', B'))\sinh(d(A', D')) \leq 1.$$

Clearly for the last inequality it suffices to assume that

$$\angle A'B'C', \angle A'D'C' \geq \angle D'A'B' = \pi/2$$

(since in the interior of such quadrilateral one can find a point C'' such that $\angle A'B'C'' = \angle A'D'C'' = \angle D'A'B' = \pi/2$, see Figure 3.2.) Now draw the diagonal $[DB]$ in the quadrilateral $[ABCD]$ and rotate the triangle $[DAB]$ around the line $[DB]$, keeping the triangle $[DCB]$ fixed until we get

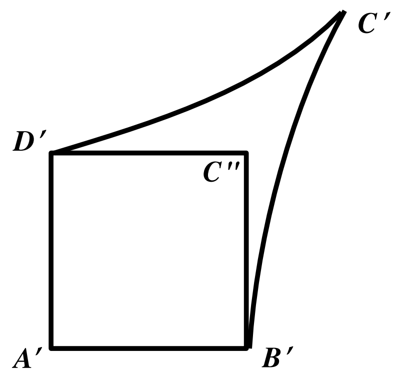

Figure 3.2: *Quadrilaterals in* $\mathbb{H}^2$.

a quadrilateral $[A'B'C'D']$ embedded in a hyperbolic plane $\mathbb{H}^2 \subset \mathbb{H}^n$ which has the same properties as the original one since $\angle D'A'B' = \angle DAB$ and $\angle A'B'C' \geq \angle ABC$, $\angle A'D'C' \geq \angle ADC$. $\square$

The following lemma shows that the nearest-point projections in $\mathbb{H}^n$ decrease length exponentially.

Lemma 3.6. *Suppose that γ is a geodesic in $\mathbb{H}^n$ and $Q \subset \mathbb{H}^n$ is a closed convex subset such that $d(\gamma, Q) \geq h \geq 0.2$. Then the diameter of the nearest-point projection of γ to Q is at most $4\exp(-h)$.*

Proof: Suppose that $A \in Q, D \in \gamma$ realize the minimal distance between Q and γ. Let $C \in \gamma - \{D\}$ be any point and $B \in Q$ be its projection. Now we apply Lemma 3.5 to the configuration A, B, C, D:

$$d(A, B) \leq \sinh(d(A, B)) \leq 1/\sinh(h).$$

However, if $h \geq 0.2$ then $1/\sinh(h) \leq 4e^{-h}$. $\square$

Corollary 3.7. *Let Q be a geodesic segment in $\mathbb{H}^n$ and α a piecewise-geodesic curve which consists of at most r segments. Suppose that $d(Q, \alpha) \geq h \geq 0.2$. Then the length of the nearest-point projection of α to Q is at most $4re^{-h}$.*

Corollary 3.8. *Suppose that $\rho_1(t), \rho_2(t)$ are geodesic rays in $\mathbb{H}^n$ which determine the same point ξ at infinity. Then there is a constant c so that*

$$|\rho_1(t) - \rho_2(t - c)| \sim \exp(-t)$$

as $t \to \infty$.

The constant c is chosen so that the points $\rho_1(t)$, $\rho_2(t - c)$ belong to the same horosphere with center at ξ.

Lemma 3.9. *Consider a sequence of triangles $[OAB] \subset \mathbb{H}^n$ with $d(O, A) = t$, $d(O, B) \sim t$ and $d(A, B) = o(t)$, as $t \to \infty$. Then given $\eta > 0$ there exists $T < \infty$ so that for all $t \geq T$ there are points $A' \in [OA], B' \in [OB]$ so that $d(A', B') \leq \eta$ and $d(A', A) = o(t), d(B', B) = o(t)$ as $t \to \infty$.*

Proof: It suffices to assume that $d(O,B) = t$ (otherwise we replace A and B by points $A_1 \in [OA], B_1 \in [OB]$ such that $d(A_1, A) \le |t - d(O,B)|$, $d(B_1, B) \le |t - d(O,B)|$ and $d(O, A_1) = d(O, B_1)$). Let $A(s) \in [OA]$, $B(s) \in [OB]$ be the points such that $d(O, A(s)) = d(O, B(s)) = s$, Let $M, M(s)$ be the mid-points of $[AB], [A(s)B(s)]$. Let $h(s) := d(M(s), M)$. Then the geodesic through the points $O, M(s), M$ is orthogonal to $[A(s)B(s)], [AB]$, the angles $\angle AA(s)M(s) = \angle BB(s)M(s)$ are obtuse, hence by applying Lemma 3.5 we get:

$$\sinh d(A(s), M(s)) \le 1/\sinh(h(s)).$$

Given $\eta > 0$ choose s so that $d(A(s), M(s)) \le \eta/2$. Note that $h(s)$ is bounded from above as $t \to \infty$. Then by triangle inequalities $s \sim t$ as $t \to \infty$. So we let $A' := A(s), B' := B(s)$. $\square$

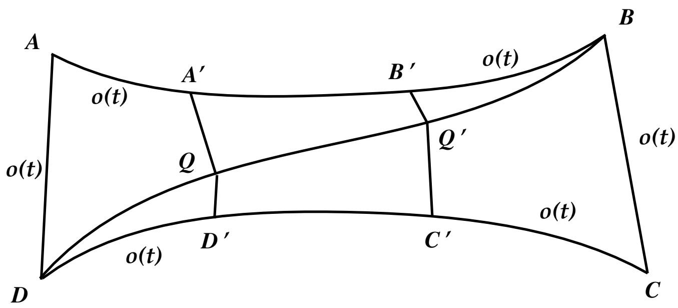

Figure 3.3:

Lemma 3.10. *Consider a sequence of quadrilaterals* $[ABCD] \subset \mathbb{H}^n$ *such that* $d(A,B) = t, d(D,C) = t$ *and* $d(A,D) = o(t), d(B,C) = o(t)$ *as* $t \to \infty$.

(1) Then, given $\epsilon > 0$ *there is* $T < \infty$ *so that for all* $t \ge T$ *there are points* $A', B' \in [AB], C', D' \in [CD]$ *with the property:*

$$d(A', D') \le \epsilon, d(B', C') \le \epsilon, d(D', C') \sim d(A', B') \sim t, \quad as \quad t \to \infty.$$

(2) Moreover, the Hausdorff distance between the segments $[A'B']$ *and* $[D'C']$ *satisfies*

$$d_H([A'B'], [D'C']) \le \epsilon.$$

Proof: Note that it suffices to prove the first assertion, the second assertion will follow from convexity of the distance function between the geodesics $[AB]$ and $[DC]$.

The triangle inequality implies that $d(A,C) \sim d(B,D) \sim t$ as $t \to \infty$. Thus Lemma 3.9 (applied to the triangle $[ABD]$) implies that we can find points $A' \in [AB], Q \in [DB]$ so that

$$d(A', Q) \le \epsilon/2, d(A, A') = o(t), d(D, Q) = o(t), \quad \text{as} \quad t \to \infty$$

(see Figure 3.3). By applying Lemma 3.9 to the triangle $[CBD]$ we find a point $Q' \in [BD]$ so that $d(Q', [DC]) \le \epsilon/2$ and $d(Q', B) = o(t)$. Let

$C' \in [DC]$ be the point nearest to Q'. It t is sufficiently large we can assume that the point Q is closer to D than the point Q'. Thus convexity of the distance function (between the geodesics $[DB]$ and $[DC]$) implies that $d(Q,[DC]) \leq \epsilon/2$, hence there is a point $D' \in [DC]$ such that $d(Q,D') \leq \epsilon/2$. We conclude that $d(A',D') \leq \epsilon$. Since $d(A,D) = o(t)$ as $t \to \infty$, the triangle inequalities imply that $d(D,D') = o(t)$ as $t \to \infty$.

This gives us the points A', D', C'. To find the point B' we repeat the above argument: $d(Q',[AB]) \leq \epsilon/2$ (by convexity of the distance function between the geodesics $[AB],[DB]$), hence we let $B' \in [AB]$ be the point nearest to Q'. $\square$

3.4. Models of the hyperbolic space

We shall use two basic models of the hyperbolic space. The first is the *upper half-space* model. Let $\mathbb{H}^n = \mathbb{R}^n_+ = \{x : x_n > 0\}$, then the hyperbolic metric is $ds = |dx|/x_n$. The geodesics in this model are the circular arcs and Euclidean rays in $\mathbb{R}^n_+$ which are orthogonal to the boundary sphere $\mathbb{R}^{n-1} \cup \{\infty\}$. For instance, if $x,y \in \mathbb{H}^n$ are points with coordinates $(0,\dots,0,t),(0,\dots,0,s)$ then $d(x,y) = |\log(s/t)|$. Let $X_i := \frac{\partial}{\partial x_i}$, $i = 1,...,n$. The covariant derivative ∇ in $\mathbb{H}^n$ satisfies:

$$\nabla_{X_i} X_i = \frac{1}{x_n} X_n, \quad \text{for } \ i \neq n$$

$$\nabla_{X_n} X_n = -\frac{1}{x_n} X_n, \quad \text{and } \ \nabla_{X_i} X_j = 0 \ \text{ for } \ i \neq j.$$

The other model is the *unit ball* model. Namely, let $\mathbb{H}^n = \mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$ with the Riemannian metric

$$ds = \frac{2|dx|}{1-|x|^2}.$$

The stereographic projection is an isometry from the unit ball model of $\mathbb{H}^n$ to the upper half-space model. Geodesics in the unit ball model are Euclidean segments and circular arcs which are orthogonal to the boundary $\mathbb{S}^{n-1}$. In the both models the closed metric balls in $\mathbb{H}^n$ are closed Euclidean balls which are contained in $\mathbb{B}^n$ and $\mathbb{R}^n_+$ respectively. The angles in $\mathbb{H}^n$ are the usual angles in $\mathbb{R}^n$ since the hyperbolic metric is conformally-Euclidean.

The space $\mathbb{H}^n$ has the natural compactification $\bar{\mathbb{H}}^n = \mathbb{H}^n \cup \partial_\infty \mathbb{H}^n$ defined in §3.2. Here we describe this compactification using models of $\mathbb{H}^n$. Identify $\mathbb{H}^n$ with the unit ball $\mathbb{B}^n$. Then the geodesic rays emanating from the origin are the Euclidean segments, thus the compactification of $\mathbb{H}^n$ is naturally homeomorphic to the closure $\bar{\mathbb{B}}^n$ of the unit ball in $\mathbb{R}^n$. The ideal boundary of $\mathbb{H}^n$ is the round sphere $\mathbb{S}^{n-1}$. If we identify $\mathbb{H}^n$ with the upper half-space then the compactification $\partial_\infty \mathbb{H}^n \cup \mathbb{H}^n$ is $\overline{\mathbb{R}^n_+} = \{x \in \mathbb{R}^n : x_n \geq 0\} \cup \{\infty\}$.

In any case the set of points at infinity is homeomorphic to the sphere $\mathbb{S}^{n-1}_\infty$ which is called the *sphere at infinity* of the hyperbolic space. We give

$\mathbb{S}^{n-1}_\infty$ the "spherical metric" (of the constant curvature $+1$). If $n = 3$ (the case we are mainly interested in) then we shall identify the sphere at infinity with the extended complex plane $\widehat{\mathbb{C}}$ (which is the one-point compactification of the complex plane).

Tubular neighborhoods of geodesic rays define another topology on $\mathbb{H}^n \cup \partial_\infty \mathbb{H}^n$ which is called the *conical topology.* Namely, retain the unit ball topology on $\mathbb{H}^n$; a sequence $x_k \in \mathbb{H}^n$ converges to a point $\xi \in \partial_\infty \mathbb{H}^n$ in the conical topology, if there is a geodesic ray r_ξ asymptotic to the point ξ and a number ϵ such that the sequence x_k is contained in the ϵ-neighborhood of r_ξ and $x_k \to \xi$ in the topology defined in §3.3. If $x_k \to \xi$ in the conical topology then we shall write this as $\lim^c_{k\to\infty} x_k = \xi$.

Exercise 3.11. $\mathbb{H}^n \cup \partial_\infty \mathbb{H}^n$ *with the conical topology is not compact.*

Horoballs in $\mathbb{H}^n$ are represented by Euclidean balls in $\mathbb{B}^n$ which are tangent to $\partial_\infty \mathbb{H}^n$. Boundaries of horoballs are called *horospheres.* Intrinsically horoballs are limits of metric balls in $\mathbb{H}^n$ whose radius tends to infinity (in the Chabauty topology). Another point of view is that horoballs are sublevel sets of *Busemann functions* in X, see §3.2.

If $\xi = \infty \in \partial_\infty \mathbb{H}^n$ and $0 = (0, \dots, 0, 1)$ is the point of normalization for the Busemann function, then

$$\beta_\xi(x) = -\log(x_n), \quad \text{for} \quad x = (x_1, ..., x_n).$$

Lemma 3.12. *The function β_ξ is convex. The Hessian $D^2\beta_\xi$ satisfies:*

$$D^2\beta_\xi(u, u) \geq 0$$

with equality if and only if the vector u is tangent to the geodesic asymptotic to ξ.

Proof: Recall that $D^2\beta_\xi(X, Y) = (XY)\beta_\xi - (\nabla_X Y)\beta_\xi$. Let $x = (x_1, ..., x_n) \in \mathbb{H}^n$, we can assume that $\xi = \infty$. Then $D^2\beta_\xi$ at x is the diagonal matrix with the diagonal entries $(1/x_n^2, \dots, 1/x_n^2, 0)$. Thus Hessian of β_ξ is positive semi-definite, its radical consists of the vectors $u = (0, \dots, 0, t), t \in \mathbb{R}$. $\square$

Exercise 3.13. *Consider the upper half-space model $\mathbb{R}^{n-1} \times \mathbb{R}_+$ of $\mathbb{H}^n$ and a bounded subset Q of $\mathbb{R}^{n-1}$. Then for every $c > 0$ the volume of $Q \times [c, \infty) \subset \mathbb{H}^n$ is finite.*

3.5. Isometries of the hyperbolic space

The easiest way to understand the group of isometries $\mathrm{Isom}(\mathbb{H}^n)$ of $\mathbb{H}^n$ is to consider different models of the hyperbolic space. We start with the unit ball model $\mathbb{B}^n$. Then the hyperbolic metric is rotationally-symmetric and thus $K := O(n) \subset \mathrm{Isom}(\mathbb{H}^n)$ is the stabilizer of the center $0 \in \mathbb{B}^n$ in $\mathrm{Isom}(\mathbb{H}^n)$. The group $O(n)$ is a maximal compact subgroup in $\mathrm{Isom}(\mathbb{H}^n)$. Now we move to the upper half-space model $\mathbb{R}^n_+$. Let B be the group which is generated by Euclidean translations in $\mathbb{R}^n$ that keep $\mathbb{R}^n_+$ invariant and Euclidean dilation $\overrightarrow{x} \mapsto k\overrightarrow{x}$, $k > 0$.

Direct calculations show that B belongs to the group $\mathrm{Isom}(\mathbb{H}^n)$. In particular, the group $\mathrm{Isom}(\mathbb{H}^n)$ acts transitively in $\mathbb{H}^n$. Moreover, for every two elements ξ, η of the orthonormal frame bundle $\mathcal{OF}(\mathbb{H}^n)$, there is an isometry g which maps ξ to η. However any isometry of a connected Riemannian manifold M, which fixes an element of the orthonormal frame bundle is the identity. Thus $\mathrm{Isom}(\mathbb{H}^n) = K \cdot B$, i.e. any element $g \in \mathrm{Isom}(\mathbb{H}^n)$ splits as the composition $g = k \circ b$, where $k \in K, b \in B$.

The group of isometries $\mathrm{Isom}(\mathbb{H}^n)$ of $\mathbb{H}^n$ is isomorphic to the subgroup of the Lorentz group $SO(n,1)$ which preserves the positive light cone; if $n = 3$ then $\mathrm{Isom}_+(\mathbb{H}^3) \cong \mathrm{PSL}(2,\mathbb{C})$ is the group of orientation-preserving isometries. In general the group $\mathrm{Isom}(\mathbb{H}^n)$ acts on $\mathbb{S}^{n-1}_\infty$ as the group of *Moebius transformations*, i.e. compositions of inversions in round spheres. For each isometry $g \in \mathrm{Isom}(\mathbb{H}^n)$ we define the *translation length* $\ell(g)$:

$$\ell(g) = \inf_{x \in \mathbb{H}^n} d(x, g(x)).$$

There are three types of conjugacy classes of nontrivial isometries of $\mathbb{H}^n$:

- loxodromic: $\ell(g) > 0$,
- parabolic: $\ell(g) = 0$ but g does not have a fixed point in $\mathbb{H}^n$,
- elliptic: g fixes a point in $\mathbb{H}^n$.

Any loxodromic element h has a unique invariant geodesic A_h in $\mathbb{H}^n$. This invariant geodesic is called the *axis* of h. The restriction of h to A_h is a nontrivial translation. The *translation length* $\ell(h)$ of a loxodromic element h is equal to the length of $A_h/\langle h\rangle$. Loxodromic elements have exactly two fixed points in $\bar{\mathbb{H}}^n$. One of these points is called *repulsive* and the other is called *attractive*. The difference is that under iterations of h any point $z \in A_h$ moves towards the attractive fixed point and away from the repulsive point (the same is actually true for every $z \in \mathbb{H}^n$). A loxodromic element h is called *hyperbolic* if it is conjugate to the dilation

$$\overrightarrow{x} \mapsto k\overrightarrow{x}$$

(where $k > 0$) in the upper half-space model of $\mathbb{H}^n$. Geometrically this means that h preserves any totally-geodesic half-plane with the boundary A_h.

Parabolic elements h have exactly one fixed point in $\bar{\mathbb{H}}^n$, thus they have no invariant geodesics. Any parabolic element is conjugate to a Euclidean isometry in the group B. Any elliptic element $g \in \mathrm{Isom}(\mathbb{H}^n)$ has a nonempty totally geodesic submanifold which is pointwise fixed by g. Any elliptic element is conjugate to an element of the group $K \cong O(n)$.

In the case of $n = 3$ each orientation-preserving isometry corresponds to an element of $PSL(2,\mathbb{C})$, hence we can define its *trace* up to the $\pm$ sign. Elements of $PSL(2,\mathbb{C})$ are classified as follows:

- *Elliptic*: $Trace(g) \in (-2, 2)$.
- *Parabolic*: $Trace(g) = \pm 2$.

- *Loxodromic*: $Trace(g) \notin [-2, 2]$.

Exercise 3.14. *Let $g \in PSL(2, \mathbb{C})$ be a loxodromic element with the trace $\pm t$. Then*

$$t = \lambda + \lambda^{-1}, \quad \ell(g) = |\log(|\lambda|)|.$$

Hint: verify this formula in the case of loxodromic elements of the form $g(z) = \lambda^2 \cdot z$.

Cross-ratio and the hyperbolic metric. For a quadruple of distinct points $x, y, z, w \in \overline{\mathbb{R}^n}$ define the *cross-ratio*

$$|x : y : z : w| = \frac{|x-y|}{|y-z|}\frac{|z-w|}{|w-x|}.$$

Exercise 3.15. *The cross-ratio is invariant under Moebius transformations of $\overline{\mathbb{R}^n}$: if $\gamma \in \mathrm{Isom}(\mathbb{H}^n)$ then*

$$|x : y : z : w| = |\gamma(x) : \gamma(y) : \gamma(z) : \gamma(w)|.$$

For a pair of distinct points $p, q \in \partial_\infty \mathbb{H}^2$ let γ_{pq} denote the geodesic in $\mathbb{H}^2$ which is asymptotic to p and q.

Lemma 3.16. *For any convex polygon P in $\mathbb{H}^2$ with the ideal vertices $x, y, z, w \in \partial_\infty \mathbb{H}^2$ and the sides $\gamma_{xy}, \gamma_{yz}, \gamma_{zw}, \gamma_{wx}$ we have:*

$$|x : y : z : w| = \frac{\sinh(d(\gamma_{wx}, \gamma_{yz})/2)}{\sinh(d(\gamma_{xy}, \gamma_{zw})/2)}.$$

Proof: We use the unit disk model for $\mathbb{H}^2$. Let $L_1 = [pq], L_2 = [rs]$ be the shortest geodesic segments connecting γ_{xy} to γ_{zw} and γ_{wx} to γ_{yz}; $p \in \gamma_{xy}$, $r \in \gamma_{wx}$. Let O be the point of intersection between L_1 and L_2. Then the polygon P is symmetric with respect to the hyperbolic reflections in L_1 and L_2. Applying an isometry of $\mathbb{H}^2$ we move the polygon P so that O is the origin. Then x, y, z, w are the vertices of a Euclidean rectangle. Let $\alpha = \angle pOx$, $\beta = \angle xOr$; then $\alpha + \beta = \pi/2$. Applying the formula (3.1) to the triangles $[pOx]$ and $[xOr]$ we get:

$$\sinh(d(p, O)) \tan(\alpha) = \sinh(d(r, O)) \tan(\beta) = 1,$$

$$\sinh(d(p, O)) = [\sinh(d(r, O))]^{-1} .$$

On the other hand,

$$|x-y|/|x-w| = \tan(\alpha) = \cot(\beta), \quad |x : y : z : w| = |x-y|^2/|x-w|^2,$$

$$|x-y|^2/|x-w|^2 = \tan^2(\alpha) = \sinh(d(r, O))/\sinh(d(p, O)) = \frac{\sinh(d(\gamma_{wx}, \gamma_{yz})/2)}{\sinh(d(\gamma_{xy}, \gamma_{zw})/2)}. \quad \square$$

Exercise 3.17. *Suppose that $x, y, z, w \in \partial_\infty \mathbb{H}^2$ are distinct points so that the geodesics γ_{xy} and γ_{wz} intersect. Show that*

$$|x : y : z : w| = \cosh^2(d(\gamma_{wx}, \gamma_{yz})/2).$$

3.6. The convergence property

Proposition 3.18. *The group $G = \mathrm{Isom}(\mathbb{H}^n)$ has the* **Cartan decomposition:** $G = KAK$.

Proof: The equality $G = KAK$ can be interpreted as follows. Pick a base-point $O \in \mathbb{H}^n$, let K be the stabilizer of O in G, i.e. K is a maximal compact subgroup of G. Choose a geodesic $L \subset \mathbb{H}^n$ passing through O, let A be the group which consists of the identity and of hyperbolic translations with the axis L. Then $G = KAK$ means that for each $g \in G$ there are elements $k, k' \in K$ and $a \in A$ so that $g = kak'$.

Recall that the group G acts transitively on $\mathbb{H}^n$ and we can identify the quotient G/K with the hyperbolic space $\mathbb{H}^n$, so that 1 projects to the point O. Let $g \cdot K = x \in \mathbb{H}^n$ be any point. Then the orbit $K(x)$ is the metric sphere with center at O passing through x. In particular we can find an element $k \in K$ so that $k(x) \in L$. Then take a translation $a \in A$ so that $ak(x) = O$. Thus $a \cdot k \cdot g \cdot K = K$ and $g \in KAK$. $\square$

For a pair of points $z, x \in \mathbb{S}^{n-1}$ we define a *quasiconstant map* $z_x : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ so that: $z_x(w) = z$ for each $w \neq x$, and z_x is undefined at x. The group $G = \mathrm{Isom}(\mathbb{H}^n)$ acts on $\mathbb{S}^{n-1}$ as the group of Moebius transformations. A sequence of elements $g_n \in G \curvearrowright \mathbb{S}^{n-1}$ converges to a quasiconstant map z_x iff g_n converges to z_x uniformly on compacts in $\mathbb{S}^{n-1} - \{x\}$. Note that $g_n \to z_x$ implies $g_n^{-1} \to x_z$. The space of quasiconstants $Q(\mathbb{S}^{n-1})$ is naturally homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$. Thus we defined a topology on $\widehat{G} = G \cup Q(\mathbb{S}^{n-1})$.

Proposition 3.19. *(Convergence property of Moebius transformations.) $\widehat{G} = G \cup Q(\mathbb{S}^{n-1})$ is compact.*

Proof: Let $G = KAK$ be the the Cartan decomposition of G. Let $\xi, \eta \in \mathbb{S}^{n-1}_\infty$ be the end-points of the geodesic L which is A-invariant. Hence for any sequence $g_i \in G$ we have: $g_i = k_i a_i c_i$, where $k_i, c_i \in K$, $a_i \in A$. The sequences k_i, c_i are subconvergent to $k \in K$ and $c \in K$; a_i is subconvergent to a, where a is either an element of A or is one of the quasiconstants ξ_η, η_ξ. Suppose that $a = \eta_\xi$, then g_i is subconvergent on $\mathbb{S}^{n-1} - c^{-1}(\xi)$ to $k(\eta)$. Thus $g_i \to x_y$, where $x = k(\eta), y = c^{-1}(\xi)$. $\square$

Let $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ be a subgroup, we define the *limit set* $\Lambda(\Gamma)$ as the collection of $x \in \partial_\infty \mathbb{H}^n$ such that for certain $y \in \mathbb{S}^{n-1}$ the quasiconstant x_y belongs to the accumulation set $A(\Gamma)$ of Γ in $Q(\mathbb{S}^{n-1})$.

Exercise 3.20. *If $g \in \Gamma$ is loxodromic or parabolic then the fixed point set of g is contained in $\Lambda(\Gamma)$.*

Exercise 3.21. *Show that $\Lambda(\Gamma)$ equals the accumulation set in $\partial_\infty \mathbb{H}^n$ of the Γ-orbit of any point $p \in \mathbb{H}^n$.*

Note that if for a quasiconstant map x_y the point x is in $\Lambda(\Gamma)$ then $y \in \Lambda(\Gamma)$ as well. Since

$$\lim_{i \to \infty} g_i = x_y \Rightarrow \lim_{i \to \infty} h g_i = h(x)_y$$

it follows that $\Lambda(\Gamma)$ is Γ-invariant. It is also clear that $\Lambda(\Gamma)$ is closed (since $A(\Gamma)$ is). The set $\Lambda(\Gamma)$ is nonempty unless Γ is relatively compact in $\mathrm{Isom}(\mathbb{H}^n)$ (in which case Γ fixes a point in $\mathbb{H}^n$). If $\Lambda(\Gamma)$ is finite then a finite-index subgroup Γ_0 of Γ fixes $\Lambda(\Gamma)$ pointwise. Hence, if $\Lambda(\Gamma)$ is finite and contains at least three distinct points, then Γ_0 has a fixed point in $\mathbb{H}^n$, which in turn implies that Γ fixes a point in $\mathbb{H}^n$ as well and $\Lambda(\Gamma)$ is empty. We conclude that $\Lambda(\Gamma)$ either contains at most two points or is infinite.

Definition 3.22. A subgroup $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ is called **elementary** if it has a (nonempty) invariant subset in $\mathbb{H}^n \cup \partial_\infty \mathbb{H}^n$ which contains at most two points. Otherwise Γ is called **nonelementary**.

It is clear that $\Lambda(\Gamma)$ is infinite for each nonelementary subgroup of $\mathrm{Isom}(\mathbb{H}^n)$.

Exercise 3.23. *Construct example of an elementary subgroup $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ so that $\Lambda(\Gamma) = \partial_\infty \mathbb{H}^n$.*

Lemma 3.24. *Let x, w be distinct points in $\Lambda(\Gamma)$, where Γ is a nonelementary group. Then there exists a sequence of loxodromic elements $f_i \in \Gamma$ with the fixed points p_i, q_i so that $\lim_i p_i = x, \lim_i q_i = w$.*

Proof: Let $x_y \in A(\Gamma), z_w \in A(\Gamma)$. Then for each $\gamma \in \Gamma$ we have:

$$x_y \circ \gamma = x_{\gamma(y)} \in A(\Gamma), \quad \gamma \circ z_w = \gamma(z)_w \in A(\Gamma).$$

Thus, since Γ-orbits of y and z are infinite, we may choose $x_y \in A(\Gamma), z_w \in A(\Gamma)$ so that the four points x, y, z, w are pairwise distinct. Let $g_i, h_i \in \Gamma$ be sequences which converge to x_y, z_w respectively. Let U_x, U_y, U_z, U_w be small disjoint balls centered at x, y, z and w respectively. Then for all sufficiently large i

$$h_i(\mathbb{S}^n - U_w) \subset U_z, \quad h_i(\mathbb{S}^n - U_y) \subset U_x \ \text{ and hence } \ g_i h_i(\mathbb{S}^n - U_w) \subset U_x.$$

It follows that each element $f_i = g_i h_i$ is loxodromic (its attractive fixed point is in U_w and its repulsive fixed point is in U_x). $\square$

Corollary 3.25. *If $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ is a nonelementary subgroup, then it contains at least two loxodromic elements with disjoint fixed point sets.*

Corollary 3.26. *If Γ is nonelementary then the fixed points of loxodromic elements of Γ are dense in $\Lambda(\Gamma)$.*

Corollary 3.27. *If Γ is nonelementary then the set $\Lambda(\Gamma)$ is perfect, i.e. each point of $\Lambda(\Gamma)$ is an accumulation point of this set.*

Proof: It suffices to show that each point p fixed by a loxodromic element $g \in \Gamma$ is an accumulation point of $\Lambda(\Gamma)$. We can assume that p is the attractive fixed point of g. Choose $z \in \Lambda(\Gamma) - Fix(g)$. Then $z_i = g^i(z) \in \Lambda(\Gamma)$ and $\lim_i z_i = p$. $\square$

3.7. Convex polyhedra in the hyperbolic space

Suppose that Φ is a locally finite closed convex polyhedron in $\mathbb{H}^n$ where all dihedral angles between faces are $\leq \pi/2$. In this case we will say that Φ has *acute* angles. For each top-dimensional face F of Φ let $\widehat{F}$ denote the hyperplane in $\mathbb{H}^n$ which contains F. The half-space in $\mathbb{H}^n$ bounded by $\widehat{F}$ and disjoint from Φ will be denoted $\widehat{F}^-$.

Theorem 3.28. *(E. Andreev, [And71a].) If Φ has acute angles then the faces F_1, F_2 of Φ intersect if and only if $\widehat{F}_1$ intersects $\widehat{F}_2$ in $\mathbb{H}^n$.*

Proof: Suppose that $F_1 \cap F_2 = \emptyset$. Without loss of generality we may assume that the distance between F_1 and F_2 within Φ is positive (otherwise move the hyperplanes $\widehat{F}_1$, $\widehat{F}_2$ a bit away from Φ and consider the resulting polyhedron). Let $\gamma \subset \Phi$ be the geodesic segment which realizes the distance between F_1, F_2 within Φ. If γ is orthogonal to both F_1, F_2, then $\hat{F}_1 \cap \hat{F}_2 = \emptyset$ and we are done. So, assume that γ is not orthogonal to F_1. Parameterize γ so that $\gamma(0) = x$ is the point of intersection between γ and F_1. Then for any vector $v \in T_x\mathbb{H}^n$ which is tangent to F_1 and is directed inside the face F_1, the angle between $\gamma'(0)$ and v is $\geq \pi/2$. Moreover, for at least one such vector v we have $\gamma'(0) \cdot v < 0$. Note that v is directed inside Φ, thus we get contradiction with the assumption that Φ has acute angles. $\square$

We now restrict to the case $n = 3$.

Theorem 3.29. *Faces F_1, F_2, F_3 of Φ intersect if and only if $\widehat{F}_1, \widehat{F}_2, \widehat{F}_3$ intersect in $\mathbb{H}^3$.*

Proof: Suppose that F is a face of Φ so that $\widehat{F}$ separates $O := \widehat{F}_1 \cap \widehat{F}_2 \cap \widehat{F}_3$ from the polyhedron Φ (see Figure 3.4). Our goal is to show that this is impossible. If such configuration of hyperplanes exists then there is a simplex $T \subset \mathbb{H}^3$ whose vertices are O, $\widehat{F}_1 \cap \widehat{F}_2 \cap \widehat{F}$, $\widehat{F}_1 \cap \widehat{F}_3 \cap \widehat{F}$ and $\widehat{F}_2 \cap \widehat{F}_3 \cap \widehat{F}$. The dihedral angles of this simplex satisfy the properties:

- All dihedral angles at the vertex O are $\leq \pi/2$.
- All dihedral angles at the face of T opposite to O are $\geq \pi/2$.

Let p, q, r denote the dihedral angles of T at the vertex O; x, y, z denote the dihedral angles at the opposite face.

The cosine formula of the spherical trigonometry implies that the planar angle $\angle OBC$ is obtuse unless $\cos z - \cos y \cos r > 0$ and $\angle OBA > \pi/2$. Since hyperbolic triangle cannot have two obtuse angles we conclude that (up to a change of notation):

$$\cos z - \cos y \cos r > 0, \quad \cos y - \cos x \cos q > 0,$$

$$\cos x - \cos z \cos p > 0.$$

Since $\cos p, \cos q, \cos r \geq 0$ it implies that $\cos r \cos p \cos q \geq 1$ which is impossible. $\square$

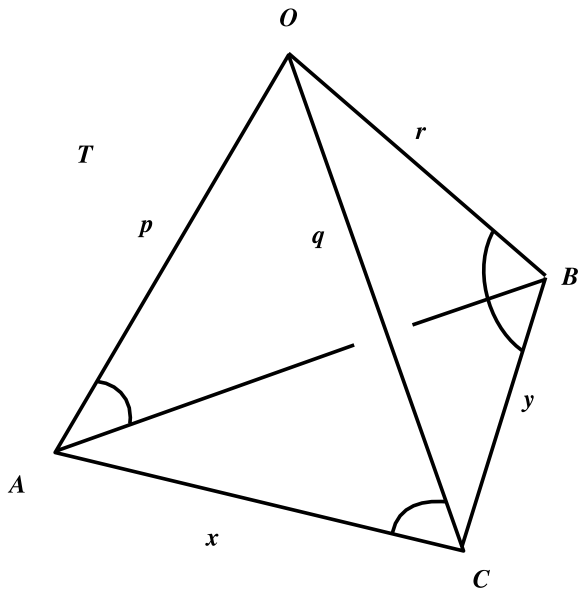

Figure 3.4:

Corollary 3.30. *Suppose that F_1, F_2, F_3, F_4 are distinct faces of Φ. Then the intersection*

$$\widehat{F}_1^- \cap \widehat{F}_2^- \cap \widehat{F}_3^- \cap \widehat{F}_4^-$$

is always empty.

3.8. Cayley graphs of finitely generated groups

Let Γ be a finitely generated group with the generating set $S = \{s_1, ..., s_n\}$, we shall assume that the identity does not belong to S. Define the *Cayley graph* $C = C(\Gamma, S)$ as follows: the vertices of C are the elements of Γ. Two vertices $g, h \in \Gamma$ are connected by an edge if an only if there is a generator $s_i \in S$ such that $h = gs_i$. Then C is a locally finite graph. Define the *word metric* d on C by assuming that each edge has the unit length, this defines the length of finite PL-paths in C, finally the distance between points $p, q \in C$ is the infimum (same as minimum) of the lengths of PL-paths in C connecting p to q. For $g \in G$ the *word length* $\ell(g)$ is just the distance $d(1, g)$ in C. It is clear that the left action of the group Γ on the metric space (C, d) is isometric.

Below are two simple examples of Cayley graphs.

Example 3.31. Let Γ be free Abelian group on two generators s_1, s_2. Then $S = \{s_i, i = 1, 2\}$. The Cayley graph $C = C(\Gamma, S)$ is the square grid in the Euclidean plane: the vertices are points with integer coordinates, two vertices are connected by an edge if and only if exactly only two of their coordinates are distinct and they differ by ± 1.

Example 3.32. Let Γ be the free group on two generators s_1, s_2. Take $S = \{s_i, i = 1, 2\}$. The Cayley graph $C = C(\Gamma, S)$ is the 4-valent tree (there are four edges incident to each vertex).

3.9. Quasi-isometries

Definition 3.33. Let X, Y be complete metric spaces. A map $f : X \to Y$ is called a (k, c)-**quasi-isometric embedding** if

$$k^{-1} d_X(x, x') - c \le d_Y(f(x), f(x')) \le k d_X(x, x') + c \tag{3.5}$$

for any $x, x' \in X$. Note that a quasi-isometric embedding does not have to be an embedding in the usual sense, however distant points have distinct images.

A (k, c)-quasi-isometric embedding is called a (k, c)-**quasi-isometry** if it admits an **approximate inverse** map $\bar{f} : Y \to X$ which is a (k, c)-quasi-isometric embedding so that:

$$d_X(\bar{f} f(x), x) \le c, \quad d_Y(f \bar{f}(y), y) \le c \tag{3.6}$$

for each $x \in X, y \in Y$.

In the most cases the *quasi-isometry constants* k, c do not matter, so we shall use the words *quasi-isometries* and *quasi-isometric embeddings* without specifying constants. If X, Y are spaces such that there exists a quasi-isometry $f : X \to Y$ then X and Y are called *quasi-isometric.* In applications X and Y will be nonempty, however, by working with *relations* instead of maps one can modify this definition so that the empty set is quasi-isometric to any bounded metric space.

Exercise 3.34. *If $f : X \to Y$ is a quasi-isometry and g is within finite distance from f (i.e. $\sup d(f(x), g(x)) < \infty$) then g is also a quasi-isometry.*

Exercise 3.35. *A subset S of a metric space X is said to be r-**dense** in X if the Hausdorff distance between S and X is at most r. Show that if $f : X \to Y$ is a quasi-isometric embedding such that $f(X)$ is r-dense in X for some $r < \infty$ then f is a quasi-isometry. Hint: construct an approximate inverse $\bar{f}$ to the map f by mapping point $y \in Y$ to $x \in X$ such that*

$$d_Y(f(x), y) \le d_Y(f(X), y) + 1.$$

For instance, the cylinder $X = \mathbb{S}^n \times \mathbb{R}$ is quasi-isometric to $Y = \mathbb{R}$; the quasi-isometry is the projection to the second factor.

Exercise 3.36. *Show that quasi-isometry is an equivalence relation between (nonempty) metric spaces.*

Lemma 3.37. *Let X be a proper geodesic metric space. Let G be a group acting isometrically properly discontinuously cocompactly on X. Pick a point $x_0 \in X$. Then the group G is finitely generated; for some choice of finite generating set S of the group G the map $f : G \to X$, given by $f(g) = g(x_0)$, is a quasi-isometry. Here G is given the word metric induced from $C(G, S)$.*

Proof: Our proof follows [GdlH91, Proposition 10.9]. Let $B = B_R(x_0)$ be the closed ball of radius R in X with the center at x_0 such that $B_{R-1}(x_0)$ projects onto X/G. Since the action of G is properly discontinuous, there are only finitely many elements $s_i \in G - \{1\}$ such that $B \cap s_i B \neq \emptyset$. Let S be the subset of G which consists of the above elements s_i (it is clear that s_i^{-1} belongs to S iff s_i does). Let

$$r := inf\{d(B, g(B)), g \in G - (S \cup \{1\})\}.$$

Clearly $r > 0$. We claim that S is a generating set of G and that for each $g \in G$

$$\ell(g) \leq d(x_0, g(x_0))/r + 1 \tag{3.7}$$

where ℓ is the word length on G (with respect to the generating set S). Let $g \in G$, connect x_0 to $g(x_0)$ by the shortest geodesic γ. Let m be the smallest integer so that $d(x_0, g(x_0)) \leq mr + R$. Choose points $x_1, ..., x_{m+1} = g(x_0) \in \gamma$, so that $x_1 \in B$, $d(x_j, x_{j+1}) < r$, $1 \leq j \leq m$. Then each x_j belongs to $g_j(B)$ for some $g_j \in G$. Let $1 \leq j \leq m$, then $g_j^{-1}(x_j) \in B$ and $d(g_j^{-1}(g_{j+1}(B)), B) \leq d(g_j^{-1}(x_j), g_j^{-1}(x_{j+1})) < r$. Thus the balls $B, g_j^{-1}(g_{j+1}(B))$ intersect, which means that $g_{j+1} = g_j s_{i(j)}$ for some $s_{i(j)} \in S \cup \{1\}$. Therefore

$$g = s_{i(1)} s_{i(2)} .... s_{i(m)}.$$

We conclude that S is indeed a generating set for the group G. Moreover,

$$\ell(g) \leq m \leq (d(x_0, g(x_0)) - R)/r + 1 \leq d(x_0, g(x_0))/r + 1.$$

The word metric on the Cayley graph $C = C(G, S)$ of the group G is left-invariant, thus for each $h \in G$ we have:

$$d(h, hg) = d(1, g) \leq d(x_0, g(x_0))/r + 1 = d(h(x_0), hg(x_0))/r + 1.$$

Hence for any $g_1, g_2 \in G$

$$d(g_1, g_2) \leq d(f(g_1), f(g_2))/r + 1.$$

On the other hand, the triangle inequality implies that

$$d(x_0, g(x_0)) \leq t\ell(g)$$

where $d(x_0, s(x_0)) \leq t \leq 2R$ for all $s \in S$. Thus

$$d(f(g_1), f(g_2))/t \leq d(g_1, g_2).$$

We conclude that the map $f : G \to X$ is a quasi-isometric embedding. Since $f(G)$ is R-dense in X, it follows that f is a quasi-isometry. $\square$

Corollary 3.38. *Let S_1, S_2 be finite generating sets for a finitely generated group G and d_1, d_2 be the word metrics on G corresponding to S_1, S_2. Then the identity map $(G, d_1) \to (G, d_2)$ is a quasi-isometry.*

Proof: The group G acts isometrically cocompactly on the proper metric space $(C(G,S_2),d_2)$. Thus the map $id : G \to C(G,S_2)$ is a quasi-isometry. $\square$

Lemma 3.39. *Let X be a locally compact path-connected topological space, let G be a group acting properly discontinuously cocompactly on X. Let d_1, d_2 be two proper geodesic metrics on X (consistent with the topology of X) both invariant under the action of G. Then the group G is finitely generated and the identity map $id : (X,d_1) \to (X,d_2)$ is a quasi-isometry.*

Proof: The group G is finitely generated by Lemma 3.37, choose a word metric d on G corresponding to any finite generating set (according to the previous corollary it does not matter which one). Pick a point $x_0 \in X$, then the maps

$$f_i : (G,d) \to (X,d_i), \quad f_i(g) = g(x_0)$$

are quasi-isometries, let $\bar{f}_i$ denote their approximate inverses. Then the map $id : (X,d_1) \to (X,d_2)$ is within finite distance from the quasi-isometry $f_2 \circ \bar{f}_1$. $\square$

A (k,c)-quasigeodesic segment in a metric space X is a (k,c)-quasi-isometric embedding $f : [a,b] \to X$; similarly, a complete (k,c)-quasigeodesic is a (k,c)-quasi-isometric embedding $f : \mathbb{R} \to X$. By abusing notation we will refer to the image of a (k,c)-quasigeodesic as a *quasi-geodesic.*

Corollary 3.40. *Let d_1, d_2 be as in Lemma 3.39. Then any (complete) geodesic γ with respect to the metric d_1 is also a quasigeodesic with respect to the metric d_2.*

Suppose that (X_n, x_n) is a sequence of complete *pointed* metric spaces (x_n are *base-points* in X_n).

Definition 3.41. (X_n, x_n) is convergent to the metric space (X,x) in the **quasi-isometric topology** (the other names are **Gromov-Hausdorff** topology and **geometric** topology)[2] if the following is satisfied:

For each $\epsilon > 0, r > 0$ there is a number $n_0 < \infty$ such that for every $n > n_0$ there exists a $(1+1/n, \epsilon)$-quasi-isometry

$$f_n : B_r(x_n) \to B_r(x).$$

The basic example of such convergence is the following. Let Y be a complete geodesic metric space and (X_n, x_n) be a sequence of pointed closed subspaces in Y (with the induced metric) so that $\lim_{n\to\infty} x_n = x$ and $X \subset Y$ is a closed subspace (with the induced metric) such that for every $r > 0$

$$\lim_{n\to\infty} d_H(X_n \cap B_r(x), X \cap B_r(x)) = 0$$

where d_H is the Hausdorff distance in Y. The quasi-isometries f_n are given by the nearest-point projections $X_n \to X$.

[2]In §8.2 we will refine the definition of the quasi-isometric topology in the case of hyperbolic manifolds.

Exercise 3.42. *Suppose that* (X_n, x_n) *converges to* (X, x) *in the quasi-isometric topology and* X *has finite diameter. Then for large* n *the spaces* X_n *also have finite diameters.*

Quasigeodesics in the hyperbolic space. The important property of (k,c)-quasigeodesics in $\mathbb{H}^n$ (and, more generally, in *Gromov-hyperbolic spaces*, [GdlH90]) is that they are always within bounded distance from a geodesic. This property was first stated and used by Mostow [Mos73]. Recall that d_H denotes the Hausdorff distance between subsets of $\mathbb{H}^n$.

Lemma 3.43. *There is a function* $\tau(k,c)$ *such that for any* (k,c)-*quasigeodesic* $f : [0,t] \to \mathbb{H}^n$ *we have:*

$$d_H(f([0,t]), [f(0), f(t)]) \le \tau(k,c)$$

where $k > 1, c > 1$.

Proof: I will give here a direct proof of this assertion, an alternative proof is given in Lemma 9.8 as an application of ultralimits.

Let γ denote the geodesic in $\mathbb{H}^n$ connecting the $f(0)$ to $f(t)$. We first replace f by a piecewise-geodesic curve φ as follows. Let $h = 2kc$, $N := [t/h]$, $t_i := hi$, $0 \le i \le N$. Let $D := 2kh$, note that

$$D > 1 > \log(4k/h) = \log(2/c). \tag{3.8}$$

Then in each interval $[t_i, t_{i+1}]$ we choose geodesic map $\varphi : [t_i, t_{i+1}] \to \mathbb{H}^n$ which has the same boundary values as f. The distance between f and φ is at most $kh + c$, thus it suffices to prove the assertion for the mapping φ. Let $x_i := \varphi(t_i)$. Then $d(x_i, x_{i+1}) \le kh + c \le 2kh$ since f is (k,c)-quasigeodesic. Consider the curve $\alpha_{ij} = \varphi([t_i, t_j])$, $i < j$. Then

$$length(\alpha) = \sum_{s=i}^{j-1} d(x_s, x_{s+1}) \le 2k|t_j - t_i|.$$

The fact that f is (k,c)-quasigeodesic implies that

$$|t_j - t_i| \le kc + kd(x_i, x_j).$$

Note that

$$d(x_i, x_j) \ge h|j-i|/k - c \ge \frac{h|j-i|}{2k} \tag{3.9}$$

and

$$length(\alpha_{ij}) \le 2kh|j-i|. \tag{3.10}$$

Let $proj_\gamma : \mathbb{H}^n \to \gamma$ denote the nearest-point projection to γ. Suppose for a moment that a certain part α_{ij} of the curve $\varphi([0, t_N])$ lies outside of the D-neighborhood of γ, but $d(x_i, \gamma) \le 2D, d(x_j, \gamma) \le 2D$. Our goal is to estimate $m = |j - i|$ from above in terms of k. The curve α_{ij} consists of m geodesic segments; hence, according to Corollary 3.7,

$$length(proj_\gamma(\alpha_{ij}) \le e^{-D} m.$$

Combining this with the triangle inequality we get

$$d(x_i, x_j) \le d(proj_\gamma(x_i), proj_\gamma(x_j)) + 4D \le e^{-D} m + 4D.$$

Now apply the lower estimate (3.9)

$$\frac{hm}{2k} \le e^{-D} m + 4D,$$

$$m(\frac{h}{2k} - e^{-D}) \le 4D.$$

The inequality (3.8) implies that

$$\frac{h}{4k} < \frac{h}{2k} - e^{-D},$$

therefore

$$m \le \frac{16Dk}{h} = 32k^2.$$

Applying (3.10) we get

$$length(\alpha_{ij}) \le 2khm \le 2kh \cdot 32k^2 = 2^7 k^4 c.$$

Therefore the whole curve α_{ij} is contained in the $2D + 2^7 k^4 c$-neighborhood of γ. This implies that the piecewise-geodesic curve $Image(\varphi)$ is contained in the $4k^2 c + 2^7 k^4 c$-neighborhood of γ. Since the distance between f and φ is at most $kh + c$ we conclude that

$$Image(f) \subset Nbd_{\tau(k,c)}(\gamma)$$

where $\tau(k, c) = 2^8 k^4 c$.

The composition $proj_\gamma \circ \varphi$ is continuous, this shows that γ is contained in the $\tau(k, c)$-neighborhood of the image of f. $\square$

Corollary 3.44. *Let $Q \subset \mathbb{H}^n$ be a (k, c)-quasigeodesic ray or a complete (k, c)-quasigeodesic. Then there is Q^* which is either a geodesic ray or a complete geodesic in $\mathbb{H}^n$ so that Q is contained in $Nbd_{\tau(k,c)+1}(Q^*)$.*

Proof: I will consider only the case of quasigeodesic rays $q : [0, \infty) \to Q \subset \mathbb{H}^n$ and leave the case of complete quasigeodesics to the reader. Consider the sequence of (k, c)-quasigeodesic segments

$$q_j = q\Big|_{[0,j]} : [0, j] \to \mathbb{H}^n.$$

Let γ_j denote the geodesic segment $[q(0), q(j)] \subset \mathbb{H}^n$. Then $d_H(q([0, j]), \gamma_j) \le \tau(k, c)$. Consider the points of intersection z_j of γ_j with the unit sphere $S_1(q(0))$. For large i, the spherical distance between z_i and z_{i+1} is at most $4e^{-i/k}$, therefore $\{z_i\}$ is a Cauchy sequence in $S_1(q(0))$. Hence the geodesic segments γ_j converge to a geodesic ray $Q^* = Image(\gamma^*)$. We now verify that $Q \subset Nbd_{\tau(k,c)+1}(Q^*)$. Pick a point $x = q(t)$ in Q and let r denote $d(q(0), x)$. Then for $j > t$ take the point $y_j \in \gamma_j$ closest to x. The point y_j is contained in the metric ball $B_r(q(0))$. Since

$$\lim_{j\to\infty} \gamma_j \Big|_{[0,2t]} = \gamma^* \Big|_{[0,2t]}$$

it follows that $d(y_j, Q^*) \leq 1$ for large j. □

3.10. Quasiconformal mappings

Definition 3.45. Suppose that X, Y are metric spaces, $f : X \to Y$ is a homeomorphism. The mapping f is called **quasiconformal** if the function

$$H_f(x) = \limsup_{r\to 0} \frac{\sup\{d(f(z), f(x)) : d(x,z) = r\}}{\inf\{d(f(z), f(x)) : d(x,z) = r\}}$$

is bounded from above in X. A quasiconformal mapping is called K-quasiconformal if the function H_f is bounded from above by K a.e. in X.

We will discuss properties of quasiconformal homeomorphisms in more details in Sections 3.11, 5.2. The notion of quasiconformality does not work well in the case when the domain and range are 1-dimensional. It is replaced by

Definition 3.46. Let $C \subset \mathbb{S}^1$ be a closed subset. A homeomorphism $f : C \to f(C) \subset \mathbb{S}^1$ is called **quasimoebius** if there exists a constant K so that for any quadruple of mutually distinct points $x, y, z, w \in \mathbb{S}^1$ their cross-ratio satisfies the inequality

$$K^{-1} \leq \frac{\lambda(|f(x) : f(y) : f(z) : f(w)|)}{\lambda(|x : y : z : w|)} \leq K \tag{3.11}$$

where $\lambda(t) = |\log(t)| + 1$.

Note that if f is K-quasimoebius then for any pair of Moebius transformations α, β the composition $\alpha \circ f \circ \beta$ is again K-quasimoebius.

If $D \subset \partial_\infty \mathbb{H}^n$ is a subset, we metrize D using the restriction of the spherical metric from $\mathbb{S}^{n-1}_\infty$. For a subset $X \subset \mathbb{H}^n$, let $\partial_\infty X$ denote the set of accumulation points of X in $\partial_\infty \mathbb{H}^n$.

Theorem 3.47. *(V. Efremovich, E. Tihomirova [ET64], see also [Tuk85b].) Suppose that $X, Y \subset \mathbb{H}^n$ are closed convex subsets, $f : X \to Y$ is a (k,c)-quasi-isometry. Then f has a continuous extension $h : \partial_\infty X \to \partial_\infty Y$ which is a quasiconformal homeomorphism between $\partial_\infty X$ and $\partial_\infty Y$ (if $n \geq 3$) and quasimoebius (if $n = 2$).*

Proof: Let $\gamma : [0, \infty) \to X$ be an infinite geodesic ray in X asymptotic to $\eta \in \partial_\infty X$. Then the image of γ in Y is a (k, c)-quasigeodesic ray. According to Corollary 3.44 there is a geodesic ray $\phi(\gamma) \subset X$ so that the Hausdorff distance between $f(\gamma)$ and a $\phi(\gamma)$ is at most $\tau(k, c) + 1$. Define $h(\eta)$ to be the point in $\partial_\infty \mathbb{H}^n$ represented by the geodesic ray $\phi(\gamma)$. The reader will verify (using Corollary 3.44) that $f \cup h : X \cup \partial_\infty X$ is a continuous mapping. Since f has an approximate inverse $\bar{f} : Y \to X$, the mapping h is a bijection. Therefore h is a homeomorphism.

I will verify quasiconformality of h for $n \geq 3$ and will leave the case $n = 2$ to the reader (use the same arguments as for $n \geq 3$ in combination with Lemma 3.16 and Exercise 3.17). According to the definition, it is enough to verify quasiconformality at each particular point x with uniform estimates on the function $H_h(x)$. Thus, after composing h with Moebius transformations, we can take $x = 0 = h(x)$, $h(\infty) = \infty$, where we consider the upper half-space model of $\mathbb{H}^n$.

Take a Euclidean sphere $S_r(0)$ in $\mathbb{R}^{n-1}$ with the center at the origin. This sphere is the ideal boundary of a hyperplane $P_r \subset \mathbb{H}^n$ which is orthogonal to the vertical geodesic $L \subset \mathbb{H}^n$, connecting 0 and ∞. Let $x_r = L \cap P_r$. Let $proj_L : \mathbb{H}^n \to L$ be the nearest point projection. The hyperplane P_r can be characterized by the following equivalent properties:

$$P_r = \{w \in \mathbb{H}^n : proj_L(w) = x_r\}$$

$$P_r = \{w \in \mathbb{H}^n : d(w, x_r) = d(w, L)\}.$$

Since quasi-isometric images of geodesics in $\mathbb{H}^n$ are uniformly close to geodesics, we conclude that

$$diam[proj_L(f(P_r))] \leq Const$$

where $Const$ depends only on the quasi-isometry constants of f. The projection π_L extends naturally to $\partial_\infty \mathbb{H}^n$. We conclude:

$$diam[proj_L(h(S_r(0)))] \leq Const.$$

Thus $h(S_r(0))$ is contained in a spherical shell

$$\{z \in \mathbb{R}^{n-1} : \rho_1 \leq |z| \leq \rho_2\}$$

where $\log[\rho_1/\rho_2] \leq Const$. This implies that the function $H_h(0)$ is bounded from above by $K := \exp(Const)$. We conclude that the mapping h is K-quasiconformal. $\square$

3.11. Distortion of distance by quasiconformal maps

In the case of domains in $\mathbb{R}^n$ $(n \geq 2)$ there is another (analytical) description of quasiconformal mappings. Suppose that D, D' are domains in $\mathbb{R}^n$. A homeomorphism $f : D \to D'$ is called *quasiconformal* if it has distributional partial derivatives in $L^2_{loc}(D)$ and the ratio

$$R_f(x) := \|f'(x)\|/|J_f(x)|^{1/n}$$

is uniformly bounded from above a.e. in D. Here $\|f'(x)\|$ is the operator norm of the derivative $f'(x)$ of f at x. The essential supremum of $R_f(x)$ in D is denoted by $K_O(f)$ and is called the *outer dilatation* of f. Let us compare $H_f(x)$ and $R_f(x)$. Clearly it is enough to consider positive-definite diagonal matrices $f'(x)$. Let Λ be the maximal eigenvalue of $f'(x)$ and λ be the minimal eigenvalue. Then $\|f'(x)\| = \Lambda$, $H_f(x) = \Lambda/\lambda$. We will restrict ourselves to the case $n = 3$, then

$$R_x \le H_f(x) \le R_x^3.$$

Two definitions of quasiconformality (using H_f and R_f) coincide (see for instance [Vuo88]) and we have:

$$K_O(f) \le K(f) \le K_O(f)^3.$$

Note that quasiconformality of mappings and the coefficients of quasiconformality $K(f)$, $K_O(f)$ do not change if instead of the Euclidean metric we consider a conformally-Euclidean metric in D.

Theorem 3.48. *(See [Vuo88, Theorem 11.19].) Suppose that $f : \mathbb{H}^n \to \mathbb{H}^n$ is a quasiconformal mapping, let $k^{-1} = K_O(f)$. Then*

$$\tanh \frac{1}{4} d(f(x), f(y)) \le 2(\tanh \frac{1}{4} d(x, y))^k,$$

$$\tanh \frac{1}{4} d(f(x), f(y)) \le 2(\tanh \frac{1}{4} d(x, y))^{K(f)^{-1/3}}$$

for every $x, y \in \mathbb{H}^n$.

This means that there are continuous functions $\theta(s,t), \zeta(s,t)$ such that:

$$\zeta(K(f), d(x,y)) \le d(f(x), f(y)) \le \theta(K(f), d(x,y))$$

$$\lim_{t\to 0} \theta(s,t) = 0.$$

(Moreover, the function $\theta(s, \bullet)$ is Hölder.) Thus we have a (nonlinear) bound on distortion of the hyperbolic metric by quasiconformal mappings.

3.12. Harmonic functions on the hyperbolic space

Suppose that $\lambda : \partial_\infty \mathbb{H}^n \to \mathbb{R}$ is a measurable function which belongs to L^1. The function λ admits a *harmonic extension* $ext(\lambda) = f : \mathbb{H}^n \to \mathbb{R}$ which is defined as follows. Pick a point $x \in \mathbb{H}^n$ and consider it as a base-point of $\mathbb{H}^n$. This gives the identification of $\partial_\infty(\mathbb{H}^n, x)$ and S_x, which is the unit sphere in the tangent space to $\mathbb{H}^n$ at x. Then define

$$f(x) := ext(\lambda) = \int_{S_x} \lambda.$$

The function f satisfies the following properties:

- f is harmonic with respect to the Laplace operator of the hyperbolic metric, in particular this is a real-analytic function.
- There is a measurable subset $E \subset \partial_\infty \mathbb{H}^n$ of the full measure such that $f \cup \lambda : \mathbb{H}^n \cup E \to \mathbb{R}$ is continuous in the conical topology.
- $ext(\lambda) \circ \gamma = ext(\lambda \circ \gamma)$ for all $\gamma \in \mathrm{Isom}(\mathbb{H}^n)$.

The function f is called the *harmonic extension* of λ. For details see [Ahl81], [Rei85].

Chapter 4

Kleinian Groups

4.1. Nilpotent groups

Our discussion of nilpotent groups follows [War76]. If G is a group with two subsets A, B then $[A, B]$ denotes the subgroup generated by all elements of the form $[a, b]$, $a \in A, b \in B$. In the special case $A = B = G$, the subgroup $[G, G]$ is called the *commutator subgroup* of G. Suppose that Γ is a group. Recall that a *central series* of Γ is a descending chain of subgroups

$$... \subset \Gamma_3 \subset \Gamma_2 \subset \Gamma_1 = \Gamma,$$

where each Γ_{i+1} is normal in Γ_i and $[\Gamma, \Gamma_i] \subset \Gamma_{i+1}$. A group is *nilpotent* if it admits a finite central series

$$1 = \Gamma_{n+1} \subset ...\Gamma_3 \subset \Gamma_2 \subset \Gamma_1 = \Gamma. \tag{4.1}$$

The smallest number n for which this property holds is called the *nilpotent class* of Γ. If Γ is nilpotent of the class n then Γ is said to be $n+1$-step nilpotent. Of special importance are two types of central series:

(a) The *lower central series* $\Gamma_{i+1} := [\Gamma, \Gamma_i]$.

(b) The *upper central series*, where the subgroups $\Gamma_i := Z_i(\Gamma)$ are defined as follows. $Z_n(\Gamma)$ is the center of Γ. Inductively, $Z_{i-1}(\Gamma) := \{x \in \Gamma : xZ_i(\Gamma) \in \text{center}(\Gamma/Z_i(\Gamma))\}$.

It is elementary that Γ is nilpotent iff either of these two central series is finite and the class n of Γ is the number of nontrivial subgroups in either central series.

Lemma 4.1. *(R. Baer, [Bae45].) If Γ is a finitely generated nilpotent group then each group Γ_i in the (lower) central series (4.1) of Γ is finitely generated.*

Proof: Suppose that Γ is 2-step nilpotent, $S = \{x_1, ..., x_N\}$ is a generating set of G. The reader will verify that $[x_i x_j, x_k] = [x_i, x_k][x_j x_k]$ for any $1 \le i, j, k \le N$. Thus the commutators $[x_i, x_j]$, $1 \le i, j \le N$, generate the subgroup $\Gamma_2 = [\Gamma, \Gamma]$ of Γ. Similarly, if Γ is n-step nilpotent, for each k the subgroup $\Gamma_{k+1}/\Gamma_{k+2}$ is generated by the k-fold commutators:

$$[....[[x_{i_1}, x_{i_2}], x_{i_3}], ..., x_{k+1}], \quad \text{where} \quad x_j \in S.$$

This implies that Γ_n is finitely generated, hence by the reverse induction each Γ_i is finitely generated as well, $1 \leq i \leq n$. □

Recall that a group Γ is said to satisfy the *ascending chain* condition if every chain of subgroups of Γ

$$G_1 \subset G_2 \subset G_3 \subset ...$$

eventually stabilizes, i.e. $G_{i+1} = G_i$ for all sufficiently large i.

Theorem 4.2. *If Γ is a finitely-generated almost nilpotent group then each subgroup of Γ is finitely generated and Γ satisfies the ascending chain condition.*

Proof: We will prove the second assertion, the proof of the first assertion is similar. First of all, it clearly suffices to consider only the case when Γ is nilpotent. The proof of the ascending chain condition is by induction on the nilpotent class of Γ. If Γ is abelian the assertion follows easily from the classification of finitely generated abelian groups. If Γ is n-step nilpotent consider the intersection G_{ij} of G_i with the member Γ_j of the (lower) central series (4.1). Then G_{in} eventually stabilizes since Γ_n is a finitely generated abelian group. Arguing inductively suppose that G_{ik} eventually stabilizes. Since Γ_{k-1}/Γ_k is finitely generated abelian, then $G_{i(k-1)}/\Gamma_k$ eventually stabilizes as well. This implies eventual stabilization of $G_{i(k-1)}$. □

Lemma 4.3. *(A. Malcev, [Mal49].) If Γ is a nilpotent group with the upper central series (4.1) such that Γ has torsion-free center then:*

(a) Each quotient $Z_i(\Gamma)/Z_{i+1}(\Gamma)$ is torsion-free ($i \leq n$).

(b) Γ is torsion-free.

Proof: (a) We argue by induction. It clearly suffices to prove the assertion for the quotient $Z_{n-1}(\Gamma)/Z_n(\Gamma)$. We will show that for each nontrivial element $\bar{x} \in Z_{n-1}(\Gamma)/Z_n(\Gamma)$ there exists a homomorphism $\varphi \in Hom(Z_{n-1}(\Gamma)/Z_n(\Gamma), Z_n(\Gamma))$ such that $\varphi(\bar{x}) \neq 1$. Since $Z_n(\Gamma)$ is torsion-free this would imply that $Z_{n-1}(\Gamma)/Z_n(\Gamma)$ is torsion-free. Let $x \in Z_{n-1}(\Gamma)$ be the element which projects to $x \in Z_{n-1}(\Gamma)/Z_n(\Gamma)$. Thus $x \notin Z_n(\Gamma)$, therefore there exists an element $g \in \Gamma$ such that $[g, x] \in Z_n(\Gamma) - \{1\}$. Define the map $\tilde{\varphi} : Z_{n-1}(\Gamma) \to Z_n(\Gamma)$ by

$$\tilde{\varphi}(y) := [y, g].$$

Obviously $\tilde{\varphi}(x) \neq 1$ and, since $Z_n(\Gamma)$ is the center of Γ, the map $\tilde{\varphi}$ descends to a map $Z_{n-1}(\Gamma)/Z_n(\Gamma) \to Z_n(\Gamma)$. We leave it to the reader to verify that $\tilde{\varphi}$ is a homomorphism.

(b) In view of (a), for each i, $m \geq 0$ and each $x \in Z_i(\Gamma) - Z_{i+1}(\Gamma)$ we have: $x^m \notin Z_{i+1}(\Gamma)$. Thus $x^m \neq 1$. By induction it follows that Γ is torsion-free. □

Theorem 4.4. *For each linear connected Lie group G (over $\mathbb{R}$) there is an upper bound $m = m(G)$ on classes of torsion-free nilpotent subgroups of G.*

Proof: It suffices to consider the case $G = GL(k, \mathbb{R})$. Suppose that $\Gamma \subset GL(k, \mathbb{R})$ is a nilpotent subgroup of class c. Consider Zariski closure $\bar{\Gamma}$ of Γ in $GL(k, \mathbb{R})$, i.e. the smallest algebraic subgroup of $GL(k, \mathbb{R})$ which contains Γ. The group $\bar{\Gamma}$ is also nilpotent since the algebraic equation

$$[\dots[[g_1, g_2], g_3]\dots, g_{c+1}] = 1 \tag{4.2}$$

satisfied by all elements of Γ also holds in $\bar{\Gamma}$. Thus we have the upper central series

$$1 \subset Z_{c+1}(\bar{\Gamma}) \subset \dots \subset Z_2(\bar{\Gamma}) \subset \bar{\Gamma}$$

where each $Z_i(\bar{\Gamma})$ is Zariski closed and $\overline{Z_i(\Gamma)} \subset Z_i(\bar{\Gamma})$ (argue by induction). Suppose one of the quotients $\mathbb{Z}_i(\bar{\Gamma})/Z_{i+1}(\bar{\Gamma})$ is finite. Then there exists $m > 0$ such $\gamma^m \in Z_{i+1}(\bar{\Gamma})$ for each $\gamma \in Z_i(\Gamma)$. This would imply that $\gamma^m \in Z_{i+1}(\Gamma)$ which contradicts Lemma 4.3. Therefore each quotient $\mathbb{Z}_i(\bar{\Gamma})/Z_{i+1}(\bar{\Gamma})$ is an (abelian) Lie group of positive dimension. It follows that the dimension of $\bar{\Gamma}$ is at least c, i.e. $c \leq m = dim(GL(n, \mathbb{R})) = (dim(G))^2$. $\square$

We refer the reader to [Weh73, Chapter 8] for the better upper bounds on the nilpotent class of torsion-free subgroups in Lie groups. To see that the above theorem fails for groups with torsion consider the finite dihedral group

$$D_n = \langle a, b | a^2 = 1, b^n = 1, aba^{-1} = b^{-1} \rangle.$$

Every such group embeds into $O(2)$: the image of a is a reflection and the image of b is the order n rotation. On the other hand, if $n = 2^p$ then D_n is p-step nilpotent.

4.2. Residual finiteness and Selberg lemma

Recall that a group G is called *residually finite* if there exists a chain of finite-index subgroups $G_i \subset G$ such that

$$\bigcap_{i=1}^{\infty} G_i = \{1\}.$$

A group G is called *Hopfian* if any epimorphism $f : G \to G$ is injective.

Exercise 4.5. *Show that every finitely generated residually finite group is Hopfian (see [Mal40]).*

Not every group is residually finite, for instance the Baumslag-Solitar group

$$G = \langle a, b | ab^2a^{-1} = b^3 \rangle$$

is not; see Exercise 10.5. However all finitely generated subgroups of $SL(n, \mathbb{C})$ are residually finite according to the following theorem known as "Selberg Lemma":

Theorem 4.6. *Suppose that G is a finitely generated subgroup of $SL(n,\mathbb{C})$. Then G is residually finite (A. Malcev [Mal40]) and there exists a finite-index subgroup $G_0 \subset G$ which is torsion-free (A. Selberg [Sel60]).*

This lemma allows (in many cases) reduction of the discussion of finitely generated discrete subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ to the case of torsion-free subgroups.

Exercise 4.7. *Let Γ be the fundamental group of a closed hyperbolic surface, F a finite group. Show that any group G which fits into short exact sequence*

$$1 \to F \to G \to \Gamma \to 1 \tag{4.3}$$

is residually finite. Hint: first consider the case when F is a finite cyclic group. The obstruction to splitting the sequence (4.3) belongs to $H^2(\Gamma, F)$; use Poincaré duality with coefficients in F to show that this obstruction virtually vanishes.

Note that residual finiteness of Γ alone is not enough to prove this assertion: there are examples of finitely generated linear groups Γ and groups G which fit into the sequence (4.3) so that G is not residually finite, see [Mil79], [Rag84].

Exercise 4.8. *Analyze the extensions (4.3) if Γ is the fundamental group of a closed hyperbolic 3-manifold.*

One of the corollaries of Thurston's Hyperbolization Theorem is the following

Theorem 4.9. *(W. Thurston, J. Hempel [Hem87].) The fundamental group of every Haken manifold is residually finite.*

There are other classes of groups which are known to be residually finite, for instance the *Teichmüller modular group* Mod_S. More generally, if G is a finitely generated residually finite group then the group $Aut(G)$, which consists of automorphisms of G, is also residually finite, see [Bau63].

4.3. Representation varieties

Let Γ be a finitely generated group, G be a semi-simple algebraic Lie group (the most interesting for us cases are: $G = \mathrm{Isom}(\mathbb{H}^n)$, $G = PSL(2,\mathbb{C})$, $G = SL(2,\mathbb{C})$). Let G^0 denote the connected component of the identity in G (with respect to the usual topology). We give $Hom(\Gamma, G)$ the structure of an algebraic variety as follows:

Let $x_1, ..., x_n$ denote generators of Γ and $R_1, ..., R_r, \ldots$ the corresponding relators. The representation variety $Hom(\Gamma, G)$ is isomorphic to the algebraic subvariety

$$\{(g_1, ..., g_n) \in G^n | R_1(g_1, ...g_n) = e, \ldots, R_r(g_1, ...g_n) = e, \ldots\}$$

where e is the identity in G. According to Hilbert's Nullstellensatz it is enough to consider only finitely many relators $R_1, ..., R_r$. It turns out that the isomorphism type of the variety $Hom(\Gamma, G)$ is independent on the choice of presentation of Γ, see [LM85], [KM99].

Remark 4.10. It is important to think of $Hom(\Gamma, G)$ not just as a set, but as an algebraic variety, more precisely, a scheme. Informally speaking, we have to remember not just the set $Hom(\Gamma, G)$ but also the equations which define it. More precisely, suppose that $G \subset \mathbb{C}^N$ is a complex affine Lie group. For instance

$$SL(2, \mathbb{C}) \subset GL(2, \mathbb{C}) \subset \mathbb{C}^4,$$

the group $PSL(2, \mathbb{C})$ can be also be regarded a complex affine group via its adjoint representation $ad : PSL(2, \mathbb{C}) \hookrightarrow GL_3(\mathbb{C}) \subset \mathbb{C}^9$. Then the equations

$$f_1(x_1, ..., x_N) = 0, ..., f_m(x_1, ..., x_N) = 0$$

defining $Hom(\Gamma, G)$ yield the ring $H := \mathbb{C}[x_1, ..., x_N]/I$, where I is the ideal generated by the polynomials $f_1, ..., f_m$. Then the isomorphism class of the variety $Hom(\Gamma, G)$ is the isomorphism class of the ring H. For instance, the subvarieties $\{z^2 = 0\}$ and $\{z = 0\}$ in $\mathbb{C}$ are non-isomorphic.

Let ad denote the *adjoint* action of G on itself by conjugation: $ad(g)h = g^{-1}hg$. Consider the quotient

$$R(\Gamma, G) = Hom(\Gamma, G)/G^0$$

where G^0 acts on homomorphisms by conjugations: $\rho \mapsto ad(g) \circ \rho$. The quotient space $R(\Gamma, G)$ is not particularly nice, for instance in many interesting cases it is not Hausdorff. However it turns out that one can modify the definition so that the "quotient" becomes an algebraic variety. Namely, instead of taking the usual quotient of topological spaces one can take so called *Mumford quotient*, so that $R(\Gamma, G)$ becomes an algebraic variety, called the *character variety* of Γ (see for instance [JM87, LM85, CS83, Mor86]). I will use the notation

$$\mathcal{R}(\Gamma, G) = Hom(\Gamma, G)//G^0$$

for this quotient. Precise definition of of this quotient will not be essential here, the idea is to define first the ring of polynomial functions of the quotient $\mathcal{R}(\Gamma, G)$ as the subring of polynomial functions of the algebraic variety $Hom(\Gamma, G)$ which are invariant under the action of G^0. From this ring one reconstructs the algebraic variety $\mathcal{R}(\Gamma, G)$.

A representation $\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^n)$ is said to be *nonelementary* if its image is a nonelementary subgroup in $\mathrm{Isom}(\mathbb{H}^n)$. A group Γ is said to be *non-radical* if it does not contain infinite normal nilpotent subgroups. Let G be an algebraic Lie group, then a representation $\rho : \Gamma \to G$ is said to be *non-radical* if $\rho(\Gamma)$ is non-radical.

Let $Hom^0(\Gamma, G)$ denote the space of all non-radical representations. It turns out that the spaces $\mathcal{R}(\Gamma, G)$ and $R(\Gamma, G)$ are not much different, the set-theoretic quotient $Hom^0(\Gamma, G)/G^0$ has structure of an algebraic variety which is naturally isomorphic to the Mumford quotient $\mathcal{R}^0(\Gamma, G) := Hom^0(\Gamma, G)//G^0$, see [JM87]. The variety $\mathcal{R}(\Gamma, G)$ need not be a manifold.

However if we restrict ourselves to the case $\Gamma = \pi_1(S)$, where S is a closed orientable hyperbolic surface of genus g and G is one of the groups:

$$\mathrm{Isom}(\mathbb{H}^3), PSL(2,\mathbb{C}), SL(2,\mathbb{C}),$$

then $\mathcal{R}^0(\Gamma, G)$ is a smooth complex manifold of the complex dimension $6g-6$ (see the next section). This explains one of the reasons why we divide the space of homomorphisms by G^0 rather than by G itself: in the case of $G = SL(2,\mathbb{C}), PSL(2,\mathbb{C})$ it is all the same, but if $G = \mathrm{Isom}(\mathbb{H}^3)$, then the group G does not act freely on $Hom^0(\Gamma, G)$ and this action is not holomorphic (because of orientation-reversing elements). If $G = PSL(2,\mathbb{R})$ then $Hom^0(\Gamma, G)/G$ is a smooth manifold of the (real) dimension $6g-6$.

Similar results hold for *relative parabolic* representation varieties of the fundamental groups of punctured surfaces. First I give the general definition. Suppose Γ is a finitely generated group and $H := \{H_1, ..., H_q\}$ is a collection of its subgroups. Let $G := \mathrm{Isom}(\mathbb{H}^n)$, or $G = PSL(2,\mathbb{C})$, or $G = SL(2,\mathbb{C})$. Consider the *relative parabolic representation variety*

$$Hom_{par}(\Gamma, H; G) = \{\Gamma \xrightarrow{\rho} G : \rho(\gamma) \text{ is parabolic for all } \gamma \in H_j, j = 1, ..., q\}.$$

Elements of $Hom_{par}(\Gamma, H; G)$ will be called *relative parabolic* representations. Define the *relative parabolic character variety* as the quotient:

$$\mathcal{R}_{par}(\Gamma, H; G) := [Hom_{par}(\Gamma, H; G) \cap Hom^0(\Gamma, G)]/G^0.$$

Then $\mathcal{R}_{par}(\Gamma, H; G)$ naturally sits in $\mathcal{R}(\Gamma, G)$ as an open subset of an algebraic subvariety of $\mathcal{R}(\Gamma, G)$.

Convention 4.11. If Γ is a subgroup of G then we shall use the notation

$$Hom_{par}(\Gamma, G) := Hom_{par}(\Gamma, H; G), \quad \mathcal{R}_{par}(\Gamma, G) := \mathcal{R}_{par}(\Gamma, H; G).$$

where H is the collection of all maximal parabolic subgroups of Γ.

Similarly, let $\rho_0 \in Hom(\Gamma, G)$, define the *relative representation variety*

$$Hom_{\rho_0}(\Gamma, H; G) = \{\rho : \Gamma \to G : \quad \rho|H_j \in ad(G)(\rho_0|H_j) \text{ for all } j = 1, ..., q\}.$$

In other words, the restriction of each $\rho \in Hom(\Gamma, H; G)$ to each H_j differs from $\rho_0|H_j$ by a conjugation via an element of G. Then the *relative character variety* is the quotient:

$$\mathcal{R}_{\rho_0}(\Gamma, H; G) := [Hom_{\rho_0}(\Gamma, H; G) \cap Hom^0(\Gamma, G)]/G^0.$$

I will omit the index ρ_0 from the above notation in the case when the choice of ρ_0 is clear.

Example 4.12. Let $\Gamma = \mathbb{Z}^2 = \langle a \rangle \oplus \langle b \rangle$, $\rho_0 : \Gamma \to G = PSL(2,\mathbb{C})$,

$$\rho_0(a) : z \mapsto z+1, \rho_0(b) : z \mapsto z+\tau$$

where $\tau \in \mathbb{C} - \mathbb{R}$. Let $H_1 := \langle a \rangle$, $H := \Gamma$. Then the germs of $Hom_{par}(\Gamma, H, G)$ and $Hom_{\rho_0}(\Gamma, H_1, G)$ at ρ_0 are equal.

Now we specialize our discussion to the case when Γ is the fundamental group of a hyperbolic surface S of genus g with q punctures. Let H_j be the cyclic groups generated by loops going around punctures of S. Choose ρ_0 such that $\rho_0(H_j)$ are nontrivial groups. Then $\mathcal{R}(\Gamma, H; G)$ is a smooth complex manifold of the complex dimension $6g-6+2q$ if $G = \mathrm{Isom}(\mathbb{H}^3), PSL(2,\mathbb{C}), SL(2,\mathbb{C})$; $\mathcal{R}(\Gamma, H; G)$ is a smooth manifold of the real dimension $6g-6+2q$ if $G = PSL(2,\mathbb{R})$. See for instance [Wei64] (cf. Proposition 4.24). Note that if $\rho_0(H_j)$ are parabolic groups for each j then

$$\mathcal{R}_{\rho_0}(\Gamma, H; G) = \mathcal{R}_{par}(\Gamma, H; G).$$

If $G = SL(2,\mathbb{C})$ (and Γ is any finitely generated group), there is a particularly nice description of the variety $\mathcal{R}(\Gamma, SL(2,\mathbb{C}))$. Namely, for each element $\gamma \in \Gamma$ we have the function

$$Tr_\gamma : Hom(\Gamma, SL(2,\mathbb{C})) \to \mathbb{C}\ , \quad Tr_\gamma(\rho) = Trace(\rho(\gamma))$$

(the *character*). It turns out that there is a finite collection of elements $\gamma_1, ..., \gamma_m \in \Gamma$ so that the vector-function:

$$T : Hom^0(\Gamma, SL(2,\mathbb{C}))/SL(2,\mathbb{C}) \to \mathbb{C}^m, T(\rho) = (Tr_{\gamma_1}(\rho), ..., Tr_{\gamma_m}(\rho))$$

is injective and the affine variety equal to the closure of its image in $\mathbb{C}^m$ is naturally isomorphic to the Mumford quotient $Hom(\Gamma, SL(2,\mathbb{C}))//SL(2,\mathbb{C})$. See [CS83], [GAM93] for an explicit construction of the elements γ_j and [Mor86] for discussion of the case of representations to $\mathrm{Isom}(\mathbb{H}^n)$ (see also Remark 5.7). As a corollary we have the following:

Proposition 4.13. *There is a finite collection of elements $\alpha_1, ..., \alpha_k \in \Gamma$, so that the map*

$$L : \mathcal{R}^0(\Gamma, \mathrm{Isom}(\mathbb{H}^3)) \to \mathbb{R}^k_+, \quad L([\rho]) = (\ell(\rho(\alpha_1)), ..., \ell(\rho(\alpha_k)))$$

is proper. (I recall that $\ell(g)$ stands for the translation length of the element $g \in \mathrm{Isom}(\mathbb{H}^3)$.)

Proof: There is a slight difference between representations into $\mathrm{Isom}(\mathbb{H}^3)$ and $SL_2(\mathbb{C})$: not every representation to $\mathrm{Isom}(\mathbb{H}^3)$ has image in $PSL_2(\mathbb{C})$ and not every representation to $PSL_2(\mathbb{C})$ lifts to $SL_2(\mathbb{C})$. However after we take an index 4 subgroup in Γ this difference disappears and its clearly enough to check convergence on a finite-index subgroup. So we will work with representations to $SL_2(\mathbb{C})$.

Recall that the translation length of loxodromic elements $g \in SL_2(\mathbb{C})$ can be recovered from the trace t of g as:

$$t = \lambda + \lambda^{-1}, \quad \ell(g) = |\log(|\lambda|)|.$$

Thus, if $\ell(g)$ is bounded by a constant C, then $1 \le \max(|\lambda|^{-1}, |\lambda|) \le e^C$ and therefore t belongs to a bounded domain in $\mathbb{C}$ as well. If g is elliptic or parabolic then $|Tr(g)| \le 2$ and there is nothing to check. $\square$

Functorial properties of representation varieties. Let M be a compact connected manifold, let M_3 be a connected hypersurface in M which splits M open into connected submanifolds with boundary M_1, M_2. Pick a base-point $* \in M_3$. Below we will use the fundamental groups of the manifolds M_i with the base-point $*$. Let $\iota_j : \pi_1(M_3, *) \hookrightarrow \pi_1(M_i, *), i = 1, 2$ be the maps induced by inclusions $M_3 \hookrightarrow M_i, i = 1, 2$. These maps induce the "restriction" morphisms

$$Res_i : Hom(\pi_1(M_i), G) \to Hom(\pi_1(M_3), G).$$

Recall that the *fiber product* of $Hom(\pi_1(M_i), G), i = 1, 2$, with respect to the morphisms Res_i is defined as

$$Hom(\pi_1(M_1), G) \times_{Res_1 = Res_2} Hom(\pi_1(M_2), G) :=$$

$$\{(\rho_1, \rho_2) | \rho_i \in Hom(\pi_1(M_i), G), i = 1, 2 \quad \text{and} \quad \rho_1 \circ \iota_1 = \rho_2 \circ \iota_2\}.$$

Exercise 4.14. *There is a natural isomorphism of algebraic varieties*

$$Hom(\pi_1(M), G) \cong Hom(\pi_1(M_1), G) \times_{Res_1 = Res_2} Hom(\pi_1(M_2), G).$$

Hint: use Seifert-Van Kampen theorem to define a presentation for $\pi_1(M)$ so that the algebraic varieties above are given by the same sets of equations.

Representations and flat bundles. There is a natural correspondence between representations and flat bundles. Suppose that M is a closed smooth manifold with the fundamental group Γ, G is a Lie group as above. Let X be either a smooth manifold on which G acts smoothly and transitively or a vector space on which G acts linearly. With any representation $\rho : \Gamma \to G$ we associate the flat X-bundle $E_\rho = X \times_\rho M$ over M:

$$X \times_\rho M = X \times \tilde{M} / \Gamma$$

where $\tilde{M}$ is the universal cover of M and $\gamma \in \Gamma$ acts on $X \times \tilde{M}$ as $(\rho(\gamma^{-1}), \gamma)$. The horizontal foliation $point \times \tilde{M}$ on $X \times \tilde{M}$ descends to a horizontal foliation on E_ρ which determines flat connection on E_ρ. If ρ varies in a connected component C of $Hom(\Gamma, G)$, then all the bundles E_ρ are smoothly isomorphic to a fixed bundle E (the isomorphism of course does not preserve the flat connection). Thus we get a map $\tilde{\phi} : C \to F(E)$, where $F(E)$ is the space of flat connections on the bundle E (provided with the C^1-topology). To make ϕ a bijection we have to divide C by the adjoint action of G^0 (connected component of the identity) and the space $F(E)$ by $\mathcal{G}_0$, which is the group of gauge transformations homotopic to the identity. It is easy to verify that $\tilde{\phi}$ projects to a bijection

$$\phi : C/G^0 \to F(E)/\mathcal{G}_0$$

which is an open map.

Theorem 4.15. *ϕ is a homeomorphism.*

Proof see in [GM88, Section 6], [FKK94]. (Actually C/G^0, $F(E)/\mathcal{G}_0$ have structure of analytic varieties and ϕ is an isomorphism of analytic varieties.)

4.4. Cohomology of groups and sheaves

In this section we review briefly cohomology of groups and Čech cohomology. We refer the reader to Brown's book [Bro82] for detailed discussion of various definitions and properties of the group cohomology. Let Γ be a group and E be a $k\Gamma$-module, where $k = \mathbb{R}$ or $k = \mathbb{C}$. An important example for us will be the case when $E = \mathfrak{g}$, the Lie algebra of a Lie group G. The action of Γ on $\mathfrak{g}$ is defined as follows. Fix a representation $\rho : \Gamma \to G$. Then the composition of ρ with the adjoint representation defines a linear action of Γ on $\mathfrak{g}$, so $\mathfrak{g}$ becomes a $k\Gamma$-module $\mathfrak{g}_\rho$.

Define the *inhomogeneous* bar-complex $C_*(\Gamma)$ as follows:

$C_n(\Gamma)$ is the space of formal linear combinations of n-tuples $[g_1, ..., g_n], g_i \in \Gamma$ with integer coefficients. Let

$$\partial := \sum_{i=0}^{n} (-1)^i d_i \ , \quad \text{where} \quad d_0[g_1, ..., g_n] = [g_2, ..., g_n]$$

$$d_i[g_1, ..., g_n] = [g_1, ..., g_i g_{i+1}, ..., g_n] \ , \quad \text{if} \quad 0 < i < n$$

$$d_n[g_1, ..., g_n] = [g_1, ..., g_{n-1}].$$

We let $C_{-1}(\Gamma) := 0$ and $d_{-1} := 0$.

Then $(C_*(\Gamma), \partial)$ is a chain complex. Define the cochain complex $(C^*(\Gamma, E), \delta)$ as $C^*(\Gamma, E) = Hom_{\mathbb{Z}}(C_*(\Gamma), E)$ and δ as dual to ∂. The cochain complex $(C^*(\Gamma, E), \delta)$ gives rise to the cohomology groups $H^*(\Gamma, E)$ of the group Γ with the coefficients in E. We now describe in details what happens in the dimensions 0 and 1. We let E^Γ denote the fixed-point set of Γ in E. Then

$$C^i(\Gamma, E) = k\Gamma - \text{module generated by maps } \Gamma^i \to E,$$

where Γ^i is the i-fold product of Γ; we let $\Gamma^0 := \{*\}$ be a one-point set. Thus $C^0(\Gamma, E) \cong E$. The coboundary homomorphisms are:

$$\delta_0 : E \to C^1(\Gamma, E), \ \ \delta_0(\xi)(\gamma) = \xi - \gamma\xi, \ \gamma \in \Gamma, \ \delta_1 : C^1(\Gamma, E) \to C^2(\Gamma, E),$$

$$\delta_1(c)(\alpha, \gamma) = c(\alpha) + \alpha c(\gamma) - c(\alpha\gamma), \ \alpha, \gamma \in \Gamma, c \in C^1(\Gamma, E) \ .$$

We will also use the notation c_γ for $c(\gamma)$, if $c \in C^1(\Gamma, E)$. Then we get the cocycles $Z^i(\Gamma, E) := ker(\delta_i)$, and coboundaries $B^i(\Gamma, E) := image(\delta_{i-1})$, $i = 0, 1$. Finally $H^i(\Gamma, E) := Z^i(\Gamma, E)/B^i(\Gamma, E)$, $i = 0, 1$. Note that $B^0(\Gamma, E) = 0$, hence

$$H^0(\Gamma, E) = \{\xi \in E | \xi - \gamma\xi = 0, \quad \text{for all} \quad \gamma \in \Gamma\},$$

i.e. $H^0(\Gamma, E) \cong E^\Gamma$. If $\Gamma \cong \mathbb{Z}$ then $Z^1(\Gamma, E) \cong E$ and $H^1(\Gamma, E) \cong H^0(\Gamma, E)$. If $\Gamma = \Gamma_1 * \Gamma_2$, then $Z^1(\Gamma, E) \cong Z^1(\Gamma_1, E) \oplus Z^1(\Gamma_2, E)$.

Lemma 4.16. *(See [Bro82, Ch. 3, Cor. 10.2].) If Γ is a finite group then $H^p(\Gamma, E) = 0$ for each $p \geq 1$.*

We will prove this lemma only in the case when $p = 1$, Γ is a finite cyclic group of order n and E is finite-dimensional (the only case we will need in what follows). Choose a generator γ of Γ. Suppose that $c \in Z^1(\Gamma, E)$, $c(\gamma) = \xi$, then the cocycle condition reads:

$$\sum_{i=0}^{n-1} \gamma^i \xi = 0.$$

Since the group Γ is finite we can give E a Γ-invariant metric. Using this metric we split E as $E = F \oplus L$ where F is the fixed-point set for the action of γ on E and L is Γ-invariant. If $\pi_F(\xi)$ is the orthogonal projection of ξ to F then the cocycle condition becomes $n\pi_F(\xi) = 0$, i.e. ξ belongs to L. The mapping $\tau \mapsto \tau - \gamma\tau$ is injective on L. Thus for each $\xi \in L$ there exists $\tau \in L$ such that $\xi = \tau - \gamma\tau$. Applying this inductively to γ^i, $i = 2, 3, ..., n-1$, we conclude that $c(\gamma^i) = \tau - \gamma^i\tau$, i.e. c is a coboundary. $\square$

An alternative descriptions of $H^i(\Gamma, E)$ can be given in terms of the Čech cohomology. Suppose that X is a topological space, $\mathcal{U}$ is an open covering of X. For the elements $U_1, ..., U_n \in \mathcal{U}$ let $U_{i_1 \dots i_n}$ denote the intersection $U_1 \cap ... \cap U_n$. Let $\mathcal{F}$ be a sheaf of abelian groups on X. The space $C^n(\mathcal{U}, \mathcal{F})$ of Čech n-cochains on X with coefficients in $\mathcal{F}$ is defined as

$$C^n(\mathcal{U}, \mathcal{F}) := \prod_{i_0, \dots, i_n} \mathcal{F}(U_{i_0 i_1 \dots i_n}), n \geq 0,$$

$$C^n(\mathcal{U}, \mathcal{F}) := 0 \text{ for } n < 0.$$

The elements of $C^n(\mathcal{U}, \mathcal{F})$ will be denoted $f_{i_0 i_1 \dots i_n}$. Each $C^n(\mathcal{U}, \mathcal{F})$ has a natural structure of abelian group (induced from $\mathcal{F}$). The coboundary operator $d : C^n(\mathcal{U}, \mathcal{F}) \to C^{n+1}(\mathcal{U}, \mathcal{F})$ is given by

$$d_n(f_{i_0 i_1 \dots i_n}) = g_{i_0 i_1 \dots i_{n+1}} := \sum_{k=0}^{n+1} (-1)^k \rho(f_{i_0 i_1 \dots \widehat{i_k} \dots i_{n+1}})$$

where ρ is the appropriate restriction homomorphism

$$\mathcal{F}_{U_{i_0 i_1 \dots \widehat{i_k} \dots i_n}} \to \mathcal{F}_{U_{i_0 i_1 \dots i_{n+1}}}.$$

With this coboundary operator $(C^\bullet(\mathcal{U}, \mathcal{F}), d_\bullet)$ becomes a cochain complex. Its cocycles and coboundaries are $Z^i := ker(d_i), B^i := Im(d_{i-1})$. The Čech cohomology groups $H^i(\mathcal{U}, \mathcal{F})$ are the quotients Z^i/B^i. We refer the reader to [Bre97] for more details. In this book we will only deal with the following two examples of sheaves:

(a) X is a Riemann surface and $\mathcal{F}$ is the sheaf of local holomorphic sections of a holomorphic vector bundle V over X. In this case it is customary to use the notation $H^i(X, V)$ for $H^i(X, \mathcal{F})$.

(b) X is a smooth manifold, $\mathcal{F}$ is the sheaf of local parallel sections of a flat vector bundle $F \to X$.

In both cases the Čech cohomologies do not depend on the choice of open covering $\mathcal{U}$ provided that $\mathcal{U}$ is chosen so that each $U_{i_1 \dots i_n}$ is contractible. (For instance, choose a Riemannian metric on X and take $\mathcal{U}$ to consist of convex open neighborhoods of points $x \in X$.) Thus we have $H^i(X, \mathcal{F}) := H^i(\mathcal{U}, \mathcal{F})$, the Čech cohomology of X with the coefficients in $\mathcal{F}$.

Below we discuss Čech cohomology in the case of Example (b). Let M be a (smooth) connected manifold with the fundamental group Γ (this manifold may have boundary) and let E be a $k\Gamma$-module, where $k = \mathbb{R}$ or $k = \mathbb{C}$. The action of Γ on E gives rise to the vector bundle $F \to M$ with the fibers isomorphic to E. This bundle has canonical flat connection ω, compare §4.3. Consider the sheaf $\mathcal{F}$ of parallel sections of the flat bundle $F \to M$. Let $H^*(M, \mathcal{F})$ denote the Čech cohomology groups of M with coefficients in the sheaf $\mathcal{F}$. If $A \subset M$ is a submanifold then $H^*(M, A; \mathcal{F})$ denotes the sheaf cohomology of the pair (M, A). If $N \subset M$ is a submanifold then by abusing notation we will retain the name $\mathcal{F}$ for the restriction $\mathcal{F}|N$ of the sheaf $\mathcal{F}$ to M ($\mathcal{F}|N$ the sheaf of parallel sections is the pull-back of the flat bundle F to N). Below we list some properties of $H^*(M, \mathcal{F})$:

- $H^*(M, \mathcal{F})$ satisfies the functorial properties and the Eilenberg-Steenrod axioms for the usual cohomology theories.
- $H^0(M, \mathcal{F})$ is the space of horizontal sections of $\mathcal{F}$. If $E = \mathfrak{g}_\rho$ where $\mathfrak{g}$ is the Lie algebra of a Lie group G, then $dim H^0(M, \mathcal{F})$ is the dimension of the centralizer of $\rho(\Gamma)$ in G.
- $H^k(M, \mathcal{F}) = 0$ if $k > dim(M)$.
- If M is aspherical then there is a canonical and natural isomorphism
$$H^*(M, \mathcal{F}) \cong H^*(\Gamma, E).$$
- **Poincaré duality**. Suppose that M is a compact, connected, oriented n-manifold and ∂M is the disjoint union of A and B each of which is a codimension 0 submanifold in ∂M (possibly with boundary). Let $\mathcal{F}^*$ denote the dual sheaf of $\mathcal{F}$ (it is the sheaf of horizontal sections of the dual bundle of F). Then each group $H^k(M, \mathcal{F})$ is finite-dimensional and
$$H^k(M, A; \mathcal{F}) \cong H^{n-k}(M, B; \mathcal{F}^*)^*.$$
- If G is a reductive Lie group (which is the case if $G = PSL(2, \mathbb{C})$) then the Killing form on its Lie algebra $\mathfrak{g}$ is invariant under the adjoint action of G and is nondegenerate. Hence, for any $\rho : \Gamma \to G$ and $E = \mathfrak{g}_\rho$ we get:
$$H^{n-k}(M, B; \mathcal{F}^*)^* \cong H^{n-k}(M, B; \mathcal{F})^* \cong H^{n-k}(M, B; \mathcal{F}).$$
- For compact M define the Euler characteristic $\chi(M, \mathcal{F})$ of M with $\mathcal{F}$-coefficients as
$$\chi(M, \mathcal{F}) = \sum_{i=0}^{\infty} (-1)^i dim(H^i(M, \mathcal{F})).$$
Then $\chi(M, \mathcal{F}) = dim(E)\chi(M)$ where $\chi(M)$ is the usual Euler characteristic of M.

- Let S be a codimension 1 submanifold in M which splits M open into the union of submanifolds N and L, then

$$\chi(M,\mathcal{F}) = \chi(N,\mathcal{F}) + \chi(L,\mathcal{F}) - \chi(S,\mathcal{F}).$$

There is yet another way to compute the cohomology $H^*(\Gamma, E)$ in the case using de Rham theorem. Let $\Omega_F^\bullet$ denote the cochain complex of differential forms on M with values in the bundle F. If U is an open subset over which the bundle F is trivial ($F|U \cong U \times \mathfrak{g}$) then $\Omega_F^\bullet$ restricted to U equals $\mathfrak{g} \otimes \Lambda^\bullet(U)$ where $\Lambda^\bullet(U)$ is the ordinary de Rham complex of differential forms on U. We again refer the reader to [Rag72, Chapter VII, §1] for precise definitions. Let $H^*(M, \Omega_F)$ denote the de Rham cohomologies of M with coefficients in Ω_F.

Theorem 4.17. *(de Rham Theorem, see [Rag72, Chapter VII, §1], [GM88].) There is a natural isomorphism*

$$H^*(M, \Omega_F) \cong H^*(M, \mathcal{F}).$$

4.5. Group cohomology and representation varieties

Let Γ be a group with the presentation

$$\langle x_1, ..., x_n | R_1, \dots, R_r, \dots \rangle.$$

Let G be an algebraic Lie group with the Lie algebra $\mathfrak{g}$. To simplify the notation we shall assume that G is a linear group. Let $R = R^r$ denote the map $(R_1, ..., R_r) : G^n \to G^r$. Then (for large r) the representation variety $Hom(\Gamma, G)$ is isomorphic to the algebraic subvariety

$$\{(g_1, ..., g_n) \in G^n | R(g_1, ...g_n) = (e, \dots, e)\}$$

in G^n. The adjoint action of G induces the adjoint representation $Ad : G \to GL(\mathfrak{g})$ given by the derivative of $ad(g)$ at the identity $e \in G$. If $G \subset GL(N)$ is a matrix group, then $\mathfrak{g} \subset gl(N)$ and $Ad(g)\xi = g^{-1}\xi g$ for each $\xi \in \mathfrak{g}, g \in G$.

Pick $\rho_0 \in Hom(\Gamma, G)$ and consider the *Zariski tangent space* $T_{\rho_0} Hom(\Gamma, G)$ of the representation variety $Hom(\Gamma, G)$ at the point ρ_0. In the down-to-earth terms $T_{\rho_0} Hom(\Gamma, G)$ is the kernel of the derivative $D_{\rho_0} R$ where $R : G^n \to G^r$ is the map given by relators. It was first noticed by André Weil in [Wei64] (see also [Rag72, Chapter VI], [Gol84]) that the Zariski tangent space $T_{\rho_0} Hom(\Gamma, G)$ is naturally isomorphic to the space of $\mathfrak{g}$-valued cocycles $Z^1(\Gamma, \mathfrak{g}) := Z^1(\Gamma, \mathfrak{g}_{Ad\rho_0})$.

Lemma 4.18. *There is a natural isomorphism between the spaces* $T_{\rho_0} Hom(\Gamma, G)$ *and* $Z^1(\Gamma, \mathfrak{g})$*)*.

Proof: I will give an informal proof following [Gol84], to reader interested in the formal argument is referred to [Rag72, Chapter VI] or [LM85, Theorem

2.6]. The curves of representations ρ_ϵ through the point ρ_0 can be described as

$$\rho_\epsilon(\gamma) = (e + \epsilon c_\gamma + O(\epsilon^2))\rho_0(\gamma), \quad \epsilon \to 0.$$

Now we find the infinitesimal condition for ρ_ϵ to be a homomorphism:

$$\rho_\epsilon(\alpha\gamma) = \rho_0(\alpha)\rho_0(\gamma) + \epsilon c_{\alpha\gamma}\rho_0(\alpha\gamma) + O(\epsilon^2) =$$

$$= (\rho_0(\alpha) + \epsilon c_\alpha \rho_0(\alpha))(\rho_0(\gamma) + \epsilon c_\gamma \rho_0(\gamma)) + O(\epsilon^2) =$$

$$\rho_0(\alpha)\rho_0(\gamma) + \epsilon(\rho_0(\alpha)c_\gamma\rho_0(\gamma) + c_\alpha\rho_0(\alpha)\rho_0(\gamma)) + O(\epsilon^2).$$

Thus

$$c_{\alpha\gamma}\rho_0(\alpha\gamma) = (Ad(\rho_0(\alpha))(c_\gamma) + c_\alpha)\rho_0(\alpha\gamma)$$

which means that

$$c_{\alpha\gamma} = (Ad(\rho_0(\alpha))(c_\gamma) + c_\alpha.$$

The last formula is just the definition of cocycles $c \in Z^1(\Gamma, \mathfrak{g})$ where Γ acts on $\mathfrak{g}$ via the adjoint representation $Ad \circ \rho_0$). The isomorphism

$$T_{\rho_0}\mathrm{Hom}(\Gamma, G) \cong Z^1(\Gamma, \mathfrak{g})$$

is given by:

$$\frac{d}{d\epsilon}\rho_\epsilon|_{\epsilon=0} = c, \quad c : \gamma \in \Gamma \mapsto c_\gamma \in \mathfrak{g}. \qquad \square$$

Now we consider "trivial" deformations ρ_ϵ of the representation ρ_0, they are given by conjugation via elements $g_\epsilon \in G$, $g_\epsilon = e - \epsilon\xi + O(\epsilon^2)$, $\xi \in \mathfrak{g}$.

$$\rho_\epsilon(\gamma) = (e + \epsilon\xi + O(\epsilon^2))\rho_0(\gamma)(e - \epsilon\xi + O(\epsilon^2)) =$$

$$\rho_0(\gamma) + \epsilon(\xi\rho_0(\gamma) - \rho_0(\gamma)\xi) + O(\epsilon^2) =$$

$$\rho_0(\gamma) + \epsilon(\xi - Ad(\rho_0(\gamma))\xi)\rho_0(\gamma) + O(\epsilon^2).$$

Thus $c_\gamma = \xi - Ad \circ \rho_0(\gamma)\xi$ is a *coboundary*, $c \in B^1(\Gamma, \mathfrak{g})$. If the action of G^0 on $Hom(\Gamma, G)$ by conjugations is free near ρ_0 then the analytic germ of the character variety $(\mathcal{R}(\Gamma, G), [\rho_0])$ is given by taking the quotient of a neighborhood of ρ_0 in $Hom(\Gamma, G)$ and its Zariski tangent space is isomorphic to $H^1(\Gamma, \mathfrak{g}) = Z^1(\Gamma, \mathfrak{g})/B^1(\Gamma, \mathfrak{g})$. From now on we will identify $T_{\rho_0}Hom(\Gamma, G)$ with $Z^1(\Gamma, \mathfrak{g})$, and the tangent space to the orbit $ad(G)\rho_0 \subset Hom(\Gamma, G)$ at ρ_0 with $B^1(\Gamma, \mathfrak{g})$.

An application of this is the following "local rigidity test":

Lemma 4.19. *(A. Weil, [Wei64].) Suppose that $\rho_0 : \Gamma \to G$ is* **infinitesimally rigid**, *i.e. $H^1(\Gamma, \mathfrak{g}) = 0$. Then the representation ρ_0 is* **locally rigid**: *each representation $\rho : \gamma \to G$ sufficiently close to ρ_0 is actually conjugate to ρ_0, i.e. $\rho \in ad(G)\rho_0$.*

A special case of the above lemma is when Γ is a finite group. Namely, any representation $\rho : \gamma \to G$ is locally rigid since $H^1(\Gamma, \mathfrak{g}_\rho) = 0$ according to Lemma 4.16. In particular:

Lemma 4.20. *Any (connected) Lie group G contains only finitely many conjugacy classes of elements of the given order m.*

Proof: As all maximal compact subgroups of G are conjugate, it suffices to consider only elements $g \in G$ of order m that belong to a compact Lie subgroup $K \subset G$. Suppose that g_j is a sequence of elements of order m in K none of which is conjugate (in G) to another; by taking a subsequence we may assume that $\lim_j g_j = g \in K$. By continuity, $g^m = 1$ as well. This gives us a family of representations

$$\rho_j : \mathbb{Z}_m \to K, \rho_j(z) = g_j$$

where z is the generator of $\mathbb{Z}_m$. The limit of representations ρ_j is the representation ρ of $\mathbb{Z}_m$ which sends z to g. Local rigidity of ρ implies that the representations ρ_j are conjugate to each other for large j, hence all but finitely many g_j's are conjugate to g. Contradiction. ☐

Smoothness of the representation variety. One can ask when the given tangent vector $c \in Z^1(\Gamma, \mathfrak{g})$ can be "extended" to a formal power series

$$\rho_\epsilon = \rho_0 + \epsilon c + \epsilon^2 c_2 + \ldots + \epsilon^n c_n + \ldots \tag{4.4}$$

where $c_j(\gamma) \in \mathfrak{g}$, which formally satisfies the homomorphism condition $\rho_\epsilon(\alpha\gamma) = \rho_\epsilon(\alpha)\rho_\epsilon(\gamma)$ (the latter is the product of formal power series). It turns out that there is an infinite sequence of obstructions to doing so. These obstructions are certain higher-order Massey products, they belong to the 2-nd cohomology group $H^2(\Gamma, \mathfrak{g})$. This was first discovered by Kodaira and Spencer in 1950-s in the context of deformations of complex structures. For the proof in the case of representation varieties see [GM87b], [GM88]. The first in this series of obstructions is $[o(c)]$ which is the composition of the Lie bracket and the cup-product

$$([c],[c]) \in H^1(\Gamma, \mathfrak{g}) \otimes H^1(\Gamma, \mathfrak{g}) \xrightarrow{\cup} H^2(\Gamma, \mathfrak{g}\otimes\mathfrak{g}) \xrightarrow{[\,,\,]} H^2(\Gamma, \mathfrak{g}).$$

The class $[c \cup c]$ is represented by the 2-cocycle:

$$o(c) : (\alpha, \beta) \mapsto [c(\alpha), Ad(\rho_0\alpha)\beta] \in Z^2(\Gamma, \mathfrak{g}).$$

It turns out that if all these obstructions vanish, then c_j's can be chosen so that the formal power series (4.4) converges in G for small ϵ; hence the tangent vector c is actually tangent to a smooth curve in $Hom(\Gamma, G)$. This is a corollary of a deep theorem of M. Artin [Art68], see [GM88]. Moreover,

Theorem 4.21. *(See [GM90].) Suppose that $H^2(\Gamma, \mathfrak{g}) = 0$. Then near the point ρ_0 the variety $Hom(\Gamma, G)$ is smooth and its dimension equals the dimension of $Z^1(\Gamma, \mathfrak{g})$.*

However the representation variety $Hom(\Gamma, G)$ in general is not smooth.

Exercise 4.22. *Let $\Gamma = \mathbb{Z}^2$, $\rho_0 : \Gamma \to Isom(\mathbb{R}^3)$ be a discrete and faithful representation. Thus the images of generators act as linearly independent translations in $\mathbb{R}^3$. Let G be the group of Moebius transformations of $\mathbb{S}^3$, it contains $Isom(\mathbb{R}^3)$. Show that $Hom(\Gamma, G)$ is not smooth at ρ_0. Hint: either find a cocycle with nonvanishing cup-product or show directly that $Hom(\Gamma, G)$ near ρ is not even a topological manifold.*

In the above example the analytical germ $(Hom(\Gamma, G), \rho_0)$ has quadratic singularity at ρ_0. There are examples of closed hyperbolic 3-manifolds M and nonelementary representations $\rho_0 : \pi_1(M) \to PSL(2, \mathbb{C})$ so that the singularity of the analytical germ $(Hom(\pi_1(M), PSL(2, \mathbb{C})), \rho_0)$ is worse than quadratic, see [KM96].

Exercise 4.23. *Let $\Gamma = \mathbb{Z}^2 = \langle a \rangle \oplus \langle b \rangle$, $\rho_0 : \Gamma \to Isom(\mathbb{R}^2)$ be a discrete and faithful representation. Let G be $PSL(2, \mathbb{C})$, it contains $\rho_0(\Gamma)$. Show that near ρ_0 the representation variety $Hom(\Gamma, G)$ is a smooth 4-dimensional complex manifold. Hint: first show that $dim_{\mathbb{C}} Z^1(\Gamma, psl(2, \mathbb{C}))$ equals 4. Then find loxodromic representations ρ_ϵ near ρ_0 and show that they form a smooth manifold of complex dimension 4. Thus dimension of the Zariski tangent space at ρ_0 equals to actual dimension of $Hom(\Gamma, G)$ near ρ_0, which means that ρ_0 is a smooth point. Alternatively, describe $Hom(\Gamma, G)$ by explicit equations and then use the implicit function theorem. Note however that the commutator map $[\cdot, \cdot] : G \times G \to G$ is* **not** *a submersion at the point $(\rho_0(a), \rho_0(b))$.*

As an application of the discussion in §4.4 we now prove that under appropriate conditions the representation variety of a surface group is smooth and will compute its dimension.

Proposition 4.24. *Let S be a closed orientable hyperbolic surface with the fundamental group Γ. Let G be a semi-simple algebraic Lie group with the Lie algebra $\mathfrak{g}$ and $\rho : \Gamma \to G$ be a representation whose image in G has finite centralizer. Then $Hom(\Gamma, G)$ is smooth near ρ and its dimension near ρ equals $-dim(G)\chi(S) + dim(G)$.*

Proof: This is a special case of a more general theorem of A. Weil [Wei64] (who also treats the case of relative representation varieties of punctured Riemann surfaces), our proof follows [Gol84]. We let $\mathfrak{g} := \mathfrak{g}_\rho$. Recall that $H^0(\Gamma, \mathfrak{g}) = \mathfrak{g}^\Gamma$, the fixed-point set for the adjoint action of Γ on $\mathfrak{g}$. Thus $dim H^0(\Gamma, \mathfrak{g})$ equals dimension of the centralizer of $\rho(\Gamma)$ in G, hence it is zero by our assumptions. By the Poincaré duality

$$0 = H^0(\Gamma, \mathfrak{g}) \cong H^2(\Gamma, \mathfrak{g}^*)^* \cong H^2(\Gamma, \mathfrak{g})$$

since G is semi-simple. Thus $Hom(\Gamma, G)$ is smooth near ρ and

$$\chi(\Gamma, \mathfrak{g}) = dim H^0(\Gamma, \mathfrak{g}) - dim H^1(\Gamma, \mathfrak{g}) + dim H^2(\Gamma, \mathfrak{g}) = -dim H^1(\Gamma, \mathfrak{g}).$$

Therefore $-dim H^1(\Gamma, \mathfrak{g}) = \chi(\Gamma, \mathfrak{g}) = dim(G)\chi(\Gamma) = dim(G)\chi(S)$. Since the centralizer of $\rho(\Gamma)$ in G is finite, it follows that δ_0 has zero kernel and $B^1(\Gamma, \mathfrak{g}) \cong \mathfrak{g}$. Therefore near ρ

$$dim Hom(\Gamma, G) = dim Z^1(\Gamma, \mathfrak{g}) = dim H^1(\Gamma, \mathfrak{g}) + dim G = \\ -dim(G)\chi(S) + dim(G). \quad \square$$

4.6. Basics of discrete groups

In what follows we shall often use the fact that a closed subgroup of a Lie group is a Lie subgroup. A subgroup Γ of a Lie group G is called *discrete* if it is discrete in G as a topological subspace. In particular, a subgroup Γ is discrete iff its closure in G, $cl_G(\Gamma)$ is zero-dimensional, i.e. the connected component of $\{1\}$ in $cl_G(\Gamma)$ is $\{1\}$.

Exercise 4.25. *Suppose that H is a connected Lie subgroup of $SL(2,\mathbb{C})$. Then either $H = SL(2,\mathbb{C})$ or it is conjugate to $SL(2,\mathbb{R})$ or to $SU(2)$ or it is conjugate to a subgroup of the group B which consists of upper-triangular matrices. Hint: if the unipotent radical R of H is nontrivial then R fixes a unique point in $\widehat{\mathbb{C}}$ which is therefore fixed by H as well. Apply similar arguments to analyze reductive Lie subgroups H which are not semi-simple. Then show that if $\mathfrak{h}$ is a simple (nonabelian) proper Lie subalgebra in $sl(2,\mathbb{C})$ then its complexification is the whole G and $\mathfrak{h}$ is totally-real (i.e. it is transversal to $\sqrt{-1}\mathfrak{h} \subset sl(2,\mathbb{C})$). This implies that $\mathfrak{h}$ is conjugate either to $sl_2(\mathbb{R})$ or to $su(2)$.*

An application of this is that each nondiscrete nonelementary subgroup of $PSL(2,\mathbb{C})$ is either dense in $PSL(2,\mathbb{C})$ or its closure is conjugate to $PSL(2,\mathbb{R})$, or to a $\mathbb{Z}_2$-extension of $PSL(2,\mathbb{R})$). See [Gre62] for a more detailed and general discussion.

Definition 4.26. Let Γ be a group of homeomorphisms of a topological space X. Then Γ acts **properly discontinuously** on X if for every compact $K \subset X$ the intersections $K \cap hK$ is empty for all but finitely many $h \in \Gamma$.

Definition 4.27. Let Γ be as above and Γ_0 be its subgroup. Then a subset S of X is called **precisely invariant** under Γ_0 in Γ if $h(S) = S$ for all $h \in \Gamma_0$ and $g(S) \cap S = \emptyset$ for all $g \in \Gamma - \Gamma_0$.

Recall that $\mathbb{S}^{n-1}_\infty$ is the sphere at infinity of the hyperbolic space $\mathbb{H}^n$.

Definition 4.28. Let $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$. Then $x \in \mathbb{S}^{n-1}_\infty$ is a **point of discontinuity** for Γ if there is a neighborhood U of x such that $U \cap gU \neq \emptyset$ only for finitely many $g \in \Gamma$. Usually, this means that $U \cap gU \neq \emptyset$ for all $g \in \Gamma \setminus \{1\}$; the exceptional case is: x is a fixed point of a finite subgroup $F \subset \Gamma$. The **domain of discontinuity** $\Omega(\Gamma)$ consists of all points of discontinuity.

It is clear that $\Omega(\Gamma)$ is an open subset of $\mathbb{S}^{n-1}_\infty$. It follows from the convergence property of Moebius transformations (see §3.6) that Γ always acts properly discontinuously on $\Omega(\Gamma)$.

Definition 4.29. A discontinuous (the other name is **Kleinian**) group is a subgroup of $\mathrm{Isom}(\mathbb{H}^n)$ with nonempty domain of discontinuity in $\mathbb{S}^{n-1}_\infty$.

Clearly any Kleinian group is discrete. Discreteness of Γ implies that it acts properly discontinuously (by left translations) on the Lie group $G = \mathrm{Isom}(\mathbb{H}^n)$. However, $\mathbb{H}^n = G/K$ where K is a maximal compact subgroup of $\mathrm{Isom}(\mathbb{H}^n)$. Therefore, compactness of K implies that any discrete group Γ acts properly discontinuously on $\mathbb{H}^n$.

Exercise 4.30. *Suppose that $g, h \in \mathrm{Isom}(\mathbb{H}^n)$ are loxodromic elements which share one but not both fixed points. Show that the subgroup of $\mathrm{Isom}(\mathbb{H}^n)$ generated by g and h is not discrete.*

If Γ is discrete then the accumulation set (in $\mathbb{H}^n \cup \partial_\infty \mathbb{H}^n$) of the Γ-orbit of each point $x \in \mathbb{H}^n$ is contained in $\partial_\infty \mathbb{H}^n$. Thus the *limit set* $\Lambda(\Gamma)$ of a discrete group $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ (see §3.6) is the set of accumulation points for the orbit Γx for some (any) point $x \in \mathbb{H}^n$. If Γ is Kleinian then $y \in \Lambda(\Gamma)$ iff for some (any) point $x \in \Omega(\Gamma)$ there is an infinite sequence of elements $\gamma_i \in \Gamma$ such that $\lim_{i\to\infty} \gamma_i x = y$.

An infinite discrete group $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ is *elementary* iff its limit set consists of either one or two points. In the latter case Γ contains an infinite cyclic subgroup $\Gamma_a := \langle \gamma \rangle$ of finite index, where γ is a loxodromic element.

The situation in the former case is more complicated. By conjugating the group Γ we get that $\Gamma(\infty) = \infty$ where we identify $\mathbb{S}^{n-1}_\infty$ with $\overline{\mathbb{R}^{n-1}}$. Thus $\Gamma \subset \mathrm{Isom}(\mathbb{R}^{n-1})$. Hence by the Bieberbach theorem (see [Thu97a, §4.2]), Γ contains a free Abelian subgroup Γ_a of finite index. The group Γ_a has rank r which is between 1 and $n-1$. The number r will be called *the virtual rank* of Γ.

Example 4.31. The subgroup $PSL(2,\mathbb{Z}) \subset PSL(2,\mathbb{C})$ is Kleinian. The limit set of this subgroup is the extended real line $\mathbb{S}^1 = \mathbb{R} \cup \{\infty\}$. The group $PSL(2,\mathbb{Z}[i]) \subset \mathrm{Isom}(\mathbb{H}^3)$ is discrete but is not Kleinian, its limit set is the whole sphere $\widehat{\mathbb{C}}$.

More generally, a subgroup $F \subset \mathrm{Isom}(\mathbb{H}^3)$ will be called *Fuchsian* if it is finitely generated, the limit set of F is a round circle (or an extended straight line) and F preserves the orientation on $\Lambda(F)$.

Remark 4.32. Classically such groups are called **Fuchsian groups of the first kind**.

Example 4.33. The subgroup $PSL(2,\mathbb{Z}) \subset \mathrm{Isom}(\mathbb{H}^3)$ is Fuchsian.

Lemma 4.34. *Let $G \subset PSL(2,\mathbb{C})$ be a nonelementary subgroup generated by the parabolic elements A, B such that BA is parabolic. Then G is a Fuchsian group.*

Proof: After conjugating G via elements of $PSL(2,\mathbb{C})$ we can assume that G lifts to a 2-generated subgroup $\tilde{G} \subset SL(2,\mathbb{C})$ which is generated by the matrices:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$$

so that $b \neq 0$ and the trace of the matrix BA equals ± 2. Thus $b = -4$ and the group generated by A and B is contained in $SL(2,\mathbb{Z})$. Moreover it is easy to see that G has finite index in $PSL(2,\mathbb{Z})$ and hence is Fuchsian. ☐

Suppose that F is a Fuchsian group, $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a homeomorphism such that $Q = hFh^{-1} \subset \mathrm{Isom}(\mathbb{H}^3)$. Then Q is again a Kleinian group, the limit set of Q is the image of the round circle (the limit set of F) under the homeomorphism h. The group Q is a *quasifuchsian* group.

Definition 4.35. Let $\Gamma \subset PSL(2, \mathbb{C})$ be a finitely generated Kleinian subgroup whose limit set is a topological circle in $\widehat{\mathbb{C}}$ and which preserves each component of $\Omega(\Gamma)$. Then Γ is called **quasifuchsian**.

Example 4.36. Suppose that we are given a collection of disjoint closed round disks

$$B_1, \dots, B_k, B_1', \dots, B_k'$$

in $\mathbb{C}$ and a family of Moebius transformations $g_j : \mathrm{int}(B_j) \to \mathrm{ext}(B_j')$. Then the group $\Gamma = \langle g_1, \dots, g_k \rangle$ is a free Kleinian group of the rank k. The group G is called a **classical Schottky group**. If instead of round disks we take topological disks then the result is called a **Schottky group**. It is also a free Kleinian group. The limit set of this group is a Cantor set (if $k \geq 2$). It is obtained by removing from the complex plane the Γ-orbit of the domain

$$D = \widehat{\mathbb{C}} - int(B_1) \cup int(B_2) \ldots \cup int(B_k) \cup int(B_1') \cup \ldots \cup int(B_k')$$

(See Figure 4.1.)

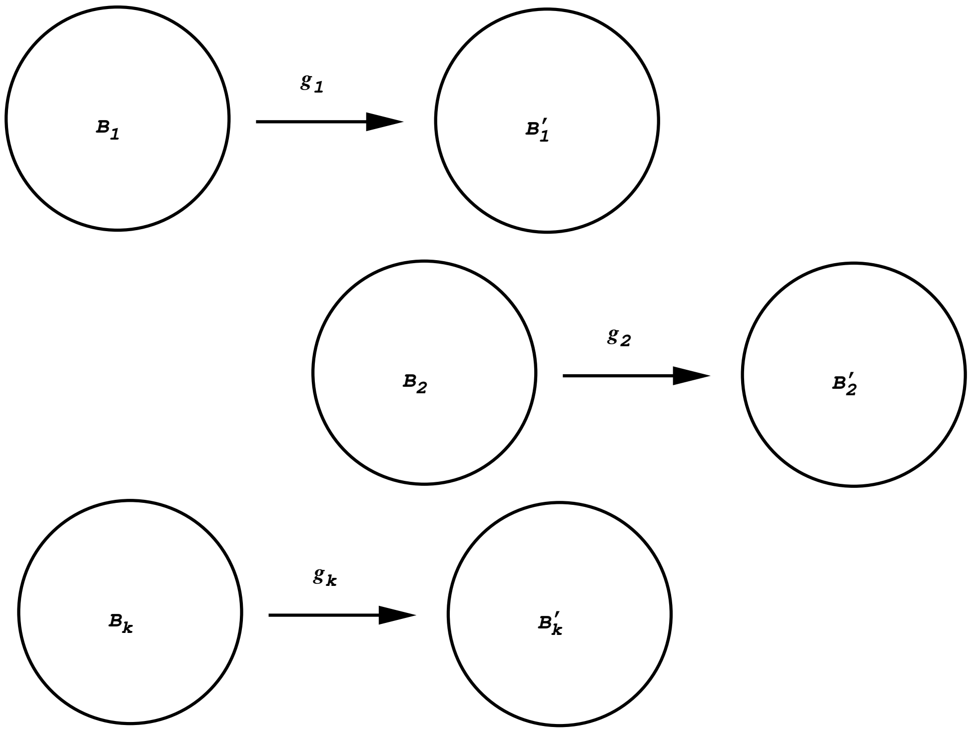

Figure 4.1: *A Schottky group.*

4.7. Properties of limit sets

Suppose that Γ is a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^n)$. The following properties of the limit set follow immediately from our general discussion of limit sets in §3.6

- The limit set is either perfect (i.e. each point is an accumulation point) or consists of at most two points. (In the latter case the group is *elementary.*)
- The limit set and discontinuity domain are Γ-invariant.
- $\Lambda(\Gamma) = \partial_\infty \mathbb{H}^n - \Omega(\Gamma)$.
- If Γ is nonelementary then the loxodromic fixed points are dense in $\Lambda(\Gamma)$.

The closed convex hull in $\mathbb{H}^n$ of the limit set $\Lambda = \Lambda(\Gamma)$ of a discrete group $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ will be denoted $C(\Lambda)$ (or $C\Lambda(\Gamma)$), it is called the *Nielsen's convex hull of* Γ. We will be mainly interested in the case $n = 3$, when the boundary of the convex hull $C(\Lambda)$ is a union of disjoint complete geodesics in $\mathbb{H}^3$ and 2-dimensional hyperbolic polygons with vertices at $\widehat{\mathbb{C}}$ (these polygons could have infinitely many vertices).

Example 4.37. Let $\Lambda \subset \mathbb{C}$ be an ellipse which is not a circle. Then the boundary of $C(\Lambda)$ has two components each of which is foliated by complete geodesics.

Suppose that $\Gamma \subset \mathrm{Isom}(\mathbb{H}^3)$ is a nonelementary discrete group and $G \subset \mathrm{Isom}(\mathbb{H}^3)$ contains Γ as a normal subgroup. Then G must be also discrete. Indeed, by continuity, the group Γ is normal in the closure $cl(G)$ of G in $\mathrm{Isom}(\mathbb{H}^3)$. The action by conjugations of the identity component $cl(G)^0$ on Γ is trivial. Therefore $cl(G)^0$ fixes pointwise the limit set of Γ. However, since $\Lambda(\Gamma)$ contains more than 2 points, the group $cl(G)^0$ must be trivial. Therefore G is discrete, otherwise $cl(G)$ is a Lie subgroup of positive dimension and $cl(G)^0 \neq \{1\}$.

4.8. Quotient spaces of discrete groups

For each discrete group $\Gamma \subset \mathrm{Isom}(\mathbb{H}^3)$ we define the quotient-spaces $M(\Gamma) = \mathbb{H}^3/\Gamma$ and $CM(\Gamma) = C(\Lambda(\Gamma))/\Gamma$.

If Γ is torsion-free, then $M(\Gamma)$, $CM(\Gamma)$ are manifolds, if it has torsion, then these spaces have more complicated *orbifold structure* which we shall discuss later. We shall mainly consider the case of torsion-free groups. The space $CM(\Gamma)$ is called the *convex core* of $M = M(\Gamma)$. If Γ is torsion-free, then the convex core may be defined as the minimal closed convex subset of M whose embedding into M is a homotopy-equivalence.

Suppose that $\Omega(\Gamma) \neq \emptyset$. Then we can attach to $M(\Gamma)$ the ideal boundary:

$$S(\Gamma) := \Omega(\Gamma)/\Gamma \quad \text{and} \quad \overset{\bullet}{M}(\Gamma) = M(\Gamma) \cup S(\Gamma)\ .$$

Example 4.38. Let G be a Schottky group of the rank r. Then $\overset{\bullet}{M}(G)$ is the handlebody of the genus r. Let $F \subset PSL(2, \mathbb{R})$ be a Fuchsian group. Then $\overset{\bullet}{M}(F) \cong [0, 1] \times (\mathbb{H}^2/F)$.

4.9. Conical and parabolic limit points

Let $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ be a discrete nonelementary group. In this section we discuss several possibilities for behavior of limit points of Γ, it will be used later in sections 4.14, 4.15.

Theorem 4.39. *(A. Beardon and B. Maskit [BM74].)* [1] *The following conditions are equivalent for each $z \in \Lambda(\Gamma)$:*

(1) There exists a sequence $\{g_j \in \Gamma\}$ such that for each $x \in \mathbb{S}^{n-1} - \{z\}$ the sequence $\{(g_j x, g_j z)\}$ is relatively compact in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} - Diagonal$.

(2) There exists a sequence $\{g_j \in \Gamma\}$ such that for some subset $W \subset \mathbb{S}^{n-1} - \{z\}$, which consists of at least two points, and for every point $x \in W$ the sequence $\{(g_j x, g_j z)\}$ is relatively compact in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} - Diagonal$.

(2') There exists a sequence $\{g_j \in \Gamma\}$ such that for every point $x \in \Lambda(\Gamma) - \{z\}$ the sequence $\{(g_j x, g_j z)\}$ is relatively compact in $\Lambda(\Gamma) \times \Lambda(\Gamma) - Diagonal$.

(3) For each (for some) point $O \in \mathbb{H}^n$ there exists a sequence $\{g_j \in \Gamma\}$, a number $r < \infty$ and a geodesic σ emanating from z such that $g_j^{-1}(O)$ converges to z inside the tubular neighborhood $U_r = Nbd_r(\sigma)$ (i.e. $\lim_j^c g_n^{-1}(O) = z$).

Proof: The implications $(1) \Rightarrow (2)$, $(1) \Rightarrow (2')$ are obvious. Let us prove $(2 \Rightarrow 3)$. Let σ be any geodesic as in (3) which is asymptotic to the points z and $y \in W$. Then the sequence of geodesics $g_j\sigma$ intersects certain ball $B(O,r)$ in $\mathbb{H}^n$. Thus, $d(g_j^{-1}O, \sigma) \le r$. The sequence $g_j^{-1}O$ cannot accumulate at any point of $\mathbb{H}^n$, suppose that a subsequence $g_{j_k}^{-1}O = O_k$ accumulates at y. Then we can take another geodesic σ' emanating from z and asymptotic to $y' \in W - \{y\}$. Then again we must have: O_k accumulates to y' or z, which contradicts the assumption that it accumulates to y. This contradiction proves that O_k converges to z. The proof of the implication $(2' \Rightarrow 3)$ is similar.

We now prove $(3 \Rightarrow 1)$. Take $\mathbb{S}^{n-1} \ni x \ne z$ and a geodesic σ' asymptotic to z and x. Since σ, σ' are asymptotic at z we conclude that O_j converges to z within the $2r$-neighborhood σ'. This implies that geodesics $g_j(\sigma')$ intersect $B(O,r)$ for all sufficiently large j. The property (1) follows. $\square$

Definition 4.40. A point $p \in \Lambda(\Gamma)$ is called a **conical limit point** (the other name is a **point of approximation**) if one of the equivalent conditions in Theorem 4.39 is satisfied.

Example 4.41. Let p be a fixed point of a loxodromic element $g \in \Gamma$. Then p is a conical limit point. In the contrast, parabolic fixed points are never conical limit points.

Remark 4.42. In the Property (2) of Theorem 4.39 it is necessary to assume that W contains at least two points, otherwise (2) does not imply (3).

Another way to describe conical limit points is to look at the quotient. Let $l = l_z \subset \mathbb{H}^3$ be a geodesic ray asymptotic to $z \in \Lambda(\Gamma)$. Let $\ell = \ell_z$ be

[1] See also the paper of B. Bowditch [Bow95].

the projection of l to the manifold $M(\Gamma)$. Then (3) is equivalent to the property:

(4) *there exists a compact subset $C \subset M(\Gamma)$ such that the ray ℓ intersects C "infinitely many times" (i.e. there exists a sequence $u_n \in \ell$ such that distance between u_o and u_n in ℓ tends to infinity but their distance in $M(\Gamma)$ is bounded). Such rays in $M(\Gamma)$ are called* **recurrent**.

WARNING. Recurrence of ℓ does not mean that the ray ℓ is relatively compact in $M(\Gamma)$. For example, suppose that Γ is a nonuniform lattice in $\mathbb{H}^n$ (i.e. the quotient $\mathbb{H}^n/\Gamma$ is not compact but has finite volume). Then almost all points of $\mathbb{S}^2 = \widehat{\mathbb{C}}$ are points of approximation but for almost all points $z \in \widehat{\mathbb{C}}$ the rays ℓ_z are not relatively compact (it follows from ergodicity of the geodesic flow of $M(\Gamma)$).

Definition 4.43. A point $p \in \Lambda(\Gamma)$ is called a **cusped parabolic point** if its stabilizer in Γ is an almost Abelian group A containing parabolic elements such that $\Lambda(\Gamma)/A$ is compact.

In particular, if the virtual rank of A is $n-1$ then the point p is always a **cusped parabolic point**. The other name for such points is **bounded parabolic fixed points**, see [Bow95].

4.10. Fundamental domains

Suppose that a group G acts properly discontinuously on a topological space X. Then a subset $F \subset X$ is called a *fundamental set* of G if:

(i) the orbit GF is equal to X,

(ii) for each $\gamma \in G$ we have $\gamma F \cap F = \emptyset$ unless γ does not act freely, in which case we require $\gamma F \cap F = Fix_X(\gamma) \cap F$.

The concept of a *fundamental domain* is more subtle. Let X be either $\mathbb{H}^3$, or $\mathbb{H}^3 \cup \Omega(\Gamma)$ or $\Omega(G)$. By a *polygon* in $\Omega(G)$ we shall mean a polygon bounded by circular arcs.

Definition 4.44. A fundamental domain (polyhedron) D for the action of a discrete group G on X is a codimension zero piecewise-smooth submanifold (subpolyhedron) of X such that:

(i)

$$\bigcup_{g\in G} g(cl_X D) = X \tag{4.5}$$

$$g(intD) \cap intD = \emptyset, \quad \text{for all} \quad g \in \Gamma - \{1\} \tag{4.6}$$

(ii) The boundary of D in X is a piecewise-smooth (polyhedral) submanifold in X and it is divided in a union of smooth submanifolds (convex polygons), which are called **faces**; for each face S there another face T and an element $g = g_{ST} \in G - 1$ so that $gS = T$ (g is called a **face-pairing transformation**); $g_{ST} = g_{TS}^{-1}$.

(iii) The orbit GD is locally finite in X, i.e. each compact in X intersects only finitely many members of the family $\{\gamma D : \gamma \in G\}$.

Remark 4.45. Faces of D are not necessarily maximal smooth (totally geodesic) submanifolds of ∂D. If $X = \Omega(G)$, then by abusing notation we shall say that D is a fundamental domain for the action of G on $\widehat{\mathbb{C}}$.

It follows that the quotient $cl_X D/G$ is homeomorphic to X/G. It is easy to see that the face-pairing transformations generate the group G. One can also "read off" the defining relations of G by looking at the fundamental domain.

Thus the collection $g(D), g \in G$ generates a tessellation of the domain of discontinuity $\Omega(G)$ (or of $\mathbb{H}^3$, or $\mathbb{H}^3 \cup \Omega(G)$) so that for different elements $g, h \in G$ the "tiles" $g(D), h(D)$ do not overlap, however they might be adjacent. Below[2] is an example of such tessellation of the hyperbolic plane by the fundamental triangles of a discrete group G acting on $\mathbb{H}^2$, the group G is generated by reflections in the edges of a fundamental triangle, it has the presentation

$$\langle x, y, z | x^2 = y^2 = z^2 = (xy)^3 = (yz)^4 = (zx)^5 \rangle.$$

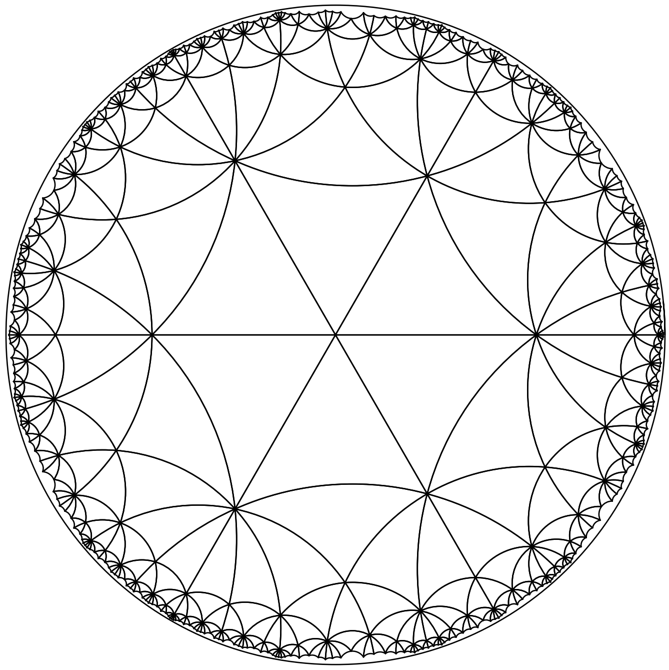

Figure 4.2: *Tessellation of the hyperbolic plane by the images of a fundamental domain.*

Theorem 4.46. *Every discrete subgroup $G \subset \mathrm{Isom}(\mathbb{H}^3)$ has a fundamental domain.*

[2]Figure 4.2 was generated with the Mathematica software package "Hyperbolic" written at the Geometry Center, University of Minnesota.

Proof: I will give a proof in the case when G acts freely on X, otherwise one has to take into account the *orbifold structure* of X/G. The quotient X/G is a smooth manifold, hence it admits a triangulation T. Take a 1-dimensional graph L in X/G which is dual to this triangulation (i.e. vertices of L are centers of top-dimensional simplices of T and two vertices are connected by an edge in L if the corresponding simplices share a common face of codimension 1). Let R be a maximal subtree in L. Remove from T all closed codimension 1 simplices which are disjoint from R. The rest is simply-connected, so it has a homeomorphic lift D into X. It is easy to see that $cl_X(D)$ is a fundamental domain. $\square$

There are other, more useful, ways to produce fundamental domains.

Definition 4.47. Let $O \in \mathbb{H}^3$ be a point which is not a fixed point for any nontrivial element of the discrete group $G \subset Isom(\mathbb{H}^3)$. Define

$$D(O,G) = \{x \in \mathbb{H}^3 : d(O,x) \leq d(x,gO), \quad \text{for all} \quad g \in G - \{1\}\}. \qquad (4.7)$$

Then $D(O,G)$ is a fundamental domain for the action of G in $\mathbb{H}^3$ which is called the **Dirichlet fundamental polyhedron** of the group Γ with the center at O. See [Bea83] or [Mas87].

Note that each Dirichlet fundamental polyhedron is convex.

Example of a fundamental domain: Take a Schottky group G. The complement to the union of disks which we used to define this group, is a fundamental domain for its action in $\widehat{\mathbb{C}}$.

4.11. Poincaré theorem on fundamental polyhedra

In this section we will give a general theorem which describes conditions under which a convex polyhedron in $\mathbb{H}^3$ is a fundamental domain. This theorem is originally due to Poincaré (who proved it for the hyperbolic plane), later (more general) versions were proven by Alexandrov [Ale54] and Maskit [Mas87]. We will present here a version valid in the hyperbolic 3-space $\mathbb{H}^3$ following [Mas87] (see also [EP94]).

Start with a convex polyhedron $P \subset \mathbb{H}^3$. As in Section 3.7 this polyhedron is the complement to a countable collection of hyperbolic half-spaces $\widehat{F}_j^-$ which form a locally-finite family in $\mathbb{H}^3$ (i.e. any compact intersects only finitely many of them). The hyperplanes $\widehat{F}_j$ bounding these half-spaces define *geometric faces* of P. We define *faces* of P to be convex domains in geometric faces of P so that each geometric face contains at most two faces. The *edges* of P are the 1-dimensional intersections between faces. The *vertices* are 0-dimensional intersections of faces.

We assume that P satisfies the following axioms:

Axiom 1. For each face S of P we are given another face R (*pared with* S) and an element $g_{SR} \in Isom(\mathbb{H}^3) - \{1\}$ so that $g_{SR}S = R$ and $g_{SR} = g_{RS}^{-1}$ (as in the definition of fundamental domain). The isometries g_{SR} are called the *face-pairing transformations* of P.

Axiom 2. $g_{SR}int(P) \cap int(P) = \emptyset$.

Example 4.48. Suppose that P itself is a hyperbolic half-space bounded by a hyperplane H, which is the only geometric face of P. Pick a geodesic $e \subset H$, it separates H into two *faces*. Now let $g \in PSL(2, \mathbb{C})$ be the rotation of order 2 around e: it necessarily pairs these faces.

If pared faces S, R are equal, then the Axiom 2 implies that $g_{SR}^2 = 1$, which is called *the reflection relation* since necessarily g_{SR} is the reflection in S.

Face-pairing transformations induce an equivalence relation on P generated by: $g_{SR}(x) \sim x$. Note that each interior point of P is equivalent only to itself. Let P^* be the quotient of P by $\sim$, which is given the quotient topology, let $p : P \to P^*$ be the natural projection.

Axiom 3. $p^{-1}(y)$ is finite for each $y \in P^*$.

Next we define *cycles* corresponding to P. Suppose that e_1 is an edge of P. Then e_1 is adjacent to a face F_1 of P. Take the face-pairing transformation $g_1 := g_{F_1,F_1'} : F_1 \to F_1'$ and $e_2 = g_1(e_1)$. Then we proceed as follows: take $F_2 \neq F_1'$ be the face adjacent to e_2, find a face-pairing transformation $g_2 := g_{F_2,F_2'} : F_2 \to F_2'$ and continue until we return back to the original face-edge pair: $(F_1, e_1) = (F_n', e_{n+1})$. The ordered collection of edges $(e_1, e_2, ...e_{n-1})$ is called an *edge cycle*, the number n is called the *period* of this cycle. Note that we allow $e_1 = e_k$ for some $1 < k < n$ in this cycle. The composition

$$T_{e_1} := g_n \circ ... \circ g_1$$

is called the *cycle transformation* at e_1. Clearly T_{e_1} keeps e_1 and the face F_1 invariant. However a priori it could be a hyperbolic translation along e_1.

Axiom 4. We require $T_e = 1$ for each edge e.

The space P^* has a natural path metric d^* induced from $\mathbb{H}^3$. The last axiom is:

Axiom 5. The metric space (P^*, d^*) is complete.

Remark 4.49. This axiom is clearly satisfied by all compact polyhedra P.

Note that all but the last axiom are local in nature, to verify them we need to look only at each individual face and edge. To the contrary, the Axiom 5 requires working with P^* as a global object. Below we shall describe some conditions which imply the Axiom 5.

Theorem 4.50. *(See [Mas87].) Suppose that P satisfies the Axioms 1 through 5. Then P is a fundamental polyhedron for the group G generated by face-pairing transformations. The subgroup $G \subset$ Isom$(\mathbb{H}^3)$ is discrete and has the presentation:*

$$\langle g_{SR} \mid g_{SS}^2 = 1, T_e = 1 \rangle$$

where S, R, e ran through the list of faces and edges of P.

Suppose that P satisfies the Axioms 1 through 5, let $\bar{P}$ be the compactification of P in $\overline{\mathbb{H}^3} = \mathbb{H}^3 \cup \mathbb{S}^2_\infty$, P_∞ is the interior of the subset $\partial_\infty P := \bar{P} \cap \mathbb{S}^2_\infty$. Then it is easy to derive from Theorem 4.50 and the definition of fundamental polyhedra that P_∞ is a fundamental domain for the action of G on $\widehat{\mathbb{C}} = \mathbb{S}^2_\infty$.

Reflection groups. Suppose that each face of P is pared to itself only (each face-pairing transformation is a reflection in that face), dihedral angles of P between adjacent faces F_i, F_j are π/n_{ij} where $n_{ij} > 1$ are integers. If faces F_i, F_j are not adjacent we put $n_{ij} = 0$. Then P satisfies the Axioms 1 through 5 and the group G generated by reflections $g_j := g_{S_j S_j}$ has the following presentation:

$$\langle g_j, j = 1, 2, ... | (g_j)^2 = 1, (g_i g_j)^{n_{ij}} = 1 \rangle.$$

Indeed, the only tricky condition is the Axiom 5, however in our case $P^* = P$ is obviously complete as a closed subset of $\mathbb{H}^3$.

Consider the following example: take three round disks D_1, D_2, D_3 in $\widehat{\mathbb{C}}$ such that the exterior angles of their intersection are: $\pi/n, \pi/m, \pi/q$, where $n, m, q = 2, 3, 4, ...\infty$. The complement of their union $\widehat{\mathbb{C}} - (D_1 \cup D_2 \cup D_3)$ is a pair of triangles $T, T' \subset \widehat{\mathbb{C}}$ bounded by arcs of circles (if $n = m = q = \infty$ then these are *ideal triangles*). Let G be the group generated by reflections in the circles $\partial D_1, \partial D_2, \partial D_3$. This group is called the *Schwarz triangle group* $T(n, m, q)$, its fundamental domain in $\widehat{\mathbb{C}}$ is $P_\infty := T \cup T'$. The convex hull of P_∞ is a fundamental domain P of G in $\mathbb{H}^3$.

From now on we restrict ourselves to polyhedra with finitely many faces. Suppose that $x_1 \in \partial_\infty P$ is a point of tangency between two faces F_1, F_n of P. Then we form a cycle corresponding to x similarly to the construction of edge cycles of P:

Take a side-pairing transformation $g_1 := g_{F_1 F_1'} : F_1 \to F_1'$ and $x_2 = g_1(x_1)$. If x_2 is not a point of tangency between two faces, we stop. Otherwise, let $F_2 \neq F_1'$ be the face tangent to F_1' at e_2, find a face-pairing transformation $g_2 := g_{F_2 F_2'} : F_2 \to F_2'$ and continue. We stop if either x_{n+1} is not a point of tangency or if $x_1 = x_{n+1} = g_n x_n$. If the latter happens we call

$$T_{x_1} := g_n \circ ... \circ g_1$$

infinite cycle transformation at x_1. Clearly $T_{x_1}(x_1) = x_1$, a priori it could be a loxodromic element.

For instance, let P be bounded by two hyperplanes H_1, H_2 which are tangent at a point $x \in \mathbb{S}^2_\infty$ and $g_{12} : H_1 \to H_2$ be the face-pairing transformation. Then $T_x = g_{12}$.

Axiom 5′. All *infinite cycle transformations* are parabolic.

Non-example. Let $Q = [xyzw]$ be a convex quadrilateral in $\mathbb{R}^2 = \mathbb{C}$ so that the sides $[xy]$ and $[zw]$ are parallel, and the sides $[xw]$ and $[yz]$ are not but $|x - w| = |z - y|$. Then there is a dilation α and rotation β (both fix the point of intersection between the lines through $[xw]$ and $[yz]$) so that

$$\alpha(x) = w, \alpha(y) = z; \beta(x) = y, \beta(w) = z.$$

Let $P \subset \mathbb{H}^3 = \mathbb{R}^3_+$ be the product $Q \times \mathbb{R}_+$. The transformations α, β pair the faces of P so that the Axioms 1 through 4 are satisfied. However Axiom $5'$ fails because, say, α is infinite cycle transformation at the infinite point $\infty \in \mathbb{C}$.

Theorem 4.51. *(B. Maskit [Mas87].) If the Axioms 1 through 4 are satisfied (and P has only finitely many faces), then the Axiom 5 is equivalent to the Axiom $5'$.*

To construct a concrete example take a configuration of four unit round disks D_1, D_2, D_3 and $D_4 \subset \mathbb{C}$ so that:

- These disks are orthogonal to a common circle C in $\mathbb{C}$.
- The disks D_i, D_{i+1} are tangent at points x_{ii+1} $(i = 1, ..., 4 \pmod 5)$.
- The configuration of disks admits a symmetry of order 4 which cyclicly permutes these disks.

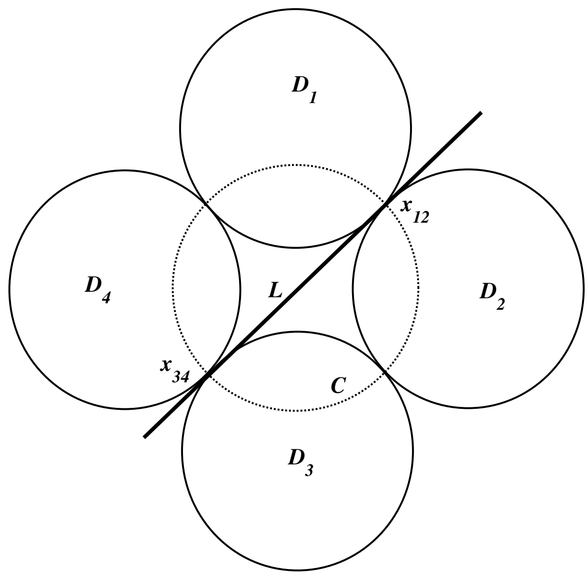

Figure 4.3: *Example of a fundamental polygon.*

Now choose two Moebius transformations

$$g_{12} : D_1 \to (\widehat{\mathbb{C}} - int(D_2)), \quad g_{34} : D_3 \to (\widehat{\mathbb{C}} - int(D_4))$$

as follows: let L be straight line tangent to D_1, D_2 at x_{12} and to D_3, D_4 at x_{34}. Let g_{12} be composition of the symmetry in L followed by the inversion in ∂D_2, similarly let g_{34} be composition of the symmetry in L followed by the inversion in ∂D_4. Let P_∞ be the complement to the union of interiors of $D_1, ..., D_4$, let P be the convex hull of P_∞ in $\mathbb{H}^3$.

Then the **Axiom 5$'$** is satisfied for the polyhedron P and the group G generated by the transformations g_{12}, g_{34}.

4.12. Kazhdan-Margulis-Zassenhaus theorem

In this section we discuss several results which are fundamental for studying discrete groups acting on symmetric spaces. The first one was proven by H. Zassenhaus in 1938, the second by D. Kazhdan and G. Margulis in 1968. Both results remained largely unnoticed[3] by the people working the the theory of Kleinian groups until Thurston came to the field in 1970-s.

Theorem 4.52. *(H. Zassenhaus, [Zas38].) Let G be a Lie group, then there is a neighborhood of identity $U_Z \subset G$ such that for each discrete subgroup $\Gamma \subset G$ the group generated by $\Gamma \cap U_Z$ is nilpotent.*

Proof: Let us compute the derivative of the commutator map $[\cdot,\cdot] : G\times G \to G$ at the identity; it is enough to consider the case when G is linear since the center of G does not matter in this computation. Let η, ζ be elements of the Lie algebra of G, $t, s \in \mathbb{R}$. Then

$$[1+t\eta, 1+s\zeta] = 1 + ts(\eta\zeta - \zeta\eta) + \quad \text{higher order terms}$$

which implies that the derivative of $[\cdot,\cdot]$ at $(1,1)$ is identically zero. Hence the map $[\cdot,\cdot]$ is a strict contraction with respect to the both variables in a neighborhood U of the identity. Thus, for any elements $x_1, ..., x_m \in \Gamma \cap U$ the iterated commutators

$$y_n := [\ldots[x_1, x_2], \ldots, x_n]$$

belong to U and moreover $\lim_n y_n = 1$. Since the group Γ is discrete, any such sequence must eventually terminate at the identity. Therefore there is a number N such that for any choice of $x_1, ..., x_N \in \Gamma \cap U$ the iterated commutator y_N equals 1. We claim that this implies that the group Γ is nilpotent. The proof is the inverse induction on n. Let Γ_n be the subgroup of Γ generated by the n-fold commutators y_n. For any generator x_i of Γ and any $y_n \in \Gamma_n$ we have: $[y_n, x_i] = y_n x_i y_n^{-1} x_i^{-1} \in \Gamma_{n+1}$, hence $x_i y_n^{-1} x_i^{-1} \in \Gamma_n$ which implies that Γ_n is a normal subgroup of Γ. The subgroup Γ_N is trivial, hence it is in the center of Γ. To show that Γ_{n-1}/Γ_n is in the center of Γ/Γ_n it is enough to verify that for elements z and w of generating sets of Γ_{n-1}/Γ_n and Γ/Γ_n the commutator $[z, w]$ is trivial. The group Γ is generated by $x_1, ..., x_m \in \Gamma \cap U$, the group Γ_{n-1} is generated by the $n-1$-fold commutators y_{n-1} of the above elements x_i. Thus Γ_{n-1}/Γ_n and Γ/Γ_n are generated by the projections $\bar{x}_i, \bar{y}_{n-1}$ of the elements x_i, y_{n-1}. By definition of Γ_n we have: $[y_{n-1}, x_i] \in \Gamma_n$, thus $[\bar{y}_{n-1}, \bar{x}_i] = 1$. □

The neighborhood $U = U_Z$ is called *Zassenhaus* neighborhood.

Let $X = G/K$ be the homogeneous space associated to G, where K is a maximal compact subgroup of G. We assume that G is given a left-invariant Riemannian metric which is also right-invariant under K. Then X becomes a Riemannian manifold and G acts (on the left) by isometries of X.

[3] Instead, various replacements of these results were proven by Shimizu, Keen, Jorgensen and others via methods specific to $SL(2,\mathbb{C})$.

Theorem 4.53. *(Kazhdan–Margulis Theorem, [KM68].) There exists a constant $\eta = \eta(X)$ which satisfies the following property. Let $x \in X$, suppose that the elements $g_1, ..., g_\ell \in G$ generate a discrete group Γ and $d(x, g_j(x)) \le \eta$. Then the group Γ is a finite extension of a nilpotent subgroup $\Gamma' \subset \Gamma$; the group Γ' is generated by $\Gamma \cap U_Z \subset Isom(X)$, where U_Z is a* **Zassenhaus** *neighborhood in G.*

Proof: We shall identify X with a Borel subgroup P of G. We can assume that $U = U_Z$ is the ϵ-neighborhood $Nbd_\epsilon(1)$ with respect to the metric on G. Since X is homogeneous we will identify the point x with the projection of $1 \in G$ to X, then $K = Fix_G(x)$.

Lemma 4.54. *There exist numbers $\eta < \epsilon_1 < \epsilon$ and an integer N so that if $g_1, ..., g_\ell \in G$ are such that $d(x, g_j(x)) \le \eta$, then:*

(1) K contains a $\epsilon_1/2$-dense subset which consists of N elements.

(2) Each word $w = w(g_1, ..., g_\ell)$ in the generators $g_1, \ldots, g_\ell$ of the length $\le N$ has the property: $d(x, wx) \le \epsilon_1$.

(3) For each w as above we have:

$$w(Nbd_{3\epsilon_1}(1))w^{-1} \subset Nbd_\epsilon(1).$$

Proof: We start with $\epsilon_1 = \epsilon/5$. Find a number N such that (1) is satisfied (recall that the group K is compact), then choose η such that (2) holds. Now each word w above is a product pk where $k \in K$ and $d(p, 1) \le \epsilon_1$ (according to (2)). Therefore, if $\delta \in Nbd_{3\epsilon_1}(1)$ and $\tau := k\delta k^{-1}$ then $\tau \in Nbd_{3\epsilon_1}(1)$ since the metric on G is bi-invariant under K. Thus

$$d(1, pk\delta k^{-1}p^{-1}) = d(1, p\tau p^{-1}) \le 2d(1, p) + d(1, \tau) \le 5\epsilon_1 \qquad \square$$

We now choose η and N as in the above lemma, we suppose that $g_1, ..., g_\ell \in G$ generate a discrete subgroup so that $d(x, g_i(x)) \le \eta$. Let Γ' denote the subgroup of Γ generated by all elements $\gamma \in \Gamma \cap Nbd_\epsilon(1)$ (a priori this subgroup can be trivial). Let

$$\Gamma = \bigcup_{j=1}^{\nu} \gamma_j \Gamma'$$

be the coset decomposition of Γ. We shall use $l(\cdot)$ to denote the word length in the group Γ with respect to the generating set $\{g_1, ..., g_\ell\}$.

Lemma 4.55. $\nu < \infty$.

Proof: Let $\gamma_j = w = g_{i_1} ....g_{i_M}$ be a word of the length $M > N$. Then $w = w_1w_2 = w_3w_4$ where $l(w_2) < l(w_4) \le N$, $w_j = p_jk_j$ and $d(k_2, k_4) \le \epsilon_1$ (according to the Part (1) of Lemma 4.54). Let

$$\delta := w_4w_2^{-1} = p_4k_4k_2^{-1}p_2^{-1}.$$

However, $d(1, k_4k_2^{-1}) \le \epsilon_1$. On another hand (according to the Part (2) of Lemma 4.54),

$$\epsilon_1 \ge d(w_jx, x) = d(p_jx, x).$$

Therefore, $d(1,\delta) \leq 3\epsilon_1$. This implies that

$$w = w_3 \delta w_2 \quad , \quad l(w_3 w_2) < M.$$

Consider the element

$$(w_3 w_2)^{-1} w = w_2^{-1} w_3^{-1} w_3 \delta w_2 = w_2^{-1} \delta w_2.$$

Then the Part (3) of Lemma 4.54 implies that this element belongs to $Nbd_\epsilon(1) \cap \Gamma \subset \Gamma'$. Thus, $w_3 w_2$ and w belong to the same coset (mod Γ'), but the length of $w_3 w_2$ is strictly smaller than M. The induction argument thus implies that for all cosets (mod Γ') we can find representatives γ_j such that $l(\gamma_j) \leq N$. This implies that ν is finite. □

Thus $|\Gamma : \Gamma'| < \infty$. According to Theorem 4.52, the group Γ' is nilpotent, therefore, Γ is finite extension of a nilpotent group. □

Other versions of this proof see in [Thu97a], [BP92]. For generalization of this theorem to the case of manifolds of nonpositive sectional curvature see [BGS85], [Fuk93].

Corollary 4.56. *There exist constants μ_n which satisfy the following property. Let $x \in \mathbb{H}^n$, elements $g_j \in Isom(\mathbb{H}^n)$ generate a discrete group Γ and $d(x, g_j(x)) \leq \mu_n$. Then the group Γ is almost Abelian and is a finite extension of an almost Abelian subgroup $\Gamma' \subset \Gamma$; the group Γ' is generated by $\Gamma \cap U_Z \subset \mathrm{Isom}(\mathbb{H}^n)$, where U_Z is* **Zassenhaus** *neighborhood in* $\mathrm{Isom}(\mathbb{H}^n)$.

Definition 4.57. The number $\mu := \mu_3$ is called the **Margulis constant** for $\mathbb{H}^3$.

4.13. Geometry of Margulis tubes and cusps

In this section we consider geometry of Margulis tubes and cusps in 3-dimensional hyperbolic manifolds although part of the discussion remains valid in $\mathbb{H}^n$ $(n \geq 4)$ and in other symmetric spaces. Let μ be the Margulis constant for $\mathbb{H}^3$. For a discrete subgroup Γ of $PSL(2,\mathbb{C})$ we let Γ_μ denote the subset of $\mathbb{H}^3$ consisting of all points z such that there exists an element of infinite order $h \in \Gamma$ such that $d(z, hz) \leq \mu$. The set Γ_μ is Γ-invariant. The stabilizer of each component of Γ_μ is an elementary subgroup of Γ by Corollary 4.56. We now analyze these components assuming for simplicity that Γ is torsion-free (otherwise we first pass to a torsion-free finite index subgroup in the stabilizer of a component of Γ_μ).

Suppose that U is a component of Γ_μ, $x \in U$; thus there exists $h \in \Gamma - \{1\}$ such that $d(x, h(x)) \leq \mu$.

Case 1. h is a loxodromic isometry of $\mathbb{H}^3$ with the axis A. The nearest-point projection $proj_A$ to the geodesic A is h-equivariant and does not increase the distance; therefore $d(proj_A(x), h(proj_A(x))) \leq \mu$ and $proj_A(x) \in \Gamma_\mu$. It follows that $A \subset \Gamma_\mu$. Convexity of the distance function between the geodesics $[x\ proj_A(x)], [h(x)\ h(proj_A(x))]$ implies that for each point $z \in [x\ proj_A(x)]$ we have: $d(z, h(z)) \leq \mu$. Therefore $[x\ proj_A(x)] \subset \Gamma_\mu$ and the union $A \cup [x\ proj_A(x)] \subset U$. So far our arguments were general: we used only the fact that $\mathbb{H}^n$ is nonpositively curved. We now use the property of $\mathbb{H}^3$ which fails for higher-dimensional hyperbolic spaces: the centralizer $Z(h)$ of h in $PSL(2,\mathbb{C})$ (which is isomorphic to $\mathbb{C}^*$) acts transitively on the surface $C = \{z \in \mathbb{H}^3 : d(z, A) = r = d(x, A)\}$. Note that if $g \in Z(h)$ then $d(g(x), hg(x)) = d(g(x), gh(x)) = d(x, h(x)) \leq \mu$. Therefore $C \subset U$. Combining this with the previous remark that $[z\ proj_A(z)] \subset U$ for each $z \in C$ we obtain:

$$Nbd_r(A) \subset U.$$

Let $\Gamma_U := Z(h) \cap \Gamma$, i.e. the stabilizer of U in Γ. We conclude that

$$\{y \in \mathbb{H}^3 : \text{ for some } g \in \Gamma_U - \{1\}, d(y, g(y)) \leq \mu\}$$

equals $Nbd_r(A)$ for some $r \in \mathbb{R}_+ \cup \{\infty\}$. If $r = \infty$ then $\langle h \rangle$ contains elements with arbitrarily small translation lengths which contradicts discreteness; thus $r \in \mathbb{R}$. On the other hand, if $g \in \Gamma - \{1\}$ is such that $d(x, g(x)) \leq \mu$, $x \in U$, then g and some $f \in \Gamma_U - \{1\}$ generate an elementary group. Discreteness of Γ implies that $g \in \Gamma_U$. Thus the connected component U of Γ_μ that we started with, equals $Nbd_r(A)$ for some $r \in \mathbb{R}_+$ and U is precisely invariant under Γ_U in Γ.

Therefore the projection of U to $M(\Gamma)$ is the solid torus U/Γ_U, such solid tori are called *Margulis tubes* in $M(\Gamma)$.

Similar arguments apply in the second case, when h is a parabolic isometry.

Case 2. h is parabolic which fixes a point $\xi \in \mathbb{S}^2$. Then the same convexity argument as above implies that the geodesic ray ρ from x to ξ is contained in U. The centralizer $Z(h)$ of h in $PSL(2,\mathbb{C})$ is isomorphic to $\mathbb{C}$ and acts simply transitively on each horosphere centered at ξ. Therefore

$$\{z \in \mathbb{H}^3 : d(z, h(z)) \leq \mu\}$$

is a horoball centered at ξ. Let Γ_U denote the centralizer of h in Γ; this subgroup is the same as the stabilizer of U in Γ. Thus $U = \{z \in \mathbb{H}^3 : d(z, g(z)) \leq \mu\}$ is a horoball centered at ξ. This horoball is precisely invariant under Γ_U, so its projection to $M(\Gamma)$ is naturally diffeomorphic to U/Γ_U.

Since $\Gamma_U \subset \mathbb{C}$ is a discrete subgroup only two subcases may occur:

(2a) $\Gamma_U \cong \mathbb{Z}^2$ is an Abelian group of rank 2. The quotient of U by Γ_U is diffeomorphic to $T^2 \times \mathbb{R}_+$, where T^2 is the 2-dimensional torus.

(2b) $\Gamma_U \cong \mathbb{Z}$ is infinite cyclic. The quotient of U by Γ_U is diffeomorphic to $D^* \times \mathbb{R}$, where $D^* = \{z \in \mathbb{C} : 0 < |z| \leq 1\}$ is the punctured disk. The product $D^* \times \mathbb{R}$ is called a *punctured solid cylinder*.

In the cases (2a), (2b) we shall refer to the horoball U as a *precisely invariant horoball*, the projection of U to $M(\Gamma)$ is called a *Margulis cusp* of rank 1 or 2 (depending on the rank of Γ_U).

Exercise 4.58. *Show that for every parabolic fixed point p of Γ there is a horoball $B \subset \mathbb{H}^3$ with center at p which is* **precisely invariant** *under the stabilizer of p in Γ. The horoball B is a component of Γ_μ. This implies that every parabolic fixed point is never a conical limit point.*

The above discussion shows that Γ_μ is a disjoint union of horoballs and tubular neighborhoods of geodesics in $\mathbb{H}^3$. Stabilizer of each component of Γ_μ is an elementary subgroup of Γ. The projection of Γ_μ to $M(\Gamma)$ is denoted $M_{(0,\mu/2]}$, it is called the *thin* part of the manifold $M = M(\Gamma)$. The complement $M_{(\mu/2,\infty)} := M - M_{(0,\mu/2]}$ is called the *thick* part of M. We will also use the notation $M_{[\epsilon,\infty)}$ for the closure of $M_{(\epsilon,\infty)}$ in M. The decomposition of M into the union of $M_{(\mu/2,\infty)}$ and $M_{(0,\mu/2]}$ is called the *thick-thin* decomposition of M. If Γ is torsion-free, then points of the thin part can be characterized by the property: the injectivity radius at each point of $M_{(0,\mu/2]}$ is at most $\mu/2$.

Let $\Gamma \subset PSL(2,\mathbb{C})$ be a cyclic subgroup generated by a loxodromic isometry g, let $M = \mathbb{H}^3/\Gamma$. Let γ denote the simple closed geodesic $A_g/\langle g\rangle$ in M. Then for each $\epsilon > 0$ the subset $M_{(0,\epsilon]} \cup \gamma$ is nonempty. Let $q \subset M$ be a piecewise–geodesic loop homotopic to γ which consists of r geodesic segments.

Lemma 4.59. *Under the above conditions we have:*

$$d(q, M_{(0,\epsilon]} \cup \gamma) \le 2r - \log(\epsilon/4),$$

where d denotes the minimal distance.

Proof: Let $x \in q$ be one of the vertices, connect this point to itself by a geodesic segment α in M which is homotopic to q (rel $\{x\}$). The loop $q * \alpha^{-1}$ lifts to a polygonal loop $\beta \subset \mathbb{H}^3$ with the consecutive vertices $x_0, x_1, ..., x_r$ so that the geodesic segment $\tilde{\alpha} := [x_0 x_n]$ covers α. Let $\tilde{q}$ denote the union of edges of β distinct from $\tilde{\alpha}$. Let α_j denote the geodesic segment $[x_0 x_j]$, $2 \le j \le r$ (so $\tilde{\alpha} = \alpha_r$). By the thin triangles property of the hyperbolic space, α_2 is contained in the 2-neighborhood of $[x_0x_1] \cup [x_1x_2]$, α_3 is contained in the 2-neighborhood of $\alpha_2 \cup [x_2x_3]$, etc. Hence α_r is contained in the $2(r-1)$-neighborhood of the piecewise-geodesic loop $\tilde{q}$; it follows that $\alpha \subset Nbd_{2(r-1)}(q)$.

Suppose that $M_{(0,\epsilon]} \neq \emptyset$. The lift $U = \Gamma_\epsilon$ of $M_{(0,\epsilon]}$ to $\mathbb{H}^3$ is closed and convex. Let h be the minimal distance between $M_{(0,\epsilon]}$ and α (equivalently, $h = d(\tilde{\alpha}, U)$). If $h \le 0.2$ then $d(q, M_{(0,\epsilon]}) \le 0.2 + 2(r-1) \le 2r$ and we are done. Thus we assume that $h \ge 0.2$. Then the length of the nearest–point projection $proj_U(\tilde{\alpha})$ of $\tilde{\alpha}$ to U is at most $4\exp(-h)$ (see Lemma 3.6). The projection $proj_U$ is Γ-equivariant, thus the end–points $\bar{x}_0, \bar{x}_r$ of the curve $proj_U(\tilde{\alpha})$ are identified by the isometry g. Hence

$$\epsilon \ge d(g(\bar{x}_0), g(\bar{x}_r)) \le 4\exp(-h)$$

which implies that $h \le -\log(\epsilon/4)$. We conclude that the minimal distance between q and $M_{(0,\epsilon]}$ is at most $2r - \log(\epsilon/4)$. The same argument works in the case when $M_{(0,\epsilon]} = \emptyset$, in which case we take $U := A_g$. $\square$

Suppose now that the loop q has a single vertex z. Let $d(\gamma, q) = h$ be the minimal distance between γ and q.

Lemma 4.60. *Under the above conditions the closed geodesic γ is contained in the $(h+4)$-neighborhood of the loop q.*

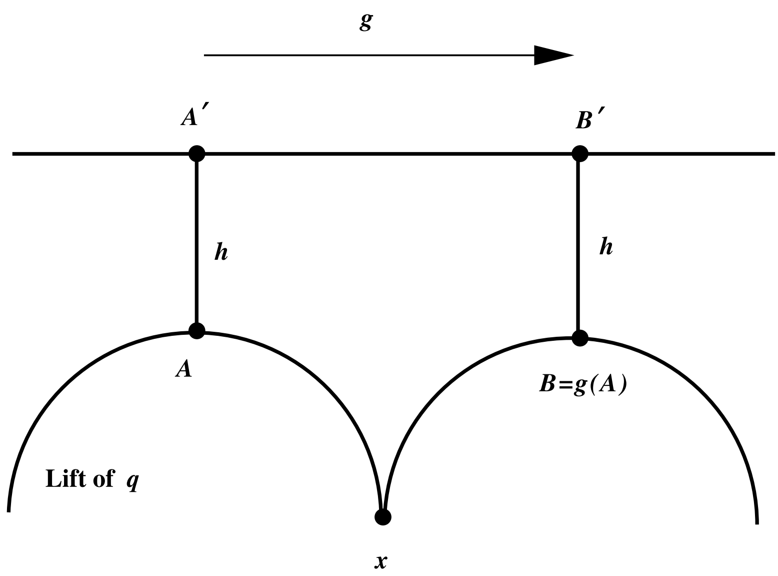

Figure 4.4:

Proof: Lift the curve q to the universal cover $\mathbb{H}^3$ and choose a point A in the lift which is the nearest to A_g. Let $B = g(A)$ and $A', B' \in A_g$ be the nearest points to A, B. Then the projection of the segment $[A'B']$ to $\mathbb{H}^3/\langle g\rangle$ is the geodesic loop γ. (See Figure 4.4). The *thin triangle property* of the hyperbolic space (see Section 3.3) implies that $[A'B']$ is within $(4 + h)$-neighborhood of the lift of q. $\square$

Another description of cusped parabolic fixed points. Recall that if p is a fixed point of a rank 2 parabolic subgroup of Γ, then p is necessarily a cusped parabolic fixed point.

Exercise 4.61. *Suppose that the stabilizer A of $p \in \Lambda(\Gamma)$ in Γ is a parabolic subgroup of rank 1. Show that p is a cusped parabolic fixed point iff there exists a pair of disjoint disks $D, D' \subset \widehat{\mathbb{C}}$ which are tangent at p so that $D \cup D'$ is precisely invariant under the subgroup A of Γ. Hint: use the Exercise 4.58.*

Let $p \in \widehat{\mathbb{C}}$ be a cusped parabolic fixed point as above. Without loss of generality we can assume that $p = \infty$, D, D' are half-planes in $\mathbb{R}^2$ given by the inequalities

$$D = \{z \in \mathbb{R}^2 : x_2 > 1\}, \quad D' = \{z \in \mathbb{R}^2 : x_2 < -1\} \tag{4.8}$$

Then we can consider a domain $V(p)$ in $\mathbb{R}^3_+$

$$V(p) = \{x \in \mathbb{R}^3_+ : x_2^2 + x_3^2 > 1\} \tag{4.9}$$

which is the convex hull of $D \cup D'$ in $\mathbb{H}^3$. The union $D \cup D'$ is precisely invariant iff V is precisely invariant. The domains like $V(p)$ are called *cuspidal* *neighborhoods* of the parabolic fixed points. By abusing notation we shall call the projection of $V(p)$ to $\overset{\bullet}{M}(\Gamma)$ a *cuspidal neighborhood* corresponding to the Margulis cusp U/A.

Similarly, if $\overline{V(p)}$ is the union of the convex hull $V(p)$ with $D \cup D'$ then we shall use the notation $\overline{V(p)}$ for its projection to $\overset{\bullet}{M}(\Gamma)$. Topologically, $V(p)$ is the product of $\Delta^* \times I$ where Δ^* is the open punctured disk; $\Delta^* \times 0$ is the projection of D to $S(\Gamma)$, $\Delta^* \times 1$ is the projection of D' to $S(\Gamma)$. The product $\Delta^* \times I$ is called the *punctured solid cylinder.* Note that if $\Gamma \subset PSL(2, \mathbb{C})$ is a geometrically finite group then each parabolic fixed point is cusped; therefore each rank 1 Margulis cusp in $M(\Gamma)$ is contained in a punctured solid cylinder $\Delta^* \times I \subset \overset{\bullet}{M}(\Gamma)$.

We now consider the intersection of precisely invariant horoballs for parabolic points (of rank 2) with the convex hulls $C(\Lambda(\Gamma))$. We again assume that $p = \infty$ is the fixed point for the group A generated by $f : z \mapsto z + 1$ and $h : z \mapsto z + a$, $a \in \mathbb{C} - \mathbb{R}$. Suppose that the discrete group Γ is nonelementary, therefore there is at least one limit point $z \in \mathbb{C}$. Then $C(\Lambda(\Gamma))$ contains the convex hull $C(Az)$ of the orbit Az. Let r denote the Euclidean radius of a maximal open disk contained in $\mathbb{C} - Az$. Suppose that $x \in \mathbb{H}^3$ is such that $d_{\mathbb{E}^3}(x, \mathbb{C}) > r$, then $x \in C(Az)$ (here $d_{\mathbb{E}^3}$ is the Euclidean distance). Therefore $C(Az)$ contains a precisely invariant horoball O with the center at ∞. Thus, the whole Margulis cusp O/A is contained in $C(\Gamma) = C(\Lambda(\Gamma))/\Gamma$.

Lemma 4.62. *(1) There is 1-1 correspondence between the Margulis cusps of $M(\Gamma)$ and the Γ-conjugacy classes of the maximal parabolic subgroups of Γ.*

(2) Two Margulis cusps C_1, C_2 in $M(\Gamma)$ contain homotopically nontrivial freely homotopic loops iff $C_1 = C_2$.

Proof: The first assertion is clear so we consider the second. Suppose that $\gamma_j \subset C_j$ are homotopic loops representing $\alpha_j \in \pi_1(C_j)$, $j = 1, 2$. Lift C_j to horoballs O_j in $\mathbb{H}^3$ with the stabilizers $A_j = \pi_1(C_j) \supset \alpha_j$. Then there is an element $g \in \Gamma$ such that $g\alpha_1 g^{-1} = \alpha_2$, therefore the fixed point sets of A_2 and gA_1g^{-1} coincide. Hence A_2 and gA_1g^{-1} generate a parabolic subgroup in Γ. However the groups A_2 and gA_1g^{-1} are maximal parabolic subgroups of Γ, thus $A_2 = gA_1g^{-1}$ and we conclude that $C_1 = C_2$. $\square$

Corollary 4.63. *Hyperbolic 3-manifolds are acylindrical and atoroidal.*

Lemma 4.64. *Suppose that M is an n-dimensional hyperbolic manifold such that the complement M_c to the union of Margulis cusps in M is compact. Then $vol(M) < \infty$.*

Proof: Since M_c is compact, the number of Margulis cusps in M is finite and each Margulis cusp has rank $n - 1$. Now use finiteness of the volume of Margulis cusps of rank $n - 1$, see Exercise 3.13. $\square$

Lemma 4.65. *Suppose that M is a hyperbolic 3-manifold with finitely generated fundamental group, M_c is a compact core of M such that the boundary of M_c consists only of tori and Klein bottles. Then $vol(M) < \infty$.*

Proof: Let $M = \mathbb{H}^3/\Gamma$ where Γ is a discrete torsion-free group of isometries of $\mathbb{H}^3$. Let A_i be the fundamental group of a boundary component T_i of M_c. Since A_i is an (almost) abelian group of rank 2, it follows that $A_i \subset \Gamma$ acts on $\mathbb{H}^3$ as a parabolic subgroup. Let $V_i \subset M$ be the Margulis cusp corresponding to A_i. Then T_i and ∂V_i bound a compact submanifold (actually homeomorphic to $T_i \times I$) in M. This implies that $M - \cup_i V_i$ is compact. Finiteness of volume of M now follows from Lemma 4.64. $\square$

Corollary 4.66. *Suppose that M is a compact Haken 3-manifold different from the 3-ball such that $\chi(M) = 0$ and $\Gamma \subset Isom(\mathbb{H}^3)$ is a discrete torsion-free subgroup isomorphic to $\pi_1(M)$. Then $vol(\mathbb{H}^3/\Gamma) < \infty$.*

Proof: Let $N := \mathbb{H}^3/\Gamma$ and let N_μ denote the complement to the Margulis cusps in N. Then $\chi(N_\mu) = \chi(M) = 0$ (since these manifolds are homotopy-equivalent) which in turn implies that ∂N_μ consists only of tori and Klein bottles. Therefore, by the above lemma, $vol(N) < \infty$. $\square$

4.14. Geometrically finite groups

A discrete subgroup $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ is called a *lattice* if $\mathbb{H}^n/\Gamma$ has finite volume. A lattice is called *uniform* if the quotient is compact. Examples of lattices are given by *arithmetic* groups, however for each $n \geq 2$ there are uniform lattices $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ which are not arithmetic [GPS88]. Instead of the general definition I will give several examples of arithmetic groups.

(1) The subgroup $SL(2, \mathbb{Z})$, which consists of matrices with integer coefficients, is an arithmetic subgroup of $SL(2, \mathbb{R})$.

(2) Recall that the Gaussian integers $\mathbb{Z}[i]$ are the complex numbers where both real and imaginary parts belong to $\mathbb{Z}$. The subgroup $SL(2, \mathbb{Z}[i])$, which consists of matrices with Gaussian integer coefficients, is an arithmetic subgroup of $SL(2, \mathbb{C})$.

The groups $SL(2, \mathbb{Z})$ and $SL(2, \mathbb{Z}[i])$ act as nonuniform lattices on $\mathbb{H}^2$ and $\mathbb{H}^3$ respectively.

(3) Consider the quadratic form $\varphi = x_1^2 + \dots x_n^2 - \sqrt{p}x_{n+1}^2$, where p is a prime number. The group of linear automorphisms $Aut(\varphi)$ of φ is isomorphic to $SO(n, 1)$. Now, let Γ be the group of matrices $SL(n+1, \mathbb{Z}) \cap Aut(\varphi)$. One can show that Γ is a uniform lattice in $Aut(\varphi)$. See [Vin92] for details.

The concept of a lattice is generalized by the notion of a *geometrically finite group.*

Definition 4.67. Suppose that $\Gamma \subset \mathrm{Isom}(\mathbb{H}^3)$ is a discrete group. Then Γ is called **geometrically finite** if for some (any) $\epsilon > 0$ the ϵ-neighborhood $Nbd_\epsilon(CM(\Gamma))$ of the convex core in $M(\Gamma)$ has finite volume.

As we shall see later (§4.15) there are several alternative descriptions of geometrically finite groups.

Definition 4.68. A discrete group $\Gamma \subset \mathrm{Isom}(\mathbb{H}^3)$ is called geometrically finite if it has a convex fundamental polyhedron $P \subset \mathbb{H}^3$ with finitely many faces.

Remark 4.69. Here by "face" we mean a subset of ∂P which appears in the definition of a fundamental domain.

Thus, each group satisfying the Definition 4.68 is finitely presentable.

Definition 4.70. A discrete group $\Gamma \subset \mathrm{Isom}(\mathbb{H}^3)$ is geometrically finite if for some (any) $O \in \mathbb{H}^3$ the Dirichlet fundamental polyhedron $D(O, \Gamma) \subset \mathbb{H}^3$ has finitely many faces.

Definition 4.71. A discrete subgroup $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ is **geometrically finite** if every limit point of Γ is either a point of approximation or is a cusped parabolic point.

Remark 4.72. C. Bishop [Bis96] proved that if each limit point of $\Gamma \subset PSL(2, \mathbb{C})$ is a point of approximation or a parabolic fixed point then Γ is geometrically finite.

Note that both points of approximation and cusped parabolic points are defined purely topologically in terms dynamics of Γ on its limit set. (Use the Condition (2') in Theorem 4.39 to define the points of approximation.) In particular, if Γ, Γ' are discrete subgroups of $\mathrm{Isom}(\mathbb{H}^n)$, the group Γ is geometrically finite and $f : \Lambda(\Gamma) \to \Lambda(\Gamma')$ is a homeomorphism which conjugates the action of Γ to the action of Γ', then Γ' is also geometrically finite. We will sketch a proof of equivalence of various definitions of geometrical finiteness in §4.15.

4.15. Criteria of geometric finiteness

Theorem 4.73. *Let $\Gamma \subset \mathrm{Isom}(\mathbb{H}^3)$ be a discrete group. Then the following conditions are equivalent:*

(i) Γ is geometrically finite (in the sense of the Definition 4.67).

(ii) The space $CM(\Gamma)_{[\mu,\infty)}$ is compact.

(iii) Each limit point of Γ is either a point of approximation or a cusped parabolic point.

(iv) Γ has a convex fundamental polyhedron with finitely many faces.

(v) Γ is finitely generated and $Vol(CM(\Gamma)) < \infty$.

Warning 4.74. If in the property (v) we do not assume Γ to be finitely generated, then it will fail to imply geometric finiteness. Moreover, there are examples constructed by E. Hamilton [Ham98], of discrete subgroups Γ of $\mathrm{Isom}(\mathbb{H}^4)$ which cannot be finitely generated but $Vol(Nbd_\epsilon CM(\Gamma)) < \infty$. These groups contain elliptic elements of arbitrarily high orders.

Proof: I give here only a partial proof of this theorem, for details see [BM74], [Bow93a] (see also [Bow95] for the discussion of geometrical finiteness in the case of discrete groups acting on spaces of negatively pinched curvature). We will assume that Γ is torsion-free. We first prove $(i) \Rightarrow (ii)$. Let $\delta < \min(\epsilon, \mu)$ where μ is the Margulis constant. There exists a maximal set $\{x_k \in CM(\Gamma)_{[\mu,\infty)}, k = 1, ..., n\}$ such that the balls $B_{\delta/2}(x_k)$ are disjoint (since the volume of $Nbd_\epsilon(CM(\Gamma))$ is finite and each ball contributes a definite amount of volume). Thus by maximality, the balls $B_\delta(x_k)$ cover the whole space $CM(\Gamma)_{[\mu,\infty)}$.

Next we prove $(ii) \Rightarrow (iii)$. Note that compactness of $CM(\Gamma)_{[\mu,\infty)}$ implies that each parabolic fixed point of Γ is cusped. Suppose that we have a nonrecurrent geodesic ray ρ in $CM(\Gamma)$. We can assume that ρ is entirely contained in a component of $CM(\Gamma)_{(0,\mu]}$. This component cannot be compact. Therefore, a component of the preimage of ρ in $\mathbb{H}^3$ is contained in a horoball with center at a parabolic fixed point. Hence each limit point is either a point of approximation or a cusped parabolic fixed point. This proves (iii).

Let us prove the implication $(iii) \Rightarrow (iv)$ assuming that Γ contains no parabolic elements.

If Γ has a Dirichlet fundamental polyhedron D with infinitely many faces then any point z of accumulation of faces cannot be a conical limit point, otherwise we can take a geodesic ray $\sigma \subset D$ which is asymptotic to z. The Γ-orbit of this ray cannot intersect a compact set infinitely many times (by local finiteness of fundamental polyhedra). Recall that the collection of faces of D is locally finite in $\mathbb{H}^3$. Thus we have a sequence of faces S_α of D which is Hausdorff-convergent to z (in the metric which determines the topology of $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$). The faces S_α pair with the faces S'_α via pairing transformations $g_\alpha \in \Gamma$ and (after taking a subsequence) the faces S'_α Hausdorff-converge to a point $w \in \widehat{\mathbb{C}}$. Then g_α converges to the quasiconstant w_z which means that z is a limit point of Γ. Contradiction. If there are parabolic points, then one has to work a bit more (see [BM74]). This proves $(iii) \Rightarrow (iv)$.

Now suppose that a convex fundamental polyhedron D of Γ has only finitely many sides. Then any point of $\partial_\infty D$ is either parabolic fixed point or a point of discontinuity. Again let us assume that there are no parabolic fixed points. Thus $\partial_\infty D$ consists only of the points of discontinuity. This implies that to get the intersection $D \cap C(\Lambda(\Gamma))$ we must "chop off" $\partial_\infty D$ via a hypersurface which has compact intersection with D. Thus $D \cap C(\Lambda)$ is compact, whence $CM(\Gamma)$ (which is the quotient of $D \cap C(\Lambda(\Gamma))$) is compact.

We refer to [Bow93b] for the proof of the implication $(v) \Rightarrow (i)$. $\square$

Exercise 4.75. *Use the above theorem to show that every finitely generated discrete subgroup of $PSL(2,\mathbb{R})$ is geometrically finite. An elementary proof of the fact that each finitely generated discrete subgroup of $PSL(2,\mathbb{R})$ has a convex fundamental polygon with finitely many sides can be found in [CB88].*

Definition 4.76. A discrete subgroup $\Gamma \subset \mathrm{Isom}(\mathbb{H}^3)$ is called **convex-cocompact** if it is geometrically finite and has no parabolic elements.

Thus Theorem 4.73 implies that a subgroup $\Gamma \subset \mathrm{Isom}(\mathbb{H}^3)$ is convex–cocompact if and only if $CM(\Gamma)$ is compact.

Definition 4.77. Let (M,P) be a compact pared 3-dimensional manifold. Suppose that $G \subset \mathrm{Isom}(\mathbb{H}^3)$ is a (torsion-free) geometrically finite Kleinian group so that there is a homeomorphism $f : M \to Nbd_\epsilon CM(G)_{[\nu,\infty)}$ for some $\epsilon > 0$, $0 < \nu < \mu_3$ and this homeomorphism carries P onto

$Nbd_\epsilon CM(G)_{[\nu,\infty)} \cap Nbd_\epsilon CM(G)_{(0,\nu]}$. Then we say that (M, P) admits a **geometrically finite hyperbolic structure**.

4.16. Kleinian groups and Riemann surfaces

Recall that a *Riemann surface* is a 1-dimensional complex manifold. Each Riemannian metric on a 2-dimensional surface S is conformally flat, hence (if S is oriented) each Riemannian metric on S determines a complex structure on S. If S is a Riemann surface and ds^2 is a Riemannian metric on S which determines the given complex structure on S, then ds^2 is said to be *compatible* with S.

A Riemann surface S is said to have *finite type* if it is conformally equivalent to a Riemann surface which is obtained from a compact Riemann surface $\widehat{S}$ by removing a finite number of points. The points removed from $\widehat{S}$ are called the *punctures* in S. The genus of $\widehat{S}$ is called the genus of S. Let S be a Riemann surface of finite type of genus g with n punctures. If $g = 0, n = 0$ then S is said to have the *spherical* conformal type, if $g = 0, 0 < n < 2$ or $(g, n) = (1, 0)$ then S is said to have the *Euclidean* conformal type. In all other cases S has the *hyperbolic* conformal type.

Theorem 4.78. *(The Uniformization Theorem.) (1) If S is a Riemann surface of* **hyperbolic type** *or* **spherical type** *then S admits a complete metric of the curvature ± 1 (negative in the hyperbolic case and positive in the spherical case) of finite area compatible with the complex structure on S. (2) If S has* **Euclidean** *type then S admits a complete flat metric compatible with the complex structure on S.*

Moreover, every simply-connected Riemann surface is either conformal to $\widehat{\mathbb{C}}$ or to $\mathbb{C}$ or to the unit disk in $\mathbb{C}$.

The most interesting case for us is when S has hyperbolic type. Then S is conformally isomorphic to $\mathbb{H}^2/\Gamma$, where $\mathbb{H}^2$ is the hyperbolic plane and Γ is a subgroup of $PSL(2, \mathbb{R})$ acting freely properly discontinuously on $\mathbb{H}^2$. (Here we identify $PSL(2, \mathbb{R})$ with the group of orientation-preserving isometries of $\mathbb{H}^2$.) The groups Γ which appear above are *Fuchsian groups.* See §4.6. The definition of Riemann surfaces of hyperbolic type makes sense even if S has infinite type.

Definition 4.79. Suppose that S is a Riemann surface whose universal cover is conformal to the unit disk in $\mathbb{C}$. Then S is called hyperbolic. [4]

Exercise 4.80. *Let $\Gamma \subset PSL(2, \mathbb{R})$ be a subgroup acting freely properly discontinuously on $\mathbb{H}^2$ so that $\mathbb{H}^2/\Gamma$ has finite area. Then the Riemann surface $S = \mathbb{H}^2/\Gamma$ has finite hyperbolic type. (Hint: first show that the* **thick part** *of S is compact, see Theorem 4.73.)*

In what follows we will need some facts about conformal geometry of Riemann surfaces. Suppose that R is the rectangle with the hight h and

[4] Actually, the only Riemann surfaces of infinite type are the ones which are hyperbolic.

width w. Then the *conformal modulus* of R is defined as h/w. Suppose that A is a Riemannian product $A = (-b, b) \times \mathbb{S}^1$, where $0 < b \le \infty$, and $\mathbb{S}^1 = \mathbb{S}^1_\ell$ is the circle of the length $\ell > 0$. Define the *conformal modulus* $\mu(A)$ to be the ratio $2|b|/\ell$. If M is a Riemann surface homeomorphic to A then, according to the uniformization theorem, M is conformally equivalent to a cylinder A' with the product metric. Then $\mu(M) := \mu(A')$. It is easy to see that the modulus is a conformal invariant.

Let S be a Riemann surface and γ be a loop on S whose free homotopy class will be denoted $[\gamma]$. Fix a metric ds in the conformal class of S, denote the area element of this metric by $dvol$. Suppose that ρ is a nonnegative Borel measurable function on S, we equip S with a (measurable) metric ρds and define the ρ-length of γ as

$$L_\rho(\gamma) = \inf_{\alpha \in [\gamma]} \int_\alpha \rho ds. \tag{4.10}$$

For the metric ρds define the area $A_\rho(S)$ as the integral

$$A_\rho(S) = \int_S \rho^2 dvol. \tag{4.11}$$

Define the *extremal length* $E_S(\gamma)$ of the curve γ on S as

$$E_S(\gamma) := \sup_\rho \{L_\rho(\gamma)^2 / A_\rho(S) : 0 < A_\rho(S) < \infty\} \tag{4.12}$$

The extremal length $E_S(\gamma)$ depends only on the conformal structure of S and on the homotopy class of γ. It is easy to see from the definition of extremal length that if S is conformally embedded in a Riemann surface M then $E_S(\gamma) \le E_M(\gamma)$ (just take an extremal metric ρ on S and extend it by zero to the rest of the surface M).

Lemma 4.81. *(See [Ahl73].) Choose the product metric on the annulus $A = \mathbb{S}^1_\ell \times (-b, b)$, let $[\gamma]$ be the generator of $\pi_1(A)$. Then this flat metric is extremal in the definition of the extremal length of γ and the extremal representative α of γ is any horizontal loop. If S is a flat torus and $\gamma \subset S$ is simple, homotopically nontrivial loop then the extremal length $E_S(\gamma)$ is equal to the length of a geodesic representative of γ in S.*

This lemma implies that the conformal modulus of an annulus is superadditive in the following sense. Let A be an annulus, $\gamma \subset int(A)$ be a simple homotopically nontrivial loop which separates A into two annuli A', A''. Then $\mu(A) \ge \mu(A') + \mu(A'')$.

Now suppose that $\gamma \subset S$ is a simple homotopically nontrivial loop. We define the *conformal modulus* $mod(\gamma)$ to be the supremum of conformal moduli of all embedded annuli in S which are homotopic to γ. The following theorem establishes relation between the extremal length and the modulus:

Theorem 4.82. *(See [Ahl73].) If $\gamma \subset S$ is a simple homotopically nontrivial loop on a hyperbolic surface S then*

$$mod(\gamma)^{-1} = E_S(\gamma). \tag{4.13}$$

Example 4.83. Suppose that S is the annulus $\mathbb{H}^2/\langle h\rangle$ where $h \in PSL(2,\mathbb{R})$ is a hyperbolic isometry. Let $\gamma \subset S$ be a simple homotopically nontrivial closed geodesic. Then

$$mod(\gamma) \le \mu(S) = \pi/length(\gamma).$$

Proof: The inequality $mod(\gamma) \le \mu(S)$ follows from superadditivity of the conformal modulus. Using the exponential map we represent S as the quotient of the Euclidean rectangle R of the width equal π and the height $L := length(\gamma)$ so that the vertical sides of R are identified by a translation. Then

$$\mu(S) = [L^2/(\pi L)]^{-1} = \pi/L.$$

□

Suppose that $\Gamma \subset PSL(2,\mathbb{C})$ is a nonelementary Kleinian group and $D \subset \Omega(\Gamma)$ is a simply–connected component with the stabilizer Γ. Then D is conformally–equivalent to the upper half-plane $\mathbb{H}^2$. Therefore $S = D/\Gamma$ has to be a Riemann surface of hyperbolic conformal type. Let h denote the hyperbolic metric on S induced by the conformal map between D and $\mathbb{H}^2$. Suppose that $g \in \Gamma$ is an element of infinite order representing a geodesic loop α on S of the length $\ell(\alpha)$ (with respect to the metric h). Recall that $\ell_{\mathbb{H}^3}(g)$ denotes the translation length of the isometry g acting on $\mathbb{H}^3$.

Theorem 4.84. *(L. Bers, B. Maskit.) For each Kleinian subgroup $\Gamma \subset PSL(2,\mathbb{C})$ and each geodesic loop α on $S = S(\Gamma)$ corresponding to an element of infinite order $g \in \Gamma$ we have:*

$$\ell_{\mathbb{H}^3}(g) \le 2\ell(\alpha) \tag{4.14}$$

Proof: First note that the assertion is true if g is parabolic. Consider the torus (or annulus in the parabolic case) $T = \Omega(\langle g\rangle)/\langle g\rangle$ which contains the annulus $A = D/\langle g\rangle$. Therefore $mod_T(\alpha) \ge mod_A(\alpha) = \pi/\ell(\alpha)$. If α is homotopic to a puncture in S then $mod_T(\alpha) = \infty$ and thus T is biholomorphic to $\mathbb{C}^*$ which means that g is parabolic. Consider the loxodromic case. Then using the exponential map we represent T as the parallelogram P with the base 2π and of the height $\ell(g)$ with identified sides. Thus $mod_T(\alpha) = mod(T-\alpha)$ and $T-\alpha$ is the annulus of the modulus at most $2\pi/\ell(g)$. Whence $\pi/\ell(\alpha) \le 2\pi/\ell(g)$. □

Corollary 4.85. *Suppose that in Theorem 4.84 the loop α is homotopic to a puncture on S. Then $\ell(g) = 0$, i.e. g is parabolic.*

Suppose that $\Gamma \subset PSL(2,\mathbb{C})$ is a geometrically finite torsion-free Kleinian group and $S = S(\Gamma)$ has a puncture q with a small punctured disk neighborhood U. Then, according to the above corollary, the puncture q corresponds to a punctured solid cylinder $\overline{V(p)} \subset \dot{M}(\Gamma)$ (see §4.13 for the notation and definitions), $\overline{V(p)} = \Delta^* \times I$. We identify U with "the bottom" of this cylinder, $\Delta^* \times 0 \subset \overline{V(p)}$. Then the other punctured disk $U' = \Delta^* \times 1 \subset \overline{V(p)}$ also gives rise to a puncture p' on the surface S (of course, $p \neq p'$), so that boundary loops of U, U' are freely homotopic in the manifold $\overset{\bullet}{M}(\Gamma)$. The conclusion of this discussion is that for a geometrically finite torsion-free Kleinian group $\Gamma \subset PSL(2, \mathbb{C})$ the punctures on the surface $S(\Gamma)$ "come in pairs", connected in $\overset{\bullet}{M}(\Gamma)$ by punctured solid cylinders.

4.17. The convex hull and the domain of discontinuity

Let Γ be a discrete nonelementary subgroup of $\mathrm{Isom}(\mathbb{H}^3)$. The boundary of the convex hull $C(\Lambda(\Gamma))$ has the property: each point $x \in C(\Lambda(\Gamma))$ is either contained in a complete hyperbolic geodesic $\gamma \subset C(\Lambda(\Gamma))$ or it lies in the interior of a convex hyperbolic polygon with ideal vertices. Connected components of the boundary of the convex hull $C(\Lambda(\Gamma))$ have the natural path-metric d_C induced from the ambient hyperbolic space. It turns out that the universal cover of each component of $(\partial C(\Lambda(\Gamma)), d_C)$ is isometric to the hyperbolic plane. This is quite obvious if $C(\Lambda(\Gamma))$ is bounded by a locally finite family of faces, the general case is more subtle and requires some approximation arguments (see [EM87]). In this section we consider the *nearest-point projection* from $(\mathbb{H}^3 \cup \Omega(\Gamma)) - C(\Lambda(\Gamma))$ to $\partial C(\Lambda(\Gamma))$, our discussion mainly follows [ECG87].

Suppose that $C \subset \mathbb{H}^3$ is a closed convex nonempty subset (we will use $C = C(\Lambda)$ or $C = Nbd_\epsilon C(\Lambda)$). Let $\Lambda := \partial_\infty C$, $\Omega := \widehat{\mathbb{C}} \setminus \Lambda$.

Let $r = r_C : \mathbb{H}^3 - C \to \partial C$ be the nearest-point projection, we know that it is a distance-decreasing mapping. Our goal is to extend r to the domain Ω. Choose a base-point $x_0 \in C$, we will normalize all Busemann functions on $\mathbb{H}^3$ at this point. According to the definition, for each $x \in \mathbb{H}^3 - C$ we have:

$$d(x, r(x)) = \min\{d(x, y) | y \in C\}.$$

Hence the point $r(x)$ also gives minimum to the normalized distance function:

$$\alpha_x(y) := d(x, y) - d(x_0, x), \quad d(x, r(x)) = \min\{\alpha_x(y) | y \in C\}.$$

The normalized distance functions $\alpha_x(y)$ have limits as $x \to \xi \in \partial_\infty \mathbb{H}^3$, the limits are the *Busemann functions* β_ξ normalized at x_0. Hence for each $\xi \in \Omega \subset \partial_\infty \mathbb{H}^3$ we let

$$r(\xi) \in \partial C \text{ be the point such that } \beta_\xi(r(\xi)) = \min\{\beta_\xi(y) | y \in C\}.$$

Then $r = r_C : \Omega \cup \mathbb{H}^3 \setminus C \to \partial C$ is a continuous mapping. Geometrically one can describe the mapping r as follows. Let $x \in \mathbb{H}^3 \setminus C$, take $B(x) \subset \mathbb{H}^3$ be the largest metric ball with center at x such that its interior is disjoint from C. Since the distance function in $\mathbb{H}^3$ is strictly convex, the ball $B(x)$ touches ∂C at a single point and this point is $r_C(x)$. The Chabauty-limits of metric balls in $\mathbb{H}^3$ are horoballs, thus for each $\xi \in \Omega$ we take the maximal horoball $B(\xi)$ with center at ξ whose interior is disjoint from C. Then $r(\xi)$ is the point of where ∂C touches $B(\xi)$.

Proposition 4.86. *The map $r : \Omega \to \partial C$ is onto.*

Proof: Suppose that $w \in \partial C$. There is a supporting hyperplane P for C which contains the point w. This plane bounds two closed hyperbolic half-spaces $P^+ \supset C$ and P^-. Then $int(P^-)$ does not intersect C and it is clear that there is a horoball $B \subset P^-$ which touches the convex set $C \cap P$ at the point w. We let ξ be the center of the horoball B, then $r(\xi) = w$. $\square$

Consider the function $d(x) = d(x, C)$.

Proposition 4.87. *(See [ECG87].) The function $d(x)$ is differentiable in $\mathbb{H}^3 - C$.*

Proof: Let $y = r(x)$, $x \in \mathbb{H}^3 - C$. Let P denote a supporting plane for C at the point y. Introduce two functions: $f_1(z) = d(z, P)$, $f_2(z) = d(z, y)$. Then $f_1 \le d \le f_2$, $f_1(x) = d(x) = f_2(x)$ and the derivatives of f_1, f_2 at x coincide. Thus, d is a differentiable function at x. $\square$

Corollary 4.88. *Let $\epsilon > 0$ and $U_\epsilon = Nbd_\epsilon C$ be the ϵ–neighborhood of C. Then the boundary of U_ϵ is differentiable.*

Lemma 4.89. *If U is a closed convex smooth submanifold with boundary in $\mathbb{H}^3$, then $r_U : \Omega \to \partial U$ is injective.*

Proof: Suppose that $r_U|_\Omega$ is not injective. Then there are two distinct point $\xi, \eta \in \Omega$ such that the horoballs $B(\xi), B(\eta)$ are tangent to ∂U at the same point y. These horoballs are distinct and their boundaries are not tangent at y (since U is a smooth submanifold). The sets U, $B(\xi)$, $B(\eta)$ are convex; thus there are (supporting) hyperplanes $P_\xi, P_\eta \subset \mathbb{H}^3$ which "separate U from $B(\xi), B(\eta)$ at x", i.e.:

- $U \subset P_\xi^+, \quad U \subset P_\eta^+$.
- $B(\xi) \subset P_\xi^-$, $B(\eta) \subset P_\eta^-$.

Here $P_\eta^\pm, P_\xi^\pm$ are the closed half-spaces bounded by P_ξ, P_η. Thus the smooth submanifold U has two distinct tangent planes P_ξ, P_η at the point x which is impossible. $\square$

Corollary 4.90. *Suppose that $C = C(\Lambda) \subset \mathbb{H}^3$ is the convex hull of a closed set $\Lambda \subset \widehat{\mathbb{C}}$, $U = Nbd_\epsilon C$ is the ϵ-neighborhood of C for some $\epsilon > 0$. Then the function $r_U : \Omega \to \partial U$ is a homeomorphism.*

Proof: The function $r_U|_\Omega$ is injective, surjective and continuous. Both ∂U and Ω are 2-dimensional manifolds. Thus r is a homeomorphism. $\square$

As above let U be an ϵ-neighborhood of C, then we have three projections:

$$r_U : \Omega \to \partial U, \quad \pi := r_C|_{\partial U} : \partial U \to \partial C, \quad r_C : \Omega \to \partial C.$$

Proposition 4.91. $r_C = \pi \circ r_U$.

Proof: It is enough to prove that for each $x \in \mathbb{H}^3 - U$ we have $\pi \circ r_U(x) = r_C(x)$. Note that $D := d(x, r_U(x)) + d(r_U(x), \pi(r_U(x))) \geq d(x, r_C(x))$. Suppose that there is a point $y \in \partial C$ such that $d(x,y) < D$. Then the geodesic segment $[xy]$ intersects ∂U in a point z such that:

$$d(x,z) \geq d(x, \partial U), \quad d(z, \partial C) \geq \epsilon = d(r_U(x), \pi(r_U(x))).$$

Contradiction. $\square$

From now on we restrict ourselves to the case when C has nonempty interior. Whence C is a (topological) submanifold with boundary in $\mathbb{H}^3$. Clearly the projection $r_C : cl(U - C) \to \partial C$ is a deformation retraction and the inclusion $U - C \to cl(U - C)$ is a homotopy-equivalence. On the other hand, $U - C \cong (0, \epsilon] \times \partial U$. We conclude that $r_C : \Omega \to \partial C$ is a homotopy-equivalence. Suppose that Γ is a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^3)$ which keeps $\Lambda = \partial_\infty C$ invariant, hence $\Lambda(\Gamma) \subset \Lambda$. Then $\Omega = \widehat{\mathbb{C}} \setminus \Lambda \subset \Omega$ is Γ-invariant as well, we shall assume that Γ acts freely on Ω. The retractions r_C, r_U are Γ-invariant, thus we have a retraction

$$\bar{r} : \Omega/\Gamma \to \partial C/\Gamma.$$

By the same arguments as above we conclude that $\bar{r}$ is a homotopy-equivalence. However most of the time the mapping $\bar{r}$ is very far from being a homeomorphism since C has "corners". Nevertheless we will find a portion of Ω where r_C is injective. Consider the upper half-space model $\mathbb{R}^3_+ = \mathbb{C} \times \mathbb{R}_+$ for $\mathbb{H}^3$, where $\mathbb{R}_+ := \{t : t > 0\}$; we assume that $\Lambda \subset \widehat{\mathbb{C}}$ contains the point ∞ and $\Lambda \cap \mathbb{C}$ lies in $\{z : Im(z) \leq c\}$. Suppose also that the region

$$L = L_{c,a} := \{(z,t) \in \mathbb{R}^3_+ : Im(z) = c, t \geq a > 0\}$$

lies entirely in ∂C. (Note that L is the intersection of a horoball centered at ∞ and geodesic hyperplane asymptotic to ∞.)

Let $D_+ := \{z : Im(z) \geq c + a\}$. It is easy to see that the restriction of r_C to D_+ is the Euclidean rotation in $\mathbb{E}^3$ by $\pi/2$ around the line $\{(z,t) : t = 0, Im(z) = c\}$. Hence $r_C : D_+ \to \partial C$ is injective, moreover it is a conformal mapping between D_+ and $r(D_+) \subset \partial C$, where the conformal structure on ∂C is given by the path-metric induced from $\mathbb{H}^3$.

Now we assume that Ω is a component of $\Omega(\Gamma)$, Γ_Ω is the stabilizer of Ω in Γ and $S := \Omega/\Gamma_\Omega$ is a Riemann surface of finite type (i.e. it is obtained from a compact Riemann surface by removing a finite number of points: punctures of S). According to Theorem 4.84, elements of $\pi_1(S)$ corresponding to loops going around punctures, are mapped to parabolic elements of Γ under the epimorphism $\pi_1(S) \to \Gamma$. Let γ be one of the parabolic elements corresponding to a puncture P. We assume that $\infty \in \widehat{\mathbb{C}}$ is the fixed point of γ; thus after lifting a punctured disc around P into Ω (and conjugating Γ by a Euclidean rotation) we get a domain $D \subset \mathbb{C}$ which is precisely invariant under $A := \langle\gamma\rangle$ in Γ and D contains the region $D_q := \{z : Im(z) \geq q\}$ for some q. The element γ acts on $\mathbb{C}$ as the translation

$z \mapsto z + a$, $a > 0$. $D_q \cap \Lambda = \emptyset$, let $D_c := \{z : Im(z) \geq c\}$ denote the maximal closed Euclidean half-plane in $\mathbb{C}$ whose interior is disjoint from Λ. Its boundary $\{z : Im(z) = c\}$ contains at least one point $x \in \Lambda$, therefore it also contains the whole A-orbit of x. This means that $C(\Lambda) \cap \{(z,t) : Im(z) = c, t \in \mathbb{R}_+\}$ contains the convex hull of the A-orbit of x, therefore it contains the domain $L := \{(z,t) : Im(z) = c, t \geq a\}$. This is the situation that we had analyzed above. We conclude that

Proposition 4.92. *Assume that $S = \Omega/\Gamma_\Omega$ is a Riemann surface of finite type and Γ_Ω acts freely on Ω. Then there exists a collection of disjoint punctured discs D_j around punctures of S so that restriction of the retraction $\bar{r} : S \to \partial C(\Lambda(\Gamma))/\Gamma$ to the union $\cup_j D_j$ is injective and conformal.*

Hence $\bar{r} : S \to \partial C(\Lambda(\Gamma))/\Gamma_\Omega$ is a homotopy-equivalence onto its image which is a homeomorphism near punctures of S. Nielsen Realization Theorem implies that the surfaces $S, \bar{r}(S)$ are quasiconformally homeomorphic via a homeomorphism homotopic to $\bar{r}$. Moreover, $\bar{r}(\Omega)$ is a component of $\partial C(\Lambda(\Gamma_\Omega))$.

Corollary 4.93. *If Ω/Γ_Ω has finite conformal type, then the corresponding component Σ of $\partial CM(\Gamma)$ has finite hyperbolic area equal to $2\pi|\chi(\Sigma)|$.*

A Kleinian group $\Gamma \subset PSL(2,\mathbb{C})$ is said to be *analytically finite* if it is nonelementary and $\Omega(\Gamma)/\Gamma$ is a finite collection of Riemann surfaces of finite conformal type.

Corollary 4.94. *Suppose that $\Gamma \subset PSL(2,\mathbb{C})$ is analytically finite and $\Lambda(\Gamma)$ is not contained in a round circle in $\mathbb{S}^2_\infty$. Then the manifolds $C(\Lambda(\Gamma))/\Gamma$ and $\overset{\bullet}{M}(\Gamma)$ are homeomorphic. The homeomorphism carries Margulis cusps to Margulis cusps.*

Proof: We already know that the complement $\overset{\bullet}{M}(\Gamma) - intC(\Lambda(\Gamma))/\Gamma$ is naturally homeomorphic to $I \times \Omega(\Gamma)/\Gamma$. This homeomorphism carries the intersection of Margulis cusps of $M(\Gamma)$ with $\overset{\bullet}{M}(\Gamma) - intC(\Lambda(\Gamma))/\Gamma$ to $P \times I$, where P is a union of punctured discs around punctures of $\Omega(\Gamma)/\Gamma$. Since $\Lambda(\Gamma)$ is not contained in a round circle, the convex hull $C(\Lambda(\Gamma))$ and the convex core $CM(\Gamma)$ are a 3-dimensional manifolds (with boundary). Attaching $\overset{\bullet}{M}(\Gamma) - intC(\Lambda(\Gamma))/\Gamma$ to $CM(\Gamma)$ amounts to attaching a collar to $CM(\Gamma)$ which does not change the topology. $\square$

Corollary 4.95. *A discrete subgroup $\Gamma \subset \mathrm{Isom}(\mathbb{H}^3)$ is convex–cocompact if and only if $\overset{\bullet}{M}(\Gamma)$ is compact.*

More generally,

Corollary 4.96. *(A.Marden, [Mar74]) A discrete group $G \subset \mathrm{Isom}(\mathbb{H}^3)$ is geometrically finite if and only if there is a finite disjoint collection U of cuspidal neighborhoods of cusps in $\overset{\bullet}{M}(\Gamma)$ so that the complement*

$$\overset{\bullet}{M}(\Gamma) - int(U)$$

is compact.

4.18. The combination theorems

The combination theorems provide a convenient tool to construct a great deal of examples of Kleinian groups. In this section we describe several such theorems, the first of them was discovered by Klein in the late 19-th century, the other combination theorems were proven by Maskit in the 1960's. If G, H are subsets of $\mathrm{Isom}(\mathbb{H}^n)$ then in what follows we shall use the notation $\langle G, H\rangle$ for the subgroup of $\mathrm{Isom}(\mathbb{H}^n)$ generated by $G \cup H$.

1. Klein combination

We start with a simple example of a combination theorem which is called the *Klein Combination.* Suppose that G_1, G_2 are two Kleinian groups in $\mathrm{Isom}(\mathbb{H}^3)$ with the fundamental domains $D_1, D_2 \subset \widehat{\mathbb{C}}$, which satisfy the following condition:

$$int D_1 \supset \widehat{\mathbb{C}} - D_2, \quad int D_2 \supset \widehat{\mathbb{C}} - D_1 \tag{4.15}$$

See the Figure 4.5.

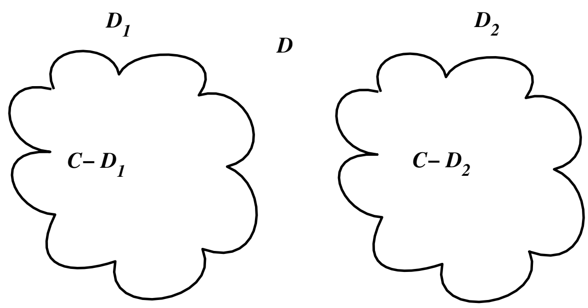

Figure 4.5: *Klein Combination.*

Theorem 4.97. *Under the above conditions the subgroup $G = \langle G_1, G_2\rangle \subset$ $\mathrm{Isom}(\mathbb{H}^3)$ generated by G_1, G_2 is a Kleinian group which is isomorphic to the free product $G_1 * G_2$. The domain $D = D_1 \cap D_2$ is a fundamental domain for the action of G on $\widehat{\mathbb{C}}$.*

Proof: Choose any point $x \in D$ and apply to x any reduced word

$$w = g_1 g_2 \dots g_{n-1} g_n$$

where $g_{even} \in G_2$, $g_{odd} \in G_1$. Then the induction argument shows that $w(x) \notin D$. For details see [Mas87]. $\square$

A typical example of application of this theorem is the *Schottky group* where we inductively apply Theorem 4.97.

Proposition 4.98. *Suppose that $G \subset \mathrm{Isom}(\mathbb{H}^3)$ is a nonelementary group (not necessarily discrete). Then G contains a non-Abelian Schottky subgroup.*

Proof: Since G is nonelementary, it contains two loxodromic elements g, h whose fixed-point sets $Fix(g), Fix(h) \subset \widehat{\mathbb{C}}$ are disjoint (see Corollary 3.25). Let D, E be two sufficiently small disjoint closed round disks centered at the repulsive fixed points x, y of g, h. The convergence property of Moebius transformations implies that we can find $n \gg 1$ so that $g^n(D) \supset D \cup E$, $h^n(E) \supset D \cup E$ and the exteriors $D' := \widehat{\mathbb{C}} - g^n(D)$, $E' := \widehat{\mathbb{C}} - h^n(E)$ are disjoint. Thus the complement to $D \cup E \cup D' \cup E'$ is a fundamental domain for the Schottky group $\langle g^n,\ h^n \rangle$. $\square$

Remark 4.99. The same conclusion holds for nonelementary subgroups of $\mathrm{Isom}(\mathbb{H}^n)$.

Similarly we get:

Proposition 4.100. *Suppose that G is a discrete subgroup of* $\mathrm{Isom}(\mathbb{H}^3)$, $h \in \mathrm{Isom}(\mathbb{H}^3)$ *is a loxodromic elements whose fixed point set $Fix(h)$ is disjoint from the limit set $\Lambda(G)$. Then:*

1. *If for every element $g \in G$ we have: $g(Fix(h)) \cap Fix(h) = \emptyset$, then there is $n \gg 1$ so that the subgroup of* $\mathrm{Isom}(\mathbb{H}^3)$ *generated by $G, \langle h^n \rangle \subset$* $\mathrm{Isom}(\mathbb{H}^3)$ *is a Kleinian subgroup isomorphic to the free product $G * \langle h^n \rangle$.*

2. *There is a finite-index subgroup $G^0 \subset G$ and $n \gg 1$ so that the subgroup of* $\mathrm{Isom}(\mathbb{H}^3)$ *generated by $G^0, \langle h^n \rangle \subset$* $\mathrm{Isom}(\mathbb{H}^3)$ *is a Kleinian group isomorphic to the free product $G^0 * \langle h^n \rangle$.*

Proof: Consider (1). Since $g(Fix(h)) \cap Fix(h) = \emptyset$ for every $g \in G$, we can find a fundamental domain D_1 for the action of G in $\widehat{\mathbb{C}}$ so that $Fix(h) \subset D_1$. As in the proof of Proposition 4.98, if n is sufficiently large, we find a fundamental domain $D_2 := \widehat{\mathbb{C}} - (D \cup D')$ for the cyclic group $\langle h^n \rangle$ so that: $\widehat{\mathbb{C}} - D_1 \subset D_2, \widehat{\mathbb{C}} - D_2 \subset D_1$. Hence we can apply Klein Combination Theorem, which implies the Assertion (1).

Now consider (2). Since $Fix(h) \cap \Lambda(G) = \emptyset$, there is only a finite number of elements $g_1, ..., g_m \in G - \{1\}$ such that $g_j(Fix(h)) \cap Fix(h) \neq \emptyset$. By Selberg Lemma (Theorem 4.6) there exists a finite-index subgroup $G^0 \subset G$ which does not contain any of the elements $g_1, ..., g_m$. Hence we can apply the Assertion (1). $\square$

2. Maskit combination

Before starting discussion of this type of combination theorems I would like to give a topological motivation for the conditions which will appear below. Let M be a PL-manifold which is the union of two submanifolds with boundary A, B such that $A \cap B = S$ is a connected hypersurface and embedding of each A, B, S into M induces a monomorphism of the fundamental groups. According to Seifert– Van Kampen theorem, the fundamental group of M splits as $\pi_1(A) *_{\pi_1(S)} \pi_1(B)$. Let $\tilde{A}, \tilde{B}, \tilde{S}$ denote the universal covers of A, B, S. Now we lift the decomposition $A \cup_S B$ of M to the universal cover $\tilde{M}$. We get a tessellation of $\tilde{M}$ by copies of the universal covers $\tilde{A}, \tilde{B}$ separated by the copies of $\tilde{S}$. Choose two representatives X, Y of $\tilde{A}, \tilde{B}$ separated by a copy Z of $\tilde{S}$. The stabilizers of X, Y in $G = \pi_1(M)$ are the groups G_A, G_B which are isomorphic to $\pi_1(A), \pi_1(B)$. The surface Z is precisely invariant under $J = \pi_1(S)$ in both G_A and G_B (see Definition 4.27). Let X^+ (resp. Y^+) be the component of $\tilde{M} - Z$ which contains X (resp. Y). It follows that the domain X^+ is precisely invariant under J in G_B and Y^+ is precisely invariant under J in G_A. The idea behind Maskit Combination Theorems is to reverse this picture. Namely, one starts with a pair of groups G_A, G_B acting on a space W so that $G_A \cap G_B$ contains a subgroup J. Then one finds a decomposition of W into domains similar to X^+, Y^+ which are separated by Z so that Z is stabilized by J and X^+, Y^+ have precise invariance properties as above. From this one would like to conclude that the group $G = \langle G_A, G_B \rangle$ generated by G_A and G_B is discrete, isomorphic to $G_A *_J G_B$ and has a fundamental domain of special kind (or/and the quotient space of the group G splits like in Seifert– Van Kampen theorem). This is briefly the 1-st combination theorem. The second combination theorem treats the case when G is an HNN-extension. On the level of Seifert–Van Kampen theorem this corresponds to a nonseparating hypersurface S. I recommend you to consider this case yourself, find a decomposition of the space $\tilde{M}$ and conclude which domains in this decomposition have precise-invariance properties under subgroups of G. Then, read our formulation of the 2-nd Maskit Combination Theorem or Maskit's book [Mas87, Chapter VII].

There are many different versions of Maskit Combination Theorems depending on how much properties we require from Kleinian groups and how strong conclusion we want. We shall present here four particular cases which will suffice for our needs.

Theorem 4.101. *(The 1-st Maskit Combination Theorem: quasifuchsian amalgamation.) Let G_2, G_1 be a pair of Kleinian groups such that $G_2 \cap G_1 \supset H$. Suppose that H is a quasifuchsian group with the limit set $\Lambda(H)$ so that $\widehat{\mathbb{C}} - \Lambda(H) = \Omega_2 \cup \Omega_1$. Assume that the domain Ω_j is precisely invariant under H in G_j ($j = 1, 2$). Then the group G generated by G_1, G_2 is isomorphic to $G_1 *_H G_2$ and is discrete. If G_1, G_2 are geometrically finite, then so is G. Under the isomorphism $G \to G_1 *_H G_2$, the image of any parabolic element of G is either conjugate into one of the groups G_1, G_2 or it commutes with a parabolic element of a conjugate of H. The surface $S(G) = \Omega(G)/G$ is naturally conformally equivalent to $(S(G_1) - \Omega_1/H) \cup (S(G_2) - \Omega_2/H)$.*

Proof: The proof can be found in [Mas87, Chapter VII]. One way to prove this theorem is as follows. Let me assume that H is not contained as a subgroup of the index 2 in any subgroup of G_1, G_2 and that H does not contain parabolic elements.

Precise invariance of Ω_j implies that the convex hulls $C(\Lambda(G_j))$ intersect along a simply-connected region Σ in $\mathbb{H}^3$ which is the convex hull of $\Lambda(H)$ so that $\Sigma/H \cong \Omega_1/H \times [0, 1]$. In general it is not true that Σ is precisely invariant in both G_1, G_2 under H. However, there exists a unique least area surface $S' \subset \Sigma/H$ (see [Uhl83]) and the projections of S' to both $C(\Lambda(G_j))/G_j$ are embeddings (see [FHS83]). Then we take a lift $\tilde{S}'$ of S' in Σ. Another way to argue is to use harmonic functions as in [Mor84]:

Lemma 4.102. *Consider the characteristic function $\lambda : \widehat{\mathbb{C}} \to \{0,1\}$ of the domain Ω_1. Let $h = ext(\lambda)$ be the harmonic extension of λ to $\mathbb{H}^3$ as in Section 3.12. Then the level set $\tilde{S} := \{x | h(x) = 0.5\}$ is precisely invariant under H in the both groups G_1, G_2.*

Proof: If $x \in \mathbb{H}^3$, S_x is the unit tangent sphere at x, we define the "exponential map"

$$Exp_x : S_x \to \mathbb{S}^2_\infty$$

by sending each unit tangent vector $v \in S_x$ to the geodesic ray ρ emanating from x in the direction v; the ray ρ determines a point at infinity $\xi = Exp_x(v)$. Since λ is H-invariant, $\tilde{S}$ is H-invariant as well (see Property 3 in Section 3.12). Suppose that there is an element $g \in G_1 - H$ such that $x \in \tilde{S} \cap g\tilde{S}$. By the construction of $ext(\lambda)$, the value $h(x)$ is the Lebesgue measure of the domain $\exp_x^{-1}(\Omega_1)$ on the unit tangent sphere S_x. Thus on the sphere S_x we get two domains: $Exp_x^{-1}(\Omega_1), Exp_x^{-1}(g\Omega_1)$, both having $1/2$ of the total area of S_x. These domains must be disjoint since Ω_1 is precisely invariant under G_1. Hence $\Omega_2 = g\Omega_1$ and the group $\langle H, g\rangle$ is a $\mathbb{Z}/2$-extension of H in G_1, which contradicts our assumptions. Similarly we prove that $\tilde{S}$ is precisely invariant under G_2. $\square$

The level set $\tilde{S}$ could be singular, but by Sard's theorem nearby there is a regular level set $\tilde{S}' := h^{-1}(0.5 \pm \epsilon)$ which is still precisely invariant. Note that the maximum principle for harmonic functions implies that $\tilde{S}'$ has no compact components. It seems to be true that the surface $\tilde{S}'$ has to be connected, but I do not know how to prove this. It is clear however that any connected component of $\tilde{S}'$ is H-invariant, so we retain the notation $\tilde{S}'$ for such connected component.

Then for each j the G_j–orbit of $\tilde{S}'$ is a disjoint family of connected surfaces that cuts from $C(\Lambda(G_j))$ an open domain D_j that is invariant under G_j. We construct a graph T whose vertices are the domains gD_j, $g \in G, j = 1, 2$. Two vertices are connected by an edge iff either the corresponding domains are adjacent along a translate of $\tilde{S}'$. Then the precise invariance of $\tilde{S}'$ in both G_j implies that:

(a) The graph T is a tree which is properly embedded in $\mathbb{H}^3$.

(b) Stabilizers of vertices are conjugates of the groups G_j and stabilizers of edges are the conjugates of H in G.

We conclude that the group G is discrete and has the structure of amalgamated free product $G_1 *_H G_2$ (see Section 10.2).

Moreover there is a natural isometry between the manifold $M(G) = \mathbb{H}^3/G$ and the manifold obtained by gluing D_1/G_1 to D_2/G_2 along $S = \tilde{S}'/H$ (the gluing map is determined by the property that it is covered by the identity map $\tilde{S}' \to \tilde{S}'$). This isometry extends to a conformal diffeomorphism between $\overset{\bullet}{M}(G)$ and

$$[D_1/G_1 \cup (S(G_1) - \Omega_1/H)] \cup_S [D_2/G_2 \cup (S(G_2) - \Omega_2/H)].$$

This implies the rest of the assertions of theorem. $\square$

To illustrate the first combination theorem consider the following example. Let M_1 be a compact convex hyperbolic 3-manifold with a single boundary component S which is totally-geodesic. Let M_2 be another copy of M_1, now "double" M_1 along S by gluing M_1 to M_2 via the identity map $S \to S$. The result is a closed hyperbolic 3-manifold M. Let $\Omega(G_1)$ be the domain of discontinuity for the action of $G_1 = \pi_1(M_1)$ on $\mathbb{H}^3$. The components of $\Omega(G_1)$ are round disks. Let Ω_1 be one of these disks, its stabilizer in G_1 is a Fuchsian subgroup H isomorphic to $\pi_1(S)$. Then Ω_1 is precisely invariant under H in G_1. Let σ be the inversion in the boundary circle of Ω_1. Define G_2 as $\sigma G_1 \sigma$, $\Omega_2 := \sigma(\Omega_1)$. It is clear now that the tripe (G_1, G_2, H) satisfies the assumption of Theorem 4.101.

Theorem 4.103. *(The 2-nd Maskit Combination Theorem: quasifuchsian amalgamation.) Let G_0 be a Kleinian group such that $G_0 \supset H_1, H_2$, where H_j are quasifuchsian groups which stabilize different connected components Ω_1, Ω_2 of $\Omega(G_0)$. Let $A \in \mathrm{Isom}(\mathbb{H}^3)$ be an element such that $A(\Omega_1) = \widehat{\mathbb{C}} - cl(\Omega_2)$ and $AH_1A^{-1} = H_2$ induces an isomorphism ϕ of H_1 and H_2. Then the group G generated by G_0 and A is discrete and isomorphic to the HNN extension $G_0 *_\phi$ of G_0 via ϕ. If G_0 is geometrically finite, then so is G. Under the isomorphism $G \to G_0 *_\phi$, the image of any parabolic element in G is either conjugate into the group G_0 or it commutes with a parabolic element of a conjugate of H_1. The surface $S(G) = \Omega(G)/G$ is naturally conformally equivalent to $S(G_0) - (\Omega_1/H_1 \cup \Omega_2/H_2)$.*

The proof can be found in [Mas87, Chapter VII] and it is essentially the same as in the case of the first combination (with slight modifications the reader can repeat the arguments from the above proof of Theorem 4.101).

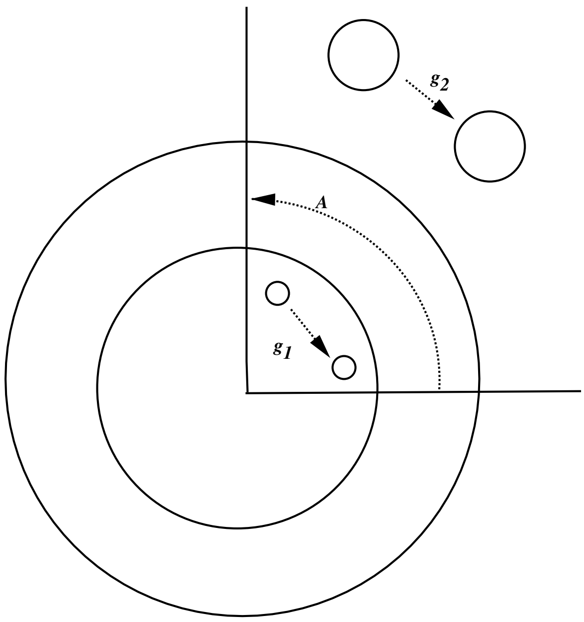

Figure 4.6: *The 1-st Maskit Combination with cyclic amalgamation.*

Now we describe two combination theorems where instead of quasifuchsian amalgamated subgroups we take *cyclic* subgroups. These cases of combination are easier to visualize and we will use them in §19.5 in the case when the amalgamated subgroups are elliptic. Recall that if G acts on a set X and $S \subset X$ then $G(S)$ denotes the G-orbit of S in X.

Theorem 4.104. *(The 1-st Maskit Combination Theorem: cyclic amalgamation.) Let G_1, G_2 be a pair of Kleinian groups such that $G_1 \cap G_2 = H$, where H is a cyclic subgroup. Let Φ_j be fundamental domains for the actions of G_j on $\widehat{\mathbb{C}}$ ($j = 1,2$). Let B_1, B_2 be open disks in $\widehat{\mathbb{C}}$ so that: $J = cl(B_1) \cap cl(B_2) = \partial B_1 = \partial B_2$ is a topological circle. Suppose that:*

- *B_j is precisely invariant under H in G_j ($j = 1,2$).*
- *$\Phi'_j := \Phi_j \cap G_j(B_j) \subset B_j$ ($j = 1,2$).*
- *$\Phi'_1 \cap \Phi_2, \Phi_1 \cap \Phi'_2$ have nonempty interiors.*

Then the subgroup $G \subset \mathrm{Isom}(\mathbb{H}^3)$ generated by G_1, G_2 is Kleinian and is isomorphic to $G_1 *_H G_2$. If G_1, G_2 are geometrically finite, then G is also geometrically finite. The quotient $\Omega(G)/G$ is naturally conformally isomorphic to

$$\Omega(G_1 - G_1(B_1))/G_1 \cup_L \Omega(G_2 - G_2(B_2))/G_2$$

where the gluing is along $L = [J \cap \Omega(H)]/H$. Any parabolic element in G is conjugate either into G_1 or into G_2 or it is conjugate to an element commuting with a parabolic element of H.

Theorem 4.105. *(The 2-nd Maskit Combination Theorem: cyclic amalgamation.) Let G_0 be a Kleinian group, H_1, H_2 be its cyclic subgroups. Let Φ_0 be a fundamental domain for the actions of G_0 on $\widehat{\mathbb{C}}$. Let B_1, B_2 be open disks in $\widehat{\mathbb{C}}$ and $A \in \mathrm{Isom}(\mathbb{H}^3)$ be a Moebius transformation so that $AH_1A^{-1} = H_2$, this conjugation induces an isomorphism $\phi : H_1 \to H_2$.*

Suppose that:

- *B_j is precisely invariant under H_j in G_0 ($j = 1,2$).*
- *$A(B_1) \cap B_2 = \emptyset$, $A(\partial B_1) \cap \partial B_2 = J$ is a topological circle.*
- *$gB_1 \cap B_2 = \emptyset$ for all $g \in G_0$.*
- *$\Phi_0 \cap (\widehat{\mathbb{C}} - G_0(B_1 \cup B_2))$ has nonempty interior.*

Then the subgroup $G \subset \mathrm{Isom}(\mathbb{H}^3)$ generated by G_0, A is Kleinian and is isomorphic to the HNN extension $G_0 *_{\phi : H_1 \to H_2}$ of G_0 via ϕ. If G_0 is geometrically finite, then G is also geometrically finite. The quotient $\Omega(G)/G$ is naturally conformally isomorphic to

$$\sim \backslash [\Omega(G_0) - G_0(B_1 \cup B_2)]/G_0$$

where the identification is such that $[J \cap \Omega(H_2)]/H_2$ is identified with $[A^{-1}(J) \cap \Omega(H_1)]/H_1$ via the projection of A. Any parabolic element in G is conjugate either into G_0 or it is conjugate to an element commuting with a parabolic element in H_j.

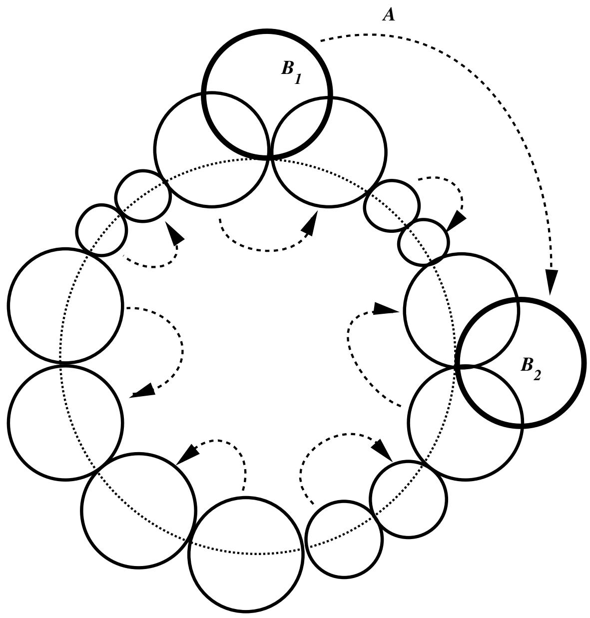

Figure 4.7: *The 2-nd Maskit Combination with cyclic amalgamation.*

Now I give two examples of the combination with cyclic amalgamation.

Example 4.106. Take A to be the rotation of the order $n > 1$ around the origin in $\mathbb{C}$; let $R := \{z : 0.5 < |z| < 2\}$. The cone $P := \{z : 0 < arg(z) < 2\pi/n\}$ is a fundamental domain for the group $J = \langle A \rangle$. Pick two Kleinian groups $\Gamma_1, \Gamma_2 \subset \mathrm{Isom}(\mathbb{H}^3)$ with the fundamental domains F_1, F_2 such that $\mathbb{C} - F_1$ is contained in $P \cap \{z : |z| < 0.5\}$, $\mathbb{C} - F_2$ is contained in $P \cap \{z : |z| > 2\} \cup \{\infty\}$. See Figure 4.6, where Γ_1, Γ_2 are cyclic groups generated by loxodromic elements g_j. Now let $G_j = \langle \Gamma_j, J \rangle$ be the subgroups of $\mathrm{Isom}(\mathbb{H}^3)$ generated by Γ_j and J $(j = 1, 2)$. The group G generated by G_1, G_2 is the result of the 1-st Maskit Combination of G_1, G_2, in particular $G \cong G_1 *_J G_2$. The precisely invariant disks are $\{|z| > 1\} \cup \{\infty\}, \{|z| < 1\}$.

Example 4.107. Start with a Fuchsian group G_0, it keeps the exterior E of the unit disk invariant. A fundamental domain Φ_0 of G_0 is the exterior of a collection of discs covering the unit circle, this domain and side-pairing are described in the Figure 4.7. Let H_1, H_2 be maximal parabolic subgroups of G_0 which are not conjugate in G_0. We identify E with the hyperbolic plane, then each H_j has a closed precisely invariant horoball $B_j \subset E$, G_0-orbits of these horoballs are disjoint. There exists a Moebius transformation $A : B_1 \to cl(\widehat{\mathbb{C}} - B_2)$ which conjugates H_1 to H_2. Let $G := \langle G_0, A \rangle$, this group is the result of the 2-nd Maskit Combination.

4.19. Ahlfors finiteness theorem

Recall that a Riemann surface X has finite type if it can be obtained from a compact complex curve by removing a finite number of points. The following theorem is one of the most fundamental facts about finitely generated Kleinian subgroups of $PSL(2, \mathbb{C})$.

Theorem 4.108. *(Ahlfors Finiteness Theorem.) Let G be a nonelementary finitely generated Kleinian subgroup of $PSL(2, \mathbb{C})$. Then $S(G) = \Omega(G)/G$ is a finite union of Riemann surfaces of finite hyperbolic type.*

Proofs of this theorem see in [Ahl64], [Kra72], [Gre77], [Ber87], [MT98, §4.2], [MS98]. We will give a proof of Theorem 4.108 in Section 8.14.

Corollary 4.109. *If G is a nonelementary finitely generated Kleinian subgroup of $PSL(2, \mathbb{C})$ and G_Ω is the stabilizer in G of a component $\Omega \subset \Omega(G)$ then $\Lambda(G_\Omega) = \partial\Omega$ and G_Ω is finitely generated. In particular, no component $\Omega \subset \Omega(G)$ has trivial stabilizer.*

Ahlfors' theorem has several extensions that we shall discuss later.

Corollary 4.110. *Suppose that G is a finitely generated nonelementary Kleinian subgroup of $PSL(2, \mathbb{C})$ which does not contain parabolic elements. Then the boundary $\partial CM(G)$ of the convex core $CM(G)$ is a compact surface.*

Proof: If the surface $S(G) = \Omega(G)/G$ has a puncture, then this puncture corresponds to a parabolic element of G (see Corollary 4.85). Thus $S(G)$ is a compact surface. Recall that either $C(\Lambda(G))$ has empty interior (in which case $CM(G)$ is compact) or $\partial CM(G)$ is homeomorphic to $S(G)$. □

Theorem 4.111. *(W. Thurston.) Let G be a geometrically finite* **Kleinian** *subgroup of $PSL(2, \mathbb{C})$ (i.e. $\Omega(G) \neq \emptyset$). Suppose that F is any finitely generated subgroup of G. Then F is geometrically finite.*

Proof: I give here a proof only in the case when G has no parabolic elements, for the general case see [Mor84, Proposition 7.1]. Under this assumption the convex core $N_G := CM(G)$ is compact and $diam(N_G) = d < \infty$. Without loss of generality we can assume that $int(CM(G)) \neq 0$, otherwise G is either elementary or is conjugate to a subgroup of $PSL(2, \mathbb{R})$, in which case all finitely generated groups are geometrically finite and there is nothing to prove. Since $diam(N_G) = d$, for each point $x \in N_G$ there exists a path ℓ of the length at most d which connects x to ∂N_G. Now consider the covering $p : \mathbb{H}^3/F \to \mathbb{H}^3/G$. For each point $y \in p^{-1}(x)$ a component ν of the path $p^{-1}(\ell)$ connects y and $p^{-1}\partial N_G$. But $p^{-1}\partial N_G \cap int N_F = \emptyset$, where $C(\Lambda(F))/F = N_F$. Therefore, ν must intersect ∂N_F which is compact. Hence $dist_H(N_F, \partial N_F) \leq d$, $diam(N_F) < \infty$, it follows that N_F is compact.
□

Corollary 4.112. *Suppose that in the situation above F is the stabilizer in G of a component Ω_0 of $\Omega(G)$. Then F is geometrically finite.*

Proof: Let $S_0 = \Omega_0/F$. Then, according to Ahlfors' finiteness theorem, $\pi_1(S_0)$ is finitely generated. Thus the group F (which is the quotient $\pi_1(S_0)$) of must be finitely generated as well. $\square$

The notion of convex-cocompact Kleinian group admits group-theoretic generalization, namely *Gromov-hyperbolic* group:

Definition 4.113. Let X be a geodesic metric space, $\delta \in \mathbb{R}_+$. The space X is called δ-**hyperbolic** if it has δ-thin triangles property: for every triangle $\Delta \subset X$ with geodesic sides, each side is contained in the δ-neighborhood of the union of two other sides.

For instance, $\mathbb{H}^n$ is 2-hyperbolic.

Definition 4.114. Let G be a finitely generated group, $C = C(G, S)$ a Cayley graph of G, where S is a finite generating set for G. The group G is called δ-**hyperbolic** if C is δ-hyperbolic.

A group which is δ-hyperbolic for some δ is called *Gromov-hyperbolic.* The notion of being Gromov-hyperbolic is quasi-isometry invariant (see for instance [Gro87] or [GdlH90]).

Exercise 4.115. *Show that each convex-cocompact subgroup of* $\mathrm{Isom}(\mathbb{H}^n)$ *is Gromov-hyperbolic.*

The notion of convex-cocompact subgroup admits another generalization:

Definition 4.116. Let G be a Gromov-hyperbolic group with the Cayley graph C, $H \subset G$ is a subgroup. Then H is called **quasiconvex** if for some (every) vertex $x \in C$ the H-orbit of x has the property: there is a constant K so that for any two points $v, w \in H(x)$ and geodesic segment $[vw]$ connecting v to w, the segment $[vw]$ is contained in the K-neighborhood of $H(x)$.

Exercise 4.117. *Let G be a convex-cocompact subgroup of* $\mathrm{Isom}(\mathbb{H}^3)$. *Then a subgroup $H \subset G$ is quasiconvex if and only if H is convex-cocompact.*

With this in mind, the following results are group-theoretic analogs of Theorem 4.111:

Theorem 4.118. *(S. Gersten [Ger96].) Let G be a Gromov-hyperbolic group of the cohomological dimension 2. Then each finitely presentable subgroup of G is also Gromov-hyperbolic.*

Theorem 4.119. *(J. McCammond, D. Wise [MW97].) Given g, r and a finite set of words $R_1(x_1, ..., x_g), ..., R_r(x_1, ..., x_g)$ where each R_i is reduced, cyclically reduced word in the generators $x_1, ..., x_g$, no R_i and R_k have conjugate powers, there exists a number N with the following property. Let G be the group with the presentation*

$$\langle x_1, ..., x_g | R_1^{q_1}, ..., R_r^{q_r} \rangle$$

where $q_i \geq N, i = 1, ..., r$. Then each finitely generated subgroup of G is quasiconvex. [5]

[5]The group G is known to be Gromov-hyperbolic for sufficiently large N, see [Gro87].

All known proofs of the Ahlfors finiteness theorem eventually require either doing complex analysis on $S(G)$ or hyperbolic geometry on $M(G)$. However Kleinian groups in $\mathrm{Isom}(\mathbb{H}^3)$ admit a natural generalization where none of these tools is applicable:

Definition 4.120. Let $G \subset Homeo(\mathbb{S}^2)$ be a discrete subgroup (with respect to the compact-open topology). The group G is said to be a **convergence group** if it satisfies the **convergence property** of Moebius transformations.

The class of convergence groups was introduced by Gehring and Martin [GM87a] as a generalization of Kleinian groups (instead of $\mathbb{S}^2$ one can also consider $\mathbb{S}^n$ or even more general compact topological spaces). For such groups one can define the domain of discontinuity and the limit set in the same way they are defined for the Kleinian groups. On can define *parabolic elements* of G as elements which have exactly one fixed point in $\mathbb{S}^2$.

Question 4.121. Suppose that $G \subset Homeo(\mathbb{S}^2)$ is a discrete finitely generated torsion-free convergence subgroup. Is it true that the surface $\Omega(G)/G$ is homotopy-equivalent to a compact surface? Is it true that G contains only finitely many conjugacy classes of maximal parabolic subgroups? Is it true that G is finitely presentable? Is it true that G is isomorphic to the fundamental group of a 3-manifold? Is it true that the action of G on $\mathbb{S}^2$ is topologically conjugate to the action of a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^3)$? (Compare Conjecture 20.13.)

See [KK98] for the discussion of the case when each limit point of G is a *point of approximation*, in which case G is Gromov-hyperbolic according to [Bow98].

4.20. Extensions of the Ahlfors finiteness theorem

Recall that a *cusp* of the hyperbolic manifold $M(G) = \mathbb{H}^3/G$ is a subset which is the quotient of a precisely invariant horoball in $\mathbb{H}^3$ by a maximal parabolic subgroup of G. If G is finitely generated discrete group with torsion then one can define cusps algebraically as to G-conjugacy classes of maximal infinite virtually parabolic subgroups of G (see §4.62).

Theorem 4.122. *(Sullivan Finiteness Theorem.) If G is a discrete finitely generated subgroup of* $\mathrm{Isom}(\mathbb{H}^3)$*, then G has only finitely many cusps.*

The reader can find analytical proofs in [Sul81b], [Kra84] and topological proofs in [FM87], [KS89]. Note that by purely topological methods one cannot prove finiteness of the number of simply-connected and annular components of $S(G)$ (see [Ka92b]).

Theorem 4.123. *(Ahlfors Recurrence Theorem.) Let G be a finitely generated discrete subgroup of* $\mathrm{Isom}(\mathbb{H}^3)$*. Then the action of G on $\Lambda(G)$ is recurrent, i.e. for every Lebesgue measurable subset $A \subset \Lambda(G)$ of nonzero* *planar measure, there are infinitely many elements $g \in G$ such that $mes(A \cap gA) \neq 0$.*

We shall prove this result later, in Section 8.13.

Theorem 4.124. *(Scott Compact Core Theorem[6], [Sco73a, Sco73b].) Suppose that M is a 3-manifold with the finitely generated fundamental group. Then: (a) the group $\pi_1(M)$ is finitely presentable. (b) There exists a compact submanifold M_c in M so that the inclusion $M_c \hookrightarrow M$, is a homotopy-equivalence.*

The submanifold M_c is called a *Scott compact core* of the manifold M.

Example 4.125. Let $M = \mathbb{R}^3 - L$, where L is a straight line. Choose a round circle $C \subset \mathbb{R}^3$ which links L nontrivially. Then for small ϵ the solid torus $M_c := Nbd_\epsilon(C)$ is a compact core in M. Now "tie a small nontrivial knot" on C within a metric ball disjoint from L. Let K denote the resulting knot in $\mathbb{R}^3$, then (for small δ) $N_c := Nbd_\delta(K)$ is also a compact core in M. Note that the complements $M - M_c$ and $M - N_c$ are not homeomorphic. The inclusion $\partial N_c \hookrightarrow cl(M - N_c)$ is not a homotopy-equivalence.

A stronger version of the Scott compact core theorem, that we shall need later, was proven by D. McCullough [McC86] (see also [KS89]):

Theorem 4.126. *Suppose that M is a 3-manifold with boundary and $\pi_1(M)$ is finitely generated, and $Q \subset \partial M$ is a compact subsurface. Then one can choose a Scott compact core $M_c \subset M$ such that $M_c \cap \partial M = Q$.*

We will refer to M_c in the above theorem as a *relative* compact core.

Theorem 4.127. *(D. McCullough, A. Miller and G. Swarup [MMS85].) Let M be an irreducible orientable noncompact 3-manifold with finitely generated fundamental group. Suppose that N_1 and N_2 are irreducible Scott compact cores of M. Then there is a homeomorphism $f : N_1 \to N_2$ such that $\iota_{1*} = \iota_{2*} \circ f_*$, where $\iota_j : N_j \to M$ are the inclusions and $f_*, \iota_{1*}, \iota_{2*}$ are homomorphisms of the fundamental groups induced by the maps f, ι_1, ι_2.*

Theorem 4.128. *(Feighn–Mess Finiteness Theorem [FM91].) Let G be a finitely generated discrete subgroup of* $\mathrm{Isom}(\mathbb{H}^3)$*. Then G has only finitely many conjugacy classes of finite order elements.*

This theorem is proven in [FM91] in two different ways. First, via the *Smith's theory*; the second proof is based on the following:

Theorem 4.129. *(M. Feighn, G. Mess.) Suppose that N is an open 3-manifold with the finitely generated fundamental group, F is a finite group of PL homeomorphisms acting on N. Then there exists a Scott compact core $N_c \subset N$ which is invariant under F.*

To derive Theorem 4.128 note that G contains a normal torsion-free subgroup of finite index Γ. Let $N := \mathbb{H}^3/\Gamma$. Then the finite group $F = G/\Gamma$

[6] This theorem was independently proven by P. Shalen.

acts on N with the quotient $M(\Gamma)$. Thus F acts on N_c with a compact quotient which implies that G has only finitely many conjugacy classes of finite order elements.

The Ahlfors Finiteness Theorem and the rest of its "companions" (with the possible exception of Theorem 4.123) fail for discrete subgroups of $\mathrm{Isom}(\mathbb{H}^4)$ even in the topological setting. The first counterexamples were constructed in [KP91b] and [KP91a] (Kleinian groups in those examples had parabolic elements); the examples without parabolic elements were constructed in [Pot94], [BM94]. The key idea in all these examples however is the same: use existence of hyperbolic 3-manifolds of finite volume which fiber over $\mathbb{S}^1$.

Question 4.130. Is there a finitely generated Kleinian group $G \subset Isom(\mathbb{H}^n)$ $(n \geq 4)$ so that $\Omega(G)$ contains a component Ω with trivial stabilizer? Does every finitely generated discrete subgroup of $\mathrm{Isom}(\mathbb{H}^n)$ $(n \geq 4)$ act recurrently on its limit set? Is there an analogue of the Ahlfors finiteness theorem for finitely generated discrete subgroups of $PU(2,1)$ with nonempty domain of discontinuity in $\mathbb{S}^3$? Is there a finitely presentable discrete subgroup $G \subset PU(2,1)$ which is not geometrically finite? (Note that there are non-finitely presentable finitely generated discrete subgroups of $PU(2,1)$, [Ka98b].)

Properties of compact cores. We will assume that M is an aspherical 3-manifold with finitely generated fundamental group, S is the boundary of M, $Q \subset S$ is a compact subsurface, M_c is the compact core of M relative to Q. Let $M' := cl(M - M_c)$.

Proposition 4.131. *If Σ is a component of ∂M_c then the inclusion $\Sigma \to M'$ is π_1-injective.*

Proof: Suppose that $D \subset M'$ is a compressing disk for Σ, $\partial D = \gamma \subset \Sigma$. Since M_c is π_1-injective in M, the simple loop γ also bounds a disk $D' \subset M_c$. The union $D \cup D'$ is an essential 2-sphere in M which implies that M is not aspherical. Contradiction. $\square$

Proposition 4.132. *Suppose that a component Σ_j of $cl(\Sigma - S)$ is incompressible in M_c. Then the inclusion $\Sigma_j \hookrightarrow E_j$ is a homotopy-equivalence, where E_j is the component of M' adjacent to Σ_j.*

Proof: The inclusion $\Sigma_j \hookrightarrow M$ is π_1-injective, and the map $\pi_1(M_c) \to \pi_1(M)$ is an isomorphism. Thus Seifert-Van Kampen theorem implies that the inclusion $\Sigma_j \hookrightarrow E_j$ is also π_1-surjective. $\square$

Remark 4.133. Note that if there is a component Σ_j of $cl(\partial M - S)$ which is compressible in M_c, then $\pi_1(M_c)$ splits as a nontrivial free product. Moreover this decomposition is such that for each component S_i of $\dot{M} - S$, the image of $\pi_1(S_i)$ in $\pi_1(M_c)$ is conjugate into one of the vertex subgroups of this decomposition.

4.21. Limit sets of geometrically finite groups

We recall that action of a group G on a measure space X is called *ergodic* if each G-invariant measurable function on X is a.e. constant.

Theorem 4.134. *(L. Ahlfors.) Let $G \subset Isom(\mathbb{H}^3)$ be a geometrically finite group. Then either the limit set of G is $\widehat{\mathbb{C}}$ or it has Lebesgue measure zero. In any case, G acts ergodically on its limit set.*

Proof: Recall that each limit point of G is either a point of approximation or a cusped parabolic limit point. In particular, a.e. limit point of G is a point of approximation. Let $h : \widehat{\mathbb{C}} \to \mathbb{R}$ be any measurable function invariant under G. There exists a harmonic extension $ext(h) : \mathbb{H}^3 \to \mathbb{R}$ (see Section 3.12). It follows that $ext(h)$ is G-invariant. For almost every limit point $z \in \Lambda(G)$ we have: $\lim^c_{w\to z} ext(h)(w) = h(z)$. Here $\lim^c_{w\to z}$ means the conical limit, i.e w is convergent to z inside of a tubular neighborhood of some geodesic ray.

Since $ext(h)$ is G–invariant, we conclude that $ext(h)(0) = h(z)$ for each $0 \in \mathbb{H}^3$ and almost every conical limit point $z \in \Lambda(\Gamma)$. By (a.e.) continuity of $ext(h)$ we conclude that $h(p) = h(z)$ for almost every $p \in \widehat{\mathbb{C}}$. This proves that h must be a.e. constant. So, suppose that $mes(\Lambda(G)) \neq 0$ and $\Lambda(G) \neq \widehat{\mathbb{C}}$. Then we define a measurable G-invariant function h on $\widehat{\mathbb{C}}$ to be 0 on $\widehat{\mathbb{C}} - \Lambda(G)$ and to be equal to 1 on $\Lambda(G)$. Contradiction. $\square$

Note that the same proof works for geometrically finite subgroups of $\mathrm{Isom}(\mathbb{H}^n)$.

Conjecture 4.135. *(Ahlfors measure zero conjecture). Let $G \subset \mathrm{Isom}(\mathbb{H}^n)$ be a finitely generated discrete subgroup ($n \geq 3$). Then the Lebesgue measure of $\Lambda(G)$ is zero unless $\Lambda(G) = \partial_\infty \mathbb{H}^n$.*

This conjecture was first formulated by Ahlfors around 1966, however it remains open even if $n = 3$. Thurston proved this conjecture in [Thu81] under the assumption that G is *geometrically tame* (see definition 14.14). Geometrical tameness of freely indecomposable torsion-free discrete subgroups of $\mathrm{Isom}(\mathbb{H}^3)$ was established by Bonahon in [Bon86]. Further progress in this direction was achieved by Canary, Minsky and Ohshika, see §14.4.

If $G \subset \mathrm{Isom}(\mathbb{H}^3)$ is torsion-free and freely indecomposable or $\mathbb{H}^3/G$ admits a compactification to a manifold with boundary, then the Ahlfors measure zero conjecture has positive solution, see [Bon86, Can93].

4.22. Groups with Kleinian subgroups

Theorem 4.136. *Let $G \subset \mathrm{Isom}(\mathbb{H}^3)$ be a finitely generated nonelementary Kleinian group; H is a group such that $\mathrm{Isom}(\mathbb{H}^3) \supset H \supset G$ and $\Lambda(G)$ is invariant under H. Then:*

(1) H is discrete provided that the limit set of G is not a round circle.

(2) If H is discrete then $|H : G| < \infty$.

Proof: (1) Suppose that H is not discrete. Then the connected component of identity $cl(H)^0$ in the closure $cl(H)$ of H in $\mathrm{Isom}(\mathbb{H}^3)$ is either conjugate to $\mathrm{PSL}(2,\mathbb{R})$ or equals $PSL(2,\mathbb{C})$ (see Exercise 4.25). Since $\Lambda(G)$ is H-invariant, it is also $cl(H)^0$-invariant. The only closed $PSL(2,\mathbb{R})$-invariant subsets in $\widehat{\mathbb{C}}$ are $\emptyset$, $\widehat{\mathbb{C}}$ and a round circle in $\widehat{\mathbb{C}}$. Thus $cl(H)^0 = PSL(2,\mathbb{C})$. However $PSL(2,\mathbb{C})$ does not have any proper invariant subsets in $\widehat{\mathbb{C}}$, which shows that H is discrete.

(2) Clearly $\Lambda(H) = \Lambda(G)$, since the limit set of H is the smallest nonempty H-invariant subset of $\widehat{\mathbb{C}}$. Thus H is a Kleinian group. It is enough to consider the case when G is torsion-free. Consider the ramified covering $p : \Omega(G)/G \to \Omega(H)/H$ induced by the inclusion $G \subset H$. According to the Ahlfors Finiteness Theorem, the hyperbolic area of $\Omega(G)/G$ is finite. This implies that p has finite multiplicity, therefore $|H : G| < \infty$. $\square$

Note that the assumption that $\Lambda(G)$ is not the round circle is necessary in (1), consider for instance $H = \mathrm{PSL}(2,\mathbb{R})$.

4.23. Ends of hyperbolic manifolds

If you are familiar with the definition of an *end* of a topological space, you can disregard the following discussion of this definition. Otherwise, before reading the definition, first read the definition of a *a based filter* in §9.1.

First recall the general definition of *ends* of noncompact manifolds. Suppose that M is a topological manifold (possibly with boundary). Let $\mathcal{S}$ denote the collection of subsets $S \subset M$ so that:

- Each S is the interior of a nonempty submanifold $\bar{S} \subset M$ which has compact boundary.
- Each $\bar{S}$ has exactly one noncompact component.

Definition 4.137. A (topological) **end** of the manifold M is a filter $\mathcal{E}$ based on $\mathcal{S}$, which satisfies the extra condition:

$$\bigcap_{U \in \mathcal{E}} U = \emptyset.$$

Any set $U \in \mathcal{E}$ is called a *neighborhood* of the end $\mathcal{E}$. This looks like an awful definition, in practice however we will have to deal only with simple examples of ends, here is the one. Let N be a compact manifold with boundary, $L \subset \partial N$ be a compact subsurface (it could be the entire boundary). Let $M := N - L$. Suppose L consists of connected components $L_1, ..., L_k$. Then the ends of M are in 1-1 correspondence with the components L_j. To construct a system of neighborhoods for each L_j take the system of compact codimension zero submanifolds with boundary $N_{\alpha j} \subset N$ each of which contains L_j, and let $\{U_{j\alpha} := int(N_{\alpha j})\}$. This collection forms a base of the corresponding filter $\mathcal{E}_j$. Ends which appear this way are called *topologically tame*.

Suppose that M is a 3-manifold with the finitely generated fundamental group, M^c is a compact core of M. Then there is 1-1 correspondence between the components N_j of $M - M^c$ and the ends of M (each N_j is a neighborhood of one and only one end). Thus, by abusing notation, I will say that N_j is an end of M.

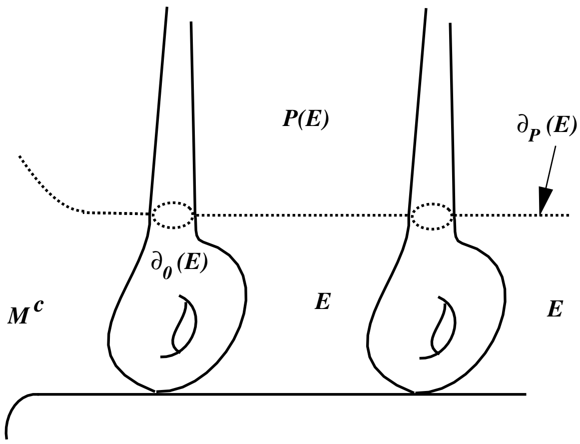

Figure 4.8:

Let M be a complete hyperbolic 3-manifold with the finitely generated fundamental group, for $\epsilon < \mu_3$ let M^0_ϵ be the union of $M_{[\epsilon,\infty)}$ and Margulis tubes in $M_{(0,\epsilon]}$. The manifold with boundary M^0_ϵ has a Scott compact core M^c_ϵ defined in Theorem 4.124. We will call the connected components of $M^0_\epsilon - M^c_\epsilon$ the *geometric ends* of the manifold M. In what follows we shall refer to the geometric ends as simply *ends*. Thus, ends of M are in 1-1 correspondence with the boundary components of $M^c_\epsilon - \partial M^0_\epsilon$. The *parabolic extension* $P(E)$ of the end E of M is the union of E and those cusps in $M_{(0,\epsilon]}$ which are adjacent to the boundary of E. Let $\partial_P E$ denote the frontier of E in $P(E)$ and $\partial_0 E := \partial E - \partial_P E$; $\partial_0 M^c_\epsilon$ is defined as the union of $\partial_0 E$ over all ends E of M^0_ϵ. See Figure 4.8. Note that a priori the inclusion $\partial_0 E \hookrightarrow E$ is not a homotopy-equivalence. However each such embedding is a homotopy-equivalence provided that M satisfies the following condition introduced by F. Bonahon in [Bon86]:

Condition (B): *The fundamental group of M does not split as a nontrivial free product so that each parabolic subgroup of $\pi_1(M)$ is conjugate into one of the vertex subgroups of this decomposition. Equivalently, $\partial_0 M^c_\epsilon$ is incompressible in M^c_ϵ.*

(See Proposition 4.132 and Remark 4.133.)

Let M be a complete hyperbolic 3-manifold with the finitely generated fundamental group $G \subset \mathrm{Isom}(\mathbb{H}^3)$. Let CM denote the convex core of M. An end E of M^0_ϵ is said to be *geometrically finite* if $E \cap CM$ is compact. Another way to express this is to say that the subset $E \subset M^0$ is precompact in $\overset{\bullet}{M} = M \cup \Omega(G)/G$. Note that G is geometrically finite if and only if all ends of M are geometrically finite.

Now consider the following special case. Take a complete hyperbolic surface Σ of finite area, which has genus g and m punctures. Consider a complete hyperbolic 3-manifold M so that $\Gamma := \pi_1(M) \cong \pi_1(\Sigma)$, where the isomorphism $\rho : \pi_1(S) \to \pi_1(M)$ is a *parabolic representation.* Thus, images of peripheral loops in Σ are parabolic elements in Γ, we call them *regular* parabolic elements of Γ. Let $P := P_1 \cup ... \cup P_m$ be the union of cusps in M which correspond to regular parabolic elements (we call them *regular cusps*). However, it could happen that some other (nonperipheral) elements γ in $\pi_1(\Sigma)$ are also mapped to parabolic elements. Such elements $\rho(\gamma)$ are called *accidental parabolic elements.* Let $Q := Q_1 \cup ... \cup Q_n$ denote the complete collection of Margulis cusps in M which correspond to accidental parabolic elements (we call them *accidental cusps*).

Our current goal is to understand the structure of accidental cusps in M. Any Scott compact core M_c of M is homeomorphic to $\Sigma_c \times [-1, 1]$, where Σ_c is a compact subsurface in Σ obtained by removing disjoint punctured disks around cusps. Theorem 4.126 implies that we can choose M_c so that:

M_c intersects the boundary of each cusp Q_j, P_i along a single annulus A_j, B_i which carries the fundamental group of Q_j, P_i.

The union $M_c \cup P$ splits the manifold M in two pieces: the "positive part" $M^+ \subset M - M_c \cup P$ (adjacent to $\Sigma_c \times \{+1\}$) and the "negative part" $M^- = M - M_c \cup P - M^+$. Therefore all accidental cusps in M group in two collections: the *positive* cusps Q_j^+ (so that $A_j^+ \subset M^+$) and the *negative* cusps Q_j^- (so that $A_j^- \subset M^-$). The annuli $A_j^\pm$ correspond to simple loops $\gamma_j^\pm \subset \Sigma$ which are homotopic to $A_j^\pm$. The distinct loops γ_j^+, γ_i^+ (and γ_j^-, γ_i^-) are disjoint. Thus we obtain two systems of simple disjoint loops $\Pi = \Pi^+ \cup \Pi^-$ on Σ such that each accidental cusp in $M^\pm$ is represented by a curve in $\Pi^\pm$.

Lemma 4.138. *No two curves in the collections* Π^+, Π^- *are homotopic.*

Proof: Suppose that $\gamma^+ \in \Pi^+$ is homotopic to $\gamma^- \in \Pi^-$. Then $\gamma^\pm$ represent cusps $Q^\pm \subset M^\pm$. The cusps $Q^\pm$ are disjoint and their fundamental groups are conjugate in $\pi_1(M)$. This contradicts Lemma 4.62. $\square$

Corollary 4.139. *The cardinality of the collection* Π *is at most* $2(3g-3+m)$.

Chapter 5

Teichmüller Theory of Riemann Surfaces

For detailed discussion of the Teichmüller theory and moduli spaces see [Abi80], [Gar87], [HM98], [Leh87], [Nag88], [Tro92], etc., here I give only a brief introduction, mostly without proofs.

5.1. Tensor calculus on Riemann surfaces

We first review formalism of the tensor calculus in the complex plane and Riemann surfaces which is used for the rest of the book.

Recall that $dz := dx + idy, d\bar{z} := \overline{dz} := dx - idy$ are the differential forms and $\partial_z := \frac{1}{2}(\partial_x - i\partial_y)$, $\partial_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y)$ are vector fields, $z \in \mathbb{C}$. Thus $dz \wedge d\bar{z} = -2idx \wedge dy$ is the multiple of the Euclidean area form. The vector fields $\partial_z, \partial_{\bar{z}}$ are thought of as sections of two different complex line bundles $T^{1,0}$ (the holomorphic line bundle) and $T^{0,1}$ (the antiholomorphic line bundle). Since the bundle $T^{1,0}$ is 1-dimensional, there are natural isomorphisms between the tensor product $(T^{1,0})^{\otimes n}$, the symmetric product $S^n(T^{1,0})$ and the exterior product $\bigwedge^n T^{1,0}$, in what follows we will identify these bundles. The same applies to the bundle $T^{0,1}$ and to the duals: holomorphic and antiholomorphic cotangent bundles $(T^{1,0})^*$, $(T^{0,1})^*$ whose sheaves of local sections are generated by the forms $dz, d\bar{z}$.

It is therefore a convenient custom in the complex analysis to skip the tensor product notation and to use $(dz)^{-1}, (d\bar{z})^{-1}$ to denote the vector fields $\partial_z, \partial_{\bar{z}}$. Thus $d\bar{z}/dz$ stands for $d\bar{z} \otimes \partial_z$, $dz^2 = dz \otimes dz$ and

$$\frac{1}{2}(dz\overline{dz} + \overline{dz}dz) = dx^2 + dy^2$$

is the Euclidean metric. Another standard convention is to contract to 1 the tensors $dz\partial_z$ and $d\bar{z}\partial_{\bar{z}}$. For instance, $f(z)dz^2\partial_z = f(z)dz$, etc. With these conventions, if U is an open disk in $\mathbb{C}$ then the tensor algebra generated by the sections of $T^{1,0}(U), T^{0,1}(U)$ and their duals becomes isomorphic to the tensor product $F(U) \otimes A(z, w)$, where $A(z, w)$ is the commutative algebra of Laurent polynomials in the variables z, w and $F(U)$ is the space of complex-valued functions on U. (We will specify later on $F(U)$ to be the space of holomorphic or smooth, or measurable functions on U.) Hence this tensor algebra is isomorphic (as a module over $F(U)$) to the direct sum

$$\bigoplus_{(p,q)\in\mathbb{Z}^2} A_{p,q}$$

where $A_{p,q}$ consists of the *differentials* $f(z)dz^p d\bar{z}^q$ of type (p,q).

If $f : U \subset \mathbb{C} \to \mathbb{C}$ is a differentiable function (we will also use the functions whose distributional derivatives belong to $L_\infty(U)$) then $\partial f := \partial_z f dz$ and $\bar{\partial} f := \partial_{\bar{z}} f \overline{dz}$ are the differential forms and the fraction

$$\bar{\partial} f/\partial f = \frac{\partial_{\bar{z}} f}{\partial_z f} d\bar{z}/dz$$

is a differential of the type $(-1,1)$. The same conventions will be used when we deal with Riemann surfaces. If S is a (hyperbolic) Riemann surface (with fixed compatible hyperbolic metric) then we let $L^r_{p,q}(S)$ denote the space of (p,q)-differentials $f(z)dz^p d\bar{z}^q$ such that $f \in L^r(S)$ and let $\Omega_{p,q}(S)$ denote the sheaf of holomorphic (p,q)-differentials $f(z)dz^p d\bar{z}^q$ (i.e. $f(z)$ is holomorphic). In particular, $\Theta_S = \Omega_{-1,0}(S)$ will denote the sheaf of holomorphic vector fields on S and $\Omega_S = \Omega_{1,0}(S)$ the sheaf of *holomorphic differentials* $f(z)dz$ on S. The tensor product $\Omega(S) \otimes \Omega(S)$ of the sheaf $\Omega(S)$ is the sheaf $\Omega_{2,0}(S)$ of holomorphic quadratic differentials on S. The cohomology with coefficients in these sheaves are related by *Serre's duality theorem*:

$$H^1(S,\Theta_S)^* \cong H^0(S,\Omega_{2,0}(S)) = Q_S,$$

the latter is the space of holomorphic quadratic differentials on S. In the case when S is a Riemann surface with punctures, Q_S will denote the space of holomorphic quadratic differentials on S which have at worst simple poles at the punctures; equivalently, these are the elements of $L^1_{2,0}(S) \cap \Omega^2(S)$. In the latter case instead of $H^1(S,\Theta_S)$ one has to consider only cohomology classes represented by L^∞-vector fields.[1]

Below is more general Serre's duality theorem for Riemann surfaces, see for instance, [Nar92]. If V is a holomorphic vector bundle on a compact Riemann surface S then

$$H^1(S,V)^* \cong H^0(S,V^* \otimes \Omega).$$

This duality is essentially the Poincaré duality on S: sections of the bundle $L^* \otimes \Omega(S)$ are holomorphic 1-forms $\zeta \otimes \omega$ ($\omega \in \Omega(S)$) with coefficients in L^*. Using identification of $H^1(S,V)$ with Dolbeault cohomology, choose a representative for $[\xi] \in H^1(S,V)$ which is a $\overline{\partial}$-closed 1-form $\tau \otimes \alpha$ with coefficients in V, where α is a $(0,1)$-differential on S. Thus, in the case when V is the holomorphic tangent bundle, $\tau \otimes \alpha$ locally is $v(z)\partial_z \overline{dz}$. Finally, take the wedge

$$\tau \otimes \alpha \wedge \zeta \otimes \omega = \zeta(\tau)\alpha \wedge \omega \in \Lambda^2(S)$$

[1] If $\mathfrak{X}_S$ is the sheaf of smooth vector fields on S then the natural mapping $H^1(S,\Theta_S) \to H^1(S,\mathfrak{X}_S)$ is zero, so each cohomology class in $H^1(S,\Theta_S)$ is the coboundary of a smooth vector field on S.

and integrate this 2-form over S. The result is the pairing $\langle \tau\otimes\alpha, \zeta\otimes\omega\rangle \in \mathbb{C}$ between $H^1(S,V)$ and $H^0(S,V^*\otimes\Omega)$.

Another space of special significance is $L^\infty_{-1,1}(S)$; we have well-defined norm $\|\nu(z)d\bar{z}/dz\| = \|\nu(z)\|_{L^\infty}$ on this space. The elements $L^\infty_{-1,1}(S)$ satisfying $\|\nu(z)d\bar{z}/dz\| < 1$ are called *Beltrami differentials* on S, the set of Beltrami differentials on S is denoted $Belt(S)$.

Suppose that $h : S \to S$ is a conformal automorphism, then it acts via pull-back on $L^r_{p,q}(S)$, this action is given by the formula:

$$h^*(f(z)dz^p d\bar{z}^q) = h^*(f(z))dz^p d\bar{z}^q, \text{where}$$

$$h^*(f(z)) := f(h(z))(h'(z))^p(\overline{h'(z)})^q.$$

Suppose now that Γ is a discrete group of conformal automorphisms of S, then $L^r_{p,q}(S,\Gamma)$ denotes the collection of elements of $L^r_{p,q}(S)$ invariant under Γ (the most interesting case for us is when S is conformally equivalent to the unit disk). Similarly one defines $Belt(S,\Gamma) = Belt(S)\cap L^\infty_{-1,1}(S,\Gamma)$ and $Q(S,\Gamma) := L^1(S,\Gamma)\cap Q_S$.

We have a pairing

$$\langle \nu, \varphi\rangle := \int_S \nu\varphi, \quad \nu \in L^\infty_{-1,1}(S), \varphi \in L^1_{2,0}(S) \tag{5.1}$$

between $L^\infty_{-1,1}(S)$ and $L^1_{2,0}(S)$. (Note that $\nu\varphi$ has type of the area form, so it could be integrated.) One can verify easily that this pairing is nondegenerate.

The Weil-Petersson pairing on Q_S. Suppose that S is a Riemann surface of (hyperbolic) finite type, $\varphi(z)dz^2, \psi(z)dz^2$ belong to Q_S. Give S the complete hyperbolic metric of finite area compatible with the conformal structure of S. This metric can be locally written as $\rho(z)dz^2$. The product of tensors

$$\varphi(z)dz^2\overline{\psi(z)dz^2}(\rho(z)dz^2)^{-1} = \frac{\varphi(z)\overline{\psi(z)}}{\rho(z)}dz\overline{dz}$$

has type of the area form, so it could be locally integrated on S. One then verifies that

$$\langle \varphi dz^2, \psi dz^2\rangle_{WP} := \int_S \frac{\varphi(z)\overline{\psi(z)}}{\rho(z)}dz\overline{dz} < \infty$$

and the bilinear form $\langle \varphi dz^2, \psi dz^2\rangle_{WP}$ is a Hermitian metric on Q_S, called the *Weil-Petersson pairing*.

Historic remark. The Weil-Petersson pairing was introduced by H. Petersson in [Pet49] in the context of automorphic forms on the hyperbolic plane and was used later (in 1957) by A. Weil [Wei79] to define a Kähler metric on the Teichmüller space (this metric will be discussed below, in §5.5).

5.2. Properties of quasiconformal maps

We begin our discussion with general properties of quasiconformal homeomorphisms (defined in Section 3.10) between domains in $\mathbb{S}^n$ ($n \geq 2$) and then will restrict ourselves to the case $n = 2$. Suppose D is a domain in $\mathbb{S}^n$, $f : D \to D'$ is a K-quasiconformal homeomorphism. We will use the notation ∂f for $\partial f/\partial z$ and $\overline{\partial} f$ for $\partial f/\partial \bar{z}$.

List of properties of quasiconformal homeomorphisms:

1. f is differentiable almost everywhere in D.

2. Partial derivatives of f are locally in $L_2(D)$.

3. Volume is absolutely continuous function under quasiconformal maps. Thus, the Jacobian determinant J_f is nonzero almost everywhere in D.

4. $K(f) = K(f^{-1})$, $K(f \circ g) \leq K(f) \cdot K(g)$ and $K_O(f \circ g) \leq K_O(f) \cdot K_O(g)$ for any pair of quasiconformal mappings f, g.

5. Suppose that $\{f_j\}$ is a sequence of K_j-quasiconformal mappings of D into $\mathbb{S}^n$ so that:

   (a) f_j fix three distinct points $a_1, a_2, a_3 \in D$;

   (b) $K_j \leq K < \infty$.

   Then $\{f_j\}$ contains a subsequence which is uniformly convergent to a K-quasiconformal homeomorphism of D.

   Now we assume that $n = 2$, $D \subset \widehat{\mathbb{C}} = \mathbb{S}^2$.

6. An orientation-preserving homeomorphism $f : D \to D'$ is K-quasiconformal iff the following conditions are satisfied:

   - f is absolutely continuous on almost every coordinate line in D.
   - $|\overline{\partial} f| \leq k|\partial f|$ almost everywhere in D, where $k = \frac{K-1}{K+1} < 1$.

   The measurable function $\mu(z) = \overline{\partial} f/\partial f$ is called the *Beltrami differential* of f. *Orientation reversing* quasiconformal homeomorphisms $f : D \to D'$ are compositions of quasiconformal homeomorphisms and orientation reversing Moebius transformations.

7. Consider the *Beltrami equation*: $\mu(z) = \overline{\partial} f/\partial f$, where $\mu \in L_\infty(D)$ and $||\mu||_{L_\infty} \leq k < 1$. This equation has a solution f which is an orientation-preserving quasiconformal homeomorphism of D; the mapping f is uniquely determined up to postcomposition with conformal mappings.

8. We normalize solutions of the Beltrami equation with the Beltrami differential μ by requiring solutions f to fix three points. Then the normalized solutions f_μ of the Beltrami equation depend complex-analytically on μ.

9. Suppose that μ is a Beltrami differential in a domain $D \subset \widehat{\mathbb{C}}$, h is a *conformal* automorphism of D. Recall that h acts on μ by

$$h^*\mu(z) = \mu(hz)\overline{h'(z)}/h'(z).$$

If h is an *anticonformal* automorphism of D then we let

$$h^*\mu(w) = \bar{\mu}(h(w))\bar{\partial}h(w)/\overline{\bar{\partial}h(w)}.$$

Let f be a solution of the Beltrami equation with the Beltrami differential μ, let h be a conformal or anticonformal automorphism of D. Then f conjugates h to a conformal (resp. anticonformal) automorphism fhf^{-1} of $f(D)$ if and only if μ is h-invariant, i.e. $h^*\mu = \mu$ (see [SS92, page 169]).

Remark 5.1. There is a slight ambiguity in the definition of quasiconformal homeomorphisms in $\widehat{\mathbb{C}}$: usually they are required to be orientation-preserving. Most of the time we will deal with such mappings. The only exceptions are Theorems 1.44, 3.47 and 8.16.

Historic remark. The existence theorem for quasiconformal mappings with prescribed measurable Beltrami differential μ and analytic dependence of the solution on μ is absolutely critical for the Teichmüller theory. The existence theorem was first proven by Morrey [Mor38] in 1938 but his proof was forgotten for a long time, this theorem was reproven by Bers and Nirenberg [BN55] in 1954. Shortly afterwards yet another proof was found by Boyarskii [Boy57], who applied integral operators and the Calderon-Zygmund inequality. It turned out that these were precisely the tools needed to prove analytic dependence of solution on μ, that was done by Ahlfors and Bers in [AB60].

A family of mappings $f_n : \mathbb{S}^1 \to \mathbb{S}^1$ is said to be *uniformly quasimoebius* if each f_n is quasimoebius for the same constant K. Similarly to the convergence property of quasiconformal mappings one has the following convergence property of quasimoebius mappings $\mathbb{S}^1 \to \mathbb{S}^1$:

If $\{f_n\}$ is a uniformly quasimoebius family so that there are three distinct points $a, b, c \in \mathbb{S}^1$ each of which is fixed by f_n, then the sequence $\{f_n\}$ contains a subsequence which converges to a quasimoebius mapping.

Suppose that X, Y are Riemann surfaces. A homeomorphism $f : X \to Y$ is called *locally quasiconformal* if it is quasiconformal in the local conformal coordinates of X and Y. The coefficient of quasiconformality $K(f)$ is the supremum of coefficients of quasiconformality of the corresponding quasiconformal homeomorphisms $u_j \circ f \circ v_i^{-1}$ between domains in $\mathbb{C}$ (where u_j, v_i are local conformal coordinates on X, Y). If $K = K(f) < \infty$ then f is K-quasiconformal. Similarly one defines the *Beltrami differential* $\mu_f = \mu(z)\overline{dz}/dz$ of f. Thus $\|\mu_f\| = k$, where $K = \frac{k+1}{k-1}$. Recall that the differential $\mu(z)\overline{dz}/dz$ is a measurable section of the bundle $\bar{\kappa} \otimes \kappa^{-1}$, where $\kappa = \kappa_X$ is the canonical bundle of X.

5.3. Geometry of quadratic differentials

Suppose that $\widehat{S}$ is a compact Riemann surface. Let $P \subset \widehat{S}$ be a finite subset. We will assume that the surface $S = \widehat{S} - P$ has hyperbolic type. Let $\phi = \phi(z)dz^2$ be a nonzero (holomorphic) quadratic differential on S. We require ϕ to have at worst *simple poles* at the punctures $p \in P$. Let $C := C(\phi)$ denote the *critical set* of ϕ, i.e. the finite collection of points $z \in S$ where $\phi(z) = 0$. Choose a base-point $z_0 \in S \setminus C$. On the surface $S \setminus C$ we have a multivalued locally-univalent holomorphic function

$$f : S \setminus C \to \mathbb{C}, \quad f(z) = \int_{z_0}^{z} \sqrt{\phi(z)}dz.$$

We define a Riemannian metric on the surface $S \setminus C$ by the formula:

$$\rho(z)|dz| := |\phi(z)|^{1/2}|dz|.$$

The function $f(z)$ is a local isometry between this metric and the flat metric in $\mathbb{C}$. The metric $\rho(z)|dz|$ extends to a path-metric d_ρ on the whole surface S. Near each critical point $z_j \in C$ the total angle of d_ρ around z_j equals $\pi(k_j + 2) > 2\pi$, where k_j is the order of zero of $\phi(z)$ at z_j. Thus $\rho(z)|dz|$ is a Euclidean metric with conical singularities so that the total angle at each singular point is larger than 2π. If $P \neq \emptyset$ then the metric d_ρ on S is not complete (because ϕ has at worst simple poles at the punctures), however it has finite area. If $S = \widehat{S}$ is compact then the lift of d_ρ to the universal cover $\tilde{S}$ of S defines a $CAT(0)$-metric on $\tilde{S}$.

Besides the metric $\rho(z)|dz|$, the quadratic differential ϕ defines a pair of foliations of $S - C$ by *horizontal* and *vertical* geodesics of the metric $\rho(z)|dz|$. They are called *horizontal* and *vertical* foliations of ϕ. Namely, pull-back the family of horizontal (resp. vertical) lines from $\mathbb{C}$ to $S - C$ via the map f and call them *horizontal* (resp. *vertical*) *trajectories* of ϕ. Different branches of the map f define the same family of horizontal/vertical trajectories. Below we will modify this definition in the case of trajectories of ϕ which lead to critical points.

Near each critical point $z_j \in C$ the horizontal/vertical foliation becomes singular: if the order of z_j equals $k = k_j$, then there are $k + 2$ horizontal (and the same number of vertical) trajectories of ϕ emanating from z_j. Besides nonsingular horizontal (vertical) trajectories of ϕ we will also use the critical ones: those which contain critical points. Consider a metric graph R and a locally isometric map $q : R \to (S, d_\rho)$ whose image is contained in the union of intervals of horizontal trajectories of ϕ and critical points. If q is so that $q(R)$ is maximal (with respect to the inclusion) among such maps, then we call q a *critical horizontal trajectory* of ϕ. After being lifted to the universal cover $\tilde{S}$, the mapping q becomes a piecewise-smooth embedding of R into $\tilde{S}$ whose image is the union of geodesics with respect to the metric $\tilde{d}_\rho$. Similarly we define the critical vertical trajectories of ϕ.

Definition 5.2. Let $f : \mathbb{R} \to (S, \rho)$ be a locally-isometric map whose image is contained in a (horizontal) trajectory of ϕ. We assume that f can be approximated (uniformly on compacts) by nonsingular trajectories of ϕ.

Then the map f will be called a **line in the horizontal foliation** of ϕ. If L is the image of a lift of f to the universal cover $\tilde{S}$ then we will refer to L as a **line** as well.

Note that lines $L \subset \tilde{S}$ can be characterized as follows. Let $\tilde{\phi}$ denote the lift of ϕ to $\tilde{S}$. Let $\tilde{S}_+, \tilde{S}_-$ be the two connected components of $\tilde{S} - L$. Then either $\tilde{S}_+$ or $\tilde{S}_-$ is disjoint from the critical trajectory of $\tilde{\phi}$ which contains L. It is clear that if $f_n : \mathbb{R} \to L_n$ is a sequence of lines in $\tilde{S}$ which converge uniformly on compacts to a map f_∞ then f_∞ is again a line.

If S is compact then the group $F = \pi_1(S)$ acts on $(\tilde{S}, \tilde{d}_\rho)$ isometrically with compact quotient. Identify $\tilde{S}$ with $\mathbb{H}^2$. Thus $\tilde{d}_\rho$ is quasi-isometric to the hyperbolic metric (see §3.9), in particular, lifts of lines of ϕ to $\mathbb{H}^2$ are quasigeodesics of the hyperbolic metric. We conclude that there is a canonical homeomorphism between the ideal boundary $\partial_\infty \mathbb{H}^2$ (with respect to the hyperbolic metric) and $\partial_\infty \tilde{S}$ (with respect to the metric $\tilde{d}_\rho$).

5.4. Teichmüller spaces of Riemann surfaces

Suppose that S, S' are connected Riemann surfaces of finite type, $\psi : \pi_1(S) \to \pi_1(S')$ is an isomorphism. This isomorphism is said to *preserve the peripheral structure of these surfaces* if for any element $\gamma \in \pi_1(S)$ corresponding to a puncture of S, its image $\psi(\gamma)$ corresponds to a puncture of S'. If S, S' are oriented then we call an isomorphism $\psi : S \to S'$ *admissible* if the induced isomorphism $H_2(S, \partial S; \mathbb{Z}) \to H_2(S', \partial S'; \mathbb{Z})$ sends the fundamental class to the fundamental class and ψ preserves the peripheral structure. Suppose that $S = \mathbb{H}^2/F, S' = \mathbb{H}^2/F'$ are hyperbolic surfaces where $F, F' \subset PSL(2, \mathbb{R})$ are Fuchsian groups. An isomorphism $\psi : F \to F'$ corresponds to an admissible isomorphism of the fundamental groups provided that it is *type-preserving*, i.e.: it sends parabolic elements to parabolic, hyperbolic elements to hyperbolic and respects the orientation [2].

Exercise 5.3. *A conformal automorphism of a hyperbolic Riemann surface of finite area is homotopic to the identity iff it equals the identity.*

Theorem 5.4. *(Nielsen Realization Theorem.) Any admissible isomorphism*

$$\psi : \pi_1(S) \to \pi_1(S')$$

is induced by an orientation-preserving quasiconformal homeomorphism $h : S \to S'$.

More generally:

Theorem 5.5. *Suppose that $F, F' \subset Isom(\mathbb{H}^2)$ are Fuchsian groups (they could have torsion), such that $\mathbb{H}^2/F, \mathbb{H}^2/F'$ have finite area and $\psi : F \to F'$ is a type-preserving isomorphism. Then there exists a ψ-equivariant quasiconformal homeomorphism $f : \mathbb{H}^2 \to \mathbb{H}^2$.*

[2] ψ is induced by a map $S \to S'$ which maps relative fundamental class to S to the relative fundamental class of S'.

Nielsen's theorem establishes existence of a quasiconformal homeomorphism in the given homotopy class; a theorem of Epstein below establishes the uniqueness of such homeomorphism up to isotopy. Epstein proved his theorem in the PL context, his result was generalized by Earle and McMullen in [EM88].

Theorem 5.6. *(D.B.A. Epstein, [Eps66]) Suppose that S_1, S_2 are hyperbolic Riemann surfaces of finite type and $f_0, f_1 : S_1 \to S_2$ are homotopic quasiconformal homeomorphisms. Then f_0 is isotopic to f_1 (through quasiconformal homeomorphisms).*

We will prove theorems 5.5 and 5.6 (and a generalization of the latter) in §8.4.

Fix once and for all a Riemann surface S of finite type, which has genus g and n punctures. Suppose that Y, Z are Riemann surfaces. Two quasiconformal homeomorphisms $f : S \to Y, g : S \to Z$ are *equivalent* if $f \circ g^{-1}$ is homotopic to a conformal map between Z and Y. The equivalence class is called a *marked Riemann surface* and denoted $[Y, f]$. The space of marked Riemann surfaces $[Y, f]$ is the *Teichmüller space* $\mathcal{T}(S)$ of the surface S. From now on we will consider Riemann surfaces of hyperbolic type (i.e. whose universal covers are conformal to the unit disc). If S has hyperbolic type, then there is the following equivalent description of the Teichmuller space $\mathcal{T}(S)$. Let $S = \mathbb{H}^2/F$ where F is a Fuchsian subgroup of $PSL(2,\mathbb{R})$. Consider the space $Hom_a(F, PSL(2,\mathbb{R}))$ of *admissible isomorphisms* between F and Fuchsian subgroups of $PSL(2,\mathbb{R})$. Take the quotient

$$\mathcal{T}_R(F) = Hom_a(F, PSL(2,\mathbb{R}))/PSL(2,\mathbb{R}) \subset \mathcal{R}_{par}(F, \{H_i\}; PSL(2,\mathbb{R}))$$

where H_i, $i = 1, ..., n$, are the cyclic subgroups of $F = \pi_1(S)$ generated by small simple loops around the punctures of S. We will refer to $\mathcal{T}_R(F)$ as the *Teichmüller space* of the Fuchsian group F.

Remark 5.7. L. Keen proved in [Kee71, Kee73] that there exists a collection of $6g - 6 + 2n$ elements $\gamma_j \in \Gamma$ (represented by simple loops in S) so that a point $[\rho] \in \mathcal{T}_R(\Gamma)$ is uniquely determined by the vector

$$(\ell(\rho(\gamma_1)), ..., \ell(\rho(\gamma_{6g-6+2n}))).$$

Recall that $\ell(h)$ is the translation length of an isometry $h : \mathbb{H}^2 \to \mathbb{H}^2$.

Nielsen's theorem provides a natural bijection between $\mathcal{T}(S)$ and $\mathcal{T}_R(F)$. In this approach $\mathcal{T}(S)$ is the collection of *marked* hyperbolic surfaces of finite area which are homeomorphic to S (the *marking* is given by a choice of an *admissible* isomorphism between the fundamental groups).

What today is known as the *Teichmüller space* was actually introduced by Fricke and Klein back in 1897–1912 [FK], although without reference to quasiconformal mappings (which were not yet invented). It also (citing Ahlfors) "requires an incredibly patient reader" to recognize $\mathcal{T}_R(F)$ in the 2-volume book [FK].

Teichmüller metric on $\mathcal{T}(S)$. If $p = [X, f_0], q = [Y, h_0] \in \mathcal{T}(S)$ then the *Teichmüller distance* between p and q is given by

$$d_{\mathcal{T}(S)}(p, q) = inf\{\frac{1}{2}\log K(f \circ h^{-1}) : (X, f) \in p, (Y, h) \in q\}$$

where $K(\tau)$ is the coefficient of quasiconformality of the quasiconformal homeomorphism τ between Riemann surfaces. The function $d_{\mathcal{T}(S)}$ is a complete metric in $\mathcal{T}(S)$. Any pair of distinct points is connected by a unique geodesic and $\mathcal{T}(S)$ is geodesically complete. However $d_{\mathcal{T}(S)}$ is not a Riemannian metric.

In what follows it will be important to distinguish surfaces with opposite orientations. If S is an oriented surface then $\bar{S}$ will denote the same surface with the opposite orientation. Each complex structure on S determines an orientation, thus if S is a Riemann surface, then $\bar{S}$ has canonical complex structure obtained by composing all the (local) coordinate maps $S \to \mathbb{C}$ with the complex conjugation in $\mathbb{C}$. Note that the canonical map $\mathcal{T}(S) \to \mathcal{T}(\overline{S})$ (which reverses orientation on each marked Riemann surface) is an isometry of Teichmüller spaces.

The infimum in the definition of Teichmüller distance $d_{\mathcal{T}(S)}(p, q)$ is always achieved. The corresponding *extremal* quasiconformal homeomorphism $\tau = f \circ h^{-1}$ is unique. Its Beltrami differential $\mu = \mu_\tau$ is described as follows (provided $p \neq q$). There exists a holomorphic quadratic differential φ on the surface Y (which is unique up to a positive scalar multiple), so that

$$\mu = t\bar{\varphi}/|\varphi| \tag{5.2}$$

where $t > 0$ is a constant. Quasiconformal homeomorphisms $\tau : Y \to X$ whose Beltrami differentials μ_τ are of the form (5.2) are called *Teichmüller maps*. The quadratic differential φ (which is defined up to a positive real multiple) is called the *initial* quadratic differential for τ. There is a unique (up to a positive real multiple) quadratic differential ψ on X so that:

- The mapping τ is affine (away from singular points) with respect to the Euclidean structures with conical singularities corresponding to φ and ψ.

- The mapping τ sends the horizontal foliation of φ to the horizontal foliation of ψ.

The quadratic differential ψ is called *the terminal quadratic differential of* τ.

The extremal quasiconformal homeomorphism τ is real-analytic on Y everywhere except at a finite collection of points which is always nonempty provided that Y is compact and τ is not conformal: these singular points are zeroes of φ. The mapping τ is not differentiable at these points.

Remark 5.8. If Y is not compact then τ can be real-analytic and nonconformal everywhere on Y. For instance, let T be a flat 2-torus, $A : T^2 \to$

T^2 an affine mapping which is not an isometry. Then for any $x \in T^2$ the restriction

$$A : T^2 - \{x\} \to T^2 - \{A(x)\}$$

is a Teichmüller mapping of $Y = T^2 - \{x\}$ which is not conformal.

Sometimes it will be important to think of $\mathcal{T}(S) = \mathcal{T}_R(F)$ as the quotient of $Belt(S) \cong Belt(\mathbb{H}^2, F)$. Namely, let $\mu \in Belt(\mathbb{H}^2, F)$ and $f_\mu = f : \mathbb{H}^2 \to \mathbb{H}^2$ be a homeomorphism which is a solution of the Beltrami equation $\overline{\partial} f = \mu \partial f$. Then f (which is unique up to postcomposition with a conformal automorphism of $\mathbb{H}^2$) is ρ-equivariant with respect to a monomomorphism $\rho : F \to PSL(2, \mathbb{R})$ whose image is a discrete subgroup $\rho(F) \subset PSL(2, \mathbb{R})$. The map $\mu \mapsto [\rho]$ determines a projection $\Phi : Belt(S) \to \mathcal{T}(S)$. Both $Belt(S)$ and $\mathcal{T}(S)$ are complex manifolds (although $Belt(S)$ is infinite-dimensional). The fact that normalized solution of the Beltrami equation depends smoothly on μ implies that the mapping Φ is smooth. It is easy to see that Φ is a submersion.

Actually, the space $\mathcal{T}(S)$ has a natural complex structure so that $\Phi : Belt(S) \to \mathcal{T}(S)$ is a holomorphic mapping; $\mathcal{T}(S)$ is biholomorphic to an open contractible domain in $\mathbb{C}^{3g-3+n}$. It is unknown if $\mathcal{T}(S)$ is *holomorphically contractible*, i.e. if there is a continuous family of holomorphic maps $f_t : \mathcal{T}(S) \to \mathcal{T}(S)$ so that $f_0 = id$, $f_1 = const$.

We now describe the tangent and cotangent spaces of $\mathcal{T}(S)$ at a point $[X, f]$. We first give a deformation-theoretic definition assuming for simplicity that X is a compact surface. Suppose that X_t is a smooth family of Riemann surfaces of finite type such that $X_0 = X$. More precisely, if $\{(U_\alpha, h_\alpha)\}$ is a finite conformal atlas on X then $\{(U_\alpha, h_{\alpha,t})\}$ is a conformal atlas on X_t, where each $h_{\alpha,t} : U_\alpha \to \mathbb{C}$ depends smoothly on the pair of variables (z, t), $z \in U_\alpha, t \in [-1, 1]$. The conformal atlas $\{(U_\alpha, h_{\alpha,t})\}$ corresponds to a collection of holomorphic transition mappings $g_{\alpha\beta,t}$ which depend smoothly on t. We thus differentiate these mappings at $t = 0$ and get a collection of holomorphic vector fields $\xi_{\alpha\beta}$ defined on $U_\alpha \cap U_\beta$. One then verifies that this collection is a cocycle $\xi \in Z^1(X, \Theta_X)$. The coboundaries $\xi \in B^1(X, \Theta_X)$ correspond to *trivial* deformations of the conformal structure on X, for such deformations there exists a smooth family of holomorphic maps $F_t : X \to X_t$, $F_0 = id$. Thus the tangent space $T_X(\mathcal{T}(S))$ is identified with the 1-st cohomology group $H^1(X, \Theta_X)$. The dual to this space is Q_X, the space of quadratic differentials, which is the cotangent bundle to $\mathcal{T}(S)$ at X.

Alternatively one identifies Q_X with the cotangent space to $\mathcal{T}(S)$ as follows. It suffices to treat the case $X = S$. Consider the projection $\Phi : Belt(S) \to \mathcal{T}(S)$. One then asks which $\nu \in L_\infty(S)$ belong to the kernel of the derivative of Φ at S:

$$\dot{f}[\nu] := \frac{d}{dt} f_{t\nu}|_{t=0} = d\Phi[\nu]$$

where $f_{t\nu} : \mathbb{H}^2 \to \mathbb{H}^2$ is the normalized solution of the Beltrami equation on $\mathbb{H}^2$. Then differentiating the Beltrami equation

$$\bar{\partial} f_{t\nu} = t\nu \partial f_{t\nu}$$

with respect to t at $t = 0$ we get:

$$\bar{\partial}\dot{f}[\nu] = \nu.$$

The condition that the deformation $f_{t\nu}$ of the conformal structure on S is infinitesimally trivial is that $\dot{f}[\nu]\partial_z$ descends to a vector field on S (for which we retain the notation $\dot{f}[\nu]\partial_z$). In particular, for each $\varphi(z)dz^2 \in Q_S$,

$$(\dot{f}[\nu]\partial_z)(\varphi(z)dz^2) = \dot{f}[\nu]\varphi(z)dz$$

is a $(1,0)$-differential on S whose exterior derivative is

$$d(\dot{f}[\nu]\varphi(z)dz) = \partial_{\bar{z}}\dot{f}[\nu]\varphi(z)\overline{dz}dz = \nu\varphi(z)\overline{dz}dz.$$

Therefore, by Stokes' theorem,

$$0 = \int_S \nu\varphi(z)dz\overline{dz} \ \text{ for each } \ \varphi(z)dz^2 \in Q_S. \tag{5.3}$$

It turns out that the condition (5.3) is necessary and sufficient for $\nu \in Ker(d\Phi)_S$ (Teichmüller Lemma). Since the pairing (5.1) is nondegenerate, it follows that Q_S is dual to the cokernel of $(d\Phi)_S$, which is the tangent space $T_S(\mathcal{T}(S))$.

A section $Q_S \to L^1_{-1,1}(S)$ to the mapping $d\Phi$ is given by

$$\varphi dz^2 \mapsto \frac{\bar{\varphi}}{\rho}\overline{dz}/dz$$

where $\rho(z)dz^2$ is the hyperbolic metric on S. The differentials of the form $\frac{\varphi}{\rho}\overline{dz}/dz$ are called *harmonic* $(-1,1)$*-differentials* on S. They are harmonic Dolbeault representatives of the cohomology classes in $H^1(S, \Theta_S)$. See [Sch90b].

The *(Teichmüller) modular group* (the other name is the *mapping class group*) Mod_S is the quotient

$$Homeo_+(S)/Homeo_0(S)$$

where $Homeo_+(S)$ is the group of orientation preserving self-homeomorphisms of S and $Homeo_0(S)$ is the subgroup of homeomorphisms homotopic to id_S. The group Mod_S acts on $\mathcal{T}(S)$ by the precomposition $Mod_S \ni g : [X, f] \mapsto [X, f \circ g]$. This action is faithful except in the following few cases:

(a) S has genus 1 and one puncture, (b) S has genus 2 and no punctures, (c) S has genus zero and four punctures, (d) S has genus zero and three punctures.

In the first three cases the kernel of the action is the center of Mod_S (isomorphic to $\mathbb{Z}/2$), in the last case Mod_S is the permutation group on three symbols and $T(S)$ is a single point.

The group Mod_S is isomorphic to

$$Out_a(\pi_1(S)) = Aut_a(\pi_1(S))/Inn(\pi_1(S))$$

where $Aut_a(\pi_1(S))$ consists of *admissible* automorphisms. It is clear that Mod_S acts isometrically on $\mathcal{T}(S)$. Each orientation-reversing homeomorphism $f : S \to S$ induces an isometry $\mathcal{T}(S) \to \mathcal{T}(\bar{S})$ (note that $f : S \to \bar{S}$ preserves the orientation).

The group Mod_S acts on $\mathcal{T}(S)$ discretely, isometrically and biholomorphically. This action is not free, there is a subgroup of finite index $\Gamma \subset Mod_S$ which acts freely (see §5.6). The quotient $\mathcal{M}(S) = \mathcal{T}(S)/Mod_S$ is called the *moduli space* of complex structures on S. It can be viewed either as the space of *unmarked* complex structures of finite type on S or *unmarked hyperbolic structures of finite area* on S. Unless $\mathcal{T}(S)$ has complex dimension ≤ 1, the quotient of Mod_S by its center is the full group of isometries and the full group of holomorphic automorphisms of $\mathcal{T}(S)$, see [Roy71].

The moduli space $\mathcal{M}(S)$ is non-compact. Given $\epsilon > 0$, consider the subset $\mathcal{M}(S)_\epsilon$ which consists of hyperbolic surfaces where the length of each homotopically nontrivial nonperipheral loop is at least ϵ.

Theorem 5.9. *(Mumford Compactness Theorem.) The space $\mathcal{M}(S)_\epsilon$ is compact.*

See [Mum71] for algebro-geometric discussion and [Abi80] for the geometric proof.

5.5. The Weil-Petersson metric on $\mathcal{T}(S)$.

Besides the Teichmüller metric, the space $\mathcal{T}(S)$ has another useful metric d_{WP}, which is called the *Weil-Petersson metric*. Recall that the Weil-Petersson pairing $\langle \cdot, \cdot \rangle_{WP}$ on Q_S determines a metric on the cotangent space $Q_S \cong T^*_S(\mathcal{T}(S))$ and therefore a metric on $T_S(\mathcal{T}(S))$. One then verifies that this metric is infinitely differentiable and hence defines a distance function d_{WP} on the Teichmüller space. The following are the basic properties of this metric:

- Weil-Petersson metric is Kähler;
- d_{WP} is incomplete;
- the manifold $(\mathcal{T}(S), d_{WP})$ has negative sectional curvature (which is not bounded away from 0 and $-\infty$);
- $(\mathcal{T}(S), d_{WP})$ admits an exhaustion by compact convex subsets;
- the modular group Mod_S acts isometrically on $(\mathcal{T}(S), d_{WP})$.

See [Mas76], [Wol75], [Wol86], [Wol87], [Tro92], for definitions and proofs. It is unknown if $\mathcal{T}(S)$ admits a $CAT(0)$ metric which is Mod_S-invariant, one can show however that most mapping class groups Mod_S do not admit isometric properly discontinuous cocompact actions on $CAT(0)$-spaces, [KL96].

5.6. Torsion in Mod_S

The following is the so called *Nielsen Realization Problem* which was was solved by S.Kerkchoff in [Ker83], different solutions were found later in [Wol87], [Tro92] and in the combination of [Gab92], [Tuk88], [CJ94].

Theorem 5.10. *Suppose that S is a complete hyperbolic surface of finite area. Then any finite subgroup $F \subset Mod_S$ has a fixed-point $[X, h] \in \mathcal{T}(S)$. This means that F can be realized as a group of conformal automorphisms of a Riemann surface X quasiconformally homeomorphic to S.*

Sketch of the proof: I will outline a proof following [Wol87]. The difficult part of the proof is to establish the properties of Weil-Petersson metric on $\mathcal{T}(S)$, once it is done the proof is finished as follows. Take a point $x \in \mathcal{T}(S)$ and consider its F-orbit $Fx = \{x_1, ..., x_n\}$. Since $(\mathcal{T}(S), d_{WP})$ admits an exhaustion by compact convex subsets, the convex hull of Fx is a compact convex subset $Q \subset \mathcal{T}(S)$. Clearly Q is a $CAT(0)$-space and F acts on Q isometrically. Thus, according to Proposition 3.4, F fixes a point in Q. $\square$

The following corollary will be used in §19.5.

Corollary 5.11. *Let $S_1, ..., S_k$ be Riemann surfaces of finite (hyperbolic) type, $\mathcal{T} = \mathcal{T}(S_1) \times ... \times \mathcal{T}(S_k)$ be the product of their Teichmüller spaces. We give $\mathcal{T}$ the Riemannian metric of direct product of Weil-Petersson metrics on the factors $\mathcal{T}(S_j)$. Then any finite group of isometries $F \subset Isom(\mathcal{T})$ fixes a point in $\mathcal{T}$.*

Proof: Each space $(\mathcal{T}(S_j), d_{WP})$ is a $CAT(0)$-space. Products of compact convex subsets in $(\mathcal{T}(S_j), d_{WP})$ exhaust $\mathcal{T}$ and we use the fact that the direct product of $CAT(0)$-spaces is again a $CAT(0)$-space. $\square$

In the discussion below we will need the **Lefschetz formula:**

If S is a compact Riemann surface and $f : S \to S$ is a finite order diffeomorphism with isolated fixed points and p is a prime so that $3 \le p \le \infty$ then

$$\sum_{n\ge 0}(-1)^n Tr(f_* : H_n(S, \mathbb{Z}_p) \to H_n(S, \mathbb{Z}_p)) = \sum_{x\in Fix_S(f)} sign(J_x(f)) \pmod p$$

where $Fix(f) = Fix_S(f)$ is the fixed-point set of f and J_x is the (real) Jacobian determinant of f at the fixed point x. If $p = \infty$ then we let $\mathbb{Z}_p = \mathbb{Z}$. Note that if f is orientation-preserving and different from the identity then the $Fix(f)$ is a finite set and $sign(J_x(f)) = 1$. In particular, $|Fix(f)| \le 2g + 2$, where $|Fix(f)|$ is the cardinality of $Fix(f)$. If f is orientation-reversing then $Fix(f)$ is either empty or is 1-dimensional.

Exercise 5.12. *Prove Lefschetz formula using f-invariant regular cell-complex structure on S, where each fixed point of f is contained in a 2-cell. Hint: if $C_*(S, \mathbb{Z}_p)$ is the associated chain-complex show that*

$$\sum_{n\ge 0}(-1)^n Tr(f_* : H_n(S, \mathbb{Z}_p) \to H_n(S, \mathbb{Z}_p)) =$$

$$\sum_{n \geq 0} (-1)^n Tr(f_* : C_n(S, \mathbb{Z}_p) \to C_n(S, \mathbb{Z}_p)) = |Fix(f)| \pmod{p}.$$

Lemma 5.13. *(A. Hurwitz.) Let S be a hyperbolic Riemann surface of finite (hyperbolic) type, suppose that $f : S \to S$ is a periodic diffeomorphism which acts trivially on $H_1(S, \mathbb{Z}_p)$ for any [3] prime $p \geq 2|\chi(S)|$. Then $f = id$.*

Proof: It suffices to consider the case of compact surfaces, the general case is proven by taking f-invariant compactification of S (add circles to each puncture) and then doubling the resulting surface. Let ds^2 be a Riemannian metric on S, let $\langle f \rangle$ denote the finite group generated by f; then the average $A(ds^2)$ of ds^2 over $\langle f \rangle$ is a Riemannian metric invariant under f. Thus f is either holomorphic or antiholomorphic with respect to the complex structure determined by $A(ds^2)$. Poincaré duality implies that f acts trivially on $H_2(S, \mathbb{Z}_p)$, thus f is necessarily holomorphic. Assume that $f \neq id$, then its fixed-point set in S is finite. Take a prime $p \geq 4g + 4$ where g is the genus of S. We will identify $\mathbb{Z}_p$ with $\mathbb{Z} \cap [-p/2, p/2]$. Then the Lefschetz formula for f implies that

$$0 > 2 - 2g = |Fix(f)|, \quad |Fix(f)| \leq 2g + 2.$$

Contradiction. □

Recall that a group G is called *locally finite* if any of its finitely generated subgroups is finite. A group G is called a *torsion group* if every element of G has finite order. Note that each locally finite group is a torsion group. Let $G := \mathbb{Q}/\mathbb{Z}$, where we consider both $\mathbb{Q}$ and $\mathbb{Z}$ as additive groups. Then the group G is locally finite. For a long time it was unknown if there are finitely generated infinite torsion groups (this is a version of the *Burnside Problem*), the first such example was constructed in [Gol64]. It is unknown if infinite torsion groups can be finitely presentable.

Corollary 5.14. *(See [ECG87].) The mapping class group contains a torsion-free subgroup of finite index. Any torsion subgroup of Mod_S is finite.*

Proof: Consider the natural homomorphism

$$\tau : Mod_S \to Aut(H_1(S, \mathbb{Z}_p))$$

where p is a sufficiently large prime. Hurwitz' lemma implies that the kernel of τ is torsion-free. The group $Aut(H_1(S, \mathbb{Z}_p))$ is finite, hence $ker(\tau)$ is a torsion-free subgroup of finite index in Mod_S. If G is a torsion group in Mod_S then $\tau : G \to Aut(H_1(S, \mathbb{Z}_p))$ is a monomorphism. Hence $G \cong \tau(G) \subset Aut(H_1(S, \mathbb{Z}_p))$ is finite. □

[3] Actually it suffices to take $p = 3$, see [ECG87] and [Iva92].

Chapter 6

Introduction to the Orbifold Theory

6.1. Definitions and examples

The notion of *orbifold* is a generalization of the notion of a *manifold* which appears naturally in the context of properly discontinuous non-free actions of groups on manifolds. Orbifolds were first invented by Satake [Sat57] in 1950-s under the name of V-manifolds, they were reinvented under the name of orbifolds by Thurston in 1970's as a technical tool for proving the Hyperbolization Theorem. Our discussion of orbifolds mainly follows [Thu81, Chapter 13] and [Sco83a]. For more detailed discussion of foundational issues of the orbifold theory we refer the reader to [Rat94, Chapter 13].

Note that many definitions that we take for granted in the manifold theory are rather mysterious in the category of orbifolds. For instance, it is unclear what is the correct definition of characteristic classes of orbifolds (compare [Kaw78] and [HH90]). Even if one can define the Euler characteristic of an orbifold it is unclear what the Betti numbers are, we will define the fundamental group of an orbifold but it is far from obvious what is the correct definition of higher homotopy groups, etc. All in all, it seems that the orbifolds live in the grey area between the "good old manifold kingdom" and the dark world populated by the "wild creatures" of the Alain Connes' *noncommutative geometry.*

Before giving a formal definition we start with the basic examples of orbifolds. Suppose that M is a smooth manifold and G is a discrete group acting smoothly, effectively and properly discontinuously on M. Then the quotient $O = M/G$ is an *orbifold*, such orbifolds are called *good*. The quotient M/G, considered as a topological space X_O, is the *underlying set* of this orbifold. If S is a set of points in M where the action of G is not free, then its projection $\Sigma = S/G$ is the *singular locus* of the orbifold O.

To be more concrete consider 2-dimensional orbifolds. Suppose that $M = \mathbb{H}^2$ and G is a discrete subgroup of $PSL(2, \mathbb{R})$. Then the quotient

$O = \mathbb{H}^2/G$ is a Riemann surface X_O with a discrete collection of *cone points* z_j which form the singular locus Σ of the orbifold O. The projection $p : \mathbb{H}^2 \to O$ is the *universal cover* of the orbifold O. The Riemann surface X_O has a natural hyperbolic metric which is singular in the discrete set Σ. Metrically the points z_j are characterized by the property that the total angles around these points are $2\pi/n_j$. The numbers n_j are the orders of cyclic subgroups G_{z_j} of G which stabilize the points in $p^{-1}(z_j)$, they are called *the local isotropy groups.* The projection p is a *ramified covering* from the point of view of Riemann surfaces. From the point of view of orbifolds this is a *covering.* Thus the singular locus of the orbifold O consists of the points z_j in Σ equipped with the extra data: the $PSL(2,\mathbb{R})$-conjugacy classes of the local isotropy groups G_{z_j} (of course each local isotropy group G_{z_j} is determined by the number n_j).

More complicated examples appear when $G \subset \mathrm{Isom}(\mathbb{H}^2)$ does not preserve orientation: such groups can contain isometric reflections. Then near each point of the quotient $\mathbb{H}^2/G$ the orbifold O can look like a piece of closed half-plane in $\mathbb{H}^2$. The local structure of the orbifold near such points is called the *boundary reflector.* More generally, reflections $R, L \in G$ can have a common fixed point $z \in \mathbb{H}^2$. In this case the group $\langle R, L\rangle$ is a finite dihedral group. The quotient of $\mathbb{H}^2$ near the projection of such point z looks like a corner with geodesic sides in $\mathbb{H}^2$. The angle at this corner equals π/n, where n is the order of the element $R \cdot L$. Another way to think about this local picture is to take a small metric disk $D_r(z)$ with the center at z. Then the group $\langle R, L\rangle$ acting on D has a fundamental domain which is a corner with the vertex z and edges fixed by the elements R and L. The local structure of the orbifold near the point z is the *corner reflector.* See Figure 6.1.

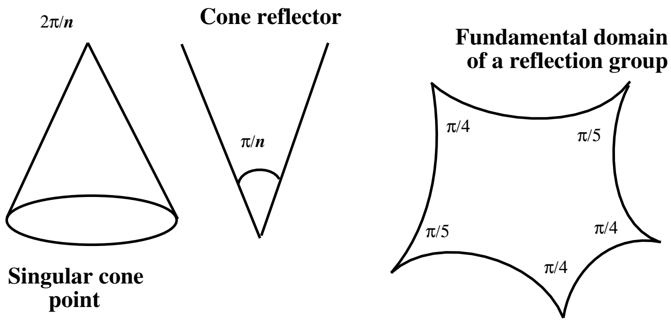

Figure 6.1: *2-dimensional orbifolds.*

Let X be either $\mathbb{H}^2$ or $\mathbb{E}^2$ or $\mathbb{S}^2$, suppose that G is a discrete subgroup of $\mathrm{Isom}(X)$ generated by reflections with the fundamental domain D bounded by geodesics fixed by reflections. Then the orbifold $O = X/G$ is naturally identified with the domain D; the orbifold data is given by the metric structure of O, namely by the angles at the corners. Such orbifold is a *good 2-dimensional reflection orbifold.*

Now we can discuss the general definition. A (smooth) n-dimensional orbifold O is a pair: a Hausdorff paracompact topological space X (which is called the *underlying space* of O and is denoted X_O or $|O|$) and a special atlas A on X. The atlas A consists of:

- A collection of open sets $U_i \subset X$, which is closed under taking finite intersections, so that $X = \bigcup_i U_i$.
- A collection of open sets $\tilde{U}_i \subset \mathbb{R}^n$.
- A collection of finite groups of diffeomorphisms Γ_j acting on $\tilde{U}_i$.
- A collection of homeomorphisms

$$\phi_i : U_i \to \tilde{U}_i/\Gamma_i.$$

We require the atlas A to behave well under inclusions. Namely, suppose that $U_i \subset U_j$, then there is a smooth embedding

$$\tilde{\phi}_{ij} : \tilde{U}_i \to \tilde{U}_j$$

and a monomorphism $f_{ij} : G_i \to G_j$ so that $\tilde{\phi}_{ij}$ is f_{ij}-equivariant. We will call U_j *coordinate* neighborhoods of the points $x \in U_j$ and $\tilde{U}_j$ their *covering coordinate* neighborhoods.

Similarly to orbifolds we define the class of *orbifolds with boundary*; just instead of *open* sets $\tilde{U}_j \subset \mathbb{R}^n$ we use open subsets in

$$\mathbb{R}^n_+ \cup \mathbb{R}^{n-1} = \{(x_1, ..., x_n) : x_n \geq 0\}.$$

The *boundary* of such orbifold consists of points $x \in |O|$ which correspond to $\mathbb{R}^{n-1}$ under the identification $U_i \cong \tilde{U}_i/G_i$. As in the case of manifolds, the boundary of each orbifold is an orbifold without boundary. By abusing notation we will call an *orbifold with boundary* simply an *orbifold.* A compact orbifold without boundary is called *closed.*

To each point $x \in X$ we associate a germ of action of a finite group of diffeomorphisms G_x at a fixed point $\tilde{x}$. If $\phi_j(x)$ is covered by a point $\tilde{x}_j \in \tilde{U}_j$, then we have the isotropy group $G_{j,x}$ of $\tilde{x}_j$ in G_j. Note that if $U_i \subset U_j$, then the map $\tilde{\phi}_{ij} : \tilde{U}_i \to \tilde{U}_j$ induces an isomorphism from the germ of the action of $G_{j,x}$ at $\tilde{x}_j$ to the germ of the action of $G_{i,x}$ at $\tilde{x}_i$. Thus we let the germ $(G_x, \tilde{x})$ be the equivariant diffeomorphism class of the germ $(G_{j,x}, \tilde{x}_j)$. The group G_x is called the *local isotropy group* of O at x. The set of points x with nontrivial local isotropy group is called *the singular locus* of O and is denoted by Σ_O. Note that the singular locus is nowhere dense in X_O. An orbifold with empty singular locus is called *nonsingular* or a *manifold.*

Since G_x acts smoothly near the fixed point $\tilde{x}$, the germ $(G_x, \tilde{x})$ is linearizable: Give a neighborhood of $\tilde{x}$ a G_x-invariant Riemannian metric, then the exponential map (with the origin at $\tilde{x}$) conjugates the orthogonal action of G_x on the tangent space $T_{\tilde{x}}\mathbb{R}^n$ to the germ of the action of G_x at $\tilde{x}$. Thus we can talk about *reflections* in the group G_x (they correspond to the reflections in $O(n)$ under the above linearization), and about the *order of rotation* for an element $g \in G_x$ which is not a reflection.

Definition 6.1. A **Riemannian metric** ρ on orbifold O is the usual Riemannian metric on $X_O - \Sigma_O$, so that after we lift ρ to the local covering coordinate neighborhoods $\tilde{U}_i$, it extends to a G_i-invariant Riemannian metric on the whole $\tilde{U}_i$.

Exercise 6.2. *Each orbifold O admits a Riemannian metric. Hint: use the partition of unity argument similar to the manifold case.*

A *homeomorphism* (resp. *diffeomorphism*) between orbifolds O, R is a homeomorphism $h : |O| \to |R|$ so that for each point $x \in O, y = h(x) \in R$ there are coordinate neighborhoods $U_x \cong \tilde{U}_x/G_x, V_y \cong \tilde{V}_y/G_y$ so that h lifts to an equivariant homeomorphism (resp. diffeomorphism)

$$\tilde{h}_{xy} : \tilde{U}_x \to \tilde{V}_y.$$

Note that to describe a smooth orbifold O up to homeomorphism it suffices to describe the topology of the pair (X_O, Σ_O) and the homeomorphic equivalence classes of the germs $(G_x, \tilde{x})$ for the points $x \in \Sigma_O$.

Example 6.3. Let O be a connected compact 1-dimensional orbifold without boundary which is not a manifold. Then O is homeomorphic to the closed interval $[ab]$ where $(G_a, \tilde{a}), (G_b, \tilde{b})$ are the germs $(\mathbb{Z}_2, 0)$ of the reflection group $\mathbb{Z}_2$ acting isometrically on $\mathbb{R}$ near its fixed point $0 \in \mathbb{R}$.

A *smooth* map between orbifolds O and R is a continuous map

$$g : O \to R$$

which can be (locally) lifted to a map of pairs of coordinate covering neighborhoods

$$\tilde{g}_{ij} : \tilde{U}_j \to \tilde{V}_i$$

as a smooth equivariant map which induces isomorphisms of local isotropy groups $G_x \to G_y, x \in O, y = g(x) \in R$. (Unfortunately there is no standard definition of morphism between orbifolds in the literature, so we have chosen one which is convenient for our purposes.) Similarly we define *immersions* and *submersions* between orbifolds as smooth maps between orbifolds which locally lift to immersions and submersions respectively.

Example 6.4. Let O be the 2-dimensional reflection orbifold so that X_O is the closed unit disk Δ^2, Σ_O is the boundary circle of O and O has the local structure of boundary reflector near each singular point. Let $D \subset \Delta^2$ be a segment connecting two boundary points. Give D the orbifold structure as in the Example 6.3. Then the inclusion $D \hookrightarrow O$ is an immersion. Let X_D be the manifold with boundary underlying D, treat X_D as orbifold with boundary with empty singular locus. Then the inclusion $\iota : X_D \to O$ is not a smooth map. Let R be the orbifold whose underlying topological space X_O is the unit square in $\mathbb{R}^2$, the vertices of this square are corner reflectors, the interiors of the edges are boundary reflectors, Σ_R is the boundary of the square. Let $j : D \hookrightarrow R$ be the inclusion of the above orbifold D as a straight-line segment connecting the opposite vertices of the square. Then j is not a smooth map.

A *suborbifold* $R \subset O$ is an orbifold R together with an *embedding* (i.e. an *injective immersion*) $\iota : R \hookrightarrow O$. We say that a suborbifold R is *properly embedded* in the orbifold O if the embedding $\iota : R \hookrightarrow O$ sends ∂R to ∂O, $\iota^{-1}(\partial O) \subset \partial R$ and written in local covering coordinates the map ι is transversal to ∂O.

Suppose that O', O are orbifolds and $p : |O'| \to |O|$ is a continuous map. The map p is called a *covering map* between the orbifolds O', O if the following property is satisfied:

For each point $x \in |O|$ there exists a chart $U = \tilde{U}/G_x$ so that for every component V_i of $p^{-1}(U)$ the restriction map $p : V_i \to U$ is a quotient map of an equivariant diffeomorphism $h_i : \tilde{V}_i \to \tilde{U}$ (if $y_i = p^{-1}(x) \cap V_i$ then h_i conjugates the action of G_{y_i} on $\tilde{V}_i$ to the action of a subgroup of G_x on $\tilde{U}$).

From now on we will assume that the orbifolds under consideration are connected.

The *universal covering* $p : \tilde{O} \to O$ of the orbifold O is the *initial object* in the category of orbifold coverings, i.e. it is a covering so that for any other covering $p' : O' \to O$ there exists a covering $\tilde{p} : \tilde{O} \to O'$ such that $p' \circ \tilde{p} = p$. If $p : \tilde{O} \to O$ is the universal covering then the orbifold $\tilde{O}$ is called the space *universal covering space* of O (by abusing notation we will also refer to $\tilde{O}$ as the universal covering).

The group $Deck(p)$ of *deck transformations* of the orbifold covering $p : O' \to O$ is the group of self-diffeomorphisms $h : O' \to O'$ such that $p \circ h = p$. A covering $p : O' \to O$ is called *regular* if $O'/Deck(p) = O$.

The *fundamental group* $\pi_1(O)$ of the orbifold O is the group of deck transformations of its universal covering. Then $O = \tilde{O}/\pi_1(O)$. An alternative definition of the fundamental group based on homotopy-classes of loops in O see in [Rat94, Chapter 13].

The following theorem was proven by Thurston in [Thu81, Chapter 13].

Theorem 6.5. *Each orbifold has a universal covering.*

Definition 6.6. An orbifold O is called **good** if its universal cover is a manifold. Orbifolds which are not good are called **bad**. An orbifold is called **very good** if is admits a finite manifold cover.

Theorem 6.7. *(D. Armstrong [Arm68].) Suppose that O is a good orbifold. Then $\pi_1(X_O) = \pi_1(O)/T$, where $T \subset \pi_1(O)$ is the normal subgroup generated by all elements which do not act freely on the universal cover $\tilde{O}$.*

As in the case of manifolds, there is a form of Seifert-Van Kampen theorem which allows to compute fundamental groups of orbifolds, we leave its proof as an exercise to the reader. (Hint: use the proof of Seifert-Van Kampen theorem which is based on the group actions on trees, see [Sco83a, Section 2] where P. Scott discusses the 2-dimensional case.)

Theorem 6.8. *Let O be a connected orbifold, and let $O_1, O_2 \subset O$ be open connected suborbifolds so that:*

- *$O = O_1 \cup O_2$.*
- *$O_1 \cap O_2$ is connected.*
- *Closures of O_1, O_2 are suborbifolds with boundary in O, so that the boundary of O_i equals the frontier of X_{O_i} in X_O.*

Let $\iota_i : O_i \hookrightarrow O$ denote the inclusion. Then the fundamental group of O is obtained from $\pi_1(O_1) * \pi_1(O_2)$ by adding the relations $\gamma_1\gamma_2^{-1} = 1$ where $\gamma_i \in \pi_1(O_i)$, $i = 1, 2$, $\gamma_i = \iota_{i*}(\gamma)$, $i = 1, 2$, for each $\gamma \in \pi_1(O_1) \cap \pi_1(O_2)$.

It is easy to see that for any compact orbifold O the fundamental group $\pi_1(O)$ is finitely generated. To prove this use either Seifert-Van Kampen theorem for orbifolds or Lemma 3.37. (Actually $\pi_1(O)$ is finitely presentable.)

The groups G_i and transition maps $\tilde{\phi}_{ij}$ which appear in the definition of smooth orbifolds may have more restrictive nature: we may require that (after restriction to open subsets) they belong to a Lie group H acting isometrically on a Riemannian manifold M. Then O is called a *geometric orbifold* modelled on the geometry (M, H). The example that we will be interested in is $M = \mathbb{H}^n$ and $H = \mathrm{Isom}(\mathbb{H}^n)$, in which case O is a *hyperbolic orbifold.* More generally, define (locally) $CAT(k)$-orbifold by taking a complete $CAT(k)$-manifold M as the model space and H as the (pseudo)-group of isometries of M.

An orbifold O is called *locally reflective* if the local isotropy group at each point is generated by reflections. For instance, 2-dimensional locally reflective orbifolds locally near each singular point look like either boundary reflectors or corner reflectors. The orbifold O is said to be a *reflection orbifold* if $\pi_1(O)$ is generated by reflections.

Any *geometric orbifold* is a Riemannian manifold outside of the singular locus, the geometric structure provides the whole space X_O with a *path-metric* d_O defined exactly as in the Riemannian case. The geometric orbifold O is *complete* if (X_O, d_O) is complete as a metric space. The same definition applies to locally $CAT(k)$-orbifolds.

Suppose that O is a geometric orbifold modelled on (M, H). Then there is a well-defined (up to conjugation in H) homomorphism $\rho : \pi_1(O) \to H$ which is called the *holonomy representation* of O and the group $\rho(\pi_1(O))$ is the *holonomy group* of O.

Proposition 6.9. *Suppose that O is a complete hyperbolic orbifold without boundary so that $\pi_1(O)$ is a finitely generated group. Then O is very good.*

Proof: We give here only a sketch of the proof, see [MM91] and [Rat94, Chapter 13] for details. As in the case of manifolds modelled on the geometry $(\mathbb{H}^n, \mathrm{Isom}(\mathbb{H}^n))$ we can define the *developing map* $dev : \tilde{O} \to \mathbb{H}^n$ on the universal cover of O. As in the Riemannian case, completeness of O implies that dev is a diffeomorphism. Thus $O = \mathbb{H}^n/G$ where G is the holonomy group of O. The proposition now directly follows from the Selberg Lemma. $\square$

Remark 6.10. To guarantee that O is a good orbifold a much weaker condition suffices: if O is a complete (locally) $CAT(0)$-orbifold then O is good. This is a version of the Cartan-Hadamard theorem due to A. Haefliger [BH99].

6.2. 2-dimensional orbifolds

Recall that according to our definition, the local isotropy group G_x of each 2-dimensional orbifold O can be conjugate (via the exponential map) to a finite group of isometries $G_x \subset \mathrm{Isom}(\mathbb{R}^2)$ which fixes the origin. Thus we have the following possibilities for the group G_x:

(1) $G_x = \{1\}$.

(2) $G_x \cong \mathbb{Z}_n$ is generated by a rotation of the order $n \geq 2$, the number n is called *the order* of the singular point x.

(3) G_x is conjugate to the group $\mathbb{Z}_2$ generated by the reflection $z \mapsto \bar{z}$.

(4) G_x is the dihedral group Dih_m generated by two reflections τ, σ whose product has order $m \geq 2$.

If only the first two possibilities occur then O is called an *orbifold of cone type*. Such orbifolds can be described combinatorially as follows. Start with a surface S (possibly with boundary), let $P \subset S$ be a discrete subset disjoint from the boundary of S. Mark each point $p \in P$ with an integer $n_p \geq 2$. Then the cone orbifold structure associated with this data is: $G_x = \{1\}$ for each $x \in S - P$, $G_p \cong \mathbb{Z}_{n_p}$ for each $p \in P$. Using Seifert-Van Kampen theorem the fundamental group of such orbifold O (if it is connected) can be computed as follows (see [Sco83a] for an alternative computation):

Take the fundamental group Γ of the surface $S-P$. For each peripheral element $\gamma_p \in \Gamma$ corresponding to a simple loop in $S - P$ which goes once around the puncture $p \in S$ we add the relation $\gamma^{n_p} = 1$ to the group Γ. The result is isomorphic to $\pi_1(O)$.

Equivalently, let surface with boundary $\dot{S}$ be the (partial) compactification of $S - P$ obtained by adding a circle γ_p for each puncture $p \in P$. Then along each boundary loop γ_p of $\dot{S}$ add a 2-cell so that the attaching map has degree $\pm n_p$. The fundamental group of the resulting CW-complex is isomorphic to $\pi_1(O)$.

Here is the complete list of *bad* 2-dimensional orbifolds (see Figure 6.2):

1. *The teardrop:* the underlying set of O is the 2-sphere and Σ_O is a single point.

2. *The double teardrop:* the underlying set of O is the 2-sphere and Σ_O is a pair of conical points with distinct orders.

3. The quotient of the teardrop by the reflection in a circle containing the singular point.

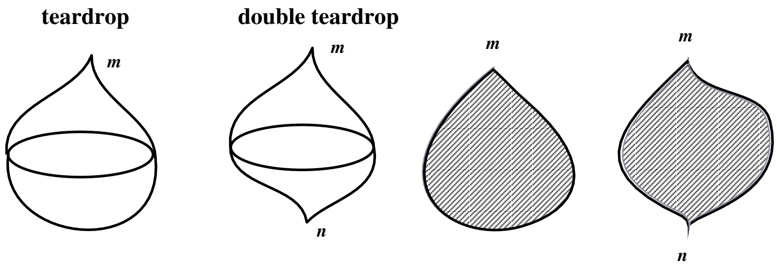

Figure 6.2: *Bad 2-dimensional orbifolds.*

4. The quotient of the double teardrop by the reflection in a circle containing both singular points.

Any good 2-dimensional orbifold (with finitely generated fundamental group) is also very good. There are 3 disjoint classes of good compact 2-dimensional orbifolds without boundary (see [Sco83a] for details):

- *Spherical orbifolds*, which are quotients of the round 2-sphere by finite groups of isometries.
- *Euclidean orbifolds*, which are quotients of the Euclidean plane by lattices.
- *Hyperbolic orbifolds*, which are quotients of the hyperbolic plane by uniform lattices.

To get a 2-dimensional compact orbifold with boundary take a 2-dimensional closed orbifold O and remove from its singular locus a finite collection of open disjoint arcs lying on ∂X_O and/or a collection of open disks from $X_O - \Sigma_O$.

Exercise 6.11. *Any compact 2-dimensional orbifold can be obtained this way.*

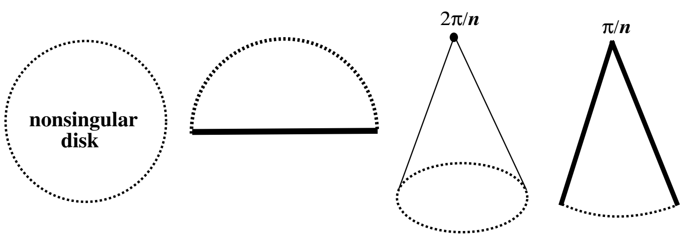

Figure 6.3: *Orbifolds covered by* D^2.

Now we will give lists of 2-dimensional orbifolds which are finitely covered by disks, annuli and tori. Suppose that O is a compact 2-orbifold which admits a manifold cover homeomorphic to the compact disk D^2. Here is the complete list of possibilities for the orbifold O (see Figure 6.3):

- $O \cong D^2$ and $\Sigma_O = \emptyset$.
- $|O| = D^2$ and O has a single singular cone point (of order $n > 1$), thus O is the quotient of D^2 by a group of rotations of order n.
- O is the quotient of D^2 by a single reflection, the singular set of O consists of a single boundary reflector arc.
- O is the quotient of D^2 by a dihedral group Dih_n. Thus $|O| = D^2$ and O has one corner reflector singular point, the rest of the singular set consists of two edges which are boundary reflectors.

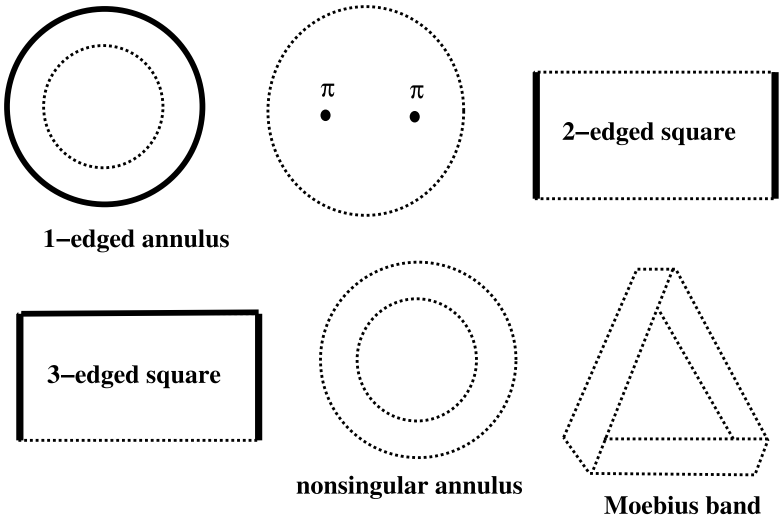

Figure 6.4: *Orbifolds covered by the annulus.*

Suppose that O is a compact 2-orbifold which admits a manifold cover homeomorphic to the annulus $A = \mathbb{S}^1 \times I$ and $O \neq A$, (see Figure 6.4). The either:

- $O \cong \mu$ is the Moebius band.
- $|O| = A$ and one boundary component of A is the boundary reflector, we call this orbifold the *1-edged annulus.*
- $|O|$ is the rectangle, two disjoint edges of this rectangle are boundary reflectors, O is the quotient of A by a single reflection; we will call this orbifold the *2-edged square.*
- $|O| = D^2$ which has two singular cone points of order 2, thus O is the quotient of A by a group of rotations of order 2.
- O is the quotient of A by the dihedral group $\mathbb{Z}/2 \times \mathbb{Z}/2$, $|O|$ is the rectangle, three edges of this rectangle are boundary reflectors. We will call this orbifold the *3-edged square.*

Finally consider orbifolds covered by the torus. We shall assume that these are locally reflective orbifolds.

- $|O|$ is the rectangle, it has 4 corner reflectors and 4 edges which are boundary reflectors; we will call this orbifold the *mirror square*.
- $|O|$ is a triangle, it has 3 corner reflectors and 3 edges which are boundary reflectors; possible angles are: $(\pi/2, \pi/4, \pi/4)$, $(\pi/3, \pi/3, \pi/3)$, $(\pi/2, \pi/3, \pi/6)$; we will call these orbifolds *Euclidean triangle orbifolds*.
- $|O|$ is the annulus, both of its boundary components are boundary reflectors; we will call this orbifold the *mirror annulus*.
- $|O|$ is the Moebius band, its boundary is the boundary reflector; we will call this orbifold the *mirror Moebius band*.

6.3. 3-dimensional locally reflective orbifolds

The class of 3-dimensional orbifolds that we will use, consists of compact locally reflective orbifolds. Underlying spaces of these orbifolds are 3-manifolds (with boundary). Below we give a combinatorial description of these orbifolds. Suppose that X is a compact 3-manifold with boundary, $\hat{\Gamma} \subset \partial X$ is a 3-valent graph. We introduce structure of a locally reflective orbifold O with the underlying set X as follows. Pick a union of components of $\partial X - \hat{\Gamma}$, its closure S will be the designated boundary ∂O. In what follows we will treat S as an orbifold, so we shall use the notation $|S|$ to denote the underlying surface with boundary. Let $\Sigma := \partial X - int(|S|)$, the designated singular locus. For each point $x \in X - \Sigma$ the local isotropy group G_x is trivial. Let $\Gamma := cl(\hat{\Gamma} - |S|)$. We count as vertices of Γ only the vertices of valence 3, the rest of vertices are *dead ends* which belong to $\partial|S|$. Choose a function

$$\theta : \text{ edges of } \Gamma \to \{\pi/2, \pi/3, ..., \pi/n, ...\}$$

so that for any vertex v of Γ and edges e_1, e_2, e_3 emanating from v satisfy the condition:

$$\theta(e_1) + \theta(e_2) + \theta(e_3) > \pi.$$

(See Figure 6.5.)

Components of $\Sigma - \hat{\Gamma}$ are the *faces* of the orbifold O. For each point x in a *face* F, the local isotropy group G_x is isomorphic to $\mathbb{Z}_2$ and the generator acts as a reflection whose fixed-point set projects to F. Edges of the graph Γ are called the *edges* of the orbifold O. If x belongs to an edge e of O but is not a vertex, then its local isotropy group G_x is a dihedral group $Dih_m = \mathbb{Z}_2 \ltimes \mathbb{Z}_m$ where $1/m = \theta(e)/\pi$. This group is generated by two isometric reflections in $\mathbb{R}^3$ so that the intersection of their fixed-point sets projects (locally) to e. Finally, if x is a vertex of Γ, then G_x is a finite Coxeter subgroup of $O(3)$ generated by three reflections, which has the presentation:

$$\langle \tau_1, \tau_2, \tau_3 : (\tau_j)^2 = 1, (\tau_i \tau_j)^{m_{ij}} = 1 \rangle$$

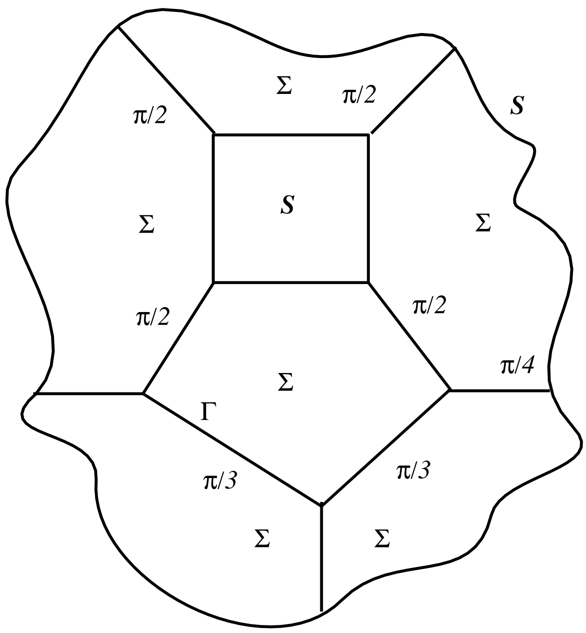

Figure 6.5: *Combinatorial structure of a locally reflective orbifold.*

where $m_{ij}^{-1} = \theta(e_{ij})/\pi$ and e_{ij} is the edge of Γ between the faces corresponding to τ_i, τ_j. The group G_x is a *spherical triangle group*.

Proposition 6.12. *Any locally reflective compact 3-dimensional orbifold O can be obtained as above.*

Proof: The underlying space $X = X_O$ is a compact manifold with boundary. Since O is locally reflective, the singular set Σ of O is contained in ∂X. The local isotropy group G_x of each point is a finite subgroup of $O(3)$ generated by reflection. The list of such groups is very small, it consists of $\mathbb{Z}/2$, dihedral groups Dih_m and spherical triangle groups above. According to these local isotropy groups we get a stratification of Σ into faces, edges and vertices. Complement to the union of faces in Σ is the graph $\hat{\Gamma}$. $\square$

Note that $S = \partial O$ inherits a locally reflective orbifold structure. The corresponding 2-dimensional orbifold has the underlying space $|S|$, its singular set consists of the boundary reflectors which are edges of $\hat{\Gamma} \cap \partial S$, besides there are corner reflectors corresponding to the end-points of these edges.

In what follows we will need to specify the *parabolic locus* on the boundary of a locally reflective orbifold, similarly to the description of the parabolic locus on the boundary of a compact 3-manifold. First, collect all components of ∂O which have the structure of Euclidean orbifolds, then we pick a collection of disjoint Euclidean suborbifolds with boundary which are contained in ∂O (in the interesting case these components are non-singular annuli and 2-edged and 3-edged squares.) The total collection of

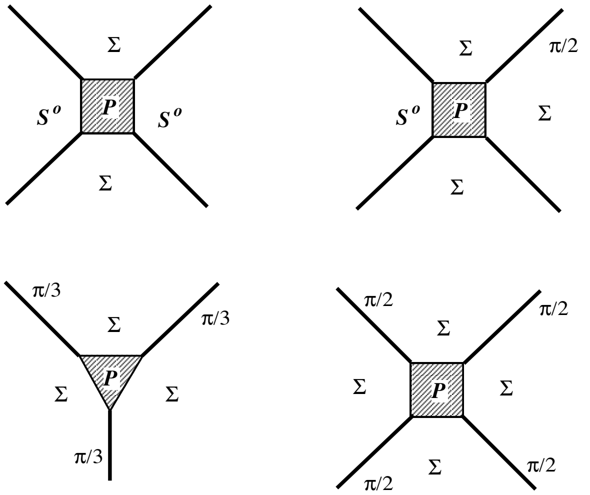

Figure 6.6: *Examples of the parabolic locus of a locally reflective orbifold.*

these suborbifolds $P = P_1 \cup ... \cup P_n$ is the parabolic locus of O. We let $S^0 := cl(S - P)$. Later on we shall impose some further restrictions on these components which generalize the definition of a *pared* 3-manifold. See Figure 6.6.

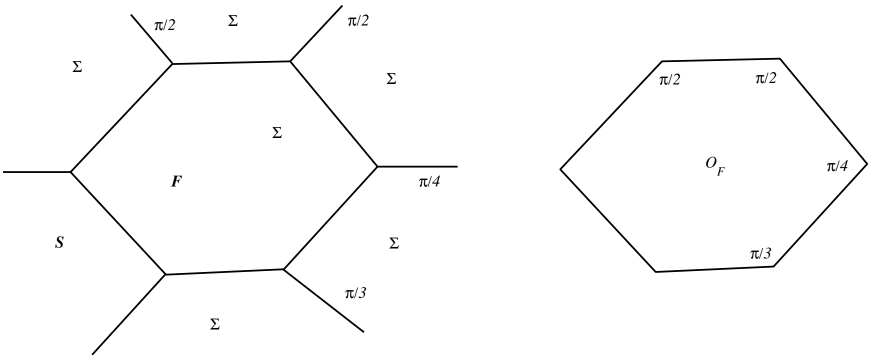

Figure 6.7: *Orbifold structure on faces of a locally reflective orbifold.*

Define the orbifold structure on the faces of locally reflective orbifolds O as follows. Each face F of O has a natural polygonal structure, where edges appear as intersections of F with other faces of O as well as with the boundary of O. We declare the former arcs to be the boundary reflectors of the orbifold O_F whose boundary is $F \cap \partial O$ and the underlying space is F. To define the angle of the corner reflector of O_F at the vertex v (between two boundary reflectors) we use the *dihedral* angle of O at the edge of Γ emanating from v. See Figure 6.7.

Definition 6.13. Suppose that O is a locally reflective orbifold. An element $g \in \pi_1(O)$ is called a **reflection** if it (locally) acts as a reflection in

$\tilde{O}$. The **face** of this reflection is the projection of the fixed-point set of g to O.

Theorem 6.14. *(J. Morgan, [Mor84], M. Kato [Kat86].) Each compact locally reflective 3-orbifold which contains no bad 2-suborbifolds is very good.*

6.4. Glossary of orbifolds: the good, the bad and ...

In the category of 3-dimensional orbifolds one can define generalizations of *compressing disks*, *essential annuli*, *irreducible*, *boundary irreducible*, *acylindrical* and *atoroidal* orbifolds, *incompressible surfaces* etc. These definitions are straightforward generalizations of corresponding definitions in the class of manifolds and we collect them in the following **Glossary** which I in part borrowed from [Dun88].

GLOSSARY

Abad 3-orbifold: a 3-orbifold which contains no bad 2-suborbifolds.

Acylindrical 3-orbifold: a 3-orbifold where each properly embedded ANNULUS A has the property: ∂A bounds an ANNULUS in ∂O.

ANNULUS: a 2-orbifold A which is finitely covered by an annulus.

Topologically atoroidal 3-orbifold: a 3-orbifold O with the following properties:

- ∂O is incompressible, no boundary component is a 2-SPHERE;
- each incompressible TORUS in O is boundary parallel;
- O is not finitely covered by the product of 2-torus and interval.

Bad orbifold: an orbifold whose universal cover is not a manifold.

BALL: a 3-orbifold which is homeomorphic to B^3/Γ, where $\Gamma \subset O(3)$ is a finite group.

Boundary parallel (or "inessential") 2-suborbifold S in a 3-orbifold O:
a properly embedded suborbifold such that a component of $O - S$ is homeomorphic to $(0,1] \times S$.

Boundary compressing DISK D for a 2-suborbifold S in a 3-orbifold O: it is a DISK such that: ∂D is a union of two arcs $\alpha \subset S, \beta \subset \partial O$ which satisfy the following property:

there is no arc $\gamma \subset \partial O \cap S$ so that $\gamma \cup \alpha$ is the boundary of a DISK in S.

Closed orbifold: a compact orbifold with empty boundary.

CIRCLE: a connected 1-dimensional closed orbifold.

Compressing ANNULUS A for a 2-suborbifold $S \subset O^3$: $\partial A = \alpha \cup \beta$ is a disjoint union of two CIRCLES, so that:

- $\alpha = A \cap S$, $\beta \subset \partial O^3$;
- α does not bound a DISK in O^3;
- $\alpha \cup \beta$ does not bound an ANNULUS in S.

Compressing DISK D for a 2-suborbifold S in a 3-orbifold O, is a DISK such that $D \cap S = \partial D$ does not bound a DISK in S.

DISK: a 2-orbifold which is diffeomorphic to D^2/Γ, where $\Gamma \subset O(2)$ is a finite group.

Good orbifold: an orbifold which is covered by a manifold.

Incompressible 2-suborbifold S **of a 3-orbifold** O: a suborbifold S which either has no compressing DISKS and boundary compressing DISKS or S is a DISK whose boundary $\partial S \subset \partial O$ does not bound any DISKS in ∂O.

Irreducible 3-orbifold: a 3-orbifold O where any SPHERE bounds a BALL.

Properly embedded 2-suborbifold S **in a 3-orbifold** O: a suborbifold such that $\partial S \cap \partial O = \partial S$ and this intersection is transversal.

Seifert 3-orbifold: a good 3-orbifold which is finitely covered by a Seifert manifold. (Note that there is also a class of bad Seifert orbifolds, see [BS85]. For example, it is natural to consider the product of $\mathbb{S}^1$ and a bad 2-orbifold as a Seifert-fibered orbifold. However we will deal only with good 3-orbifolds.)

SOLID TORUS: a 3-orbifold which is covered by the 3-dimensional solid torus.

2-SPHERE: a 2-orbifold which is covered by $\mathbb{S}^2$.

Superincompressible 2-suborbifold of a 3-orbifold: a 2-sided properly embedded incompressible 2-suborbifold which has no compressing annuli.

TORUS: a 2-orbifold covered by the 2-torus.

Very good orbifold: an orbifold which admits a finite manifold covering.

Waldhausen orbifold: a compact 3-dimensional orbifold which is irreducible, abad and contains an incompressible suborbifold.

Note that what we call *Waldhausen* orbifold is an object similar to a *Haken* 3-manifold. However in general *Waldhausen* orbifolds do not admit a *Haken hierarchy*, see [Dun88]. Nevertheless, Dunbar proves in [Dun88] the following:

Theorem 6.15. *Suppose that O is a compact orientable Waldhausen orbifold such that O does not admit incompressible Euclidean and hyperbolic suborbifolds which are spheres with three singular points. Then O is* Haken*, i.e. it admits a Haken hierarchy: a chain of orbifolds $O_i, i = 0, 1, 2, ..., h$ so that*

(a) Each O_i is Waldhausen for $i < h$.

(b) Each O_{i+1} is obtained from O_i by splitting along an incompressible suborbifold.

(c) O_h is a disjoint union of BALLs.

We note that orientability of O seems to be an essential assumption in [Dun88] since the key argument in Dunbar's proof uses the fact that a complex curve in a complex affine variety is never compact. Such conclusion is clearly false in the case of real curves in real affine varieties which appear if we consider nonorientable orbifolds (e.g. the locally reflective orbifolds). Takeuchi [Tak91] uses Dunbar's theorem to prove his homeomorphism theorem for very good orientable Haken orbifolds.

Now we give several alternative definitions:

Definition 6.16. A very good 3-dimensional orbifold O is called **irreducible**, (resp. **acylindrical**) if some (any) of its finite manifold covers is irreducible, (resp. weakly acylindrical). A very good 3-dimensional non-Seifert orbifold O is called **topologically atoroidal**, if some (any) of its finite manifold coverings is atoroidal.

Example 6.17. Here is an example of a *compressible* Euclidean suborbifold $Q \subset O$. Start with the closed upper half-space H, which will be the the underlying set of the reflection orbifold O. The singular set of O is the boundary of H. The orbifold structure of O is given by the connected graph Γ with two vertices x, y connected by an edge $[xy]$ and four infinite rays; two rays are emanating from x and from y. The function θ on the edges of Γ takes the value $\pi/2$ on each infinite ray and takes the value π/m ($m \geq 2$) on $[xy]$. This data determines the orbifold O. Take a disk $D \subset H$ whose boundary crosses each infinite ray exactly once, this disk determines a suborbifold $Q = O_D$ in O. The suborbifold Q is the mirror square which bounds a **SOLID TORUS** in O, see Figure 6.8.

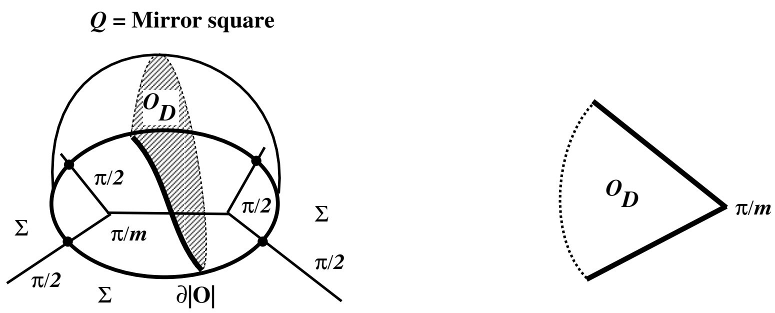

Figure 6.8: *SOLID TORUS bounded by the mirror square: example of a compressible Euclidean suborbifold Q.*

Equivalence of the definitions from GLOSSARY and the Definition 6.16 follows from the following theorems:

Theorem 6.18. *(Equivariant loop theorem.) Suppose that M is a compact manifold with boundary, $Q \subset \partial M$ is a subsurface which is compressible in*

M. Let F be a finite group of diffeomorphisms acting on (M, Q). Then there exists a compressing disk $(D^2, \partial D^2) \subset (M, Q)$ which is precisely invariant under its stabilizer in F.

Theorem 6.19. *(Equivariant sphere theorem.) Suppose that M is a compact orientable manifold, $\Sigma \subset M$ is a sphere which does not bound a ball in M. Let F be a finite group of diffeomorphisms acting on M. Then there exists a sphere $\mathbb{S}^2 \subset M$ (or a projective plane) which does not bound a ball and is precisely invariant under its stabilizer in F.*

Theorem 6.20. *(Equivariant annulus theorem.) Suppose that M is a compact manifold with boundary, $Q \subset \partial M$ is a subsurface such that the manifold $(M, M - Q)$ is not acylindrical. Let F be a finite group of diffeomorphisms acting on (M, Q). Then there exists an essential annulus (or Moebius band) $(A, \partial A) \subset (M, M - Q)$ which is precisely invariant under its stabilizer in F.*

Theorem 6.21. *(Equivariant torus Theorem.) Suppose that M is a compact irreducible 3-manifold which is not Seifert and is not atoroidal. Let F be a finite group of diffeomorphisms acting on M. Then there exists an essential torus (or a Klein bottle) $T \subset M$ which is precisely invariant under its stabilizer in F.*

Equivariant loop and sphere theorems were first proven using minimal surfaces in [MY81]. A purely combinatorial proof appeared later in [JR89]. Equivariant Torus theorem follows from uniqueness of the canonical JSJ decomposition of Haken manifolds into atoroidal and Seifert components. Similarly the Equivariant Annulus theorem follows from uniqueness of the characteristic submanifold in a Haken manifold (see also [JR89]).

The following is Thurston's formulation of a characterization of 3-dimensional hyperbolic reflection orbifolds which was proven by E. Andreev in [And70], [And71b].

Theorem 6.22. *Suppose that (O, P) is a non-Seifert pared 3-dimensional orbifold with the underlying set homeomorphic to the closed 3-ball, $\partial O = P$. Then O is hyperbolic provided that O is Haken and is topologically atoroidal.*

We will say that O is a *tetrahedron* if X_O is the 3-ball and the corresponding graph $\Gamma \subset \partial X_O$ is the complete graph on four vertices.

Similarly, O is a *prism* if X_O is the 3-ball and the graph $\Gamma \subset \partial X_O$ is isomorphic to the 1-skeleton of a convex prism in $\mathbb{R}^3$ which has triangular bases where the edges of Γ connecting the opposite triangular bases all have the label $\pi/2$.

Exercise 6.23. *Show that if O is a prism then O cannot be hyperbolic. List all the tetrahedral orbifolds which are hyperbolic.*

Thus, unless O is a tetrahedron or a prism, O is hyperbolic under the assumption that O is topologically atoroidal and irreducible. Combinatorially, the assumption that O is topologically atoroidal can be formulated as follows. (I will assume for simplicity that $P = \emptyset$.)

Suppose that:

(a) No polygonal circle on $\partial|O|$ crosses transversally precisely one edge.

(b) No polygonal curve α on $\partial|O|$ crosses transversally the graph Γ in precisely two points unless these points belong to one and same edge J and α bounds a union of two disks in $\partial|O|$ which are adjacent to J.

(c) No polygonal curve α on $\partial|O|$ crosses Γ transversally in precisely three points with the labels $\beta_1, \beta_2, \beta_3$ such that $\beta_1 + \beta_2 + \beta_3 \geq \pi$, unless these three points belong to three edges emanating from the same vertex of Γ, in which case the angle sum along α is more than π.

(d) No polygonal curve α on $\partial|O|$ crosses Γ transversally in precisely four points along edges labeled by $\pi/2$, unless either α bounds a disk in $\partial|O|$ which intersects Γ along one or two edges containing $\alpha \cap \Gamma$, or α intersects four distinct edges e_1, e_2, e_3, e_4 of Γ where $e_1 \cap e_2 = p, e_3 \cap e_4 = q$ and p, q are connected by an edge of Γ (see Figure 6.9).

Then O is hyperbolic unless it is a tetrahedron or a prism.

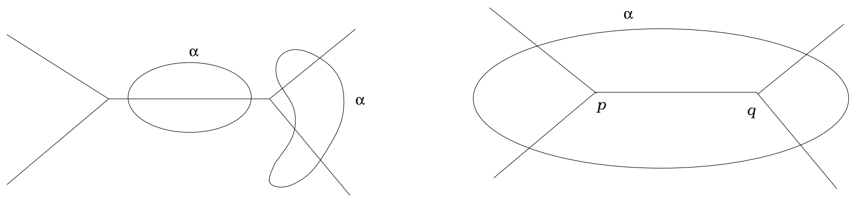

Figure 6.9:

The class of orbifolds that we will be mainly interested in consists of *all right* locally reflective 3-dimensional orbifolds, which we describe below. Suppose that O is a compact locally reflective 3-dimensional orbifold whose underlying space is orientable, $S = \partial O$ and the graphs $\Gamma, \hat{\Gamma} \subset \partial X_O$ and the function θ are as in §6.3. Define a *designated parabolic locus* P of the orbifold O to be a collection of disjoint incompressible Euclidean suborbifolds $P_i \subset S$ so that:

Axiom 1: The function θ is identically equal to $\pi/2$ and no component of P is a 1-edged annulus.
Thus each component $P_i \subset P$ is either:

(a) An incompressible TORUS $T_i \subset S$.

(b) A homotopically nontrivial (nonsingular) annulus $A_i \subset S$.

(c) A 2-edged square.

(d) A 3-edged square.

We let $P^{(1)}$ be the union of all components of P which are 2-edged squares and $\mathcal{F}(P^{(1)})$ be the union of $P^{(1)}$ and of all faces of O which are adjacent to $P^{(1)}$.

Axiom 2: For any pair of distinct connected components $P_i, P_j \subset P$, the conjugacy classes of $\pi_1(P_i), \pi_1(P_j)$ in $\pi_1(O)$ intersect by a finite subgroup. The suborbifold $S^0 := S - P$ is incompressible in O.

Axiom 3: The graph Γ has no vertices of valence ≥ 2, only *dead ends* (which belong to ∂S).

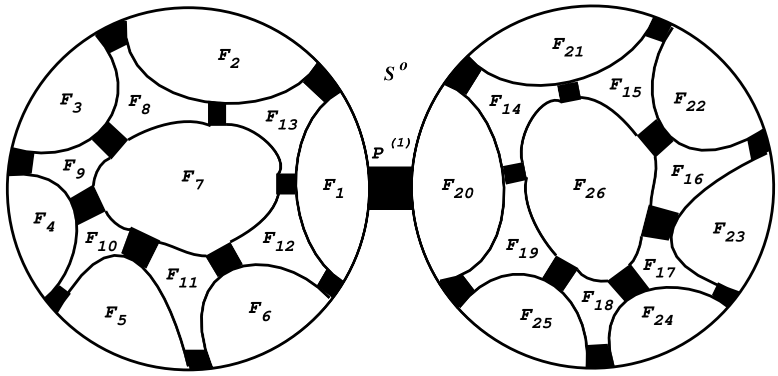

Figure 6.10: *Example of a finite type orbifold O with the underlying space B^3; here $\mathcal{F}(P^{(1)}) = F_1 \cup F_{20} \cup P^{(1)}$. Note that the orbifold O itself is reducible. Black boxes in this figure denote components of the parabolic locus, one of them is the 2-edged square. F_j's are the faces of O.*

Axiom 4: O is very good; the orbifold $O - \mathcal{F}(P^{(1)})$ is irreducible, atoroidal and acylindrical.

Definition 6.24. We say that a compact pared 3-orbifold (O, P) is **all right** if it satisfies the Axioms 1–4.

Axiom 5: Suppose that $\lambda \subset \partial X_O - \mathcal{F}(P^{(1)})$ is an arc with the end-points on ∂S^0 which is transversal to Γ and which is not homotopic into S^0 relatively to $\partial\lambda$. Then $int(\lambda)$ intersects at least 3 edges of Γ.

Definition 6.25. We say that a compact 3-orbifold O has **finite type** if it satisfies the Axioms 1–5.

Remark 6.26. There is nothing special about this class of orbifolds, it is just what we need for the induction step in Thurston's Hyperbolization Theorem. Figure 6.10 gives example of an orbifold of finite type.

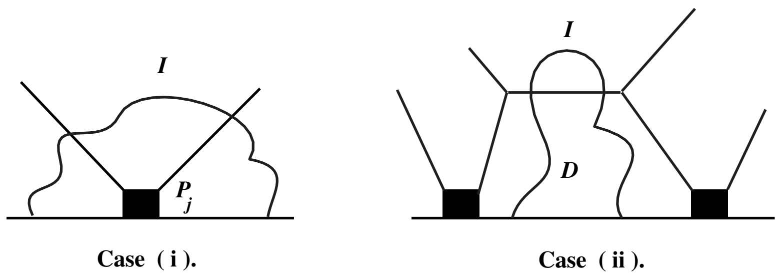

Figure 6.11: *Possible positions for the disk D.*

Suppose that I is an arc in $\partial X_O - \mathcal{F}(P^{(1)})$ with the end-points in the same component of ∂S^0, I is transversal to Γ and is homotopic into S^0 relatively to ∂I. Then we connect the end-points of I in S^0 by an embedded arc $J \subset S^0$. The union $I \cup I'$ bounds a disk $D \subset \partial X_O$. Assume that I intersects at most 4 edges of Γ. Since the orbifold $O - \mathcal{F}(P^{(1)})$ is irreducible and acylindrical, we conclude that $I \cup I'$ bounds a properly embedded disk $B \subset \partial X_O$ and this disk satisfies one of the following combinatorial possibilities described in the Figure 6.11:

(i) The disk B contains exactly two vertex of $\hat{\Gamma}$.

(ii) The disk B contains no vertices of $\hat{\Gamma}$.

Definition 6.27. An **all right** orbifold O has zero Euler characteristic $\chi(O)$ if it satisfies:

Axiom 6: $S^0 = \emptyset$, i.e. the whole boundary of O consists of incompressible tori, Klein bottles, mirror annuli, mirror Moebius bands and mirror squares.

(We take this as the definition of the orbifold with zero Euler characteristic, instead of defining the Euler characteristic intrinsically. It is clear that if $M \to O$ is a finite manifold cover of such orbifold then $\chi(M) = 0$.)

Note that in this case $P^{(1)} = \emptyset$, hence the orbifold O itself is irreducible, atoroidal and acylindrical. If O is an *all right* orbifold of zero Euler characteristic, then we say that *peripheral elements* of $\pi_1(O)$ are those which are conjugate into $\pi_1(P_j)$ for some $P_j \subset \partial O$.

Later on we shall need another class of orbifolds as well.

Definition 6.28. Suppose that (O, P) is an **all right** pared orbifold. O is said to be **all right bipolar** orbifold if it satisfies

Axiom 7: Faces of O can be colored in white and red colors so that no faces of the same color are adjacent. The coloring of faces of O induces a coloring of the edges of $S^0 \subset \partial O$. We require all edges of S^0 to have the same color.

6.5. A homeomorphism theorem for 3-orbifolds

In this section we will prove a homeomorphism theorem for the *all right* orbifolds of zero Euler characteristic. We start by collecting some basic facts about orbifolds of this class.

Lemma 6.29. *Suppose that O is an all right orbifold of zero Euler characteristic. Then for any reflection $r \in \pi_1(O)$ the centralizer of r in $\pi_1(O)$ contains $\mathbb{Z} * \mathbb{Z}$.*

Proof: Since O is irreducible, the face of the reflection r in O is an incompressible surface in $|O|$. This face F is neither Euclidean nor spherical 2-orbifold since O is atoroidal and irreducible. Thus the lift of F to the universal cover of O is simply-connected and its stabilizer in $\pi_1(O)$ is isomorphic to a nonelementary discrete subgroup of $\mathrm{Isom}(\mathbb{H}^2)$. However, this stabilizer is isomorphic to the centralizer of r in $\pi_1(O)$. $\square$

Lemma 6.30. *Suppose that O is an all right orbifold of zero Euler characteristic and Q is a very good compact irreducible 3-orbifold with the fundamental group isomorphic to $\pi_1(O)$. Then Q also is an all right orbifold of zero Euler characteristic.*

Proof: Let $\psi : \pi_1(O) \to \pi_1(Q)$ be the isomorphism. The only assertion which has to be proven is that for any reflection $r \in \pi_1(O)$ the image $\psi(r)$ is again a reflection. Note that the centralizer of the element $\psi(r) \in \pi_1(Q)$ keeps the fixed-point set $Fix(\psi(r))$ invariant. If $\psi(r)$ is not a reflection, its fixed-point set has dimension ≤ 1, which implies that the centralizer of $\psi(r)$ is almost cyclic. This contradicts Lemma 6.29. $\square$

We need to establish certain properties of all right orbifold of zero Euler characteristic. Recall that any such orbifold O is good, thus its universal covering space $\tilde{O}$ is a manifold. We denote by $\tilde{\Sigma}_O$ the lift of the singular set of O to $\tilde{O}$. Each component C of $\tilde{O} - \tilde{\Sigma}_O$ is bounded by lifts of faces of the orbifold O. The boundary of C has a natural polyhedral structure. We call a connected nonempty smooth subsurface $F \subset \partial_{\tilde{O}} C$ *a face* of C if there is a reflection $r \in \pi_1(O)$ such that F is a component of the intersection between $cl(C)$ and the fixed-point set of r. Let $\hat{F}$ denote the whole fixed-point set of the reflection r which corresponds to F. The sets $\hat{F}$ are simply-connected, properly embedded and noncompact since O is irreducible. Thus we shall refer to these sets as *planes* in $\tilde{\Sigma}_O$. Note that the *planes* in $\tilde{\Sigma}_O$ can have nonempty boundary. Choose a Riemannian metric on O and lift it to the usual Riemannian metric on $\tilde{O}$.

Lemma 6.31. *Suppose that C is a component of $\tilde{O} - \tilde{\Sigma}_O$ and F_1, F_2 are disjoint faces of C. Then $\hat{F}_1 \cap \hat{F}_2 = \emptyset$.*

Proof: Let r_j denote the reflection in $\pi_1(O)$ with the fixed-point set $\hat{F}_j$. Choose points $x_j \in F_j$. Suppose that $x \in \hat{F}_1 \cap \hat{F}_2$. Then there exists a reflection $r \in \pi_1(O)$ so that the fixed-point set $\hat{F}$ of r separates x from C, thus it also separates x from the both points x_1, x_2. Therefore $\hat{F}$ must intersect the sets $\hat{F}_1, \hat{F}_2$. Since O is an *all right* orbifold, either $\hat{F}_1 = \hat{F}_2$ or they intersect by the right angle.

In the first case we get contradiction with irreducibility of O, in the second case we get a subgroup in $\pi_1(O)$ generated by r, r_1, r_2 which corresponds to a spherical triangle orbifold $T(2,2,2)$. This is impossible by the assumption that O is an *all right* orbifold. $\square$

Introduce the following equivalence relation: two reflections $r', r'' \in \Gamma = \pi_1(O)$ will be called *connected* if their fixed-point sets belong to the same connected component of $\tilde{\Sigma}_O$. This equivalence relation can be characterized algebraically as follows. Reflections r', r'' are connected iff there exists a chain of reflections $r_1 = r', r_2, ..., r_n = r''$ so that each pair r_i, r_{i+1} generates the dihedral group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

We define a simplicial tree $T = T_O$ as follows. Blue vertices of T are components of $\tilde{O} - \tilde{\Sigma}_O$, black vertices are components of $\tilde{\Sigma}_O$. Blue and black vertices are connected by an edge iff they are adjacent in $\tilde{O}$. The group Γ acts naturally on the tree T without inversions.

Lemma 6.32. *Suppose that $g \in \pi_1(O)$ does not stabilize any black vertex of the tree $T = T_O$. Then for any reflection $r \in \pi_1(O)$ the elements $r, r' := grg^{-1}$ are not connected.*

Proof: Let F be the fixed-point set of r, $F' = g(F)$ is the fixed-point set of r'. If the elements r, r' would be equivalent then their fixed-point sets F, F' would be in the same component of $\tilde{\Sigma}_O$. $\square$

The basic fact about 3-dimensional orbifolds that we will need is the following generalization of Waldhausen-Johannson theorem:

Theorem 6.33. *Suppose that 3-orbifolds O, Q are* **all right** *orbifolds of zero Euler characteristic and their fundamental groups are isomorphic. Then O is homeomorphic to Q.*

Remark 6.34. One can weaken the assumption that O, Q **all right** orbifolds of zero Euler characteristic, however the above theorem will suffice for our purposes.

Proof: Let $\Gamma = \pi_1(O)$, $\Delta := \pi_1(Q)$ and $\psi : \Gamma \to \Delta$ be the isomorphism. First we note that ψ carries peripheral elements of Γ to peripheral elements of Δ (since O, Q have finite manifold covers, which are atoroidal, acylindrical and have incompressible boundary).

Let $T = T_O$, $S = T_Q$ denote the simplicial trees corresponding to the decompositions of $\tilde{O}, \tilde{Q}$ into components, as described above. We know that the isomorphism ψ carries reflections in the group Γ to reflections in the group Δ (see the proof of Lemma 6.30). The equivalence relation for these reflections is obviously preserved by the isomorphism. Thus, by Lemma 6.32, stabilizers of black vertices of T are mapped to stabilizers of black vertices of S. Our goal is to prove the same for blue vertices.

Proposition 6.35. *Suppose that r', r'' are* **connected** *reflections in Γ so that their fixed-point sets $\hat{F}', \hat{F}''$ correspond to faces F', F'' of a component C of $\tilde{O} - \tilde{\Sigma}_O$. Then the same is true for the reflections $\psi(r'), \psi(r'')$.*

Proof: The proof is by induction on the combinatorial length of the chain of reflections connecting r', r''. First suppose r', r'' are neighbors, i.e. that $\hat{F}' \cap \hat{F}'' \neq \emptyset$ and $r' \neq r''$. Then obviously the fixed-point sets $Fix(\psi(r'))$ and $Fix(\psi(r''))$ intersect in $\tilde{Q}$ by the right angle and we can take any component of $\tilde{Q} - \tilde{\Sigma}_Q$, which is adjacent to $Fix(\psi(r')) \cap Fix(\psi(r''))$.

The next case is when there is a neighboring face F of C between F' and F''. Let $\hat{F} = Fix(r)$, where r is a reflection in Γ. If a plane $\hat{F}''' = Fix(r''')$ intersects $\hat{F}$ transversally, then the reflection r''' belongs to the stabilizer $Stab(\hat{F})$ of $\hat{F}$ in the group Γ. The same is true for the fixed-point sets in the manifold $\tilde{Q}$. The isomorphism ψ carries $G = Stab(\hat{F})$ into $\psi(G) = Stab(Fix\psi(r))$. We claim that this isomorphism is induced by a homeomorphism $\hat{F} \to Fix(\psi(r))$. Indeed, by compactness of O and Q, the groups $G, \psi(G)$ act cocompactly on the planes $\hat{F}, Fix(\psi(r))$. Peripheral elements of $G, \psi(G)$ are also peripheral elements of $\pi_1(O), \pi_1(Q)$. Therefore by Theorem 5.5 there exists a homeomorphism $\hat{F} \to Fix(\psi(r))$ which induces the isomorphism $G \to \psi(G)$.

As an intermediate corollary we have the following assertion that will be used later on:

Corollary 6.36. *The orbifolds*

$$Fix(r)/Stab(Fix(r)) \quad and \quad Fix(\psi(r))/Stab(Fix(\psi(r)))$$

are homeomorphic for each reflection $r \in \Gamma$.

Now we go back to the planes $\hat{F}', \hat{F}''$. They are disjoint, intersect F and they are not separated by any other plane in $\tilde{\Sigma}_O$. If a plane $Fix(\psi(r'''))$ separates $Fix(\psi(r'))$ from $Fix(\psi(r''))$, then $Fix(\psi(r'''))$ must intersect the plane $Fix(\psi(r))$ and we get contradiction with existence of an equivariant homeomorphism $Fix(r) \to Fix(\psi(r))$. Thus there exists a component $B = \psi_*(C)$ of $\tilde{Q} - \tilde{\Sigma}_Q$ so that the planes $Fix(\psi(r'))$, $Fix(\psi(r))$ and $Fix(\psi(r''))$ correspond to consecutive faces of B. The general case follows by induction. $\square$

Corollary 6.37. *Suppose that an element* $g \in \Gamma$ *stabilizes an edge of the tree* T. *Then* $\psi(g)$ *will also stabilize an edge of the tree* S.

Proposition 6.38. *Suppose that* $g \in \Gamma$ *fixes a blue vertex of the tree* T.*Then* $\psi(g)$ *fixes a blue vertex of the tree* S.

Proof: We already know that stabilizers of edges and black vertices in T, S are preserved under the isomorphism ψ. Hence if $\psi(g)$ does not fix any blue vertex of S, it acts on S as a hyperbolic element (Lemma 10.2). Thus there exists an axis A for the action of $\psi(g)$ on S. Let v be any vertex on A, then the points $v, \psi(g)v$ are separated on S by a black vertex w. Choose a point $x \in \tilde{Q}$ within the component corresponding to the vertex v, there exists a *plane* $\hat{F} \subset w \subset \tilde{\Sigma}_Q$ which separates x from $\psi(g)x$. We will treat $\hat{F} = Fix(\psi(r))$ as a locally finite relative cycle in $Z_2^{lf}(\tilde{Q}, \partial\tilde{Q}; \mathbb{Z}_2)$. Let $[x, \psi(g)x]$ be a smooth arc in $\tilde{Q}$ obtained by lifting the geodesic arc $[v, \psi(g)v] \subset S$. Then the orbit $\Lambda := \langle\psi(g)\rangle([x, \psi(g)x])$ belongs to $Z_1^{lf}(\tilde{Q}; \mathbb{Z}_2)$ and the $\mathbb{Z}_2$ intersection number between $\hat{F}$ and Λ equals 1.

Recall that the element g was stabilizing a component C of $\tilde{O} - \tilde{\Sigma}_O$, choose a properly embedded real line $\Lambda' \subset C$ invariant under g. Then Λ' is disjoint from any *plane* $\hat{F}' \subset \tilde{\Sigma}_O$. This implies that the algebraic intersection number between the locally finite cycles $\Lambda' \in Z_1^{lf}(\tilde{O}; \mathbb{Z}_2)$ and $\hat{F}' \in Z_2^{lf}(\tilde{O}, \partial\tilde{O}; \mathbb{Z}_2)$ equals zero.

Now we use the fact that both orbifolds O, Q are very good. They admit a common finite regular manifold covering M (by Johannson's theorem) and we shall identify $\tilde{O}$ and $\tilde{Q}$ with the universal covering $\tilde{M}$ of M. The actions of Γ and Δ on $\tilde{M}$ descend to homotopic finite group actions on M. Thus there exists a proper ψ^{-1}-equivariant homotopy-equivalence $h : (\tilde{M}, \partial M) \to (\tilde{M}, \partial M)$. The mapping h induces isomorphisms of the locally finite homology groups. Moreover, by equivariance, we have:

$$[h(\hat{F})] = [\hat{F}'] \in Z_2^{lf}(\tilde{O}, \partial\tilde{O})$$

for the plane $\hat{F}' = Fix(\psi^{-1}(r))$ and $[h(\Lambda)] = [\Lambda']$ in $H_1^{lf}(\tilde{O})$. Thus the algebraic intersection numbers between $\Lambda, \hat{F}$ and $\Lambda', \hat{F}'$ must be equal. On the other hand, one of them is 1 and the second is zero. Contradiction. $\square$

Lemma 6.32 and Proposition 6.38 imply that ψ sends the stabilizers of blue vertices in T to the stabilizers of blue vertices in S. Consider components C and B of $\tilde{O} - \tilde{\Sigma}_O$ and $\tilde{Q} - \tilde{\Sigma}_Q$ such that

$$\psi : Stab(C) \to Stab(B).$$

The quotients $cl(C)/Stab(C)$ and $cl(B)/Stab(B)$ are naturally homeomorphic to $|O|, |Q|$ respectively. According to Corollary 6.37, an element $g \in Stab(C)$ stabilizes a boundary component of C iff $\psi(g)$ stabilizes a boundary component of B. Thus the induced isomorphism $|\psi| : \pi_1(|O|) \to \pi_1(|Q|)$ carries peripheral elements to peripheral elements. Our next problem is to construct a natural homeomorphism $\partial|O| \to \partial|Q|$ which preserves the orbifold structures and is well-behaved with respect to the homomorphisms $\pi_1(\partial|O|) \to \pi_1(|O|)$, $\pi_1(\partial|Q|) \to \pi_1(|Q|)$.

As we already know each face $F \subset Fix(r)$ of the component $C \subset \tilde{O} - \tilde{\Sigma}_O$ is simply-connected and there exists a homeomorphism $h_F : F \to F' \subset Fix(\psi(r))$ which conjugates the stabilizer of F to the stabilizer of F'.

Recall also that faces F_1, F_2 are adjacent iff the corresponding reflections satisfy the relation $(r_1 r_2)^2 = 1$. For adjacent faces (after adjusting homeomorphisms h_{F_j} via ψ-equivariant isotopy) we have:

$$h_{F_1}|_{F_1 \cap F_2} = h_{F_2}|_{F_1 \cap F_2}.$$

Thus we define the homeomorphism $h : \partial_{\tilde{O}} C \to \partial_{\tilde{Q}} B$ as the union of homeomorphisms h_{F_j}. It is clearly well-defined and extends naturally to squares and planes which are lifts of components of $\partial O, \partial Q$. The resulting homeomorphism h is ψ-equivariant and it projects to a homeomorphism $\hat{h} : \partial|O| \to \partial|Q|$. By construction, the induced isomorphisms of fundamental groups of boundary components of $\partial|O|$ and $\partial|Q|$ preserve the *peripheral structure* in the sense of §1.4. Thus by Theorem 1.30 there exists an extension of $\hat{h}$ to a homeomorphism $f : |O| \to |Q|$ which preserves the orbifold structure on the boundary because $\hat{h}$ did. $\square$

Chapter 7

Complex Projective Structures

In this section we consider complex projective structures on Riemann surfaces. These structures provide a generalization of Kleinian groups and they will be useful in our discussion of the geometric convergence in Section 8.17. I will give here only a general introduction to the theory of complex projective structures, for details and recent developments see [Hej75], [KT92], [Ka95b], [GKMar], [Gol87b], [Tan97a], [Tan97b], [McM98].

7.1. Basic definitions

Let S be a smooth surface without boundary. A *complex projective structure* σ on S is a maximal atlas such that all transition maps belong to the group $PSL(2,\mathbb{C})$. Each complex projective structure on S defines a local diffeomorphism dev from the universal covering $\tilde{S}$ to the extended complex plane $\widehat{\mathbb{C}}$. Locally, the map dev is a complex–projective diffeomorphism with respect to the complex projective structures on $\tilde{S}$ and $\widehat{\mathbb{C}}$. The map dev is called a *developing map* of σ. Consider the action of the fundamental group $F = \pi_1(S)$ on $\tilde{S}$ as the group of covering transformations. Then the there is a homomorphism $\rho : F \to PSL(2,\mathbb{C})$ so that dev is ρ-equivariant. The representation ρ is called a *monodromy*[1] *representation* of the structure σ. This representation is unique up to conjugation in $PSL(2,\mathbb{C})$. The group $\rho(F)$ is called the *monodromy group*. Conversely, if we are given a local homeomorphism $dev : \tilde{S} \to \widehat{\mathbb{C}}$ which is ρ-equivariant, then we can take the pull back $dev^*(can)$ of the canonical complex projective structure from $\widehat{\mathbb{C}}$ to $\tilde{S}$. The group F acts as a group of automorphisms of $dev^*(can)$, thus the projection of $dev^*(can)$ to $\tilde{S}/F$ is a complex projective structure σ. The map dev is a developing map of this structure. An important class of complex projective structures is given by *uniformization*. Suppose that $\Gamma \subset PSL(2,\mathbb{C})$ is a Kleinian group acting discontinuously and freely on a domain $D \subset \widehat{\mathbb{C}}$. Then the canonical complex projective structure on D

[1] or *holonomy*

projects to a complex projective structure on $S = D/\Gamma$ which is *uniformized* by Γ. In this case the monodromy representation is an epimorphism $\rho : \pi_1(S) \to \Gamma$ with the kernel $\pi_1(D)$. The developing map is the covering $\tilde{S} \to D$. However complex projective structure does not have to appear this way. In particular, the monodromy representation can be nondiscrete and image of the developing map can be the whole sphere $\widehat{\mathbb{C}}$.

Each complex projective structure σ on S corresponds to a complex structure on S. In what follows we will consider only those structures σ which correspond to Riemann surfaces of *finite type*. We will say that σ has *finite type* if the following is satisfied:

For each puncture q in S and a small punctured disk neighborhood $U \subset S$ of q the projection of the developing map *dev* to U is either a well-defined injective holomorphic map or is equivalent[2] to the logarithmic map

$$\log : \{z \in \mathbb{C} : 0 < |z| < 1\} \to \widehat{\mathbb{C}} \ .$$

In particular, the image of the small loop around q under the monodromy ρ is either trivial or a parabolic element of $PSL(2, \mathbb{C})$.

Similarly to the definition of Teichmüller space one can define the space $MCP(S)$ of all marked complex projective structures of finite type on S. Namely, we fix a complex projective structure s of finite type on S, then points in $MCP(S)$ are the equivalence classes of marked complex projective surfaces, which are triples (Σ, σ, f), where Σ is a surface homeomorphic to S, σ is a complex projective structure of finite type on Σ and $f : (S, s) \to (\Sigma, \sigma)$ is an orientation-preserving homeomorphism. We identify two triples (Σ, σ, f), (Σ', σ', f') if there is a complex projective diffeomorphism $\phi : \Sigma \to \Sigma'$ such that $\phi \circ f$ is homotopic to f'.

Proposition 7.1. *Suppose that two triples (Σ, σ, f), (Σ', σ', f') are equivalent via a mapping $\phi : \Sigma \to \Sigma'$. Let d and d' be developing maps for these marked surfaces. Then there exists a Moebius transformation $\xi \in PSL(2, \mathbb{C})$ such that $\xi \circ d' \circ \phi = d$.*

Proof: Follows directly from the definition of the developing map. □

Here is another way to think about complex projective structures. Given a representation ρ we construct the associated bundle $E = S \times_\rho \widehat{\mathbb{C}}$ as the quotient of $\tilde{S} \times \widehat{\mathbb{C}}$ by the following action of F:

$$\gamma(w, z) = (\gamma(w), \rho(\gamma^{-1})(z)), \quad w \in \tilde{S}, z \in \widehat{\mathbb{C}}, \gamma \in F.$$

The bundle $E = E_\rho$ has a natural flat connection $\omega = \omega_\rho$. Smooth ρ-equivariant maps $\tilde{S} \to \widehat{\mathbb{C}}$ naturally correspond to sections of the bundle E. Such a map is a local diffeomorphism iff the corresponding section is transversal to ω.

Properties of monodromy representations. Suppose that S is a Riemann surface of finite (hyperbolic) type, σ is a complex projective structure of finite type on S with the monodromy representation $\rho : \pi_1(S) \to PSL(2, \mathbb{C})$. Then:

[2] Up to precomposition and postcomposition with diffeomorphisms.

1. ρ lifts to a representation $\tilde{\rho} : \pi_1(S) \to SL(2, \mathbb{C})$.

2. If S is compact then ρ is always a nonelementary representation.

3. The bundle E_ρ is always trivial.

It turns out that (in the case S is compact) the first two conditions are sufficient for ρ being the monodromy of a complex projective structure on S (see [Ka95b], [GKMar]).

Therefore, instead of thinking about complex projective structures, one can work with the pair (ω, d), where ω is a flat connection on $E = S \times \widehat{\mathbb{C}}$ and d is a smooth section of E which is transversal to ω, this section will be called the *developing section* (see Theorem 4.15). This allows us to topologize the space $MCP(S)$ of marked complex projective structures on S by using C^1 topology on the space of section of E and C^1 topology on the space of connections on E.

There is an alternative point of view on complex projective structure coming from the complex analysis. Let S be a Riemann surface of finite (hyperbolic) type. Each marked complex projective structure σ on S (which is consistent with the orientation on S) also determines a marked complex structure $p(\sigma) \in \mathcal{T}(S)$. Thus we get a natural projection $MCP(S) \to \mathcal{T}(S)$ which is easily seen to be a continuous map. Now fix $\tau = p(\sigma)$ and consider the fiber $p^{-1}(\tau) \subset MCP(S)$. This fiber can be described as follows. Uniformize τ by a Fuchsian group F, $(S, \tau) = \mathbb{H}^2/F$. Then for each $\nu \in p^{-1}(\tau)$ the developing map $f = dev_\nu$ of ν is a locally injective holomorphic mapping $\mathbb{H}^2 \to \widehat{\mathbb{C}}$ which is ρ-equivariant where ρ is a monodromy homomorphism of ν. Then the *Schwarzian derivative*

$$\tilde{\varphi}(z) := \{f, z\} = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

determines a holomorphic quadratic differential $\tilde{\varphi} dz^2$ on $\mathbb{H}^2$ which is F-invariant. Hence $\tilde{\varphi} dz^2$ projects to a holomorphic quadratic differential φdz^2 on the surface (S, τ). The fact that ν has finite type implies that φ has at worst simple poles at the punctures. Vice-versa, if φdz^2 is a holomorphic quadratic differential on the surface (S, τ) then we lift it to an F-invariant holomorphic quadratic differential $\tilde{\varphi} dz^2$ on $\mathbb{H}^2$. Then solve the Schwarzian differential equation $\{f, z\} = \tilde{\varphi}$. Each solution is a locally injective holomorphic map $f : \mathbb{H}^2 \to \widehat{\mathbb{C}}$, which is unique up to post-composition with $\gamma \in PSL(2, \mathbb{C})$ and which is therefore ρ-equivariant with respect to a homomorphism $\rho : F \to PSL(2, \mathbb{C})$. This defines a map $MCP(S) \to Q(S)$ where $Q(S)$ is the affine bundle over $\mathcal{T}(S)$ whose fiber over $\tau \in \mathcal{T}(S)$ is the space of holomorphic quadratic differentials on (S, τ). One can easily show that this map is a homeomorphism. (It is actually a real-analytic diffeomorphism.) Thus we can identify $MCP(S)$ with $Q(S) \cong \mathcal{T}(S) \times Q_0$, where Q_0 is the vector space of holomorphic quadratic differentials (with at worst simple poles at the punctures) on (S, τ_0) for a fixed conformal structure τ_0 on S. Therefore topologically $MCP(S)$ is just $\mathbb{C}^{6g-6+2n}$ where g is the genus of S and n is the number of punctures.

There is a natural map $hol : MCP(S) \to \mathcal{R}(F, PSL(2,\mathbb{C}))$ which associates with each complex projective structure the conjugacy class of its monodromy representation. The following result is crucial for applications of complex projective structures in the theory of Kleinian groups. Identify F with a Fuchsian subgroup of $PSL(2,\mathbb{R})$.

Theorem 7.2. *(The Holonomy Theorem.) The map*

$$hol : MCP(S) \to \mathcal{R}_{par}(F, PSL(2,\mathbb{C}))$$

is an open mapping. If S is a closed hyperbolic surface then this mapping is a local homeomorphism.

Proof: This theorem was first proven by D. Hejhal [Hej75] in the case of compact surfaces (see [ECG87], [Gol87a] for generalizations). Our proof follows [Gol87a]. Choose a compact subsurface $S_c \subset S$ which is a deformation retract of S. Suppose that $[\rho] = hol(\sigma)$, let ω denote the corresponding flat connection and d the section of the restriction of the bundle E to S_c. Let $[\rho_n] \in \mathcal{R}_{par}(F, PSL(2,\mathbb{C}))$ be a sequence convergent to $[\rho]$. Then we have C^1–convergence of the corresponding flat connections $\omega_n \to \omega$ (see Theorem 4.15). Choose $d_n = d$ to be the developing section. Therefore for large values of n the section d is transversal to ω_n. It is easy to see that the developing sections d_n extend to $S - S_c$ so that the resulting complex projective structure $\sigma_n = [\omega_n, d_n]$ has finite type.

Here is a down-to-earth alternative proof of the first assertion of theorem which I will describe in the case of compact surfaces. Let $f : \tilde{S} \to \widehat{\mathbb{C}}$ be the developing mapping of the structure σ. Let F denote the group of deck-transformations of the universal covering $\tilde{S} \to S$. Choose a triangulation T of S so that each edge is a circular arc with respect to the complex projective structure σ and each simplex is contained in a coordinate neighborhood of σ. Lift this triangulation to a triangulation $\tilde{T}$ of the universal cover $\tilde{S}$ of S. Pick a finite collection $\Delta_1, ..., \Delta_m$ of 2-simplices in $\tilde{T}$, one for each orbit. Let $g_i, i = 1, ..., N$, be the elements of the deck-transformation group F, so that $g_i(\cup_j \Delta_j) \cap \cup_j \Delta_j \neq \emptyset$. Let C be a compact subset of $\tilde{S}$ whose interior contains both $D := \cup_j \Delta_j$ and its images under g_i's. For each ρ_n construct a continuous ρ_n-equivariant mapping $f_n : D \to \widehat{\mathbb{C}}$ so that:

(i) f_n maps each 2-simplex diffeomorphically to a 2-simplex in $\widehat{\mathbb{C}}$ bounded by circular arcs.

(ii) f_n's converge to $f|D$ in the C^1-topology as n tends to infinity.

Finally, extend each f_n to ρ_n-equivariant mapping $f_n : \tilde{S} \to \widehat{\mathbb{C}}$. It remains to show that each mapping f_n is a local homeomorphism for large n. It suffices to check this for points in D.

(a) If $x \in int(C)$ belongs to the interior of a 2-simplex in $\cup_i g_i D$, then the claim follows since each f_n is homeomorphism on each simplex. (b) Suppose x belongs to the interior of a common arc α of two 2-simplices Δ, Δ' in $\cup_i g_i D$. Since f is a local homeomorphism, $f(\Delta), f(\Delta')$ lie (locally) on different sides of the circular arc $f(\alpha) \subset \widehat{\mathbb{C}}$. Therefore the same holds for f_n if n is sufficiently large. So, f_n does not "fold" along the arc α and is a local homeomorphism at x. (c) Lastly, if x is a vertex of a simplex, then the degree of f at x equals 1, hence for large n, the degree of f_n at x is 1 and it follows from (b) that f_n is a local homeomorphism at x.

Equivariance of f_n's imply that they converge to f uniformly on compacts. A priori f_n's are not smooth and this convergence is not C^1. This is easily rectified by smoothing f_n's equivariantly near the vertices and the edges of $\tilde{T}$. We leave it to the reader to generalize this proof to the case of surfaces with punctures.

We now prove the second assertion of theorem. Suppose that σ is a complex projective structure on S with the monodromy $\rho : F = \pi_1(S) \to PSL(2, \mathbb{C})$ and developing mapping $f : \mathbb{H}^2 \to \widehat{\mathbb{C}}$. The representation ρ is nonelementary. Let Φ be a compact fundamental polygon for the the action the Fuchsian group F on $\mathbb{H}^2$. Let U denote the closure of an open neighborhood of Φ in $\mathbb{H}^2$. Suppose that σ_1, σ_2 are structures near σ with the monodromy representations ρ_1, ρ_2 which are conjugate in $PSL(2, \mathbb{C})$. Then we may assume that $\rho_1 = \rho_2$ are near ρ (here we use the fact that ρ is nonelementary). Let $f_1, f_2 : \mathbb{H}^2 \to \widehat{\mathbb{C}}$ denote the developing mappings of σ_1, σ_2. Then the restrictions of the to U are near $f|U$ in C^1-topology. Thus there exists $\epsilon > 0$ such that for each $z \in \Phi$ the set $Z := f_2^{-1} f_1(z) \cap U$ has the property:

(1) Z contains a point $z' \in U$ such that $d(z, z') \le \epsilon/2$ and for each $z'' \in Z - \{z'\}$, $d(z, z'') \ge \epsilon$.

(2) ϵ is less than the injectivity radius of the hyperbolic surface S.

Then define the mapping $h : \Phi \to U$ as $h(z) := z'$. This mapping is well-defined, continuous and is locally linear-fractional. Hence h is the restriction of a linear-fractional transformation to Φ. Since $\rho_1 = \rho_2$ we conclude that for each side-pairing transformation $g \in F$, $g(h(z)) = hg(z)$, $z \in \Phi$. Therefore h projects to a conformal automorphism $S \to S$ which is homotopic to the identity. Hence $h = id$. $\square$

As a simple corollary we prove

Theorem 7.3. *Let $F \subset PSL(2, \mathbb{R})$ be a Fuchsian subgroup. Then the Teichmüller space $\mathcal{T}_R(F)$ is open in (with respect to the classical topology) in $\mathcal{R}_{par}(F, PSL(2, \mathbb{R}))$.*

Proof: Suppose that $[\rho] \in \mathcal{T}_R(F)$ is the limit of $[\rho_j] \in \mathcal{R}_{par}(F, PSL(2, \mathbb{R}))$. Then for large j the representations ρ_j are monodromy representations of complex projective structures σ_j of finite type. Let $d_j : \tilde{S} \to \widehat{\mathbb{C}}$ be the developing maps, where $\tilde{S}$ is the universal cover of S. Then for large j the image of each d_j is contained in $\mathbb{H}^2$. Hence the pull-back of the hyperbolic structure on $\mathbb{H}^2$ via d_j is an F-invariant complete hyperbolic metric on $\tilde{S}$. This implies that d_j's are diffeomorphisms for large j. Hence $[\rho_j] \in \mathcal{T}_R(F)$. $\square$

7.2. Grafting

In this section we discuss a very basic example of complex projective structure which has surjective developing map and discrete monodromy. The first example of this kind was constructed by B.Maskit [Mas69]. We start with a hyperbolic surface S and a simple closed geodesic $\gamma \subset S$. Let σ denote the hyperbolic complex projective structure on S. For sufficiently small $\epsilon > 0$ the ϵ-neighborhood $Nbd_\epsilon(\gamma)$ of γ in S is diffeomorphic to the annulus. We shall assume that the universal cover of S is embedded in $\widehat{\mathbb{C}}$ as the hyperbolic upper half-plane $\mathbb{H}^2$. Let $g \in F = \pi_1(S)$ be the element of the fundamental group corresponding to the generator of $\pi_1(\gamma)$. The quotient $\Omega(\langle g\rangle)/\langle g\rangle$ is the torus T^2 which has complex projective structure τ uniformized by the group $\langle g\rangle$, this torus contains a copy γ' of the closed loop γ. Cut the surface S and the torus T^2 open along the geodesics γ and γ'. Now glue these surfaces together along their boundary loops using the identification $\gamma \sim \gamma'$ so that the orientations of S and T^2 agree. The resulting surface S' is diffeomorphic to S and has a new complex projective structure $gr_\sigma(\gamma)$ which restricts to the structures σ and τ on S and T^2. The structure $gr_\sigma(\gamma)$ is said to be obtained from the hyperbolic structure on S via *grafting* along γ. If we identify the fundamental groups of S and S' with the Fuchsian group F then the monodromy representation of $gr_\sigma(\gamma)$ is the identity map $F \to F \hookrightarrow PSL(2,\mathbb{C})$. The image of the developing map $dev_{gr_\sigma(\gamma)}$ of the structure $gr_\sigma(\gamma)$ intersects the limit set of $F \subset PSL(2,\mathbb{R})$ in a nonempty subset (which contains $\Omega(\langle g\rangle) \cap \Lambda(F)$). Therefore (by equivariance) $dev_{gr_\sigma(\gamma)}$ is onto $\widehat{\mathbb{C}}$. Instead of the structure τ on the torus T^2 we could have taken its n-fold ($n \geq 1$) cover $\tau(n)$ which is induced by an n-fold covering over $\Omega(\langle g\rangle)$. Then we again glue S' and the complex-projective cylinder $(T^2 - \gamma', \tau(n))$. We get a complex projective structure $gr_\sigma(n\gamma)$ on the surface S. More generally, if $\gamma_1, ..., \gamma_r$ is a collection of simple closed disjoint geodesics on S and $n_1, ..., n_r$ is a collection of positive integers we define the *grafting* $gr_\sigma(n_1\gamma_1 \cup ... \cup n_r\gamma_r)$. The structure $gr_\sigma(n_1\gamma_1 \cup ... \cup n_r\gamma_r)$ still has the identity monodromy representation and the surjective developing map $\tilde{S} \to \widehat{\mathbb{C}}$. Similarly one defines grafting for *quasifuchsian projective structures.* (The definition actually can be made much more general.)

Definition 7.4. Let G be a torsion-free quasifuchsian subgroup of $PSL(2,\mathbb{C})$ without parabolic elements, $\Omega(G) = \Omega_1 \cup \Omega_2$ is the disjoint union of two topological disks. Then the projection of the canonical complex projective structure from $\widehat{\mathbb{C}}$ to Ω_j/G is called a **quasifuchsian complex projective structure**.

Clearly we have two marked quasifuchsian structures with the monodromy G: Ω_1/G and Ω_2/G, these structures have opposite orientation. If S is a closed orientable hyperbolic surface then a representation $\rho : \pi_1(S) \to \mathrm{Isom}(\mathbb{H}^3)$ is called **quasifuchsian** if it is faithful and its image is a quasifuchsian group.

Theorem 7.5. *(W. Goldman [Gol87b].)*[3] *Suppose that μ is a complex projective structure on S with the quasifuchsian monodromy representation $\rho : F \to \rho(F) = G \subset PSL(2,\mathbb{C})$ and the same orientation as the quasifuchsian projective structure σ on S, given by Ω_1/G. Then either $\mu = \sigma$ or μ is equivalent to a structure which is obtained from σ by grafting.*

[3] See also [Ka88] for a generalization to 3-dimensional flat conformal manifolds.

The above theorem allows us to describe the topology of the space of complex projective structure on S with quasifuchsian monodromy as follows. Fix an orientation o and hyperbolic structure on S and let $QF(S)$ denote the space of marked quasifuchsian complex projective structures on S which are compatible with the orientation o. Let C denote the set whose elements are formal linear combinations

$$n_1\gamma_1 + \ldots + n_m\gamma_m$$

where n_j are positive integers, $\gamma_1, ..., \gamma_m$ are disjoint simple closed geodesics in S ($0 \leq m \leq 3g-3$ where g is the genus of S). Give C the discrete topology. Then $hol^{-1}QF(S)$ is naturally homeomorphic to to the direct product $QF(S) \times C$. Namely, each structure $\mu \in hol^{-1}QF(S)$ equals $gr_\sigma(n_1\gamma_1 \cup ... \cup n_m\gamma_m)$ where σ is a quasifuchsian structure on S with the monodromy ρ and the orientation consistent with o. Then we map μ to $(n_1\gamma_1 + \ldots + n_m\gamma_m, \sigma)$. One can easily see that this map is a homeomorphism.

At the first glance the quasifuchsian structure and its grafting seem to be thousands miles apart. It turns out however that this is not quite the case:

Theorem 7.6. *(C. McMullen, [McM98].) Let γ be a simple closed loop. Then the closures of $QF(S)$ and $gr_{QF(S)}(\gamma)$ in $MCP(S)$ have nonempty intersection.*

Chapter 8

Sociology of Kleinian Groups

8.1. Algebraic convergence of representations

Let G be a Lie group and Γ a finitely generated group. A representation $\rho : \Gamma \to G$ is said to be *discrete* if its image is a discrete subgroup of G. A representation is said to be *faithful* if it is a monomorphism.

Definition 8.1. Suppose that ρ_j is a sequence of representations of Γ to G. Then the sequence ρ_j is said to be **algebraically convergent** to a representation ρ iff for each $g \in \Gamma$ we have:

$$\lim_{n\to\infty} \rho_j(g) = \rho(g) \tag{8.1}$$

where convergence is understood in the topology of the Lie group G.

The topology of algebraic convergence is consistent with the usual topology of the representation variety $\mathcal{R}^0(\Gamma, G)$. This is the most intuitively obvious concept of convergence of representations. As we shall see, there are some other definitions of convergence which differ from this one.

Definition 8.2. If Γ is a non-radical group we let $\mathcal{D}(\Gamma, G)$ denote the space of G-conjugacy classes of discrete and faithful representations of Γ to the Lie group G (most of the time we will have $G = \mathrm{Isom}(\mathbb{H}^n)$ or $G = PSL(2, \mathbb{C})$).

Note that there is no problem here with taking quotient, since we consider only nonradical representations. Similarly we can define relative versions of $\mathcal{D}(\Gamma, G)$:

If $H = \{H_1, ..., H_m\}$ is a collection of subgroups in Γ then

$$\mathcal{D}(\Gamma, H; G) := \mathcal{D}(\Gamma, G) \cap \mathcal{R}(\Gamma, H; G).$$

If $\Gamma \subset G = \mathrm{Isom}(\mathbb{H}^n)$ is a discrete subgroup then define

$$\mathcal{D}_{par}(\Gamma, G) = \mathcal{D}(\Gamma, G) \cap \mathcal{R}_{par}(\Gamma, G).$$

Recall that $\mathcal{R}_{par}(\Gamma, G) = \mathcal{R}(\Gamma, H; G)$ where H is the collection of maximal parabolic subgroups of Γ.

Example 8.3. Let $\Gamma_n = \langle a, b_n \rangle$ be an Abelian subgroup of $\mathrm{Isom}(\mathbb{C})$, where $a : z \mapsto z+1$, $b_n : z \mapsto z+p+i/n$ where $n \in \mathbb{Z}, p \in \mathbb{R}$. Define the sequence of representations $\rho_n : \Gamma_1 \to \Gamma_n$ so that $a \mapsto a$, $b_1 \mapsto b_n$. Then the sequence $\{\rho_n\}$ consists of discrete and faithful representations. This sequence has algebraic limit ρ so that $\rho(a) = a$, $\rho(b_1) : z \mapsto z + p$. On the other hand, this limit is always either nondiscrete or nonfaithful (depending on p). It turns out that this is an exceptional situation.

Recall that in Theorem 4.4 we proved that for each Lie group G there exists a number $m = m(G)$ so that $m-1$ bounds from above the class of each torsion-free nilpotent subgroup of G; i.e. if $g_1, ..., g_{m+1} \in G$ generate a torsion-free subgroup Γ and the m-fold commutator

$$[\dots[[g_1, g_2], g_3] \dots, g_{m+1}] \tag{8.2}$$

is nontrivial, then Γ is not nilpotent.

Theorem 8.4. *Suppose that Γ is a finitely generated non-radical[1] group, ρ_i is a sequence of discrete and faithful representations of Γ into a linear Lie group G, which converges algebraically to $\rho : \Gamma \to G$. Then ρ is again discrete and faithful. In other words, $\mathcal{D}(\Gamma, G)$ is closed in $\mathcal{R}(\Gamma, G)$.*

Remark 8.5. In the case $G = \mathrm{Isom}(\mathbb{H}^n)$ this theorem is called the "Chuckrow's theorem", despite the fact that V. Chuckrow [Chu68] proved it only for Kleinian subgroups in $PSL(2, \mathbb{C})$ and her arguments cannot be generalized to higher dimensions. A better name would be probably "Wielenberg's theorem", since Wielenberg [Wie77] proved it for representations to $SO(n, 1)$.

Proof: (i) First suppose that ρ is not faithful, let K denote the kernel of ρ. The orders of finite order elements of Γ are bounded from above by Selberg lemma. Since G contains only finitely many conjugacy classes of elements of the given order (see Lemma 4.20) and

$$\lim_{i \to \infty} \rho_i(g) = 1, \quad \text{for each} \quad g \in K,$$

the subgroup K is torsion-free. Pick any finite collection of elements $g_1, ..., g_k \in K$. For all sufficiently large i the images $\rho_i(g_j), 1 \le j \le k$, belong to Zassenhaus neighborhood U_Z of the identity in G. Hence the subgroup generated by $g_1, ..., g_k$ is nilpotent. Therefore K is a torsion-free subgroup of G which is exhausted by nilpotent subgroups. The classes of these subgroups are bounded from above by $c < \infty$, hence K is nilpotent as well. We get a normal infinite nilpotent subgroup $K \subset \Gamma$, this contradicts the assumption that Γ is non-radical.

(ii) Since Γ is a finitely generated linear group we may apply Selberg lemma to Γ. Hence, after passing to a finite-index subgroup in Γ we may

[1] See §4.3.

assume that Γ and all its images $\rho_i(\Gamma)$ are torsion-free groups. Now let us assume that the group $\rho(\Gamma)$ is not discrete. The closure $H = \overline{\rho\Gamma}$ of this group in G (in the classical topology) is a Lie subgroup in G. Let H^0 be the identity component of H; it is a normal subgroup of H and $dim(H^0) > 0$. Note that $\rho(\Gamma) \cap H^0$ is dense in H^0 since H^0 is open in H, in particular $\rho(\Gamma) \cap H^0$ is infinite. If H^0 is not nilpotent then we can find elements $h_1, ..., h_{m+1} \in H^0 \cap U_Z$ such that the m-fold commutator $[\ldots[[h_1, h_2], h_3] \ldots, h_{m+1}]$ is non-trivial. (Here $m-1$ is the upper bound on the class of each torsion-free nilpotent subgroup of G). By approximating h_j's by the elements of $\rho(\Gamma)$ we find $\rho(g_1), \ldots \rho(g_{m+1}) \in H^0 \cap U_Z$ such that

$$[\ldots[[\rho(g_1), \rho(g_2)], \rho(g_3)] \ldots, \rho(g_{m+1})] \neq 1.$$

Hence for large i's the the elements $\rho_i(g_j), 1 \leq j \leq m+1$, generate a non-nilpotent subgroup of G. Since for sufficiently large i's the elements $\rho_i(g_j), 1 \leq j \leq m+1$, belong to U_Z, Zassenhaus theorem implies that the subgroup generated by $\rho_i(g_j), 1 \leq j \leq m+1$, is nilpotent. Contradiction. Thus H^0 is a nilpotent Lie group. Since the representation $\rho : \Gamma \to H$ is faithful, it follows that $\rho^{-1}(H^0)$ is an infinite normal nilpotent subgroup of Γ which again contradicts the assumption that Γ is non-radical. $\square$

Remark 8.6. See [Mar93] for the generalization of the above theorem to the case of manifolds of variable negative curvature.

8.2. Geometric convergence

The notion of algebraic convergence of representations is nice and simple, however there are situations when it does not reflect changes in geometry of the convergent sequences of groups. This fact was first recognized by L. Bers and T. Jorgensen in 1970's, which lead to the definition of *geometric convergence* below. As before, let G be a Lie group.

Definition 8.7. Suppose that $\Gamma_j \subset G$ is a sequence of closed subgroups. Then the **geometric limit** $Lim^{geo}_{j\to\infty}(\Gamma_j)$ of this sequence is a subgroup $\Gamma_\infty \subset G$ such that:

(a) For each convergent subsequence $\gamma_{j_i} \in \Gamma_j$ the limit $\lim_i \gamma_{j_i}$ belongs to Γ_∞.

(b) For each $\gamma \in \Gamma_\infty$ there is a sequence $\gamma_j \in \Gamma_j$ such that $\lim_j \gamma_j = \gamma$. In other words, the sequence $\{\Gamma_j\}$ converges to Γ_∞ in the Chabauty topology.

If $Lim^{geo}_{j\to\infty}(\Gamma_j) = \Gamma_\infty$ then we will also say that the groups Γ_j converge to Γ_∞ *geometrically*. Since the collection of closed subsets in G is compact in the Chabauty topology, every sequence of closed subgroups in G contains a convergent subsequence.

Example 8.8. Let G be a linear Lie group, Γ a finitely generated infinite subgroup. Consider the sequence of finite-index subgroups $\Gamma_n \subset \Gamma$ such that $\cap_n \Gamma_n = \{1\}$. Then $Lim^{geo}_{n\to\infty}(\Gamma_n) = \{1\}$.

The following proposition generalizes Theorem 8.4 (compare [Bel98]). I am grateful to I. Belegradek who noticed an error in the original version of this proposition.

Proposition 8.9. *Let* Γ *be a non-radical finitely generated group and* $\rho_j : \Gamma \to G = Isom(\mathbb{H}^n)$ *be a sequence of discrete and faithful representations which converge to* $\rho_\infty : \Gamma \to G$. *Assume that* $Lim^{geo}_{j\to\infty}\rho_j(\Gamma) = H$. *Then* H *is a discrete subgroup of* G.

Proof: Let $\{\gamma_1, ..., \gamma_k\}$ be a generating set of Γ; let Γ_j denote the image of ρ_j, $j = 1, 2, ...$ We argue as in the proof of Theorem 8.4: assume that H is not discrete. The geometric limit H is a closed subgroup of G; let H^0 be the identity component of H. If H^0 is not nilpotent then (as in the proof of Theorem 8.4) we can find elements $g_1, ..., g_{m+1} \in H^0 \cap U_Z$ such that the m-fold commutator

$$[\cdots[[g_1, g_2], g_3]\cdots, g_{m+1}]$$

is non-trivial. By the definition of geometric convergence, the elements g_i are approximated by $g_{i,j} \in \Gamma_j$. Hence for large j we have: $g_{i,j} \in U_Z$, therefore $g_{i,j}, i = 1, ..., m+1$ generate a nilpotent group (by Zassenhaus theorem). This contradiction implies that the group H^0 is nilpotent.

Recall that according to Selberg lemma, the orders of elliptic elements of Γ_j are uniformly bounded from above. Hence, by choosing sufficiently small Zassenhaus neighborhood U_Z, we get: $U_Z \cap \Gamma_j$ contains no elliptic elements.

Let $V \subset W \subset U_Z$ be neighborhoods of identity in G such that $cl(W) \subset U_Z$ and for each $\gamma_i, i = 1, ..., k$, we have:

$$\rho_\infty(\gamma_i)V\rho_\infty(\gamma_i)^{-1} \subset W.$$

Define $R_j := \Gamma_j \cap U_Z$ and the group N_j generated by R_j. The subgroup N_j is nontrivial, nilpotent and torsion-free, hence it has a unique fixed point p_j in $\partial_\infty\mathbb{H}^n$ or a unique invariant geodesic L_j. For large j the intersection $\Gamma_j \cap V$ is nonempty (it contains elements which the approximate elements of $H \cap V$) and for each $R \in \Gamma_j \cap V$ and $1 \le i \le k$, we have

$$\rho_j(\gamma_i)R\rho_j(\gamma_i)^{-1} \in U_Z.$$

Thus $\rho_j(\gamma_i)N_j\rho_j(\gamma_i)^{-1} \cap N_j$ contains parabolic or loxodromic elements. If N_j contains parabolic elements, then $\rho_j(\gamma_i)(p_j) = p_j$, $1 \le i \le m$. If N_j contains loxodromic elements then $\rho_j(\gamma_i)(L_j) = L_j$, $1 \le i \le k$. Thus Γ_j has either a fixed point in $\partial_\infty\mathbb{H}^n$ or an invariant geodesic, hence Γ_j is elementary. Contradiction. $\square$

Remark 8.10. The same arguments work in the case of all rank 1 Lie groups G. I do not know what happens in the case of semi-simple Lie groups of higher rank. One can easily exclude the case when H^0 is semi-simple, but it is unclear how to proceed in the case of nontrivial unipotent radical in H^0.

We now refine the notion of the quasi-isometric topology (see Definition 3.41) for the sequences of complete hyperbolic manifolds. Consider a sequence M_j of complete n-dimensional hyperbolic manifolds and elements $v_j \in \mathcal{OF}(M_j), v \in \mathcal{OF}(M)$ of the orthonormal frame bundles of M_j, M. Let x_j, x denote the projections of v_j, v to M_j, M. The sequence (M_j, v_j) is convergent to (M, v) in the *refined Gromov-Hausdorff topology* if for sufficiently large $R > 0$ and each $\kappa > 1$ there exists j_0 so that for all $j > j_0$ there are open neighborhoods U_j, U of $B_R(x_j) \subset M_j$, $B_R(x) \subset M$ and maps $\alpha_j : (U, v) \to (U_j, v_j)$ which are κ-bilipschitz diffeomorphisms. (Recall that $B_R(x)$ is the metric R-ball centered at x.)

Theorem 8.11. *(Benedetti and Petronio [BP92], Epstein, Canary and Green [ECG87, Theorem 3.2.9].) The following properties are equivalent:*

1. *A sequence of discrete torsion-free groups $G_j \subset \mathrm{Isom}(\mathbb{H}^n)$ is convergent geometrically to a group G.*

2. *The pointed manifolds $(M(G_j), x_j)$ are convergent to $(M(G), x)$ in the quasi-isometric topology for some choice of base-points.*

3. *The pointed manifolds $(M(G_j), v_j)$ converge to $(M(G), v)$ in the refined Gromov-Hausdorff topology for some choice of elements $v_j \in \mathcal{OF}(M(G_j))$, $v \in \mathcal{OF}(M(G))$.*

Moreover, if $\rho_j : \Gamma \to G_j$ are isomorphisms of finitely generated groups converging to an isomorphism $\rho_\infty : \Gamma \to \Gamma_\infty \subset G$ so that $Lim^{geo}_{j\to\infty} G_j = G$, then the maps $\alpha_j : U \to U_j$ can be chosen so that they induce the maps $\rho_j \circ \rho_\infty^{-1}$ on the subgroup Γ_∞.

Exercise 8.12. *Suppose that the sequence $(M(G_n), x_n)$ converges to (M, x) in the geometric topology and M has finite volume. Then for large n each manifold $M(G_n)$ has finite volume. (Hint: first show that the thick part of $M(G_n)$ is compact for large n.)*

Suppose that $\rho_n : \Gamma \to \Gamma_n \subset \mathrm{Isom}(\mathbb{H}^3)$ is a sequence of nonelementary discrete and faithful representations, which is algebraically convergent to a representation $\rho_\infty : \Gamma \to \Gamma_\infty$. Assume that Γ_n converge geometrically to a group G_∞. There are three essentially equivalent ways to state the basic problem which can appear.

1. The quotient manifolds $M(\Gamma_n) = \mathbb{H}^3/\Gamma_n$ do not necessarily converge in the quasi-isometric topology to the quotient manifold $M(\Gamma_\infty)$ for **any** choice of base-points.

2. The Hausdorff limit of the sequence of limit sets $\Lambda(\Gamma_n)$ can be much bigger than the limit set of the group Γ_∞.

3. The geometric limit G_∞ could be strictly larger than the algebraic limit Γ_∞.

This phenomenon was first discovered by Troel Jorgensen and I describe below the simplest example of it following [Thu81]. Start with a sequence of loxodromic representations of a cyclic group which is convergent to a discrete parabolic group. Let $w_n = n^{-2} + \pi i/n$. Define the sequence of representations $\phi_n : \mathbb{Z} \to SL(2, \mathbb{C})$ as:

$$a_n = \phi_n(1) = \begin{pmatrix} \exp(w_n) & n\sinh(w_n) \\ 0 & \exp(-w_n) \end{pmatrix}.$$

Therefore $a = \phi(1) = \lim_{n\to\infty} \phi_n(1)$ is the matrix:

$$\phi(1) = \begin{pmatrix} 1 & \pi i \\ 0 & 1 \end{pmatrix}.$$

However the limit of the sequence $\phi_n(n)$ is the matrix

$$b = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}.$$

Thus the geometric limit contains $\mathbb{Z}^2$. One can see that in this example the geometric limit is exactly the group $\mathbb{Z}^2$. However the limit sets of the groups $\phi(\mathbb{Z})$ and $\mathbb{Z}^2 = Lim^{geo}_{n\to\infty}\phi_n(\mathbb{Z})$ coincide (they are equal to a single point). To construct a more sophisticated example one can use the Klein Combination. Namely, let c be a loxodromic transformation in $\widehat{\mathbb{C}}$ which does not fix the point ∞. Then for sufficiently large power m the complements to the fundamental domains of $\langle d = c^m\rangle$ and $\langle a_n\rangle$ are disjoint for all n. A fundamental domain for the group $\mathbb{Z}^2 = Lim^{geo}_{n\to\infty}\phi_n(\mathbb{Z})$ is a rectangle R of the shape $1 \times \pi$. One can also assume that the complement to the fundamental domain Φ of $\langle d\rangle$ is contained in R. Then the groups $\rho_n(\mathbb{Z}*\mathbb{Z}) = \langle a_n, d\rangle$ are Kleinian and $\rho_n : \mathbb{Z} * \mathbb{Z} \to \langle a_n, d\rangle$ are isomorphisms. The group $H = \langle a, b, d\rangle$ is isomorphic to $(\langle a\rangle \oplus \langle b\rangle) * \langle d\rangle$. Therefore it cannot contain $\Gamma = \mathbb{Z} * \mathbb{Z}$ as a subgroup of finite index. On the other hand, H is contained in the geometric limit of the sequence $\rho_n(\mathbb{Z} * \mathbb{Z})$. One can actually prove that the geometric limit is exactly the group H. The group $\Gamma = \langle a\rangle * \langle d\rangle$ is the algebraic limit of $\rho_n(\mathbb{Z} * \mathbb{Z})$. This group is geometrically finite, therefore its limit set cannot coincide with the limit set of H (which contains Γ).

Theorem 8.13. *Suppose that Γ is a finitely generated nonelementary group, $\rho_n : \Gamma \to \mathrm{Isom}(\mathbb{H}^3)$ is a sequence of discrete and faithful representations which converges algebraically to a representation ρ_∞, while the groups $\Gamma_n = \rho_n(\Gamma)$ converge geometrically to a group $G \subset \mathrm{Isom}(\mathbb{H}^3)$. Assume that the image Γ_∞ of ρ_∞ is Kleinian and $\Lambda(\Gamma_\infty)$ is not a round circle[2]. Under these assumptions, if the Hausdorff limit of the limit sets $\Lambda(\Gamma_n)$ is the limit set $\Lambda(\Gamma_\infty)$ then the geometric limit G equals Γ_∞.*

Proof: The group G is discrete, $\Lambda(G)$ is contained in the Hausdorff limit of the limit sets $\Lambda(\Gamma_n)$, hence $\Lambda(G) \subset \Lambda(\Gamma_\infty)$. On the other hand, $G \supset \Gamma_\infty$. Thus $\Lambda(\Gamma_\infty) = \Lambda(G)$ and Theorem 4.136 implies that $|G : \Gamma_\infty| < \infty$. We need to prove that the index equals to 1.

Lemma 8.14. *Suppose that $g \in G$ is such that $\rho_\infty(h) = g^m \in \Gamma_\infty$ for a primitive loxodromic element and some $m \neq 0$. Then $g \in \Gamma_\infty$.*

[2]This assumption is actually unnecessary.

Proof: Suppose that $\gamma_i \in \Gamma_i$ is a sequence of elements converging to g. Note that

$$\lim_{i\to\infty} d(\gamma_i^m, \rho_i(h)) = 0$$

where d is a metric on the Lie group $\mathrm{Isom}(\mathbb{H}^3)$. Thus Zassenhaus Theorem implies that for large i the elements $\gamma_i^m, \rho_i(h)$ belong to the same elementary subgroup of Γ_i. Since h is primitive and loxodromic, the element γ_i belongs to the subgroup $\langle\rho_i(h)\rangle$. Thus $g \in \langle\rho_\infty(h)\rangle \subset \Gamma_\infty$. $\square$

Recall that G is discrete, thus the Abelian loxodromic subgroups of G are cyclic. Since Γ is nonelementary, if $G/\Gamma_\infty \neq 1$ there is a primitive loxodromic element $g \in G$ which does not belong to Γ_∞. Let h be the generator of $\Gamma_\infty \cap \langle g\rangle$. Then h is a primitive loxodromic element of Γ_∞ and $g^m = h$ for some m. This shows that $G = \Gamma_\infty$. $\square$

T. Jorgensen and A. Marden [JM90] and K. Ohshika [Ohs90b], proved the following converse to Theorem 8.13:

Theorem 8.15. *Suppose that the group Γ in Theorem 8.13 is the fundamental group of a complete hyperbolic surface of finite area, homomorphisms ρ_n, ρ_∞ are type-preserving and the geometric limit G of the groups Γ_n equals the algebraic limit Γ_∞. Then limits sets of Γ_n are Hausdorff-convergent to the limit set of Γ_∞.*

8.3. Isomorphisms of geometrically finite groups

The following theorem (Part (a)) was proven by D.Mostow in [Mos73] in the case of lattices, the general result is due to P.Tukia [Tuk85b].

Theorem 8.16. *(a) Let $\rho : \Gamma_1 \to \Gamma_2$ be an isomorphisms between two geometrically finite groups in $\mathbb{H}^n$, which is type-preserving (i.e. $g \in \Gamma_1$ is parabolic iff $\rho(g)$ is parabolic). Then there is a ρ-equivariant quasiconformal homeomorphism $h : \Lambda(\Gamma_1) \to \Lambda(\Gamma_2)$, which is quasimoebius if $n = 2$. (b) Moreover, if $n \geq 3$ and $X \subset \mathbb{H}^n \cup \Omega(\Gamma_1)$, $f : X \to \mathbb{H}^n \cup \Omega(\Gamma_2)$ is a ρ-equivariant quasiconformal embedding, then the extension of f via h is again quasiconformal.*

We shall prove only the part (a) of the statement in the case when Γ_1 is has no parabolic elements, a proof of the whole theorem in the general case can be found in [Tuk85b].

Proof: Our assumptions mean that the groups Γ_1, Γ_2 are convex-cocompact. Therefore they act cocompactly on the convex domains $C(\Lambda(\Gamma_i))$, $i = 1, 2$. The inclusions $C(\Lambda(\Gamma_i)) \hookrightarrow \mathbb{H}^n$ are isometric. Pick points $x_i \in C(\Lambda(\Gamma_i))$, $i = 1, 2$. According to Lemma 3.37 the maps $\Gamma_i \to \Gamma_i(x_i) \hookrightarrow C(\Lambda(\Gamma_i))$ are quasi-isometries, therefore the map

$$f : \Gamma_1(x_1) \to \Gamma_2(x_2), \quad f(\gamma x_1) = \rho(\gamma) x_2$$

is a quasi-isometry. Thus f determines a ρ-equivariant quasi-isometry

$$f' : C(\Lambda(\Gamma_1)) \to C(\Lambda(\Gamma_2)).$$

Therefore we can apply Theorem 3.47 to conclude that f' admits a quasiconformal (or quasimoebius if $n = 2$) extension $h : \Lambda(\Gamma_1) \to \Lambda(\Gamma_2)$ which (by continuity) is ρ-equivariant. $\square$

Theorem 8.17. *Suppose that $\Gamma \subset PSL(2, \mathbb{C})$ is a finitely generated Kleinian group. Then the following three assertions are equivalent:*

1. *Γ is quasifuchsian (see Definition 4.35).*

2. *Γ is geometrically finite and admits a type-preserving isomorphism ρ to a Fuchsian group $F \subset PSL(2, \mathbb{R})$.*

3. *There exists a Fuchsian group $F \subset PSL(2, \mathbb{R})$ and a quasiconformal homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $f\Gamma f^{-1} = F$.*

Proof: Let us prove (1)$\Rightarrow$ (2). Suppose that Γ is a quasifuchsian group, $\Omega(\Gamma) = \Omega_1 \cup \Omega_2$ is the disjoint union of two open disks. Let $S_i := \Omega_i/\Gamma$. Thus we have natural isomorphisms

$$\psi_i : \Gamma \to \pi_1(S_i) \subset PSL(2, \mathbb{R}), \quad i = 1, 2$$

where $\pi_1(S_i)$ acts as a Fuchsian group on $\mathbb{H}^2$. According to Ahlfors' finiteness theorem, each surface S_i has finite hyperbolic type and punctures of S_i correspond to hyperbolic elements of Γ. In particular, Γ is a surface group. Hence maximal parabolic subgroups of Γ are cyclic. For each parabolic element $g \in \Gamma$ which fixes a point $p \in \Lambda(\Gamma)$ the quotient $\Lambda(\Gamma)/\langle g \rangle$ is a topological circle, hence it is compact. This implies that each parabolic fixed point of Γ is *bounded*. In particular, the isomorphisms ψ_i preserve the type of elements. Let M^0 denote the manifold $M = M(\Gamma)$ with Margulis cusps removed and let $\dot{N}$ denote the closure of M_0 in $\dot{M}(\Gamma) = M \cup \Omega(\Gamma)/\Gamma$. Let $S_i^0 := S_i \cap \dot{N}$ and $\partial_P(\dot{N})$ denote the frontier of $\dot{N}_0$ in $\dot{M}(\Gamma)$ (this is a finite union of cylinders). We give $\partial_P(\dot{N})$ and S_1, S_2 orientations induced from $\dot{N}$. Then

$$S_1 \cup S_2 \cup \partial_P(\dot{N})$$

determines a 2-cycle σ in $\dot{N}$. Since the inclusions $S_i^0 \hookrightarrow \dot{N}$ are homotopy-equivalences, it follows that $\sigma \in B_2(\dot{N})$. Hence $\dot{N}$ is compact which means that the group Γ is geometrically finite (see Theorem 4.96). This proves (1)$\Rightarrow$ (2).

(2)$\Rightarrow$ (3). By Theorem 8.16 there exists a homeomorphism $h : \Lambda(F) \to \Lambda(\Gamma)$ which induces ρ. We can assume that h preserves orientation (otherwise we first alter ρ by composing it with an automorphism of F which is induced by an orientation-reversing homeomorphism of the surface). Thus, the limit set of Γ is a topological circle and $\Omega(\Gamma)$ is the union of two open disks Ω_1, Ω_2. Each of these disks is invariant under Γ (since the action of Γ preserves the orientation of the limit circle). Recall that Γ is finitely generated, thus the surfaces Ω_j/Γ have finite conformal type and (as in the proof of the implication (1)$\Rightarrow$ (2)) the isomorphisms $\Gamma \to S_i = \Omega_j/\Gamma$ are type-preserving. We identify $\pi_1(S_i)$ with the group F via ρ. Therefore by Theorem 5.4 there is a quasiconformal homeomorphism $\alpha : \Omega(F)/F \to S_1 \cup S_2$, which induces the isomorphism ρ of the fundamental groups of connected components. The lift of α to a quasiconformal homeomorphism of domains of discontinuity $\tilde{\alpha} : \Omega(F) \to \Omega(\Gamma)$ extends via h to a quasiconformal homeomorphism of $\widehat{\mathbb{C}}$ (by Theorem 8.16) which conjugates F to Γ. Therefore the group Γ is quasifuchsian .

The implication (3)$\Rightarrow$ (1) is obvious. $\square$

Let $\Gamma \subset PSL(2,\mathbb{C})$ be a geometrically finite group, $S_j = \Omega_j/\Gamma_j$, where Γ_j is the stabilizer of a component Ω_j of $\Omega(\Gamma)$. We have a natural epimorphism $\theta_j : \pi_1(S_j) \to \Gamma_j$. A parabolic element $\gamma \in \Gamma_j$ is called *accidental parabolic* if $\theta^{-1}(\gamma_j)$ contains no elements corresponding to peripheral loops in S_j. A parabolic element of Γ is called *accidental parabolic* if it belongs to some Γ_j and is accidental parabolic there.

Corollary 8.18. *Suppose that $\Gamma \subset PSL(2,\mathbb{C})$ is a geometrically finite Kleinian group which does not have accidental parabolic elements and $S_j = \Omega_j/\Gamma_j$ is incompressible in $\overset{\bullet}{M}(\Gamma)$ for each j. Then each Γ_j is a quasifuchsian group.*

Proof: Let Γ_j be the stabilizer of a component Ω_j of $\Omega(\Gamma)$. Then Γ_j is geometrically finite (by Corollary 4.112) and $\theta : \pi_1(S_j) \to \Gamma_j$ is an isomorphism. However $S_j = \mathbb{H}^2/F_j$, where F_j is a Fuchsian group preserving the upper-half plane $\mathbb{H}^2$. The map $F_j \to \Gamma_j$ is an isomorphism which is type-preserving by the assumptions. Thus we apply Theorem 8.17 to conclude that Γ_j is quasifuchsian. $\square$

Remark 8.19. It is easy to see that the condition $\partial\overset{\bullet}{M}(\Gamma)$ to be incompressible is necessary. For example, let Γ be a Schottky group. Then Γ satisfies all but one conditions of the Corollary, however the stabilizer of the only component of $\Omega(\Gamma)$ is Γ itself which is not quasifuchsian (since its limit set is not a topological circle).

Exercise 8.20. *Show that the group G in the Example 4.107 is geometrically finite, each component of $\Omega(G)$ is simply-connected, however G contains an accidental parabolic element.*

Theorem 8.21. *(A. Marden, W. Thurston, H. Reimann.) Suppose that $\rho : G_1 \to G_2$ is an isomorphism of two subgroups in $\mathrm{Isom}(\mathbb{H}^3)$, $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a quasiconformal homeomorphism which is ρ-equivariant. (The groups do not have to be discrete). Then h admits a ρ–equivariant quasiconformal extension $\tilde{f} : \overline{\mathbb{H}^3} \to \overline{\mathbb{H}^3}$. Moreover $K_O(\tilde{f}) \le K^3(f)$ and $\tilde{f}$ depends continuously on f (with respect to the compact-open topology).*

This theorem was originally proven by A. Marden in the case when G_1, G_2 are geometrically finite without estimate on the coefficient of quasiconformality [Mar74]. The general case was proven by Thurston, see Reimann's paper [Rei85] for a detailed proof.

The following innocent-looking conjecture is actually a deep problem in the theory of Kleinian groups; it is closely related to Thurston's *Ending Lamination Conjecture.*

Conjecture 8.22. *(W. Thurston) Suppose that G_1, G_2 are finitely generated discrete subgroups of* $\mathrm{Isom}(\mathbb{H}^3)$ *and $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is an orientation-preserving homeomorphism which conjugates G_1 to G_2. Then there exists a quasiconformal homeomorphism $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which conjugates G_1 to G_2.*

8.4. Douady-Earle extension

The main goal of this section is to prove Theorem 5.5. The proof presented here is a variation on the work of Douady and Earle [DE86], who constructed *conformally natural* quasiconformal extension to $\mathbb{H}^2$ of quasi-moebius homeomorphisms $\mathbb{S}^1 \to \mathbb{S}^1$.

Let μ be a Radon probability measure on the unit circle. The following definition is motivated by Mumford's stability condition for the action of $PSL(2,\mathbb{R})$ on $(\mathbb{S}^1)^n$ (see [DM93], [Mum94]). Note that according to this terminology, being *unstable* is not the same as not being *stable.*

Definition 8.23. The measure μ is said to be **stable** if $\mu(\xi) < 1/2$ for each $\xi \in \mathbb{S}^1$. The measure μ is said to be **unstable** if there is a point $\xi \in \mathbb{S}^1$ such that $\mu(\xi) > 1/2$. The measure μ is said to be **semi-stable** if $\mu(\xi) \leq 1/2$ for each $\xi \in \mathbb{S}^1$.

It is clear that any measure without atoms is stable. Given a Radon probability measure μ on $\mathbb{S}^1$, define the *weighted Busemann function*

$$\beta_\mu = \int_{\mathbb{S}^1} \beta_\lambda d\mu(\lambda).$$

The function β_μ on $\mathbb{H}^2$ is smooth and convex (since each β_λ is). Recall that $D^2\beta_\lambda(u,u) \geq 0$ with the equality iff the vector u is tangent to a geodesic asymptotic to λ (see Lemma 3.12). Thus the Hessian of β_μ is positive definite unless μ is supported on a 2-point set. Note that if we change the base-point $0 \in \mathbb{H}^2$ then the function β_μ will change by an additive constant. The function β_μ is conformally natural in the following sense:

If $g \in \mathrm{Isom}_+(\mathbb{H}^2)$ then $\beta_{g_*(\mu)} = \beta_\mu \circ g^{-1} + const$, where $g_*\mu$ is the push-forward of μ via g. The following is a generalization of [DE86, Proposition 1], see also [MZ96] and [LM] where measures on the ideal boundaries of other symmetric spaces are discussed.

Lemma 8.24. *If the measure μ is stable then the function β_μ has minimum in $\mathbb{H}^2$ and this minimum is unique.*

Proof: If μ is stable then the Hessian $D^2\beta_\mu$ is positive definite, hence β_μ is strictly convex. Therefore it suffices to show that the sublevel sets of function β_μ are compact. If not, then there is an unbounded sublevel set

$\beta_\mu^{-1}((-\infty, t])$. Since this subset of $\mathbb{H}^2$ is convex, it contains a complete geodesic ray ρ asymptotic to a point $\eta \in \mathbb{S}^1 = \partial_\infty \mathbb{H}^2$. Stability of μ implies that $\mu(\eta) < 1/2$, hence there is a closed arc $J \subset \mathbb{S}^1 - \eta$ such that $\mu(J) \geq 1/2$. Let C denote the closed convex hull of $\{\eta\} \cup J$ in $\mathbb{H}^2$. After increasing J if necessary we may assume that C contains ρ and the base-point $0 \in \mathbb{H}^2$ (used for normalization of Busemann functions). Then (since Busemann functions are 1-Lipschitz) we have:

$$\int_{\mathbb{S}^1 - J} \beta_\lambda(x) d\mu(\lambda) \geq \int_{\mathbb{S}^1 - J} -d(x,0) d\mu(\lambda) = -d(x,0)(1 - \mu(J)).$$

If $x \in \rho$ is sufficiently close to η, $\beta_\lambda(x) > 0$ provided that $\lambda \in J$. Let $\gamma_{x,\lambda}$ denote the geodesic ray from x to $\lambda \in J$ and let x'_λ be the point of intersection of $\gamma_{x,\lambda}$ with the horosphere centered at λ which passes through 0. Then $\beta_\lambda(x) = d(x, x'_\lambda)$ and the difference $d(x'_\lambda, x) - d(0, x)$ converges to $\beta_\eta(y_\lambda)$ as $x \to \eta$, where y_λ is the point of intersection of the horosphere centered at λ containing 0 with the geodesic connecting η and λ. Since y_λ varies in a bounded region when $\lambda \in J$, it follows that there is a constant k so that

$$\beta_\lambda(x) \geq d(0, x) - k.$$

Therefore

$$\int_J \beta_\lambda(x) d\mu(\lambda) \geq \int_J (d(0,x) - k) d\mu(\lambda) = \mu(J)(d(0,x) - k)$$

and

$$\int_{\mathbb{S}^1} \beta_\lambda(x) d\mu(\lambda) \geq \mu(J)(d(0,x) - k) - d(x,0)(1 - \mu(J)) = d(x,0)(2\mu(J) - 1) - k.$$

Recall that $2\mu(J) - 1 > 0$, thus

$$\lim_{x \to \eta} d(x,0)(2\mu(J) - 1) - k = +\infty.$$

This contradicts the assumption that the function β_μ is bounded from above along the geodesic ray ρ. ☐

Exercise 8.25. *Show that β_μ is not bounded from below if μ is unstable. Analyze the case of semi-stable measures which are not stable.*

For a stable measure μ define the *conformal barycenter* $c(\mu)$ of μ to be the point of minimum of β_μ. The point $c(\mu)$ does not depend on the choice of base-point in $\mathbb{H}^2$. Conformal naturality of β_μ implies that for each isometry $g \in \mathrm{Isom}_+(\mathbb{H}^2)$ we have:

$$c(g_* \mu) = g c(\mu).$$

Note that the fact that β_μ is strictly convex implies that $c(\mu)$ is the unique nondegenerate zero of the gradient vector field

$$V_\mu = -\nabla \beta_\mu = \int_{\mathbb{S}^1} -\nabla \beta_\lambda d\mu(\lambda).$$

Properties of the conformal barycenter.

Lemma 8.26. *The mapping $\mu \mapsto c(\mu)$ is smooth.*

Proof: Embed the space of Radon probability measures on $\mathbb{S}^1$ to the Banach space $[C^0(\mathbb{S}^1)]^*$ (the dual space of the continuous functions on $\mathbb{S}^1$). Give the space $C^1(\mathbb{H}^2)$ of C^1-smooth functions on $\mathbb{H}^2$ the topology of C^1-convergence (uniformly on compacts). Then the mapping $\mu \mapsto \beta_\mu$ is (the restriction of) a continuous linear operator. The implicit function theorem implies that the nondegenerate zero $c(\mu)$ of the vector field V_μ is an infinitely differentiable function of μ. Thus $c(\mu)$ depends smoothly on μ. □

If $J \subset \mathbb{S}^1 = \partial_\infty \mathbb{H}^2$ is a closed arc, define the convex subset $H(J) \subset \mathbb{H}^2$ which the collection of points $x \in \mathbb{H}^2$ so that the angle between the two geodesic rays from x to the end-points of J is at least $\pi/2$.

Lemma 8.27. *If $J \subset \mathbb{S}^1$ is a closed arc such that $\mu(J) \geq 2/3$, then $c(\mu) \in H(J)$.*

Proof: It suffices to show that for each point $x \in \partial H(J)$ the gradient vector field $V_\mu(x)$ is directed inward $H(J)$. Let ν be the normal unit vector to $\partial H(J)$ at x directed inward $H(J)$. We will show that $V_\mu(x) \cdot \nu > 0$. Indeed:

$$V_\mu(x) \cdot \nu = \int_J -\nabla\beta_\lambda(x) \cdot \nu d\mu(\lambda) + \int_{\mathbb{S}^1 - J} -\nabla\beta_\lambda(x) \cdot \nu d\mu(\lambda).$$

Since both ν and $\nabla\beta_\lambda(x)$ are unit vectors we have:

$$\int_{\mathbb{S}^1 - J} -\nabla\beta_\lambda(x) \cdot \nu d\mu(\lambda) \geq -\int_{\mathbb{S}^1 - J} d\mu(\lambda) = -1 + \mu(J).$$

On the other hand, for each $\lambda \in J$, $-\nabla\beta_\lambda(x) \cdot \nu \geq \sqrt{2}/2$, hence

$$V_\mu(x) \cdot \nu \geq -1 + \mu(J) + \mu(J)\sqrt{2}/2 > 0$$

provided that $\mu(J) \geq 2/3$. □

Let L be the Lebesgue measure on $\mathbb{S}^1$ normalized to have unit total mass. Realize $\mathbb{H}^2$ as the open unit disk in $\mathbb{R}^2$ centered at the origin. For $z \in \mathbb{H}^2$ choose $g \in \mathrm{Isom}_+(\mathbb{H}^2)$ so that $g(0) = z$, let $L_z := g_* L$ be the push-forward of L. Since L is invariant under rotations around the origin, the measure L_z does not depend on the choice of g. Note that L_z does not have atoms, thus its push-forward by any nonconstant continuous mapping also does not have atoms, and hence is stable.

Finally we define the Douady-Earle extension: if $h : \mathbb{S}^1 \to \mathbb{S}^1$ is a continuous nonconstant mapping, $z \in \mathbb{H}^2$, let

$$Ext(h)(z) = \tilde{h}(z) := c(h_* L_z).$$

Properties of the Douady-Earle extension:

(a) Lemma 8.26 implies that $\tilde{h} : \mathbb{H}^2 \to \mathbb{H}^2$ is smooth.

(b) Conformal naturality of the extension: if $\alpha \in \mathrm{Isom}_+(\mathbb{H}^2)$ then

$$Ext(\alpha \circ h)(z) = c(\alpha_* h_* L_z) = \alpha c(h_* L_z) = \alpha \circ \tilde{h}(z)$$

and

$$Ext(h \circ \alpha)(z) = \tilde{h}(\alpha z).$$

In particular, if $\alpha, \gamma \in \mathrm{Isom}_+(\mathbb{H}^2)$ are such that $\alpha \circ h = h \circ \gamma$, then

$$\alpha \circ \tilde{h} = \tilde{h} \circ \gamma.$$

(c) The mapping $\tilde{h} \cup h : \mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2 \to \mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$ is continuous.

Proof: The mapping $\tilde{h}$ is continuous according to (a), hence it suffices to verify that

$$\lim_{z \to \zeta} \tilde{h}(z) = h(\zeta), \quad \zeta \in \partial_\infty \mathbb{H}^2, \quad z \in \mathbb{H}^2.$$

Consider the closed arcs $J \subset \mathbb{S}^1$ whose interiors contain $h(\zeta)$. Then the sets $J \cup H(J)$ form basis of neighborhoods of $h(\zeta)$ in $\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$. Given such J there exists a neighborhood U of ζ such that for each $z \in U \cap \mathbb{H}^2$ the total mass of L_z concentrated in $h^{-1}(J)$ is at least $2/3$. Hence according to Lemma 8.27, $\tilde{h}(z) \in H(J)$. This implies continuity. $\square$

(d) Let $C^0(\mathbb{S}^1)$ denote the space of continuous mappings $\mathbb{S}^1 \to \mathbb{S}^1$ with C^0 topology induced from the space of continuous complex-valued mappings $\mathbb{S}^1 \to \mathbb{C}$. As in the proof of Lemma 8.26 we let $C^1(\mathbb{H}^2)$ denote the space of C^1-smooth maps $\mathbb{H}^2 \to \mathbb{H}^2$ with the topology of C^1-convergence (uniformly on compacts in $\mathbb{H}^2$).

Lemma 8.28. *The mapping $ext : C^0(\mathbb{S}^1) \to C^1(\mathbb{H}^2)$ is continuous.*

Proof: According to Lemma 8.26, the mapping $\mu \mapsto c(\mu)$ is smooth. Since L_z depends smoothly on z and the push-forward by h is the restriction of a bounded linear operator, for each $\ell \geq 0$ and each bounded region in $\mathbb{H}^2$, the derivative $D^\ell(\tilde{h})$ depends continuously on h. $\square$

(e) The mapping $\tilde{h} : \mathbb{H}^2 \to \mathbb{H}^2$ is an immersion provided that $h : \mathbb{S}^1 \to \mathbb{S}^1$ is a homeomorphism.

Proof: The argument below was suggested to me by Bruce Kleiner. Conformal naturality of the Douady-Earle extension implies that it is enough to consider the derivative of $\tilde{h}$ at the origin under the assumption $\tilde{h}(0) = 0$. Let $v \in T_0(\mathbb{H}^2)$ be a unit vector tangent to a geodesic γ, where $\gamma(0) = 0$. Let g_t be a family of translations along the geodesic γ so that $g_0 = id$, $g_t(0) = \gamma(t)$. Then the push-forward measures $L_{\gamma(t)}$ on $\mathbb{S}^1$ satisfy

$$L_{\gamma(t)} = |g_{-t}|' \cdot L \tag{8.3}$$

Therefore

$$\left. \frac{dL_{\gamma(t)}}{dt} \right|_{t=0} = N \cdot L \tag{8.4}$$

where $N(x)$ is a smooth function on $\mathbb{S}^1$, which equals $\frac{d}{dt}|g'_{-t}(x)|_{t=0}$. Note that $N(x)$ has two zeroes which we shall denote i and j and N is positive on an open arc $\alpha \subset \mathbb{S}^1$ which connects i to j and is negative on the complementary open arc $\omega \subset \mathbb{S}^1$. It is clear that

$$\int_{\mathbb{S}^1} N(x) dL = 0$$

since the total mass of the measure $L_{\gamma(t)}$ does not depend on t. Let μ_t denote the push-forward $h_* L_{\gamma(t)}$. Applying h_* to the equation (8.3) we get:

$$\mu_t = |g_{-t}|' \circ h^{-1} \cdot \mu = M_t(x) \cdot \mu$$

and (since $|g_{-t}|' \circ h^{-1}$ depends smoothly on t):

$$\frac{d\mu_t}{dt}\bigg|_{t=0} = \nu \cdot \mu$$

where $\nu \circ h = N$. We again have:

$$\int_{\mathbb{S}^1} \nu d\mu = 0$$

and ν is positive on $h(\alpha)$, negative on $h(\omega)$ and vanishes at $h(i), h(j)$.

Let c_t be the conformal barycenter of μ_t, then it satisfies the equation

$$0 = \int_{\mathbb{S}^1} \nabla \beta_\lambda(c_t) d\mu_t \tag{8.5}$$

(where ∇ is the gradient). Suppose that $D\tilde{h}(v) = 0$, thus $c_t'|_{t=0} = 0$. Differentiating the equation (8.5) with respect to t at $t = 0$ we get:

$$0 = \frac{d}{dt} \int_{\mathbb{S}^1} \nabla \beta_\lambda(c_t) d\mu_t \bigg|_{t=0} = \frac{d}{dt} \int_{\mathbb{S}^1} \nabla \beta_\lambda(c_t) M_t d\mu \bigg|_{t=0} =$$

$$\int_{\mathbb{S}^1} [\frac{d}{dt} \nabla \beta_\lambda(c_t)] M_0 d\mu \bigg|_{t=0} + \int_{\mathbb{S}^1} \nabla \beta_\lambda(c_0) \frac{d}{dt} M_t d\mu \bigg|_{t=0} = \int_{\mathbb{S}^1} \nabla \beta_\lambda(0) \nu d\mu$$

since $\frac{d}{dt} \nabla \beta_\lambda(c_t) \Big|_{t=0} = D^2 \beta_\lambda(0) c_t' \,|_{t=0} = 0$. Thus

$$\int_{\mathbb{S}^1} \nabla \beta_\lambda(0) \nu d\mu = 0.$$

Let W be the unit tangent vector at the origin $0 \in \mathbb{H}^2$ which (after we identify W with a point on $\mathbb{C}$) bisects the arc $h(\alpha)$. We will obtain contradiction by proving that

$$\int_{\mathbb{S}^1} (-\nabla \beta_\lambda(0) \cdot W) \nu d\mu > 0$$

(here and in what follows $\cdot$ denotes the Euclidean scalar product in $\mathbb{R}^2$). Note that for each $Z \in h(\alpha)$ we have

$$h(i) \cdot W = h(j) \cdot W < Z \cdot W.$$

Recall that hyperbolic geodesic rays from $0 \in \mathbb{H}^2$ to the points $\lambda \in \mathbb{S}^1$ are Euclidean segments and the hyperbolic metric at 0 is twice the Euclidean metric, thus for each $\lambda \in \mathbb{S}^1$,

$$-\nabla \beta_\lambda(0) = 2\lambda \quad ,$$

where we identify tangent vectors in $T_0(\mathbb{H}^2)$ with points of $\mathbb{C}$. Then

$$\int_{h(\alpha)} (-\nabla\beta_\lambda(0)\cdot W)\nu d\mu > 2\int_{h(\alpha)} (h(i)\cdot W)\nu d\mu \ .$$

Similarly, for each $Z \in h(\omega)$ we have

$$h(i)\cdot W > Z\cdot W \ .$$

Therefore (since ν is negative on the arc $h(\beta)$) we get:

$$\int_{h(\omega)} (-\nabla\beta_\lambda(0)\cdot W)\nu d\mu > 2\int_{h(\omega)} (h(i)\cdot W)\nu d\mu.$$

This implies that

$$\int_{\mathbb{S}^1} (-\nabla\beta_\lambda(0)\cdot W)\nu d\mu > 2\int_{\mathbb{S}^1} (h(i)\cdot W)\nu d\mu = 2(h(i)\cdot W)\int_{\mathbb{S}^1} \nu d\mu = 0.$$

Contradiction. □

(f) If h is a homeomorphism then $\tilde{h} : \mathbb{H}^2 \to \mathbb{H}^2$ is a diffeomorphism.

Proof: It suffices to show that $\tilde{h}$ is a covering. The mapping $\tilde{h}$ is a local diffeomorphism according to (e) and is proper according to (c). Thus $\tilde{h}$ satisfies the path-lifting property, which implies that it is a covering. □

Note that if $h : \mathbb{S}^1 \to \mathbb{S}^1$ preserves orientation, then $\tilde{h} : \mathbb{H}^2 \to \mathbb{H}^2$ is orientation-preserving as well.

(g) If $h : \mathbb{S}^1 \to \mathbb{S}^1$ is orientation-preserving quasimoebius mapping then the extension $\tilde{h} : \mathbb{H}^2 \to \mathbb{H}^2$ is quasiconformal.

Proof: Smoothness of $\tilde{h}$ implies that it is locally quasiconformal. We need to show that $K_{\tilde{h}}(x)$ is uniformly bounded for $x \in \mathbb{H}^2$. Let $x_n \in \mathbb{H}^2$ be a sequence which converges to a point $\eta \in \mathbb{S}^1 = \partial_\infty \mathbb{H}^2$. We will show that

$$\overline{\lim}_n K_{\tilde{h}}(x_n) < \infty \ .$$

Choose $\alpha_n, \gamma_n \in \mathrm{Isom}_+(\mathbb{H}^2)$ so that the mappings

$$h_n = \alpha_n \circ h \circ \gamma_n : \mathbb{S}^1 \to \mathbb{S}^1$$

fix three distinct points and $\gamma(0) = x_n$. Note that since the mappings α_n, γ_n are Moebius, the sequence of mappings h_n consists of uniformly quasimoebius mappings which fix three distinct points in $\mathbb{S}^1$. Thus (after taking a subsequence if necessary) we may assume that the sequence $h_n : \mathbb{S}^1 \to \mathbb{S}^1$ converges uniformly to an orientation-preserving quasimoebius mapping $h_\infty : \mathbb{S}^1 \to \mathbb{S}^1$. According to Lemma 8.28,

$$\lim_{n\to\infty} Ext(h_n) = Ext(h_\infty)$$

in C^1-topology (locally in $\mathbb{H}^2$). Conformal naturality of the Douady-Earle extension implies that $Ext(h_n) = \alpha_n \circ \tilde{h} \circ \gamma_n$. Hence

$$\lim_{n\to\infty} K_{\tilde{h}_n}(0) = K_{\tilde{h}_\infty}(0) < \infty.$$

On the other hand, since α_n, γ_n are conformal, it follows that $K_{\tilde{h}}(x_n) = K_{\tilde{h}_n}(0)$. We conclude that the sequence $K_{\tilde{h}}(x_n) \in [1, \infty)$ is bounded. ☐

We get

Theorem 8.29. *Suppose that $\rho : G_1 \to G_2$ is an isomorphism of two subgroups in Isom$(\mathbb{H}^2)$ (which do not have to be discrete), $h : \mathbb{S}^1 \to \mathbb{S}^1$ is a quasimoebius homeomorphism which is ρ-equivariant. Then h admits a ρ-equivariant quasiconformal extension to a map $f : \mathbb{H}^2 \to \mathbb{H}^2$. If h is orientation-preserving then f also preserves orientation.*

Now we can finish the proof of Theorem 5.5.

Theorem 8.30. *Suppose that $F, F' \subset$ Isom$_+(\mathbb{H}^2)$ are Fuchsian subgroups such that the quotients $\mathbb{H}^2/F, \mathbb{H}^2/F'$ have finite area and $\psi : F \to F'$ is a type-preserving isomorphism. Then there exists a ψ-equivariant quasiconformal homeomorphism $f : \mathbb{H}^2 \to \mathbb{H}^2$.*

Proof: The limit set of each group F, F' is the circle $\mathbb{S}^1$ and F, F' are geometrically finite groups. Thus, according to Theorem 8.16, there exists a ψ-equivariant quasimoebius homeomorphism $h : \mathbb{S}^1 \to \mathbb{S}^1$. Theorem 8.29 now implies that h extends to a ψ-equivariant quasiconformal homeomorphisms $\mathbb{H}^2 \to \mathbb{H}^2$. ☐

Below we state and prove a generalization of Theorem 5.6 due to Earle and McMullen [EM88]. Suppose that S is a hyperbolic Riemann surface, $S = \mathbb{H}^2/G$ where $G \subset PSL(2, \mathbb{R})$ is a discrete torsion-free subgroup. Let $\mathbb{S}^1 = \partial_\infty \mathbb{H}^2$. Define the *ideal boundary* ∂S of S as $\Omega(G) \cap \mathbb{S}^1/G$. The partial compactification of S is the surface with boundary $\hat{S} := (\Omega(G) \cap (\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2))/G$. Recall that each quasiconformal homeomorphism $f : S \to S'$ lifts to a quasiconformal homeomorphism of $\mathbb{H}^2$ which extends homeomorphically to the unit circle. By restricting this extension to $\mathbb{S}^1 \cap \Omega(G)$ and then projecting the result to $\hat{S}, \hat{S}'$ we get a homeomorphism $\widehat{f} : \hat{S} \to \hat{S}'$.

Definition 8.31. Two quasiconformal homeomorphisms $f_0, f_1 : S \to S'$ are said to be *homotopic* (resp. isotopic) relative to the ideal boundary of S if there exists a continuous family of continuous maps (resp. homeomorphisms) $f_t : \hat{S} \to \hat{S}'$, $t \in I$, which are all equal on ∂S.

Theorem 8.32. *(Homotopy implies isotopy, [EM88].) Suppose that $f_0, f_1 : S \to S'$ are homotopic (rel. ideal boundary) quasiconformal mappings of Riemann surfaces of hyperbolic type. Then f_0 is isotopic to f_1 (rel. ideal boundary) through quasiconformal homeomorphisms.*

Proof: It suffices to consider the case when $S = S', f_0 = id$ (by taking $f_0 := id, f_1 := f_1^{-1} \circ f_0$). Let $S = \mathbb{H}^2/G$. Let F_0, F_1 denote lifts of f_0, f_1 to the hyperbolic plane. Recall that F_0, F_1 extend to the unit circle. The extensions induce the same (up to a conjugation by elements of G) automorphism of the group G. Since f_0 is homotopic to f_1 (rel. ideal boundary) we can choose the lifts F_0, F_1 which have the identity extension to the unit circle. Let μ_0, μ_1 denote the Beltrami differentials of F_0, F_1. Let $F_t : \mathbb{H}^2 \to \mathbb{H}^2$

denote the interpolation between F_0, F_1 given by a solution of the Beltrami equation

$$\overline{\partial} F_t = ((1-t)\mu_0 + t\mu_1)\partial F_t.$$

Since the Beltrami differentials $(1-t)\mu_0+t\mu_1$ are G-invariant, the mappings F_t conjugate G to subgroups $G_t \subset PSL(2,\mathbb{R})$. However there is no (a priori) reason for these groups to be the same as G. Let $h_t : \mathbb{S}^1 \to \mathbb{S}^1$ denote the homeomorphic extension of F_t to the boundary of $\mathbb{H}^2$. Let E_t be the Douady-Earle extension of h_t to $\mathbb{H}^2$. By conformal naturality of the Douady-Earle extension, E_t conjugates G to G_t and induces the same isomorphism $\rho_t : G \to G_t$ as F_t. Note that $E_0 = E_1 = id$ since $h_0 = h_1 = id$. Now take $H_t := E_t^{-1} \circ F_t$. This is a continuous family of quasiconformal homeomorphisms $\mathbb{H}^2 \to \mathbb{H}^2$ which induce the identity isomorphism $G \to G$ (as did F_0, F_1) and $H_0 = F_0, H_1 = F_1$. By construction, the extension of each H_t to the unit circle is the identity map. Thus the projection of H_t to S defines the required isotopy of f_0 to f_1. □

8.5. Mostow rigidity theorem

Now we can prove the Mostow Rigidity Theorem (Theorem 1.44). Suppose that Γ, Γ' are lattices in $\mathrm{Isom}(\mathbb{H}^n)$, $n \geq 3$, and $\rho : \Gamma \to \Gamma'$ is an isomorphism. Then ρ is type-preserving. (Indeed, parabolic elements of Γ, Γ' are characterized by the property that their centralizers in Γ, Γ' contain $\mathbb{Z} \times \mathbb{Z}$.) Thus, by Theorem 8.16 there exists a ρ-equivariant quasiconformal homeomorphism $h : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$. Quasiconformal homeomorphisms are differentiable a.e. on $\mathbb{S}^{n-1}$ and have a.e. non-zero Jacobian determinants J_h.

Remark 8.33. The a.e. nonvanishing of J_h does not follow from equivariance, invertibility and a.e. differentiability of h. For instance, if F_1, F_2 are Fuchsian subgroups of $PSL(2,\mathbb{R})$, $\psi : F_1 \to F_2$ is an admissible isomorphism then ψ is induced by a homeomorphism $h : \mathbb{S}^1 \to \mathbb{S}^1$ which is a.e. differentiable (as any homeomorphism of $\mathbb{S}^1$). However, in view of the following theorem, the derivative of h vanishes a.e. unless h is induced by an isometry of $\mathbb{H}^2$.

Suppose that $z \in \mathbb{S}^{n-1}$ is a point of differentiability so that $J_z(h) \neq 0$. Since only countable number of points in $\mathbb{S}^{n-1}$ are fixed points of parabolic elements, we can choose z to be a point of approximation of Γ. The result will follow from the following:

Theorem 8.34. *(P. Tukia, N. Ivanov.) Suppose that Γ is a discrete nonelementary subgroup of $\mathrm{Isom}(\mathbb{H}^n)$, $z \in \Lambda(\Gamma)$ is a point of approximation, $h : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is a homeomorphism which conjugates Γ to another Moebius group $\Gamma' = h\Gamma h^{-1}$. If h is differentiable at the point z with nondegenerate derivative, then the map h is necessarily Moebius.*

Proof: Our proof will follow arguments of [Iva96] (see also [Tuk85a], [Sch95]), under the extra assumption that h is quasiconformal, which will suffice for our purposes. We consider the upper half-space model of $\mathbb{H}^n$ so that $\mathbb{S}^{n-1}_\infty \cong \overline{\mathbb{R}^{n-1}}$. By applying a conformal change of coordinates we can assume that $z = h(z) = 0 \in \mathbb{R}^{n-1}$. Let $L \subset \mathbb{H}^n$ be the "vertical" geodesic emanating from 0, pick a base-point $y_0 \in L$. Since z is a point of approximation, there is a sequence of elements $\gamma_i \in \Gamma$ so that $\gamma_i(y_0) \to z$ and $d(\gamma_i(y_0), L) \le C$ for each i. Let y_i denote the nearest-point projection of $\gamma_i(y_0)$ to L. Take the sequence of hyperbolic translations $T_i : \overrightarrow{x} \mapsto \lambda_i \overrightarrow{x}$ with the axis L, so that $T_i(y_0) = y_i$. Then the sequence $\gamma_i^{-1}T_i$ is relatively compact in $\mathrm{Isom}(\mathbb{H}^n)$ and lies in a compact $K_0 \subset \mathrm{Isom}(\mathbb{H}^n)$. Now we form the sequence

$$h_i(x) := \lambda_i^{-1} h(\lambda_i x) = T_i^{-1} \circ h \circ T_i(x).$$

Note that $\lambda_i \to 0$ as $i \to \infty$. Since the function h is assumed to be differentiable at zero, there is a linear transformation $A \in GL(n-1, \mathbb{R})$ so that

$$\lim_{i\to\infty} h_i(x) = Ax$$

for all $x \in \mathbb{R}^{n-1}$. Since the sequence h_i is K-quasiconformal, Property (4) of quasiconformal mappings in Section 5.2 implies that this convergence is uniform on compacts in $\mathbb{R}^{n-1}$. By assumption, h_i conjugates the group $\Gamma_i := T_i^{-1}\Gamma T_i \subset \mathrm{Isom}(\mathbb{H}^n)$ into a group of Moebius transformations. Since the sequence $\gamma_i^{-1}T_i$ is relatively compact in $\mathrm{Isom}(\mathbb{H}^n)$, and $\gamma_i^{-1}\Gamma\gamma_i = \Gamma$, the sequence Γ_i is subconvergent (in the Chabauty topology on $\mathrm{Isom}(\mathbb{H}^n)$) to a nonelementary group Γ_∞. After taking a subsequence we assume that $\lim_i^{geo} \Gamma_i = \Gamma_\infty$. Hence the limit A conjugates Γ_∞ to a subgroup of $\mathrm{Isom}(\mathbb{H}^n)$. Since the group Γ_∞ is nonelementary it contains an element γ such that $\gamma(\infty) \notin \{\infty, 0\}$.

Lemma 8.35. *Suppose that $\gamma \in Isom(\mathbb{H}^n)$ is such that $\gamma(\infty) \notin \{\infty, 0\}$, $A \in GL(n-1, \mathbb{R})$ is an element which conjugates γ to $A\gamma A^{-1} \in Isom(\mathbb{H}^n)$. Then A is a Euclidean similarity, i.e. it belongs to $O(n-1)\cdot\mathbb{R}_+$.*

Proof: The assertion clearly holds for $n = 2$ ($GL(1,\mathbb{R}) = O(1)\cdot\mathbb{R}_+$), thus we assume that $n \ge 3$. Suppose that A is not a similarity. According to our assumptions, $A\gamma^{-1}(\infty) \neq 0$. Let P be a hyperplane in $\mathbb{R}^{n-1}$ which contains the origin but does not contain $A\gamma^{-1}(\infty)$. Then $\gamma A^{-1}(P)$ is a round sphere Σ in $\mathbb{R}^{n-1}$. Since A is not a similarity, the image $A(\Sigma)$ is an ellipsoid which is not a round sphere. Hence the composition $A\gamma A^{-1}$ does not send planes to round spheres and therefore it is not Moebius. Contradiction. $\square$

Thus we have proved that h is conformal at the point z. To conclude that h is Moebius we need to use the fact that $h\Gamma h^{-1} \subset \mathrm{Isom}(\mathbb{H}^n)$ once again.

Pick three distinct points $a, b, c \in \mathbb{S}^{n-1}$. A *normalization* $N(f)$ of a homeomorphism $f : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is a postcomposition $g \circ f$, where $g \in \mathrm{Isom}(\mathbb{H}^n)$, so that $N(f)$ fixes $\{a, b, c\}$ pointwise. The normalization is uniquely determined by f up to postcomposition with an element of the compact subgroup $K_{abc} \subset \mathrm{Isom}(\mathbb{H}^n)$ which fixes the round circle in $\mathbb{S}^{n-1}$ containing $\{a, b, c\}$. We let $\mathcal{N}(f)$ denote the projection of $N(f)$ to the quotient $K_{abc}\backslash Homeo(\mathbb{S}^{n-1})$. Thus, $\mathcal{N}(g \circ f) = \mathcal{N}(f)$ for all $g \in \mathrm{Isom}(\mathbb{H}^n)$.

The fact that $h\Gamma h^{-1} = \rho(\Gamma) \subset \mathrm{Isom}(\mathbb{H}^n)$ means that

$$\mathcal{N}(h) \circ \gamma = \mathcal{N}(h \circ \gamma) = \mathcal{N}(\rho(\gamma) \circ h) = \mathcal{N}(h)$$

for all $\gamma \in \Gamma$. Recall that $T_i = \gamma_i \circ k_i$ where k_i are elements of a compact $K_0 \subset \mathrm{Isom}(\mathbb{H}^n)$. Thus

$$\mathcal{N}(h_i) = \mathcal{N}(T_i^{-1} \circ h \circ T_i) = \mathcal{N}(h \circ T_i) = \mathcal{N}(h \circ \gamma_i \circ k_i) = \mathcal{N}(h) \circ k_i$$

since $\gamma_i \in \Gamma$. On the other hand,

$$\lim_{i\to\infty} \mathcal{N}(h) \circ k_i = \lim_{i\to\infty} \mathcal{N}(h_i) = \mathcal{N}(A)$$

where $A \in \mathrm{Isom}(\mathbb{H}^n)$ is a Euclidean similarity. Hence there exists a relatively compact sequence $k_i' \in K_{abc}$ so that

$$\lim_{i\to\infty} k_i' \circ N(h) \circ k_i = N(A).$$

After passing to a subsequence so that $\lim_i k_i' = k'$ and taking the limit we conclude that

$$k' \circ N(h) \circ k = N(A)$$

which means that $h \in \mathrm{Isom}(\mathbb{H}^n)$. This finishes the proof of Theorem 8.34. $\square$

The following is a simple but useful corollary of Theorems 1.44 and 1.31:

Theorem 8.36. *Suppose that (M, P) is a pared Haken manifold so that one of its finite coverings (N, Q) admits a complete hyperbolic structure of finite volume. Then (M, P) also admits a complete hyperbolic structure of finite volume.*

Proof: Without loss of generality we can assume that the covering $N \to M$ is regular, let F denote the group of covering transformations. Each element $f \in F$ is homotopic to an isometry $f^{\#}$ by the Mostow Rigidity Theorem. Moreover $f^{\#} \circ g^{\#} = (f \circ g)^{\#}$ since any isometry of N which is homotopic to the identity is the identity itself. Thus we get an action of a finite group of isometries $F^{\#}$ on N which induces the same outer automorphisms of $\pi_1(N)$ as F. Note that the action of $F^{\#}$ is free. Indeed, suppose that some $f^{\#}$ has a nonempty fixed-point set. Lift f and $f^{\#}$ to the universal cover $\mathbb{H}^3$. The lift is not unique, but we can compose lifts with elements of $G = \pi_1(N) \curvearrowright \mathbb{H}^3$ to guarantee that:

- The extensions of the lifts $\tilde{f}^{\#}$ and $\tilde{f}$ to the sphere at infinity $\widehat{\mathbb{C}}$ coincide.
- $\tilde{f}^{\#}$ is an elliptic isometry of finite order m.

Thus $\tilde{f}^m$ acts trivially on $\widehat{\mathbb{C}}$, which implies that it commutes with all elements of G; thus $\tilde{f} : \mathbb{H}^3 \to \mathbb{H}^3$ has finite order. We conclude that the finite group $\langle \tilde{f} \rangle$ does not act freely on $\mathbb{H}^3$, which contradicts the fact that F acts freely on N.

Consider the total lifts of the groups $F, F^{\#}$ to $\mathbb{H}^3$: these lifts $\Gamma, \Gamma^{\#}$ are groups of diffeomorphisms of $\mathbb{H}^3$, and they are finite normal extensions of the group G. All elements of $F, F^{\#}$ induce nontrivial automorphisms of G. Extend both actions to $\widehat{\mathbb{C}}$: the restrictions of $\Gamma, \Gamma^{\#}$ to $\widehat{\mathbb{C}}$ are faithful representations $\Gamma, \Gamma^{\#} \to Homeo(\widehat{\mathbb{C}})$ and they have the same image. We conclude that the groups $\Gamma, \Gamma^{\#}$ are isomorphic. Hence the manifolds $M = N/F$ and $M^{\#} = N/F^{\#}$ have isomorphic fundamental groups ($\cong \Gamma$) and they are homeomorphic by Theorem 1.31. $\square$

8.6. Sullivan rigidity theorem

Suppose that $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$, $n \geq 3$, is a discrete group whose action on its limit set is recurrent. Let $\Gamma' \subset \mathrm{Isom}(\mathbb{H}^n)$ be another discrete group and $f : \partial_\infty \mathbb{H}^n \to \partial_\infty \mathbb{H}^n$ be a quasiconformal homeomorphism which is conformal on $\Omega(\Gamma)$ and which conjugates Γ to Γ'. The following is a deep generalization of Mostow Rigidity Theorem which is due to D. Sullivan.

Theorem 8.37. *(Sullivan Rigidity Theorem, [Sul81a], [Ahl80], [Ota96].) Under the above conditions f is a Moebius transformation.*

Note that this theorem deals with the groups which are not necessarily finitely generated, we will use Theorem 8.37 in the proof of Theorem 13.11 in the case of infinitely generated groups.

According to Ahlfors' Theorem 4.123 every finitely generated discrete subgroup of $\mathrm{Isom}(\mathbb{H}^3)$ acts recurrently on its limit set. Thus, combining Theorem 4.123 with his Theorem 8.37, Sullivan proves

Theorem 8.38. *(D. Sullivan) Suppose that $G \subset PSL(2, \mathbb{C})$ is a finitely generated discrete group and μ is a G-invariant Beltrami differential such that $supp(\mu) \subset \Lambda(G)$. Then $\mu = 0$ almost everywhere.*

The following is one of the many characterizations of recurrent groups:

Theorem 8.39. *(D. Sullivan, [Sul81a].) The action of a discrete group $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ on its limit set is recurrent if and only if for some (any) convex fundamental polyhedron P for the action of Γ in $\mathbb{H}^n$, the intersection*

$$\partial_\infty P \cap \Lambda(\Gamma) \tag{8.6}$$

has zero Lebesgue measure.

We shall apply this theorem in the situation when the intersection (8.6) is countable.

8.7. Bers isomorphism

In this section we consider discrete, nonelementary, finitely generated groups $\Gamma \subset \mathrm{Isom}(\mathbb{H}^3)$ whose limit sets are connected. Equivalently, the surface $\partial \overset{\bullet}{M}(\Gamma)$ is incompressible in $\overset{\bullet}{M}(\Gamma)$ (under the assumption that Γ is torsion-free). Most of the time we will deal with torsion-free subgroups $\Gamma \subset PSL(2, \mathbb{C})$, in any case will use only those quasiconformal homeomorphisms which preserve orientation. Define the *Teichmüller space* of the group Γ as

$$\begin{aligned}\mathcal{T}(\Gamma) = \{\rho \in Hom(\Gamma, \mathrm{Isom}(\mathbb{H}^3)) : &\text{ there is a } \rho\text{-equivariant} \\ \text{quasiconformal homeomorphism} \quad &f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}\}/PSL(2, \mathbb{C}).\end{aligned}$$

If $f\Gamma f^{-1} \subset \mathrm{Isom}(\mathbb{H}^3)$ then the homomorphism f_* defined by

$$f_*(\gamma) = f \circ \gamma \circ f^{-1}, f_* : \Gamma \to \mathrm{Isom}(\mathbb{H}^3)$$

is called the *representation induced by* f. We let $[\rho]$ denote the $PSL(2, \mathbb{C})$-conjugacy class of $\rho \in Hom(\Gamma, \mathrm{Isom}(\mathbb{H}^3))$.

For each Γ-conjugacy class of maximal parabolic subgroups in Γ choose a representative P_j. Let $P = \{P_1, P_2, ...\}$ be the collection of representatives. I recall that the relative representation variety $\mathcal{R}(\Gamma, P; \mathrm{Isom}(\mathbb{H}^3))$ consists of conjugacy classes of representations which send elements of P_j to parabolic transformations. Thus $\mathcal{T}(\Gamma) \subset \mathcal{D}(\Gamma, \mathrm{Isom}(\mathbb{H}^3)) \cap \mathcal{R}(\Gamma, P; \mathrm{Isom}(\mathbb{H}^3))$. Let $L_\infty(\Omega(\Gamma))$ denote the space of complex-valued L_∞-functions on the domain $\Omega(\Gamma)$. The condition that $f_*(\Gamma) = f \circ \Gamma \circ f^{-1}$ consists of Moebius transformations can be expressed in terms of the Beltrami differential $\mu(z)$ of the quasiconformal homeomorphism f:

$$\gamma^* \mu = \mu, \quad \text{i.e. } \mu \text{ is } \Gamma-\text{invariant} \tag{8.7}$$

(see §5.2). Define

$$B_\infty(\Gamma) := \{\mu \in L_\infty(\Omega(\Gamma)) : ||\mu||_\infty < 1, \mu \text{ is } \Gamma\text{-invariant}\}.$$

This space is the open unit ball in the Banach space

$$L_\infty(\Gamma) := \{\mu \in L_\infty(\Omega(\Gamma)) : \mu \text{ is } \Gamma\text{-invariant}\}.$$

Thus we can think about $\mathcal{T}(\Gamma)$ as the quotient of the space $B_\infty(\Gamma)$ by the following equivalence relation:

$$\begin{aligned}\mu \sim \nu \text{ iff the quasiconformal homeomorphisms } f_\mu, f_\nu \text{ induce} \\ \text{the same element of } \mathcal{R}(\Gamma, \mathrm{Isom}(\mathbb{H}^3)).\end{aligned}$$

We assume that $\mathcal{T}(\Gamma)$ has the topology of algebraic convergence induced from the usual topology of the algebraic variety $\mathcal{R}(\Gamma, \mathrm{Isom}(\mathbb{H}^3))$. Define the *Teichmüller distance* between the points $[\rho], [\phi] \in \mathcal{T}(\Gamma)$ as

$$\begin{aligned}d_{\mathcal{T}}(\rho, \phi) = \inf\{\frac{1}{2}\log K(f) | f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} &\text{ is a quasiconformal homeomorphism} \\ \text{such that } f_*(\gamma) = \phi \circ \rho^{-1}(\gamma) &\text{ for each } \gamma \in \rho(\Gamma)\}.\end{aligned}$$

From now on let we will assume that the group Γ is torsion-free: it will simplify our arguments and terminology. Let $\Sigma = \Omega(\Gamma)/\Gamma = \Sigma_1 \cup ... \cup \Sigma_n$ be the disjoint union of connected components. Define the Teichmüller space $\mathcal{T}(\Sigma)$ as the direct product of Teichmüller spaces $\mathcal{T}(\Sigma_1) \times ... \times \mathcal{T}(\Sigma_n)$. We define the Teichmüller metric on the product is the maximum of Teichmüller metrics on components. We will use the notation $\mathcal{T}_t(\Gamma)$ for the set $\mathcal{T}(\Gamma)$ given the topology induced by the Teichmüller metric.

Theorem 8.40. *(L. Bers [Ber72].) There exists a natural homeomorphism* $\beta : \mathcal{T}(\Sigma) \to \mathcal{T}_t(\Gamma)$ *which is called the* **Bers identification**. *This homeomorphism is an isometry of Teichmüller metrics. The identity map* $\mathcal{T}_t(\Gamma) \to \mathcal{T}(\Gamma)$ *is always continuous. If* Γ *is geometrically finite then the identity map* $\mathcal{T}_t(\Gamma) \to \mathcal{T}(\Gamma)$ *is a homeomorphism.*

Remark 8.41. If we drop the assumption that $\partial \overset{\bullet}{M}(\Gamma)$ is incompressible then the map β is a covering map.

Proof: The proof which we present here depends upon the Ahlfors' Finiteness Theorem. However in §8.14 we will use arguments from this proof to prove the Ahlfors' Theorem.

We can assume that the group Γ is Kleinian (otherwise the assertion follows from the Sullivan Rigidity Theorem). Let X denote the surface Σ regarded as the origin of the Teichmüller space. Let $[f : X \to Y] \in \mathcal{T}(\Sigma)$, where f is a quasiconformal map. Then $\bar{\partial} f/\partial f$ is a Beltrami differential μ on X. Lift μ to a differential $\tilde{\mu}$ on $\Omega(\Gamma)$ using the covering $\Omega(\Gamma) \to X$. Then $\tilde{\mu}$ is invariant under Γ and we extend $\tilde{\mu}$ by zero to the limit set of Γ. The result is a Beltrami differential on $\widehat{\mathbb{C}}$ which is Γ-invariant (we retain the name $\tilde{\mu}$ for this differential) and we solve the Beltrami equation

$$\bar{\partial} h/\partial h = \tilde{\mu}. \tag{8.8}$$

The quasiconformal homeomorphism h conjugates Γ to a Moebius group Γ'. Therefore, the induced representation $\rho = h_*$ of the group Γ determines an element of $\mathcal{T}(\Gamma)$. We define the map β by the formula:

$$\beta[f : X \to Y] := [\rho].$$

The right inverse to the map β is defined as follows. Suppose that $[\rho] \in \mathcal{T}(\Gamma)$, then we let $\Gamma' = \rho(\Gamma)$, $Y = \Omega(\Gamma')/\Gamma'$ and the *marking* $X \to Y$ is obtained by descending a ρ-equivariant mapping f from $\Omega(\Gamma) \to \Omega(\Gamma')$ to $X \to Y$.

Let us check that the correspondence β is well-defined. Suppose that

$$[f : X \to Y] = [g : X \to Z]$$

is a point in the Teichmüller space of X. Then there exists a biholomorphic map $\alpha : Z \to Y$ which is homotopic to $f \circ g^{-1}$. Define the representations ρ' and ρ'' using $\beta(f)$ and $\beta(g)$. The corresponding groups Γ' and Γ'' are finitely generated and have the property that the representation $\rho' \circ (\rho'')^{-1}$ is induced by a quasiconformal map $\tilde{\alpha}$ which is conformal on $\Omega(\Gamma'')$. By Sullivan's rigidity theorem, the map $\tilde{\alpha}$ is Moebius. Thus $[\rho'] = [\rho'']$. The fact that β is onto follows from existence of the right inverse to $[\beta]$. Then we need to check that β is injective. Suppose that $\beta[X, id] = \beta[f : X \to Y]$. Let $\tilde{\mu}$ be the Γ-invariant Beltrami differential in $\widehat{\mathbb{C}}$ obtained by lifting the Beltrami differential of f. As above, we let h denote the normalized solution of the Beltrami equation (8.8). The assumption $\beta[X, id] = \beta[f : X \to Y]$ is equivalent to saying that the representation $\rho : \Gamma \to PSL(2, \mathbb{C})$ induced by h is the identity. In particular, $\rho(\Gamma) = \Gamma$ and the Riemann surface Y is the quotient $\Omega(\Gamma)/\Gamma$, which equals X. The marking $f : X \to Y$ has to be homotopic to the identity since $\rho : \Gamma \to \Gamma$ is the identity automorphism and each component of $\Omega(\Gamma)$ is simply-connected. We conclude that $[X, id] = [f : X \to Y]$, which implies that β is injective.

Therefore β is a bijection.

Remark 8.42. We note that if X does not have finite conformal type, one has to modify the definition of the Teichmüller space of X to get a meaningful version of Bers' theorem. With the modified definition the Teichmüller space of the open unit disk is no longer a single point, it is the quotient of the quasi-Moebius automorphisms of the unit circle by the action of $PSL(2, \mathbb{R})$. Then the proof of injectivity of β becomes much more complicated.

Sullivan rigidity theorem implies now that β is an isometry of the Teichmüller metrics on $\mathcal{T}(\Gamma)$ and $\mathcal{T}(X)$. This proves the first part of the theorem.

The fact that $id : \mathcal{T}_t(\Gamma) \to \mathcal{T}(\Gamma)$ is continuous follows from continuity of solutions of the Beltrami equation with respect to the Beltrami differentials.

The bicontinuity of this map in the case of geometrically finite groups is less obvious. One way to prove it is to establish that the relative representation variety $\mathcal{R}(\Gamma, P; \mathrm{Isom}(\mathbb{H}^3))$ is smooth near any point $[\rho] \in \mathcal{T}(\Gamma)$ and has the same dimension as $\mathcal{T}(X)$. (We shall prove it in the Sections 8.8, 8.9.) Then continuity of β^{-1} follows from the invariance of domain theorem. The other way is to establish so called *quasiconformal stability* of geometrically finite groups using direct geometric arguments as it is done in [Mar74], [Sul85b]. $\square$

Remark 8.43. If $\Gamma \subset PSL(2, \mathbb{C})$, then Σ is orientable; in this case the Teichmüller spaces $\mathcal{T}(\Gamma) \subset \mathcal{R}(\Gamma, P; PSL(2, \mathbb{C}))$ and $\mathcal{T}(\Sigma)$ have natural complex structures. One can prove that β is biholomorphic.

8.8. Smoothness of representation varieties

Let $\Gamma \subset PSL(2, \mathbb{C})$ be a finitely generated torsion-free discrete nonelementary subgroup, $M = M(\Gamma) = \mathbb{H}^3/\Gamma$. Choose $\epsilon > 0$ smaller than the Margulis constant, then M^0_ϵ is the union of the ϵ-thick part $M_{[\epsilon, \infty)}$ and of the Margulis tubes in M. Let C denote $cl(M - M^0_\epsilon)$, the union of Margulis cusps in M. Let α denote the total number of cusps and τ denote the number of rank 2 cusps in M. Choose a compact core $N := M_c \subset M^0_\epsilon$ as in Section 4.23, $P := M_c \cap C$ is the disjoint union of annuli and tori so that for each component $C_i \subset C$ the inclusion $P \cap C_i \hookrightarrow C_i$ is a homotopy-equivalence, (M_c, P) is a *pared manifold.* For each toral component $T_i \subset P$ choose an incompressible annulus $A_i \subset T_i$; let A denote the union of annular components of P and the annuli $A_i \subset T_i$. Thus A consists of α disjoint annuli. We let B denote $P - A$, i.e. it is a disjoint union of τ annuli each of which is contained in a toral component of ∂N. Recall that $\partial_0 N := \partial N - P$. Let $\rho : \Gamma = \pi_1(N) \hookrightarrow PSL(2, \mathbb{C})$ be the identity representation. We consider the absolute and relative character varieties:

$$\mathcal{R}(\Gamma, PSL(2, \mathbb{C})) \quad \text{and} \quad \mathcal{R}_{par}(\Gamma, PSL(2, \mathbb{C})).$$

By abusing notation we let $[\rho]$ denote the projection of ρ to each of these varieties. Note that

$$\mathcal{R}_{par}(\pi_1(M), \pi_1(P); PSL(2, \mathbb{C}))$$

equals the relative character variety

$$\mathcal{R}(\pi_1(M), \pi_1(A); PSL(2, \mathbb{C})).$$

The following theorem was previously known for geometrically finite groups Γ, in the geometrically infinite case it seems to be new. If Γ is a non-uniform lattice then this theorem is the "algebraic part" of Thurston's Dehn surgery theorem (see [Thu81, Theorem 5.6], [CS83, Theorem 3.2.1], [BP98, Chapter 2]).

Theorem 8.44. *$[\rho]$ is a smooth point in both*

$$\mathcal{R}(\Gamma, PSL(2, \mathbb{C})) \quad \textit{and} \quad \mathcal{R}_{par}(\Gamma, PSL(2, \mathbb{C})).$$

The complex dimension of the first space near $[\rho]$ is $d = -\frac{3}{2}\chi(\partial M_c) + \tau$, the complex dimension of the second space near $[\rho]$ is $d - \alpha$.

Proof: I will first give a quick proof of the first assertion in the case when the group Γ contains no rank 2 parabolic subgroups. Then the closed surface $S = \partial N$ is a disjoint union of hyperbolic surfaces. Let $\mathfrak{g}$ denote the Lie algebra of the group $G = PSL(2, \mathbb{C})$; we shall consider $\mathfrak{g}$ as the Γ-module via the adjoint action of Γ. As we discussed in Section 4.5, it suffices to show that the second group cohomology $H^2(\Gamma, \mathfrak{g})$ vanishes. We will use the sheaf-cohomology and de Rham cohomology interpretation of $H^*(\Gamma, \mathfrak{g})$ in our computations (see Section 4.5):

$$H^*(\Gamma, \mathfrak{g}) \cong H^*(N, \mathcal{F}) \cong H^*(N, \Omega_F)$$

where F is the vector bundle associated to the adjoint action of Γ on $\mathfrak{g}$. By the Poincaré duality $H^2(N, \mathcal{F}) \cong H^1(M, M - N; \mathcal{F})$. However, each component S_j of ∂N is a closed hyperbolic surface (since Γ has no parabolic elements) and $\pi_1(S_j) \subset \Gamma$ operates as a nonelementary subgroup on $\mathbb{H}^3$. For each component $M_j \subset cl(M - N)$ which is bounded by a surface $S_j \subset \partial N$, the inclusion $S_j \hookrightarrow M_j$ is a homotopy-equivalence. Thus the image of $\pi_1(M_j)$ in $PSL(2, \mathbb{C})$ has trivial centralizer and for each component $M_j \subset M - N$ we have: $H^0(M_j, \mathcal{F}) = 0$. Consider the long exact sequence of the pair $(M, M - N)$:

$$\to H^0(M - N, \mathcal{F}) \to H^1(M, M - N; \mathcal{F}) \to H^1(M, \mathcal{F}) \to H^1(M - N, \mathcal{F}) \to$$

Since $H^0(M-N,\mathcal{F}) \cong \oplus_i H^0(M_i,\mathcal{F}) = 0$, the group $H^1(M, M-N;\mathcal{F})$ is isomorphic to the kernel of the restriction homomorphism

$$res : H^1(M,\mathcal{F}) \to H^1(M-N,\mathcal{F}).$$

Suppose $[\xi]$ is an element of the kernel of res. Then, using the isomorphism between sheaf-cohomology and de Rham cohomology, we can take ξ to be a differential 1-form in Ω_F so that ξ is exact in each component of $M-N$. This means that there is a smooth section $\eta : M-N \to F$ so that $\zeta = \xi - d\eta$ is zero in $M-N$. Using a partition of unity we extend η to a smooth section $\eta : M \to F$. Hence the 1-form $\zeta = \xi - d\eta$ is cohomologous to ξ in M and vanishes identically in $M-N$. The form ζ has compact support in $int(N)$, which implies that ζ is L_2-integrable on M.

The following lemma is in the heart of Calabi-Weil infinitesimal rigidity theorem:

Lemma 8.45. *Suppose that $\zeta \in \Omega^1_F$ is L_2-integrable. Then ζ is exact.*

Proof: See [Rag72, Proposition 7.57]. □

Hence the restriction morphism $res : H^1(M,\mathcal{F}) \to H^1(M-N,\mathcal{F})$ is injective and (by duality) $H^2(\Gamma,\mathfrak{g}) \cong H^1(M, M-N;\mathcal{F}) = 0$. This concludes the proof that $Hom(\Gamma, PSL(2,\mathbb{C}))$ is smooth at ρ in the case when M has no rank 2 cusps. To compute the dimension of $H^1(\Gamma,\mathfrak{g})$ we again use duality and Lemma 8.45. By dualizing the assertion that the homomorphism res is injective we conclude that the connecting homomorphism in the long exact sequence of the pair $(M, M-N)$:

$$H^1(M-N,\mathcal{F}) \to H^2(M, M-N;\mathcal{F}) \cong H^1(M,\mathcal{F})$$

is onto. Hence

$$0 \to H^1(M,\mathcal{F}) \to H^1(M-N,\mathcal{F}) \to H^2(M, M-N;\mathcal{F}) \to 0.$$

Thus $dim_{\mathbb{C}} H^1(M,\mathcal{F}) = \frac{1}{2} dim_{\mathbb{C}} H^1(M-N,\mathcal{F})$. Recall that $M-N$ is homotopy-equivalent to the surface ∂N. By Poincaré duality applied to the surface ∂N we have:

$$0 = H^0(M-N,\mathcal{F}) \cong H^2(M-N,\mathcal{F})$$

hence $dim_{\mathbb{C}} H^1(\partial N,\mathcal{F}) = -\chi(\partial N,\mathcal{F}) = -dim_{\mathbb{C}}(\mathfrak{g}) \cdot \chi(\partial N) = -3\chi(\partial N)$. Therefore

$$dim_{\mathbb{C}} H^1(M,\mathcal{F}) = -\frac{3}{2}\chi(\partial N)$$

which concludes the computation of $dim H^1(\Gamma,\mathfrak{g})$ in the case M has no rank 2 cusps.

We now consider the general case. If $\Gamma_j \cong \mathbb{Z}^2$ is a discrete rank 2 parabolic subgroup of $PSL(2,\mathbb{C})$ then $dim_{\mathbb{C}} H^0(\Gamma_j,\mathfrak{g}) = 1$, since Γ_j has 1-dimensional centralizer in $PSL(2,\mathbb{C})$. Hence our computations actually show that

$$H^2(\Gamma,\mathfrak{g}) = H^0(M-N,\mathcal{F}) \cong \oplus_j H^0(M_j,\mathcal{F}) \cong \mathbb{C}^r$$

where τ is the number of rank 2 cusps in M. Therefore we cannot use vanishing of the 2-nd cohomology group to prove smoothness of the representation variety. We will use transversality arguments instead, these arguments are similar to [Ka91] and [AM90, Chapter III].

Lemma 8.46. *There exists a system of disjoint 1-handles $H_j, j = 1, ..., t$, in the manifold N which are attached to ∂N so that $N - int(\cup_j H_j)$ is a handlebody.*

Proof: The proof is a direct generalization of the construction of a Heegaard splitting for a closed manifold. Start with a triangulation Δ of N. The complement to the regular neighborhood $Nbd(\Delta^1)$ of the 1-skeleton Δ^1 of this triangulation is a handlebody. The same is true if instead of Δ^1 we take the union Θ of those edges of Δ which are not contained in ∂N. We think of $Nbd(\Theta)$ as a union of 0-handles (corresponding to the regular neighborhoods of the vertices) and 1-handles, which are the regular neighborhoods of the edges. Finally we slide the 0-handles to the boundary of N which gives the required system of handles. $\square$

The 1-handles H_j are called the *tunnels* in N, the minimal t in the above lemma is called the *tunnel number* of N. Note that $t \geq 1$ unless N is the handlebody. Let N_1 denote the regular neighborhood of $\partial N \cup \cup_j H_j$ in N. The manifold N_1 is homotopy-equivalent to the following CW-complex:

Take the wedge of all boundary surfaces of N and take a wedge of this with the $t - b + 1$ circles, where b is the number of boundary components of N.

The frontier of N_1 in N is a connected surface Σ so that $\chi(\Sigma) = \chi(\partial N) - 2t$.

Let N_2 denote $N - int(N_1)$, this manifold is homeomorphic to the handlebody of the genus $g = t + 1 - \chi(\partial M)/2$ (i.e. the genus of Σ). Note that the fundamental group of Σ maps onto $\pi_1(N)$, hence the images of $\pi_1(\Sigma)$ and $\pi_1(N_2)$ in $PSL(2, \mathbb{C})$ are nonelementary subgroups.

Cohomology computations for $H^1(N_1, \mathcal{F}), H^1(N_2, \mathcal{F}), H^1(\Sigma, \mathcal{F})$**.**

(i) $dim_{\mathbb{C}} H^1(\Sigma, \mathcal{F}) = -3\chi(\Sigma)$ since the image of $\pi_1(\Sigma)$ in $PSL(2, \mathbb{C})$ is a nonelementary subgroup (see Section 4.5).

(ii) The fundamental group of N_2 is free on g generators, hence

$$dim_{\mathbb{C}} Z^1(\pi_1(N_2), \mathfrak{g}) = 3g, \quad dim_{\mathbb{C}} B^1(\pi_1(N_2), \mathfrak{g}) = 3$$

and

$$dim_{\mathbb{C}} H^1(\pi_1(N_2), \mathfrak{g}) = 3g - 3.$$

(iii) The fundamental group of N_1 is the free product of the following groups:

(1) The hyperbolic surface groups $\pi_1(S_j), j = 1, ..., b - \tau$, where S_j are non-toral components of ∂N.

(2) $\mathbb{Z}^2$-subgroups which are $\pi_1(T_i), i = 1, ..., \tau$, where T_i are the toral components of ∂N.

(3) $t-b+1$ copies of $\mathbb{Z}$ which correspond to the circle components of the wedge decomposition.

The representation varieties (to $PSL(2,\mathbb{C})$) of each of these groups are smooth near the restriction of ρ (see Section 4.5). Hence $Hom(\pi_1(N_1), PSL(2,\mathbb{C}))$, which is the direct product of the representation varieties of the free factors of $\pi_1(N_1)$, is smooth near ρ. If $\Gamma = \Gamma_1 * \Gamma_2$ then $Z^1(\Gamma,\mathfrak{g}) \cong Z^1(\Gamma_1,\gamma) \oplus Z^1(\Gamma_2,\gamma)$. Thus

$$dim_{\mathbb{C}} Z^1(\pi_1(N_1),\mathfrak{g}) = \sum_{j=1}^{b-\tau} dim_{\mathbb{C}} Z^1(\pi_1(S_j),\mathfrak{g}) + \sum_{i=1}^{\tau} dim_{\mathbb{C}} Z^1(\pi_1(T_i),\mathfrak{g}) + \\ +3(t-b+1) = -3\chi(\partial N) + \tau + 3t + 3$$

and

$$dim_{\mathbb{C}} H^1(\pi_1(N_2),\mathfrak{g}) = -3\chi(\partial N) + \tau + 3t \ .$$

By combining these formulae we get:

$$dim_{\mathbb{C}} H^1(\pi_1(N_1),\mathfrak{g}) + dim_{\mathbb{C}} H^1(\pi_1(N_2),\mathfrak{g}) = \\ \tau - \frac{3}{2}\chi(\partial N) + dim_{\mathbb{C}} H^1(\pi_1(\Sigma),\mathfrak{g}).$$

We now compute the dimension of $H^1(\Gamma,\mathfrak{g})$. By the same arguments as in the case when M has no cusps of rank 2 we get the short exact sequence:

$$0 \to H^1(N,\mathcal{F}) \to H^1(\partial N,\mathcal{F}) \to H^2(N,\partial N;\mathcal{F}) \to 0$$

Hence

$$dim_{\mathbb{C}} H^1(N,\mathcal{F}) = \frac{1}{2} dim_{\mathbb{C}} H^1(\partial N,\mathcal{F}) = \tau - \frac{3}{2}\chi(\partial N).$$

Let $Res_i : Hom(\pi_1(N_i), G) \to Hom(\pi_1(\Sigma), G)$ denote the *restriction* homomorphisms. Then, according to the Exercise 4.14, we have an isomorphism of algebraic varieties

$$Hom(\pi_1(N), G) \cong Hom(\pi_1(N_1), G) \times_{Res_1 = Res_2} Hom(\pi_1(N_2), G)$$

and by taking derivatives at ρ:

$$Z^1(\pi_1(N),\mathfrak{g}) \cong Z^1(\pi_1(N_1),\mathfrak{g}) \oplus_{D_\rho Res_1 = D_\rho Res_2} Z^1(\pi_1(N_2),\mathfrak{g}).$$

The varieties $Hom(\pi_1(N_1), G), Hom(\pi_1(N_2), G), Hom(\pi_1(\Sigma), G)$ are smooth near ρ, hence to verify smoothness of $Hom(\pi_1(N), G)$ is it enough to check that the smooth maps maps Res_1, Res_2 are transversal near ρ. Thus transversality follows from the formula:

$$dim_{\mathbb{C}} Z^1(\pi_1(N_2),\mathfrak{g}) + dim_{\mathbb{C}} Z^1(\pi_1(N_2),\mathfrak{g}) - dim_{\mathbb{C}} Z^1(\pi_1(\Sigma),\mathfrak{g}) = \\ dim_{\mathbb{C}} Z^1(\pi_1(N),\mathfrak{g}).$$

This proves smoothness of $Hom(\pi_1(N), G)$ near ρ. Since the group Γ has trivial centralizer in G, the character variety $\mathcal{R}(\Gamma, G)$ is also smooth near $[\rho]$ and its dimension equals

$$dim_{\mathbb{C}} H^1(N,\mathcal{F}) = dim_{\mathbb{C}} H^1(\pi_1(N),\mathfrak{g}) = \\ \frac{1}{2} dim_{\mathbb{C}} H^1(\partial N,\mathcal{F}) = \tau - \frac{3}{2}\chi(\partial N).$$

We now consider the relative case. By the definition of $\mathcal{R}(\pi_1(N), \pi_1(A); G)$, its Zariski tangent space at $[\rho]$ equals

$$H^1_{par}(N, \mathcal{F}) := ker[H^1(N, \mathcal{F}) \to H^1(A, \mathcal{F})].$$

Lemma 8.47. *The restriction map*

$$res : H^1(N, \mathcal{F}) \to H^1(\partial N - A, \mathcal{F})$$

is injective.

Proof: We convert to the de Rham cohomology and consider the restriction map

$$res : H^1(N, \Omega_F) \to H^1(\partial N - A, \Omega_F).$$

Suppose $[\xi]$ is an element of the kernel. Then take ξ to be a differential 1-form in Ω_F which is identically zero in a neighborhood of $\partial N - A$. Without loss of generality we can assume that the normal derivative of ξ on $\partial N \cap C$ is identically zero. We extend ξ by zero to the complement $M - C$. The resulting form ξ' clearly belongs to $L_2(M - C)$. If $C_j \subset C$ is a rank 1 cusp then $C_j \cong \mathbb{S}^1 \times \mathbb{R} \times \mathbb{R}_+$ where $\mathbb{S}^1 \times \mathbb{R}$ corresponds to the boundary of C_j. The form $\xi'|\partial C_j$ vanishes on the complement to the annulus $A_j = \mathbb{S} \times (-1, 1) \subset \partial C_j$. Hence we extend ξ' by zero to $(\partial C_j - A_j) \times \mathbb{R}_+$. Let ζ denote the resulting 1-form. The remaining part $C'_j := A_j \times \mathbb{R}_+$ of the cusp C_j has finite hyperbolic volume. The rank 2 cusps $C_i \cong T^2 \times \mathbb{R}_+$ also have finite volume. Thus the cohomology class $[\xi]$ is zero provided that we can extend the form ζ to a closed bounded form in each truncated rank 1 cusp C'_j and each rank 2 cusp C_i. (This extension would be a closed $L_2(M)$-form which is exact by Lemma 8.45.) We construct such extension in the case of rank 2 cusps and leave the case of C'_j to the reader.

Consider the restriction ζ to $C_i = T^2 \times \mathbb{R}_+$. Without loss of generality we can replace ζ by cohomologous 1-form on ∂C_i which is identically zero on the annulus $B_i = T^2 - A_i$. From the point of view of group cohomology this amounts to considering cocycles in $Z^1(\pi_1(C_i), \mathfrak{n})$ where $\mathfrak{n}$ is the (nilpotent) Lie subalgebra in $\mathfrak{g}$ so that the adjoint action of $\pi_1(C_i)$ on $\mathfrak{n}$ is trivial. (The subalgebra $\mathfrak{n}$ is the Lie algebra of the maximal unipotent Lie subgroup in $PSL(2, \mathbb{C})$ which contains the image of $\pi_1(C_i)$.) However the associated flat bundle $U \to C_i$ is the product bundle. Thus we use the constant extension of the form ζ to the cusp C_i. $\square$

Let S denote the union of non-toral components of ∂N, $S_0 = \partial_0 N = S - P$.

Lemma 8.48. $H^1(S, \mathcal{F}) \cong H^1(S_0, \mathcal{F})$.

Proof: $H^2(S, \mathcal{F}) \cong H^2(S_0, \mathcal{F}) = 0$ and $H^0(S, \mathcal{F}) \cong H^0(S_0, \mathcal{F}) = 0$. Thus the isomorphism $H^1(S_0, \mathcal{F}) \cong H^1(S, \mathcal{F})$ follows from the equality of Euler characteristics: $\chi(S, \mathcal{F}) = \chi(S_0, \mathcal{F})$. $\square$

Exercise 8.49. *Use Mayer-Vietoris sequence to show that the restriction homomorphism* $H^1(S, \mathcal{F}) \to H^1(S_0, \mathcal{F})$ *is not injective unless* $S_0 = S$.

Lemma 8.50. $dim_{\mathbb{C}} H^1_{par}(N, \mathcal{F}) = dim_{\mathbb{C}} H^1(N, \mathcal{F}) - \alpha$.

Proof: Recall that

$$dim_{\mathbb{C}} H^0(\partial N, \mathcal{F}) = dim_{\mathbb{C}} H^1(N, \partial N; \mathcal{F}) = dim_{\mathbb{C}} H^2(N, \mathcal{F}) = \tau.$$

The exact sequence of the pair (N, A):

$$\begin{aligned} 0 = H^0(N, \mathcal{F}) \to H^0(A, \mathcal{F}) \cong \mathbb{C}^\alpha \to H^1(N, A; \mathcal{F}) \to H^1(N, \mathcal{F}) \\ \to H^1(A, \mathcal{F}) \to H^2(N, A; \mathcal{F}) \cong H^1(N, \partial N - A; \mathcal{F}) \to H^2(N, A; \mathcal{F}) \\ \cong H^1(N, \partial N - A; \mathcal{F}) \to H^2(N, \mathcal{F}) \to H^2(A, \mathcal{F}) = 0 \end{aligned} \tag{8.9}$$

implies exactness of

$$0 \to \mathbb{C}^\alpha \to H^1(N, A; \mathcal{F}) \to H^1_{par}(N, \mathcal{F}) \to 0.$$

Hence, to prove lemma suffices to show that $H^1(N, A; \mathcal{F}) \cong H^1(N, \mathcal{F})$. Considering the sequence (8.9) again we note that

$$dim_{\mathbb{C}} H^1(N, \mathcal{F}) - dim_{\mathbb{C}} H^1(N, A; \mathcal{F}) = \tau - dim_{\mathbb{C}} H^1(N, \partial N - A; \mathcal{F}).$$

Finally, consider the long exact sequence of the pair $(N, \partial N - A) = (N, S_0 \cup B)$:

$$0 \to H^0(S_0 \cup B, \mathcal{F}) \to H^1(N, S_0 \cup B; \mathcal{F}) \xrightarrow{0} H^1(N, \mathcal{F}) \to H^1(S_0 \cup B, \mathcal{F}) \to$$

According to Lemma 8.47 the map $H^1(N, \mathcal{F}) \to H^1(S_0 \cup B, \mathcal{F})$ is injective; thus

$$\mathbb{C}^\tau \cong H^0(S_0 \cup B, \mathcal{F}) \cong H^1(N, \partial N - A; \mathcal{F}),$$

which implies that

$$dim_{\mathbb{C}} H^1(N, \mathcal{F}) - dim_{\mathbb{C}} H^1(N, A; \mathcal{F}) = \tau - dim_{\mathbb{C}} H^1(N, \partial N - A; \mathcal{F}) = 0.$$

Therefore $dim_{\mathbb{C}} H^1(N, A; \mathcal{F}) = dim_{\mathbb{C}} H^1(N, \mathcal{F})$. $\square$

We are now ready to apply transversality arguments again. Consider the restriction morphism

$$Res : Hom(\pi_1(N), G) \to \prod_{i=1}^{\alpha} Hom(\pi_1(A_i), G)$$

where $G = PSL(2, \mathbb{C})$. We identify ρ with its image under Res. We know that the variety $Hom(\pi_1(N), G)$ and each $Hom(\pi_1(A_i), G)$ are smooth near ρ. Thus it suffices to show that Res is transversal to the smooth subvariety

$$\prod_{i=1}^{\alpha} Hom_{par}(\pi_1(A_i), G)$$

where each

$$Hom_{par}(\pi_1(A_i), G)$$

is canonically isomorphic to the $ad(G)$-orbit through $\rho|\pi_1(A_i)$ (see Convention 4.11). Thus all what we need is the equality:

$$dim_{\mathbb{C}}(D_\rho Res)^{-1}(T_\rho \prod_{i=1}^{\alpha} Hom_{par}(\pi_1(A_i), G)) + dim_{\mathbb{C}} \prod_{i=1}^{\alpha} Hom(\pi_1(A_i), G) =$$

$$dim_{\mathbb{C}} \prod_{i=1}^{\alpha} Hom_{par}(\pi_1(A_i), G) + dim_{\mathbb{C}} T_\rho Hom(\pi_1(N), G) \qquad (8.10)$$

According to Lemma 8.50 we have

$$dim_{\mathbb{C}}(D_\rho res)^{-1}(T_\rho \prod_{i=1}^{\alpha} Hom_{par}(\pi_1(A_i), G)) = dim_{\mathbb{C}} Z^1_{par}(\Gamma, \mathfrak{g}) =$$

$$dim_{\mathbb{C}} Z^1(N, \mathcal{F}) - \alpha \quad .$$

Therefore the equality (8.10) follows from

$$dim_{\mathbb{C}} Z^1(N, \mathcal{F}) - \alpha + 2\alpha = \alpha + dim_{\mathbb{C}} Z^1(N, \mathcal{F}).$$

Thus we proved smoothness of $Hom_{par}(\Gamma; G)$ near ρ which in turn implies that $\mathcal{R}_{par}(\Gamma; G)$ is smooth near $[\rho]$. The formula for the dimension of $\mathcal{R}_{par}(\Gamma, \pi_1(P); G)$ near $[\rho]$ immediately follows from Lemma 8.50 since $H^1_{par}(N, \mathcal{F})$ is isomorphic to the tangent space to $\mathcal{R}_{par}(\Gamma, \pi_1(P); G)$ at $[\rho]$. □

Remark 8.51. Suppose that the manifold $M = M(\Gamma)$ has finite volume and τ cusps. Note that we proved that $\mathcal{R}(\Gamma, G)$ is a smooth complex manifold of dimension τ near $[\rho]$ and that the restriction map

$$Res : Hom(\pi_1(N), G) \to \prod_{i=1}^{\tau} Hom(\pi_1(A_i), G)$$

is transversal to the smooth subvariety

$$\prod_{i=1}^{\alpha} Hom_{par}(\pi_1(A_i), G).$$

Recall that $\rho(A_i)$ is a nontrivial subgroup of $G = PSL(2, \mathbb{C})$ for each i. Hence for each i the action of G by conjugations on $Hom(\pi_1(A_i), G)$ near ρ is a smooth bundle over complex 1-dimensional base. Thus the projection of Res to

$$[Res] : \mathcal{R}(\Gamma, G) \to \prod_{i=1}^{\tau} Hom(\pi_1(A_i), G)/G$$

is also a submersion at $[\rho]$.

8.9. Applications to quasiconformal stability

According to Theorem 8.44 the character variety $\mathcal{R}_{par}(\Gamma, PSL(2, \mathbb{C}))$ is smooth near $[id]$ and its dimension equals

$$-\frac{3}{2}\chi(\partial M_c) - \beta$$

where $\beta = \alpha - \tau$ is the number of rank 1 cusps in Γ and M_c is a compact core of $M(\Gamma)$. Now we assume that Γ is geometrically finite, thus each rank 1 cusp corresponds to two punctures in the Riemann surface $S(\Gamma) = \Omega(\Gamma)/\Gamma$. Therefore the total number of cusps in $S(\Gamma)$ equals $n = 2\beta$. Let $S(\Gamma)$ be the disjoint union of k surfaces S_i of genus g_i with n_i punctures. Recall that the complex dimension of the Teichmüller space $\mathcal{T}(S(\Gamma))$ equals

$$\sum_{i=1}^{k}(3g_i - 3 + n_i).$$

The Euler characteristic of S_i equals $2 - 2g_i - n_i$, i.e. $3g_i - 3 + n_i = -\frac{1}{2}(3\chi(S_i) + n_i)$. Also note that $\chi(S(\Gamma)) = \chi(\partial M_c)$. Hence

$$dim_{\mathbb{C}}\mathcal{T}(S(\Gamma)) = \sum_{i=1}^{k} -\frac{1}{2}(3\chi(S_i)+n_i) = -n/2 - \frac{3}{2}\chi(S(\Gamma)) = -\frac{3}{2}\chi(\partial M_c) - \beta.$$

Therefore, we conclude that (near $[id]$)

$$dim_{\mathbb{C}}\mathcal{R}_{par}(\Gamma, PSL(2,\mathbb{C})) = dim_{\mathbb{C}}\mathcal{T}(S) = -\frac{3}{2}\chi(\partial M_c) - \beta\ .$$

This implies that the complex dimension of the variety $\mathcal{R}_{par}(\Gamma, PSL_2(\mathbb{C}))$ near any representation $[\rho] \in \mathcal{T}(\Gamma)$ is equal to the dimension of $\mathcal{T}(\Gamma)$ and $\mathcal{R}_{par}(\Gamma, PSL_2(\mathbb{C}))$ is smooth at such points. Bers' theorem follows.

Theorem 8.52. *Suppose that $\Gamma \subset PSL(2,\mathbb{C})$ is a torsion–free geometrically finite group. Then the Teichmüller space $\mathcal{T}(\Gamma)$ is open in $\mathcal{R}_{par}(\Gamma, PSL(2,\mathbb{C}))$ (with respect to the topology of algebraic convergence).*

Proof: Follows from the preceding calculations since both spaces are manifolds of the same dimension. □

Corollary 8.53. *Suppose that $\Gamma \subset PSL(2,\mathbb{C})$ is a geometrically finite torsion–free group and*

$$\rho_n \in Hom_{par}(\Gamma, PSL(2,\mathbb{C}))$$

is a sequence of representations of Γ which converges to the identity representation $\rho : \Gamma \hookrightarrow PSL(2,\mathbb{C})$. Then for all sufficiently large n, the $PSL(2,\mathbb{C})$-conjugacy classes of the representations $\rho_n : \Gamma \to \rho_n(\Gamma)$ belong to the Teichmüller space $\mathcal{T}(\Gamma)$ and ρ_n are induced by quasiconformal conjugations $f_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ so that

$$\lim_{n\to\infty} K(f_n) = 1.$$

Proof: This is just another reformulation of Theorems 8.52 and 8.44. □

The above property of the group Γ is called *quasiconformal stability.* It was first established by Marden in [Mar74] whose proof was based on analysis of deformations of the Dirichlet fundamental polyhedron of Γ. Sullivan [Sul85b] gave a different proof based on symbolic dynamics. Yet another proof is contained in [ECG87], it is based on a generalization of the Holonomy Theorem. Sullivan proved that a finitely generated discrete torsion-free group $\Gamma \subset PSL(2,\mathbb{C})$ is quasiconformally stable if and only if Γ is geometrically finite [Sul85b]. Below we present a proof based on Theorem 8.8 and Sullivan rigidity theorem.

Theorem 8.54. *Suppose that Γ is a nonelementary finitely generated discrete torsion-free subgroup of $PSL(2,\mathbb{C})$. Then Γ is quasiconformally stable if and only if Γ is geometrically finite.*

Proof: We already know that any geometrically finite group is quasiconformally stable. Suppose that Γ is a quasiconformally stable group. Define $S = S(\Gamma) = \Omega(\Gamma)/\Gamma$. Theorem 8.37 implies that the space of quasiconformal deformations $\mathcal{T}(\Gamma)$ is isomorphic to the Teichmüller space $\mathcal{T}(S)$. By the assumption, this space is locally homeomorphic to the representation variety $\mathcal{R}_{par}(\Gamma, PSL(2,\mathbb{C}))$ near $[id]$. In particular, they must have the same dimension. In the following computations we will ignore the rank 2 Margulis cusps since they do not contribute to the dimensions of $\mathcal{R}_{par}(\Gamma, PSL(2,\mathbb{C}))$ and $\mathcal{T}(S)$. Take a relative Scott compact core N of the manifold $\dot{M}(\Gamma) = (\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$ so that N intersects each Margulis cusp in a single incompressible annulus. According to our calculation of dimensions

$$dim_{\mathbb{C}} \mathcal{R}_{par}(\Gamma, PSL(2,\mathbb{C})) = -\frac{3}{2}\chi(\partial N) - \beta$$

where β is the number of rank 1 cusps in M. On the other hand,

$$dim_{\mathbb{C}} \mathcal{T}(\Gamma) = -\frac{3}{2}\chi(S) - n/2$$

where n is the number of punctures on the surface S. Let $S' := \partial N - S$. Let d denote the number of boundary components of S'. Note that $\chi(S) + \chi(S') = \chi(\partial N)$. The connected components of S' correspond to geometrically infinite ends of M. Hence S' does not contain components which are annuli and triply punctured spheres (see Lemma 4.34). Thus

$$-3\chi(S') - d \geq m$$

where m is the number of components of S'. In particular, if Γ is geometrically infinite, then $-3\chi(S') - d > 0$. On the other hand, direct computation shows that

$$-\frac{3}{2}\chi(\partial N) - \beta - (-\frac{3}{2}\chi(S) - n/2) = (-3\chi(S') - d)/2 > 0.$$

Hence $dim_{\mathbb{C}} \mathcal{R}_{par}(\Gamma, PSL(2,\mathbb{C})) > dim_{\mathbb{C}} \mathcal{T}(\Gamma)$. Contradiction. Thus M has no geometrically infinite ends, which proves that Γ is geometrically finite. □

8.10. Calabi-Weil infinitesimal rigidity theorem

Let $G = \mathrm{Isom}(\mathbb{H}^n)$, $n \geq 3$, and $\mathfrak{g}$ be the Lie algebra of G. Let Γ be a lattice in G. The following theorem historically preceded Mostow rigidity theorem:

Theorem 8.55. *(E. Calabi [Cal61], A. Weil [Wei60, Wei62].) The identity representation $\rho : \Gamma \hookrightarrow G$ is infinitesimally rigid, i.e. $H^1_{par}(\Gamma, \mathfrak{g}) = 0$. In particular, the point $[\rho]$ is isolated in $\mathcal{R}_{par}(\Gamma, G)$.*

Actually, if $n \geq 4$ then the same conclusion holds if we replace $H^1_{par}(\Gamma, \mathfrak{g})$ by $H^1(\Gamma, \mathfrak{g})$, since for each discrete parabolic subgroup $P \subset G$ of maximal virtual rank, $H^1(P, \mathfrak{g}) = H^1_{par}(P, \mathfrak{g})$ (see [GR70] for details). Note that this rigidity theorem does not follow from Mostow rigidity: a point in algebraic variety can be isolated and have nonzero Zariski tangent space in the same time. (Consider for instance $\{z \in \mathbb{C} | z^2 = 0\}$.)

There are two strategies in proving Theorem 8.55. The first one was used by Calabi and later generalized by Weil (see also [Rag72]). Let $M = M(\Gamma) = \mathbb{H}^n/\Gamma$. Let $[\xi] \in H^1_{par}(\Gamma, \mathfrak{g}) \cong H^1_{par}(M, \Omega_F)$. The 1-form $\xi \in \Omega^1_F$ representing $[\xi]$ can be chosen in $L_2(M)$. Then one finds a *harmonic* L_2-form $\omega \in \Omega^1_F$ which represents $[\xi]$. Finally, Bochner-type computations show that such forms have to be identically zero, which implies that $[\xi] = 0$. $\square$

The second strategy is to follow Mostow's proof, it is realized in [Ka98a]. Recall that the Lie algebra $\mathfrak{g}$ embeds naturally in the Lie algebra of smooth vector fields on $\mathbb{H}^n$; moreover each vector field $\alpha \in \mathfrak{g}$ extends to a conformal vector field on $\mathbb{S}^{n-1}$. One can find a *quasiconformal vector field* ζ on $\mathbb{H}^n$ which induces the cocycle $c \in Z^1_{par}(\Gamma, \mathfrak{g})$, i.e.

$$\zeta - g_*\zeta = c_g \in \mathfrak{g}, \quad \text{for each} \quad g \in \Gamma.$$

The quasiconformal vector field admits a *tangential extension* ζ_∞ to the sphere at infinity so that ζ_∞ is again a quasiconformal vector field on $\mathbb{S}^{n-1}$ and that

$$\zeta_\infty - g_*\zeta_\infty = c_g \in \mathfrak{g}, \quad \text{for each} \quad g \in \Gamma.$$

Then, unless ζ_∞ is a conformal vector field, the field ζ_∞ yields a measurable Γ-invariant plane field on $\mathbb{S}^{n-1}$, which is Γ-invariant. This contradicts the ergodic properties of Γ. $\square$

8.11. Space of quasifuchsian representations

One of the most important examples of representation varieties for us will be the Teichmüller space $\mathcal{T}(\Gamma)$ of a Fuchsian group. In this case $\Omega(\Gamma) = \Omega \cup \bar{\Omega}$, where the overline means the complementary open topological disk in $\mathbb{S}^2$ (this notation will become convenient later). Thus $\Sigma = \Omega(\Gamma)/\Gamma = \Sigma \cup \bar{\Sigma} = \Omega/\Gamma \cup \bar{\Omega}/\Gamma$. We can consider $\Sigma, \bar{\Sigma}$ as two copies of one and the same Riemann surface with different orientations (they are isometric as hyperbolic surfaces via an orientation reversing isometry). Then $\mathcal{T}(\Gamma) \cong \mathcal{T}(\Sigma) \times \mathcal{T}(\bar{\Sigma})$ is a manifold of the complex dimension $6g - 6 + 2p$, where g is the genus of Σ and p is the number of punctures. There is a natural involution θ on $\mathcal{T}(\Gamma)$ which is defined as follows.

Suppose that an element $[\rho] \in \mathcal{T}(\Gamma)$ is given by a quasiconformal map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with Beltrami differential μ. Define a new Beltrami differential ν on $\widehat{\mathbb{C}}$ by the formula:

$$\nu(z) = \overline{\mu}(\bar{z})$$

where we assume that $\Lambda(\Gamma)$ is the real line $\{Im(z) = 0\} \cup \{\infty\}$. Then direct calculation shows that ν is invariant under Γ and thus corresponds to an element $\theta([\rho]) \in \mathcal{T}(\Gamma)$. In terms of the decomposition $\mathcal{T}(\Gamma) = \mathcal{T}(\Sigma) \times \mathcal{T}(\bar{\Sigma})$ (given by the Bers' theorem) one can describe θ as $\theta(\tau_1, \tau_2) = (\tau_2, \tau_1)$, where we identify the surfaces $\Sigma, \bar{\Sigma}$ using orientation reversing isometry j induced by the complex conjugation. Therefore, an element $\tau = [\rho] = \beta(\tau_1, \tau_2)$ is a Fuchsian representation iff $\tau = \theta(\tau)$. For instance, suppose that $\phi, \psi \in Out_a(\Gamma)$ are outer automorphisms which do not differ by a finite order automorphism. Then ϕ, ψ are induced by two quasiconformal homeomorphisms h_1, h_2 of the surfaces $\Sigma, \bar{\Sigma}$. Thus, (h_1, h_2) defines an element τ of of $\mathcal{T}(\Sigma) \times \mathcal{T}(\bar{\Sigma})$ as $([\Sigma, h_1 : \Sigma \to \Sigma], [\bar{\Sigma}, h_2 : \bar{\Sigma} \to \bar{\Sigma}])$. The point τ is not fixed by the involution θ. Therefore, the representation $[\rho] = \beta(\tau)$ is not Fuchsian. An example of this situation is when $\phi = \psi^{-1}$ is an outer automorphism of infinite order.

Corollary 8.56. *Suppose that Γ is a geometrically finite Kleinian group such that the manifold $\overset{\bullet}{M}(\Gamma)$ has incompressible boundary. Let $F_1, ..., F_k$ be the collection of subgroups in Γ corresponding to the components of $\Omega(\Gamma)/\Gamma$. Then $\mathcal{T}(\Gamma)$ naturally embeds in the product $\mathcal{T}(F_1) \times ... \times \mathcal{T}(F_k)$.*

Proof: Let Ω_i denote the component of $\Omega(\Gamma)$ stabilized by F_i. If $[\rho] \in \mathcal{T}(\Gamma)$ corresponds to $(\tau_1, ..., \tau_k)$, where $\tau_i \in \Omega_i/F_i$, then

$$(\tau_1, ..., \tau_k) \mapsto ((\tau_1, \bar{\tau}_1), ..., (\tau_k, \bar{\tau}_k)) \in \mathcal{T}(F_1) \times ... \times \mathcal{T}(F_k). \qquad \square$$

8.12. Distortion of the translation length

Theorem 8.57. *Suppose that $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a K-quasiconformal homeomorphism which conjugates two loxodromic elements g_1, g_2 with the translation lengths $\ell(g_1), \ell(g_2)$. Then*

$$K^{-1} \leq \ell(g_2)/\ell(g_1) \leq K \tag{8.11}$$

Proof: The quotient $\Omega(\langle g_j \rangle)/\langle g_j \rangle$ is a torus T_j. Up to conformal conjugation, the elements g_j have the form $z \mapsto \lambda_j z$, where $|\lambda_j| = \exp(\ell(g_j))$. Consider the universal covering

$$\exp : \mathbb{C} \to \mathbb{C}^* = \Omega(\langle g_j \rangle).$$

Then the preimages of the fundamental rings $\{1 \leq |z| \leq |\lambda_j|\}$ of $\langle g_j \rangle$ under the exponential map are the vertical strips $S_j = \{0 \leq Im(z) \leq \log(|\lambda_j|)\}$.

The homeomorphism f descends to a quasiconformal homeomorphism $q : T_1 \to T_2$. The coefficient of quasiconformality of q is not less than the coefficient of quasiconformality $K(h)$ of an extremal affine map $h : T_1 \to T_2$ which is homotopic to q. A lift of h to the universal cover $\mathbb{C}$ is an affine map A which sends S_1 to the strip S_2. Therefore we can use the calculations in [Abi80, Chapter I, §2], to conclude that the coefficient of quasiconformality $K(A) = K(h)$ is at least $\log(|\lambda_2|)/\log(|\lambda_1|) = \ell(g_2)/\ell(g_1)$. Thus, $K = K(f) \geq \ell(g_2)/\ell(g_1)$. The consideration of the inverse map f^{-1}, which has the same coefficient of quasiconformality K, yields the estimate. $\square$

8.13. Proof of the recurrence theorem

Proof of Theorem 4.123. We will use quasiconformal deformations of Kleinian groups to prove the Ahlfors Recurrence Theorem. Recall that Γ is a finitely generated discrete subgroup of $PSL(2,\mathbb{C})$. Suppose that there exists a subset $A \subset \Lambda(\Gamma)$ of nonzero measure such that $mes(gA \cap A) \neq 0$ only for finite number of elements $g \in \Gamma$. Then, after taking a finite subdivision of A, we can assume that

$$mes(gA \cap A) \neq 0 \iff g = 1.$$

Define the vector space $Belt(A)$ as the collection of $\mu_A \in L^\infty(A, \mathbb{C})$ such that $||\mu_A|| < 1$. Extend each μ_A by zero to $\widehat{\mathbb{C}} - \Gamma(A)$ and by $g^*(\mu_A)$ to each set $g^{-1}A$, where $g \in \Gamma$. The result of this extension is a Γ-invariant Beltrami differential μ, $||\mu|| < 1$ and the map $\mu_A \mapsto \mu$ is linear. Then for each μ_A there exists a quasiconformal map $f = f^\mu$ normalized as $f(0,1,\infty) = (0,1,\infty)$ which is the solution of the Beltrami equation $\bar{\partial} f = \mu \partial f$. The map f conjugates Γ to a discrete group $\Gamma' = f^{-1}\Gamma f \subset PSL(2,\mathbb{C})$. Therefore, like in the proof of Bers' theorem, we have a continuous embedding $\beta : Belt(A) \to \mathcal{R}(\Gamma, PSL(2,\mathbb{C}))$. (Injectivity of β follows from the fact that a quasiconformal conjugation of a discrete Moebius group is never induced by a conjugation in $PSL(2,\mathbb{C})$ unless the restriction of the quasiconformal homeomorphism to the limit set is conformal.) However, the space $Belt(A)$ is infinite–dimensional, and the representation variety $\mathcal{R}(\Gamma, PSL(2,\mathbb{C}))$ has finite dimension since Γ is finitely generated. This contradiction concludes the proof of Theorem 4.123. $\square$

8.14. Proof of the Ahlfors finiteness theorem

I now give a proof of the Ahlfors' finiteness theorem following the same line of reasoning as in the previous section. The proof presented here requires just the bare minimum of the complex analysis: (a) the existence theorem for the Beltrami equation and (b) the Rado-Cartan uniqueness theorem for holomorphic functions. However our proof does require some 3-dimensional topology and/or Greenberg's algebraic trick to deal with the triply-punctured spheres. The key ideas of the proof are due to Ahlfors [Ahl64] and Carleson & Gamelin [CG93, pp. 72–72]. The original Ahlfors' arguments, as well as the deformation arguments presented below, cannot exclude[3] the possibility that $S(G) = \Omega(G)/G$ contains infinite number of triply punctured spheres. (Few more exceptional cases appear if G has torsion.) The first proof of finiteness of the number of triply punctured spheres as found by L. Greenberg. Analytical approaches to resolving this problem are described in [Ber67], [Ahl69], [Kra72]. Alternatively one can use topological arguments based upon a version of Scott compact core theorem (as it is done in [KS89]).

In what follows we will need the following boundary version of the uniqueness theorem for holomorphic functions.

[3] In the case when G is torsion-free.

Theorem 8.58. *(T.Rado, H. Cartan, see e.g. [Nar, Ch. 11, §8, Theorem 2].) Suppose that f is a holomorphic function in a connected open subset $D \subset \mathbb{C}$ and the boundary ∂D contains a nonisolated point z_0 so that the following holds. There exists a neighborhood U of z_0 in $\mathbb{C}$ such that for each $z \in U \cap \partial D$,*

$$\lim_{w \to z} f(w) = 0.$$

Then f is identically zero in D.

We now begin the proof the Ahlfors' finiteness theorem (Theorem 4.108).

Step 1. Recall that according to Selberg Lemma, the group G contains a finite index torsion-free subgroup G'. It is clear that G' is finitely generated, hence analytical finiteness of G' would imply analytical finiteness of G. Thus we assume that G is torsion-free. We next note that it suffices to prove analytical finiteness for finitely generated Kleinian groups G such that each component of $\Omega(G)$ is contractible. To prove this implication we apply the Loop Theorem to the pair

$$(M(G), S(G))$$

where $S(G) = \Omega(G)/G$ and $M(G) = (\mathbb{H}^3 \cup \Omega(G))/G$. The Loop Theorem implies that $S(G)$ is the conformal connected sum of a finite number of surfaces $S(G_j)$, $j = 1, ..., k$, where each component of $\Omega(G_j)$ is simply-connected for every j. If we know that each $S(G_j)$ has finite conformal type, this would imply that $S(G)$ has finite conformal type as well and we are done.

Step 2.

Claim 8.59. *Suppose that $G \subset PSL(2, \mathbb{R})$ is a finitely generated Kleinian group such that $\Lambda(G) = \mathbb{S}^1$. Then the Riemann surface $\mathbb{H}^2/G$ is has finite conformal type (equivalently, this is a surface of finite hyperbolic area).*

Proof: Recall that (any) Dirichlet fundamental polygon P of G in $\mathbb{H}^2$ has finitely many sides (see Exercise 4.75). Since $\Lambda(G) = \mathbb{S}^1$ it follows that P is a finitely-sided polygon of finite area, its accumulation set in $\mathbb{S}^1$ consists of a finite number of vertices. Now the claim trivially follows. □

Step 3. This is the most interesting part of the proof.

Proposition 8.60. *For each component $\Omega_0 \subset \Omega(G)$ the stabilizer G_0 of Ω_0 in G has the property: $\Lambda(G_0) = \partial\Omega_0$.*

Proof: Suppose the assertion is false. Then there exists a point $z_0 \in \partial\Omega_0 - \Lambda(G_0)$, moreover, a whole neighborhood U of z_0 in $\mathbb{C}$ is disjoint from $\Lambda(G_0)$. It is clear that z_0 cannot be an isolated point of $\partial\Omega_0$ (since $\Lambda(G)$ is perfect).

We pick a base-point point $x \in \Omega_0$ which is not fixed by any element of G_0, then choose a sufficiently small disk $D_\epsilon \subset \Omega_0$ centered at x. (The disk is chosen so that its images under the elements of G_0 are disjoint.) Consider an infinite-dimensional space V of quasiconformal homeomorphisms f :

$D_\epsilon \to D_\epsilon$ which fix three distinct points z_1, z_2, z_3 in ∂D_ϵ and so that the restriction mapping

$$V \to Homeo(\partial D_\epsilon), \quad f \mapsto f|\partial D_\epsilon$$

is injective. (For instance, start with the infinite-dimensional space W of piecewise-linear homeomorphisms $\eta : \partial D_\epsilon \to \partial D_\epsilon$ which fix the points z_1, z_2, z_3 and take V to be the space of the radial extensions of η's.) Let μ_f denote the Beltrami differential of $f \in V$. We now argue as in the proof of Theorem 8.40. For each f extend μ_f G-invariantly from D_ϵ to the G-orbit of this disk and by zero to the rest of the 2-sphere. We will use the notation ν_f for this extension. Let h_f denote the normalized (at three limit points of G) solution of the Beltrami equation

$$\bar{\partial} h = \nu_f \partial h, \quad h_f = h.$$

Claim 8.61. *The mapping $A : f \mapsto h_f|\Lambda(G)$ is injective.*

Proof: Suppose that $f_1, f_2 \in V$ are such that $A(f_i)$ coincide, $i = 1, 2$. Let $h_i := h_{f_i}, i = 1, 2$. Recall that $\nu_i = \nu_{f_i}$ are zero on $\Sigma := \Omega_0 - G_0(D_\epsilon)$, hence each h_i is conformal in that part of Ω_0. On the other hand, the disks in $G_0(D_\epsilon)$ do not accumulate to the points of the set $U \cap \partial\Omega_0$ (since this set is disjoint from the limit set of G_0). Hence $U \cap \partial\Omega_0 = U \cap \partial\Sigma$ (provided that D_ϵ is sufficiently small).

Therefore the holomorphic function $(h_1 - h_2)|\Sigma$ tends to zero as $w \to z \in U \cap \partial\Sigma$. Applying Theorem 8.58 we conclude that the functions h_1 and h_2 are equal on Σ, in particular they are equal on ∂D_ϵ. On the other hand, $h_i|D_\epsilon$ satisfy the same Beltrami equation as f_i. It follows that $h_i|D_\epsilon = \varphi_i \circ f_i$ for conformal mappings φ_i of D_ϵ to the complex plane. Since both

$$h_2^{-1} h_1 = f_2^{-1} \varphi_2^{-1} \varphi_1 f_1 \quad \text{and} \quad f_2$$

preserve D_ϵ we get:

$$\varphi_2^{-1} \varphi_1 f_1 : D_\epsilon \to D_\epsilon.$$

Thus the mapping $\psi = \varphi_2^{-1}\varphi_1 : D_\epsilon \to D_\epsilon$ is a conformal automorphism. It follows that ψ is the identity (since it fixes three distinct boundary points z_1, z_2, z_3). We conclude that $\varphi_1 = \varphi_2$ and hence

$$f_1|\partial D_\epsilon = \varphi_1^{-1} h_1|\partial D_\epsilon = \varphi_2^{-1} h_2|\partial D_\epsilon = f_2|\partial D_\epsilon.$$

Recall that V is chosen so that if $f_1|\partial D_\epsilon = f_2|\partial D_\epsilon$ then $f_1 = f_2$. This proves injectivity of the mapping A. □

We now proceed as in the standard proof [Ahl64] of the Ahlfors finiteness theorem: the mapping A determines an embedding of the infinite-dimensional space V to the finite-dimensional algebraic variety $Hom(G, PSL(2, \mathbb{C}))$, which is a contradiction.

Corollary 8.62. *The surface $\Omega(G)/G$ contains no disks and annuli.*

Proof: If a component $S_0 = \Omega_0/G_0$ is a disk or an annulus then G_0 is either trivial or cyclic. This implies that the complement to Ω_0 in $\mathbb{S}^2$ is either empty or consists of one or two points. In any case it follows that G is elementary which contradicts our assumptions. □

Step 4. Suppose that G is a finitely generated Kleinian group and Ω_0 is a component of $\Omega(G)$ with the stabilizer G_0 in G. Then Ω_0/G_0 is conformally equivalent to the quotient $\mathbb{H}^2/\Gamma_0$ where $\Gamma_0 \subset PSL(2,\mathbb{R})$ is a subgroup whose limit set is the whole boundary circle of $\mathbb{H}^2$.

Proof. Let $x \in \Omega_0$ be a base-point and let $R : D \to \Omega_0$ be the Riemann mapping from the unit disk to Ω_0. We will identify D with the hyperbolic plane $\mathbb{H}^2$. Recall that R has radial limits a.e. on the boundary of D. Let $\Gamma_0 := R^{-1}G_0R \subset Isom(\mathbb{H}^2)$. It suffices to show that $\Lambda(\Gamma_0) = \mathbb{S}^1$. Suppose that the limit set of Γ_0 is a proper subset of the unit circle. Let $\gamma \subset \Omega(\Gamma_0)\cap\mathbb{S}^1$ be a (nondegenerate) arc. Since the Riemann mapping has radial limits a.e. in γ take a pair of "generic" distinct points $p, q \in \gamma$ so that the hyperbolic geodesic $\alpha \subset \mathbb{H}^2$ connecting them is mapped by R to a smooth arc $R(\alpha) \subset \Omega_0$ which has limit points $a, b \in \partial\Omega_0 \cap \mathbb{C}$, so that

$$a = \lim_{z\to p, z\in\alpha} R(z), \quad b = \lim_{z\to q, z\in\alpha} R(z).$$

I will use the notation γ for the part of γ between p and q since we will not need the rest of this arc. Let H denote the half-plane in $\mathbb{H}^2$ bounded by α which is adjacent to γ, by choosing γ sufficiently small we get: $x \notin R(H)$. We note that if $a = b$ and the topological circle $L := R(\alpha) \cup a$ bounds the open disk $R(H) \subset \Omega_0$, then the function $R(z) - a$ tends to zero on γ; this contradicts Theorem 8.58.

Remark 8.63. Alternatively one can use F. and M. Riesz theorem (see e.g. [Nar92]) for this part of the proof.

Therefore we can assume that either $a \neq b$ (Case 1) or $a = b$ and the topological circle $L = R(\alpha) \cup a$ bounds an open disk which contains $R(H)$ and a nonempty part E of the limit set of G_0 (Case 2). In the former case the arc α separates a part E of $\partial\Omega_0$ from the base-point x (if γ is chosen sufficiently small). See Figure 8.1.

Since $E \subset \Lambda(G_0)$, there exists a sequence $g_n \in G_0$ such that $\lim_n g_n(x) \in E$. Therefore all but finitely many members of the sequence $g_n(x)$ belong to $R(H)$. Let $h_n \in \Gamma_0$ be the elements corresponding to g_n under the isomorphism $\Gamma_0 \to G_0$ induced by R, let $y := R^{-1}(x)$. Since $\gamma \cap \Lambda(\Gamma_0) = \emptyset$, only finitely many members of the sequence $h_n(y)$ belong to the half-plane H. Contradiction. □

Step 5. Now there are several ways to argue. One can refer to [KS89] (see Theorem 4.126) which gives a purely topological proof (under the assumption that $S(G) = \Omega(G)/G$ contains no disks and annuli) that $S(G)$ has finite **topological** type, i.e. it has finite number of components each of which is homeomorphic to a compact surface with a finite number of disks removed. Given this, we conclude that $S(G) = \Omega(G)/G$ has finite conformal type.

Alternatively, one can repeat the deformation-theoretic argument, however it cannot exclude the possibility that $S(G)$ contains infinitely many

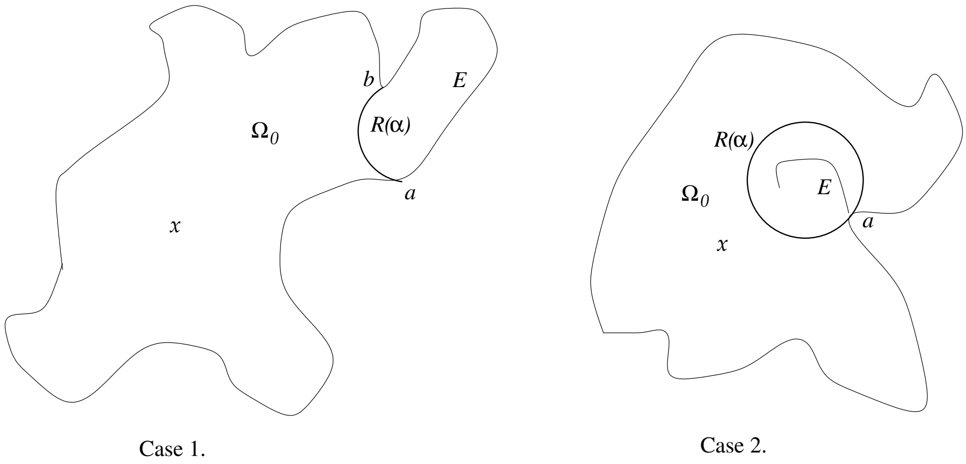

Figure 8.1:

triply punctured spheres. To finish the proof one would have to use the algebraic trick of L. Greenberg [Gre67] or use Sullivan's cusps finiteness theorem or again refer to [KS89].

The deformation-theoretic argument. Suppose that there is a component Ω_0 of $\Omega(G)$ such that Ω_0/G_0 does not have finite topological type (where G_0 is the stabilizer of Ω_0 in G). Then Ω_0/G_0 is conformally equivalent to $S_0 = \mathbb{H}^2/\Gamma_0$ where Γ_0 is a non-finitely generated Kleinian group whose limit set is the whole boundary circle of $\Delta = \mathbb{H}^2$. The Teichmüller space $\mathcal{T}(S_0)$ of the surface S_0 is infinite-dimensional. Let $R : \mathbb{H}^2 \to \Omega_0$ be the Riemann mapping. This map has (distinct) radial limits a_1, a_2, a_3 at three distinct fixed points b_1, b_2, b_3 of hyperbolic elements of Γ_0 (since R conjugates the Kleinian groups Γ_0 and G_0). We represent elements of $\mathcal{T}(S_0)$ by quasiconformal homeomorphisms $f : cl(\Delta) \to cl(\Delta)$ which fix the points b_1, b_2, b_3. These homeomorphisms form an infinite-dimensional Banach space V.

We now proceed as above, each $f \in V$ corresponds to a G_0-invariant Beltrami differential μ_f. Extend μ_f to a G-invariant Beltrami differential ν_f on $\mathbb{S}^2$. Let h_f denote the solution of the Beltrami equation $\bar{\partial} h_f = \nu_f \partial h_f$ normalized to fix the points $a_i, i = 1, 2, 3$. This determines the continuous (Bers) mapping $\beta : V \to Hom(G, PSL(2, \mathbb{C}))$. We will show that the mapping β is injective. Note that our normalization convention implies that each h_f has the property:

$$R^{-1} \circ h_f \circ R = f$$

since the mappings $R^{-1} \circ h_f \circ R$ and f differ by a conformal automorphism of $\mathbb{H}^2$ which fixes three distinct points b_1, b_2, b_3. Suppose $f_1, f_2 \in V$ are quasiconformal homeomorphisms such that $\beta(f_1) = \beta(f_2)$. Then the mapping $h := h_2^{-1} \circ h_1$ commutes with each element of G. Let $\theta : \Gamma_0 \to G_0$ denote the isomprphism induced by conjugation via R. Then the mapping

$f := f_2^{-1} \circ f_1$ satisfies:

$$f = R^{-1} \circ h \circ R$$

and f commutes with each element γ of Γ_0:

$$f\gamma f^{-1} = (R^{-1}hR)\gamma(R^{-1}h^{-1}R) = R^{-1}h\theta(\gamma)h^{-1}R = R^{-1}\theta(\gamma)R = \gamma.$$

Hence f_1, f_2 represent the same point of the Teichmüller space $\mathcal{T}(S_0)$. This proves injectivity of β. Since V is infinite-dimensional and $Hom(G, PSL(2,\mathbb{C}))$ is finite-dimenaional we get a contradiction.

This proves that each component of $S(G)$ has finite conformal type. Let $S(G)^*$ denote $S(G)$ with triply punctured spheres removed. To prove that $S(G)^*$ has finite number of components we have to repeat the same argument once again. If $S_0 = \Omega_0/G_0$ is not a triply-punctured sphere then the complex dimension of the Teichmüller space $\mathcal{T}(S_0)$ is at least 1. Let $S_i, i \in I$ denote the components of $S(G)$. Since (by the same arguments as above) the Bers mapping

$$\beta : \mathcal{T}(S(G)) \to \mathcal{T}(G) \subset Hom(G, PSL(2,\mathbb{C}))//PSL(2,\mathbb{C})$$

is injective we conclude that

$$\mathcal{T}(S(G)) = \prod_{i\in I} \mathcal{T}(S_i)$$

is finite-dimensional. Hence, I is finite. □

8.15. A generalization of the Bers' isomorphism

In the section we consider a class of Kleinian groups Γ in $\mathrm{Isom}(\mathbb{H}^3)$ which have torsion. We make the following assumptions:

- Γ is geometrically finite.
- $\Omega(\Gamma)$ consists of simply-connected components Ω_j.
- The stabilizer Γ_j of each component Ω_j has a fundamental domain $\Phi_j \subset \Omega_j$ which is either a quadrilateral or a triangle, whose edges are circular arcs tangent at the vertices.
- Each Γ_j is generated by inversions in the edges of Φ_j.

The class $\mathcal{B}$ of Kleinian groups which satisfy these conditions looks rather random, but it will be quite useful in the proof of Theorem 13.2. Let $q = q(\Gamma)$ denote the number of components of $\Omega(\Gamma)/\Gamma$ which are quadrilaterals . We will assume that $q < \infty$.

First we look at the quotient $\Omega_j/\Gamma_j = \Phi_j$. If we double Φ_j along its boundary (which does not include vertices), the result is a 3-punctured or 4-punctured sphere. Let $S_{0,4}$ denote a 4-punctured sphere which appears as the double of one of these quadrilaterals . Then $S_{0,4}$ admits an anticonformal automorphism J which fixes the union of edges of this quadrilateral . Thinking about $S_{0,4}$ as the extended complex plane $\widehat{\mathbb{C}}$ with four points removed, we conclude that J acts as an inversion in $\widehat{\mathbb{C}}$ which fixes these punctures, hence they must belong to a common round circle in $\widehat{\mathbb{C}}$. The opposite is true as well, given four points z_0, z_1, z, z_∞ in $\widehat{\mathbb{C}}$ which lie on a common round circle $\overline{\mathbb{R}}$ and a reflection J which fixes this circle, the quotient $[\widehat{\mathbb{C}} - \{z_0, z_1, z, z_\infty\}]/J$ is quasiconformal equivalent to the *ideal* quadrilateral Φ_j that we have started with. The group of Moebius transformations acts transitively on the set of triples of distinct points in $\widehat{\mathbb{C}}$. Thus we can assume that $z_0 = 0, z_1 = 1, z_\infty = \infty$ and z is a point on the ray $(1, \infty)$. The real parameter z gives a parameterization to the space of conformal structures on the ideal quadrilateral Φ_j. Alternatively we can take the conformal modulus of the quadrilateral Φ_j with marked vertices. Let R_x be the rectangle in $\mathbb{C}$ with the vertices $\{0, x, x+i, i\}$. Then take a Riemann mapping

$$f_z : \Phi_j \to R_x.$$

The number $x = m(\Phi_j) \in (0, +\infty)$ is the conformal modulus of Φ_j, which uniquely determines this quadrilateral up to conformal equivalence.

Take the subgroup Γ^0 of index 2 in Γ, which preserves the orientation in $\mathbb{H}^3$. Then $\Omega(\Gamma^0)/\Gamma^0$ is a collection of 3- and 4-punctured spheres. Now we consider the Teichmüller spaces $\mathcal{T}(\Gamma), \mathcal{T}(\Gamma^0)$. According to Bers' theorem, the Teichmüller space $\mathcal{T}(\Gamma^0)$ is canonically isomorphic to the product:

$$\prod_{j=1}^{q} \mathcal{T}(S_{0,4})$$

since 3-punctured spheres are conformally rigid. Quasiconformal deformations

$$f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}, \quad f\Gamma^0 f^{-1} \subset PSL(2, \mathbb{C})$$

which extend to quasiconformal deformations of the group Γ are exactly those whose Beltrami differentials in components Ω_j are invariant under inversions from Γ_j. This means that they are coming from quasiconformal deformations of ideal quadrilaterals Φ_j. Hence we have a natural homeomorphism

$$\beta : \prod_{j=1}^{q} \mathbb{R}_+ \to \mathcal{T}(\Gamma)$$

so that $\beta : (m(\Phi_1), ..., m(\Phi_q)) \mapsto [\rho]$.

8.16. Totally degenerate groups

Definition 8.64. Suppose that Γ is a finitely generated nonelementary Kleinian subgroup in $\mathrm{Isom}(\mathbb{H}^3)$. The group Γ is called **totally degenerate** if its limit set is simply connected.

Existence of such groups was established by Bers in the 1960's. However, so far nobody was able to give a constructive[4] example of such group. McMullen in [McM96] gives an algorithm which approximates matrices of generators for a totally degenerate group. The question about matrices with entries from $\bar{\mathbb{Q}}$ can look strange, but the fact that we do not even know if totally degenerate groups defined over $\bar{\mathbb{Q}}$ exist (I believe that they do), means that we still lack in understanding of Kleinian groups and their deformation spaces.

Below I describe an indirect construction of totally degenerate groups which is due to Bers. Fix a Fuchsian group F which is isomorphic to the fundamental group of a closed oriented hyperbolic surface S. Let $\rho_n : F \to F_n \subset PSL(2,\mathbb{C})$, $[\rho_n] \in \mathcal{T}(F)$ be a sequence of discrete, faithful and type-preserving representations. Suppose that there exists a limit $\rho = \lim_{n\to\infty} \rho_n$ and $\rho(F) = \Gamma$ contains no parabolic elements.

Theorem 8.65. *The group* Γ *constructed above satisfies one of the following three mutually exclusive conditions:*

(a) Γ *is quasifuchsian,*

(b) Γ *is totally degenerate,*

(c) the domain of discontinuity of Γ *is empty.*

Proof: Suppose that $\Omega(\Gamma) \neq \emptyset$, $D \subset \Omega(\Gamma)$ is a component with the stabilizer H. The Ahlfors' Finiteness Theorem implies that H is finitely generated. If D is not simply–connected, then the surface $D/H \subset \partial \dot{M}(\Gamma)$ is compressible in the manifold $\dot{M}(\Gamma) = \mathbb{H}^3 \cup \Omega(\Gamma)/\Gamma$. Therefore there exists an embedded disk $X \subset \dot{M}(\Gamma)$ such that $\partial X \subset D/H$ is homotopically nontrivial. This implies that $\Gamma = \pi_1 \dot{M}(\Gamma)$ splits as a nontrivial free product, which contradicts the assumption that $\Gamma \cong F \cong \pi_1(S)$. So, all components of $\Omega(\Gamma)$ are simply–connected. Suppose that $|\Gamma : H| = \infty$. Then $H \cong \pi_1(D/H)$ is a finitely generated free group, which means that D/H has punctures. Hence H has parabolic elements. This contradiction proves that stabilizers of all components of $\Omega(\Gamma)$ have finite index in Γ. Suppose that $\Omega(\Gamma)$ is not connected (i.e. Γ is not a totally degenerate group). Then Γ has a finite–index subgroup K which stabilizes all components of $\Omega(\Gamma) = \Omega(K)$. For each component $D \subset \Omega(K)$ the inclusion $D/K \hookrightarrow \dot{M}(K)$ is a homotopy–equivalence. Thus $H_2(\dot{M}(K),\mathbb{Z}) \cong \mathbb{Z}$. Therefore $\dot{M}(K)$ is compact and has exactly two boundary components $D_1/K, D_2/K$. In particular, the group K is geometrically finite and has no parabolic elements. According to Theorem 8.17, the group K is quasifuchsian. Hence Γ is also quasifuchsian. □

It is easy to see that totally degenerate groups are not geometrically finite. Indeed, suppose that Γ is totally degenerate and has no parabolic elements. Since $\Omega(\Gamma)$ is simply-connected, we have an isomorphism of a surface group to Γ:

$$\pi_1(\Omega(\Gamma)/\Gamma) \to \Gamma.$$

[4]By this I mean: to give matrices of generators whose entries belong to $\bar{\mathbb{Q}}$.

Since Γ is nonelementary we can identify $\pi_1(\Omega(\Gamma)/\Gamma)$ with a Fuchsian group $F \subset \mathrm{Isom}(\mathbb{H}^2)$. Since Γ is assumed to have no parabolic elements, the surface $S = \Omega(\Gamma)/\Gamma$ is compact, hence F has no parabolic elements as well.

If Γ were geometrically finite, then Theorem 8.16 would imply that $\Lambda(\Gamma)$ is a topological circle which contradicts the assumption that $\Omega(\Gamma)$ is connected. If Γ has parabolic elements and is geometrically finite, then each cusp in $M(\Gamma)$ corresponds to a pair of punctures in the quotient surface S (see the end of §4.16). Thus the surface S contains two loops α_1, α_2 which encircle two distinct punctures but γ_1, γ_2 are freely homotopic in the manifold $\dot{M}(\Gamma)$. Since the inclusion $S \to \dot{M}(\Gamma)$ is a homotopy-equivalence, the loops α_1, α_2 are homotopic in the surface S. This implies that S is homeomorphic to $\mathbb{C} - \{0\}$, hence Γ is elementary. Contradiction.

Historically this was the first example of a geometrically infinite Kleinian group which is finitely generated (due to L. Greenberg who had noticed this fact in 1967).

Nonconstructive example of a totally degenerate group. Start with the Fuchsian group F which corresponds to a pair of points $(\sigma, \overline{\sigma}) \in \mathcal{T}(S) \times \mathcal{T}(\overline{S}) = \mathcal{T}(F)$, where S is a closed hyperbolic orientable surface. Pick an orientation–preserving *aperiodic* homeomorphism $h : S \to S$. This means that for every nontrivial loop $\gamma \subset S$ and any $n > 0$ the loops γ and $h^n(\gamma)$ are not freely homotopic. Existence of such maps will be discussed in the Chapter 11.

Consider the sequence $[\phi_n] = (h^n\sigma, \overline{\sigma}) \in \mathcal{T}(S) \times \mathcal{T}(\overline{S})$, where the homeomorphism h^n acts as an element of the mapping class group on $\mathcal{T}(S)$. The sequence $h^n\sigma$ is not relatively compact in $\mathcal{T}(S)$ since $h_* : F \to F$ has infinite order and the mapping class group acts discretely on the Teichmüller space. The corresponding representations $\phi_n : F \to F_n$ are induced by quasiconformal conjugations via quasiconformal homeomorphism f_n, which are conformal in the unit disk Δ. Therefore the sequence $\{f_n|_\Delta\}$ subconverges and the limit f_∞ is a nonconstant holomorphic map. The Schwarzian derivative $\{f_n, z\}$ defines a quadratic differential which is F–invariant. By continuity $\{f_\infty, z\}$ is also F–invariant. Thus, f_∞ induces a representation ϕ of F into $PSL(2,\mathbb{C})$. The representation ϕ is the limit of ϕ_n. Actually one can prove that the whole sequence $\{\phi_n\}$ converges.

The representation ϕ cannot be quasifuchsian since in $\mathcal{T}(F)$ algebraic convergence implies convergence in $\mathcal{T}(S) \times \mathcal{T}(\overline{S})$. The group $\phi(F)$ has nonempty discontinuity set which contains $f_\infty(\Delta)$. One can prove that $\phi(F)$ has no parabolic elements (see the Chapter 11). Thus, Theorem 8.65 forces $\phi(F)$ to be totally degenerate.

8.17. Algebraic topology versus geometric topology

Fix a finitely generated group Γ.

Definition 8.66. A sequence of representations $\rho_n : \Gamma \to PSL(2,\mathbb{C})$ converges **strongly** to a representation ρ if $\{\rho_n\}$ converges to ρ algebraically and the sequence $\{\rho_n(\Gamma)\}$ converges to $\rho(\Gamma)$ geometrically.

Let $\Gamma \subset PSL(2,\mathbb{C})$ be a geometrically finite nonelementary subgroup so that: for each component $S \subset S(\Gamma) = \partial \overset{\bullet}{M}(\Gamma)$ the homomorphism $i_* : \pi_1(S) \to \pi_1(M)$, induced by inclusion $S \hookrightarrow \overset{\bullet}{M}(\Gamma)$, is injective and maps non-peripheral elements of $\pi_1(S)$ to loxodromic elements of Γ.

Suppose $\rho_n : \Gamma \to \Gamma_n \hookrightarrow PSL(2,\mathbb{C})$ is a sequence of representations so that $[\rho_n] \in \mathcal{T}(\Gamma)$. If Ω^S is a component of $\Omega(\Gamma)$ stabilized by $\pi_1(S)$ then we let Ω_n^S denote the corresponding component of $\Omega(\Gamma_n)$ (stabilized by $\rho_n(\pi_1(S))$). We let $S_n := \Omega_n^S/\rho_n(\pi_1(S))$ denote the corresponding component of $S(\Gamma_n)$. The homomorphism $\rho_n : \pi_1(S) \to \pi_1(S_n)$ determines a marking on the Riemann surface S_n.

Theorem 8.67. *(W. Thurston [Thu81, Chapter 9], see also papers by Ohshika [Ohs92, Ohs98], and the book of Matsuzaki and Taniguchi [MT98, §7.3.3].) Suppose that Γ is as above and $\rho_n : \Gamma \to \Gamma_n \hookrightarrow PSL(2,\mathbb{C})$ is a sequence of representations so that $[\rho_n] \in \mathcal{T}(\Gamma)$, $\lim_{n\to\infty} \rho_n = \rho$ and for each component $S \subset \partial \overset{\bullet}{M}(\Gamma)$ either $\rho|\pi_1(S)$ maps loxodromic elements to loxodromic or the sequence of points $(S_n, \rho_n|\pi_1(S)) \in \mathcal{T}(S)$ converges in $\mathcal{T}(S)$. Then:*

1. *ρ_n converges strongly to ρ.*
2. *Moreover, let Ω^0_∞ be a component of $\Omega(\Gamma_\infty)$ and Ω^0_n be the sequence of components of $\Omega(\Gamma_n)$ whose closures converge to the closure of Ω^0_∞ in the Hausdorff topology (see Theorem 8.15). Let Ω^0 denote the corresponding component of $\Omega(\Gamma)$; let Γ^0_n be the stabilizers of Ω^0_n in Γ_n ($n = 1, 2, ..., \infty$) and Γ^0 be the stabilizer of Ω^0 in Γ. Then the marked Riemann surfaces Ω^0_n/Γ^0_n converge to $\Omega^0_\infty/\Gamma^0_\infty$ in the Teichmüller space $\mathcal{T}(\Omega^0/\Gamma^0)$.*

I will omit the proof of the Assertion (1). In §14.6 we will prove the Assertion (1) under the assumption that the domain of discontinuity of Γ_∞ is empty. I will present a proof of the Assertion (2) assuming for simplicity that the surface $S = \Omega^0/\Gamma^0$ is compact.

Proof: The stabilizer Γ^0 is a quasifuchsian group, $\rho_n|\Gamma^0$ converges strongly to $\rho|\Gamma^0$ according to the Assertion (1). Thus it is enough to prove the second assertion under the assumption $\Gamma^0 = \Gamma$.

If the group Γ_∞ is quasifuchsian then it belongs to $\mathcal{T}(\Gamma)$. Therefore the conclusion follows from the Bers isomorphism: the Teichmüller topology on $\mathcal{T}(\Gamma)$ is equivalent to the algebraic topology.

Now suppose that $\Omega(\Gamma_\infty)$ consists of a single component Ω^0_∞, α is the marked complex projective structure on $\Omega^0_\infty/\Gamma_\infty$. It follows from Theorem 7.2 that there is a sequence $\alpha_n \in MCP(S)$ which is convergent to α, so that the monodromy representation of α_n is ρ_n. The developing maps $d_n : \mathbb{H}^2 \to \widehat{\mathbb{C}}$ of α_n converge to the developing map d of α uniformly on compacts in C^1-topology. Let D be a compact fundamental domain for the action of Γ on $\mathbb{H}^2$. Since d is a quasiconformal homeomorphism onto its image, $d(D)$ is a fundamental domain for the action of Γ_∞ on Ω^0_∞. Hence, Hausdorff convergence of limit sets of $\Gamma_n = \rho_n(\Gamma)$ to the limit set of Γ_∞ implies that $d_n(D) \subset \Omega_n^0$ for large n. Therefore the image of the developing map $d_n(\mathbb{H}^2)$ lies inside Ω_n^0. The map d_n projects to a local homeomorphism

$$f_n : S = \mathbb{H}^2/\Gamma \to \Omega_n^0/\Gamma_n = S_n.$$

The surface S_n is homeomorphic to S, thus the covering f_n is a quasiconformal homeomorphism. This implies that $d_n : \mathbb{H}^2 \to \Omega_n^0$ are quasiconformal homeomorphisms which converge to d uniformly on compacts in C^1 topology. $\square$

Actually, many assumptions in Theorem 8.67 are unnecessary. For instance, Γ does not have to be geometrically finite, ρ_n do not have to be induced by quasiconformal conjugations, moreover in the subsequent paper [Ohs98].

8.18. Justification of the Poincaré's continuity method

In this section I explain how to justify the Poincaré's continuity method [Poi85] of proving the Uniformization Theorem in the case of punctured spheres.[5] The reader can compare this proof with the logic behind Thurston's proof of the Hyperbolization Theorem.

Consider the moduli space $\mathcal{M}(\Sigma)$ of Riemann surfaces which are obtained from $\widehat{\mathbb{C}}$ by removing n distinct points. This space is the quotient of

$$C_{n-3} = \{(z_1, ..., z_{n-3}) \in \mathbb{C}^{n-3} : z_j \neq 0, 1, \quad \text{and} \quad z_j \neq z_j \quad \text{for} \quad j \neq k\}$$

by the action of the permutation group on $n-3$ symbols. The space $\mathcal{M}(\Sigma)$ is an orbifold, so it will be more convenient to work with C_{n-3} which is a complex manifold. By abusing notation we will still refer to the elements of C_{n-3} as *Riemann surfaces*. One can think of the elements of C_{n-3} as the Riemann surfaces S with the *homological marking* given by an isomorphism $H_1(S_0, \mathbb{Z}_2) \to H_1(S, \mathbb{Z}_2)$.

Step 1. Find a Riemann surface $S_0 \in C_{n-3}$ which is uniformizable, i.e. $S_0 = \mathbb{H}^2/\Gamma$ where Γ is a torsion-free Fuchsian group. For instance, take a finite cyclic cover over the triply-punctured sphere.

Step 2. For each $S \in C_{n-3}$ consider the Schwarzian differential equation $\{f, z\} = \varphi$ where φ is a holomorphic quadratic differential on S with at worst simple poles in the punctures. This equation has unique (up to postcomposition with elements of $PSL(2, \mathbb{C})$) solution f which is a multivalued locally injective holomorphic function. The solution f depends holomorphically on φ and on the location of punctures in $\widehat{\mathbb{C}}$. The function f determines a complex projective structure of finite type σ on S (the lift of f to the universal cover of S is the developing map). Let $[\rho]$ be the conjugacy class of

[5] A. Tyurin had challenged me to do this long time ago.

the monodromy representation of σ. This gives the complex-analytic *monodromy map* $\mu_S : \mathbb{C}^{n-3} \to \mathcal{R}_{par}(\Gamma, PSL(2,\mathbb{C}))$, where $\mathbb{C}^{n-3} \cong Q(S)$ is the space of holomorphic quadratic differentials on S which have at worst simple poles in the punctures. Note that the individual monodromy maps do not assemble to a map $C_{n-3} \times \mathbb{C}^{n-3} \to \mathcal{R}_{par}(\Gamma, PSL(2,\mathbb{C}))$: to do so we have to first pass to the universal cover of C_{n-3}. However, if $\gamma \subset C_{n-3}$ is a smoothly embedded curve in C_{n-3} starting at S_0, then we get a well-defined smooth map $\mu_\gamma : \gamma \times \mathbb{C}^{n-3} \to \mathcal{R}_{par}(\Gamma, PSL(2,\mathbb{C}))$. If $[\rho] = \mu_S(\varphi)$ is such that $\rho(\Gamma) \subset PSL(2,\mathbb{R})$ and $\mathbb{H}^2/\rho(\Gamma) = S$, then $[\rho]$ is called *the uniformization representation* and S a *uniformizable* Riemann surface.

Step 3. For each S the mapping μ_S is transversal to $\mathcal{R}^0(\Gamma, PSL(2,\mathbb{R}))$. Poincaré had completely ignored this and the following step. Transversality was established by G. Faltings [Fal83] in 1983 (see also [SW98] for more general results in this direction).

Remark 8.68. The image of μ_S is a smooth complex-analytic subvariety properly embedded in $\mathcal{R}_{par}(\Gamma, PSL(2,\mathbb{C}))$ (see [Ka95b]). However this image is not an algebraic subvariety since it intersects the real-algebraic subset $\mathcal{R}(\Gamma, PSL(2,\mathbb{R}))$ transversally in countably many points according to Tanigawa's theorem, see [Tan97a].

Step 4. Theorem A. *Suppose that $S \in \gamma$ is uniformizable and $[\rho] \in \mu_S(\mathbb{C}^{n-3}) \cap T_R(\Gamma)$ is the conjugacy class of the uniformization representation. Then there exists an open neighborhood W of $[\rho]$ in $\mathcal{R}^0(\Gamma, PSL(2,\mathbb{R}))$ so that $\mu_\gamma^{-1}(W)$ is an open neighborhood V of $\{S\} \times \mathbb{C}^{n-3}$ in $\gamma \times \mathbb{C}^{n-3}$. Moreover, V contains an open neighborhood V' of S so that for each $(S', \varphi') \in V$ the point $[\rho'] = \mu(S', \varphi')$ is a uniformization representation for S'.*

Proof: Existence of W follows directly from transversality of the intersection between $T_R(\Gamma)$ and the image of μ. Existence of V' follows from Theorem 7.3. □

Thus, the subset U of γ which consists of *uniformizable* Riemann surfaces is open.

Step 5: This is the step which Poincaré was mostly trying to justify in [Poi85]. The only tool for working with discrete groups Poincaré knew was the fundamental polygon, this was clearly insufficient.

Theorem B. *U is closed.*

Proof: Suppose that S_j is a sequence of points in U which converges to a surface $S_\infty \in C_{n-3}$. Let $[\rho_j]$ denote the conjugacy class of the uniformization representation of S_j.

(a) **The sequence $[\rho_j]$ subconverges in $\mathcal{R}^0(\Gamma, PSL(2,\mathbb{R}))$.**

It is enough to show that for each element $g_\alpha \in \Gamma$ corresponding to a loop α in S, the length $\ell_{\rho_j}(\alpha)$ is bounded from above by a constant $Const_\alpha < \infty$. It suffices to consider only simple nonperipheral loops α (see Remark 5.7). Note that the *conformal modulus* (see §4.16) $\mathrm{mod}_{S_j}(\alpha)$ of α in S_j satisfies:

$$\underline{\lim}_{j\to\infty} \mathrm{mod}_{S_j}(\alpha) \geq \mathrm{mod}_{S_\infty}(\alpha).$$

Thus there is $\epsilon > 0$ such that $\operatorname{mod}_{S_j}(\alpha) > \epsilon$. Therefore, according to the Example 4.83, we have

$$\ell_{\rho_j}(\alpha) \leq \pi/\operatorname{mod}_{S_j}(\alpha) < \pi/\epsilon.$$

(b) If $[\rho_j]$ subconverges to $[\rho_\infty]$ then $\rho_\infty : \Gamma \to PSL(2,\mathbb{R})$ is a discrete and faithful representation, see Theorem 8.4. (Poincaré unsuccessfully tried to show this by studying possible degenerations of the fundamental polygons of $\rho_j(\Gamma)$. He succeeded under the assumption that each $\rho_j(\Gamma)$ admits an index 2 extension in $\operatorname{Isom}(\mathbb{H}^2)$ which is a reflection group, however the general case was beyond his reach.)

(c) Let $\Gamma_j = \rho_j(\Gamma)$, $\Gamma_\infty := \rho_\infty(\Gamma)$. Thus the normal family of Γ_j-equivariant holomorphic covering maps $p_j : \mathbb{H}^2 \to S_j$ subconverges to a holomorphic Γ_∞-equivariant covering map $p_\infty : \mathbb{H}^2 \to S_\infty$. This implies that $S_\infty = \mathbb{H}^2/\Gamma_\infty$ and thus S_∞ is also uniformizable. $\square$

Now the uniformization theorem for the Riemann surfaces in C_{n-3} follows from the fact that C_{n-3} is path-connected.

Chapter 9

Ultralimits of Metric Spaces

Let (X_i) be a sequence of metric spaces. One can describe the limiting behavior of the sequence (X_i) by studying limits of sequences of finite subsets $Y_i \subset X_i$. Ultrafilters are an efficient technical device for simultaneously taking limits of all such sequences of subspaces and putting them together to form one object, namely an ultralimit of (X_i) (see [KL95, KL97, KKL98, Dru00] for examples of application of ultralimits to the study of quasi-isometries of metric spaces). We discuss the concept of ultralimit following [Gro93] and [KL95].

9.1. Ultrafilters

Let I be an infinite set, $\mathcal{S}$ is a collection of subsets of I. A *filter based on* $\mathcal{S}$ is a nonempty family ω of members of $\mathcal{S}$ with the properties:

- $\emptyset \notin \omega$.
- If $A \in \omega$ and $A \subset B$, then $B \in \omega$.
- If $A_1, \ldots, A_n \in \omega$, then $A_1 \cap \cdots \cap A_n \in \omega$.

If $\mathcal{S}$ consists of *all* subsets of I we will say that ω is a filter on I. Subsets $A \subset I$ which belong to a filter ω are called ω-large. We say that a property (P) holds for ω-*all* i, if (P) is satisfied for all i in some ω-large set. An *ultrafilter* is a maximal filter. The maximality condition can be rephrased as: for every decomposition $I = A_1 \cup \cdots \cup A_n$ of I into finitely many disjoint subsets, the ultrafilter contains exactly one of these subsets.

For example, for every $i \in I$, we have the *principal* ultrafilter δ_i defined as $\delta_i := \{A \subset I \mid i \in A\}$. An ultrafilter is principal if and only if it contains a finite subset. The interesting ultrafilters are of course the non-principal ones. They cannot be described explicitly but exist by Zorn's lemma: every filter is contained in an ultrafilter. Let $\mathcal{Z}$ be the *Zariski filter* which consists of complements to finite subsets in I. An ultrafilter is a nonprincipal ultrafilter, if and only if it contains $\mathcal{Z}$. For us it is not important how ultrafilters look like, but rather how they work: an ultrafilter ω on I assigns a "limit" to every function $f: I \to Y$ with values in a compact space Y. Namely,

$$\omega\text{-}\lim f = \omega\text{-}\lim_i f(i) \in Y$$

is defined to be the unique point $y \in Y$ with the property that for every neighborhood U of y the preimage $f^{-1}U$ is "ω-large". To see existence of a limit, assume that there is no point $y \in Y$ with this property. Then each point $z \in Y$ possesses a neighborhood U_z such that $f^{-1}U_z \notin \omega$. By compactness, we can cover Y with finitely many of these neighborhoods. It follows that $I \notin \omega$. This contradicts the definition of a filter. Uniqueness of the point y follows, because Y is Hausdorff. Note that if y is an accumulation point of $\{f(i)\}_{i\in I}$ then there is a non-principal ultrafilter ω with ω-lim $f = y$, namely an ultrafilter containing the pullback of the neighborhood basis of y.

9.2. Ultralimits of metric spaces

Let $(X_i)_{i\in I}$ be a family of metric spaces parameterized by an infinite set I. For an ultrafilter ω on I we define the ultralimit

$$X_\omega = \omega\text{-}\lim_i X_i$$

as follows. Let $\Pi_i X_i$ be the product of the spaces X_i, i.e. it is the space of sequences $(x_i)_{i\in I}$ with $x_i \in X_i$. The distance between two points $(x_i), (y_i) \in \Pi_i X_i$ is given by

$$d_\omega\big((x_i),(y_i)\big) := \omega\text{-}\lim\big(i \mapsto d_{X_i}(x_i,y_i)\big)$$

where we take the ultralimit of the function $i \mapsto d_{X_i}(x_i,y_i)$ with values in the compact set $[0,\infty]$. The function d_ω is a pseudo-distance on $\Pi_i X_i$ with values in $[0,\infty]$. Set

$$(X_\omega, d_\omega) := (\Pi_i X_i, d_\omega)/\sim$$

where we identify points with zero d_ω-distance.

Example 9.1. Let $X_i = Y$ for all i, where Y is a compact metric space. Then $X_\omega \cong Y$ for all ultrafilters ω.

The concept of ultralimits extends the notion of Gromov-Hausdorff limits:

Proposition 9.2. *Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of compact metric spaces converging in the Gromov-Hausdorff topology to a compact metric space X. Then $X_\omega \cong X$ for all non-principal ultrafilters ω.*

Proof: Realize the Gromov-Hausdorff convergence in an ambient compact metric space Y, i.e. embed the X_i and X isometrically into Y such that the

X_i converge to X with respect to the Hausdorff distance. Then there is a natural isometric embedding

$$X_\omega = \omega\text{-}\lim_i X_i \xrightarrow{\iota} \omega\text{-}\lim_i Y \cong Y$$

Since ω is non-principal, the ω-limit is independent of any finite collection of X_i's and we get:

$$\iota(X_\omega) \subseteq \bigcap_{i_0} \overline{\bigcup_{i \geq i_0} X_i} = X.$$

On the other hand $X \subseteq \iota(X_\omega)$, because $\iota\big((x_i)\big) = x$ if (x_i) is a sequence with $x_i \in X_i$ converging in Y to $x \in X$. Hence $\iota(X_\omega) = X$ which proves the claim. $\square$

If the spaces X_i do not have uniformly bounded diameter, then the ultralimit X_ω decomposes into (generically uncountably many) components consisting of points of mutually finite distance. We can pick out one of these components if the spaces X_i have basepoints x_i^0. The sequence $(x_i^0)_i$ defines a basepoint x_ω^0 in X_ω and we set

$$X_\omega^0 := \big\{x_\omega \in X_\omega \mid d_\omega(x_\omega, x_\omega^0) < \infty\big\}.$$

Define the *based ultralimit* as

$$\omega\text{-}\lim_i (X_i, x_i^0) := (X_\omega^0, x_\omega^0).$$

Example 9.3. For every locally compact space Y with a basepoint y_0, we have:

$$\omega\text{-}\mathit{lim}_i(Y, y_0) \cong (Y, y_0).$$

We observe that some geometric properties pass to ultralimits:

Proposition 9.4. *Let $(X_i, x_i^0)_{i \in I}$ be a sequence of based geodesic spaces and let ω be an ultrafilter. Then X_ω^0 is a geodesic space.*

If the X_i are CAT(κ)–spaces for some $\kappa \leq 0$ then X_ω^0 has the same upper curvature bound κ.

Proof: The ultralimit of geodesic segments in X_i is a geodesic segment in X_ω^0. Therefore X_ω^0 is a geodesic space. To verify the second assertion it suffices to prove that any pair of points $x_\omega = (x_i)$ and $y_\omega = (y_i)$ in X_ω^0 can be joined by a unique geodesic (then we approximate triangle Δ_ω in X_ω^0 by triangles Δ_i in X_i, the ω-limit of comparison triangles for Δ_i is the comparison triangle for Δ_ω). Suppose that $d_\omega(x_\omega, y_\omega) = s + t$ where $s, t \geq 0$. There are points z_i on the geodesic segments $[x_i y_i]$ such that for $s_i := d_i(x_i, z_i)$ and $t_i := d_i(z_i, y_i)$ we have $\omega\text{-}\lim s_i = s$ and $\omega\text{-}\lim t_i = t$. Hence $z_\omega := (z_i)$ satisfies $d_\omega(x_\omega, z_\omega) = s$ and $d_\omega(z_\omega, y_\omega) = t$. Suppose that $u_\omega = (u_i)$ is another point with the same property. Consider in the model space M_κ^2 comparison triangles $\Delta(x_i', u_i', y_i')$ with the same side-lengths as

$\Delta(x_i, u_i, y_i)$. Let z_i' be a division point on $[x_i' y_i']$ corresponding to z_i on $[x_i y_i]$. Since

$$\omega\text{-lim}(d_i(x_i, u_i) + d_i(u_i, y_i) - d_i(y_i, x_i)) = 0,$$

we have

$$\omega\text{-lim}\, d_i(u_i, z_i) \leq \omega\text{-lim}\, d_{M^2_\kappa}(u_i', z_i') = 0$$

and therefore $u_\omega = z_\omega$. Thus there is a unique point $z_\omega \in X_\omega$ with $d_\omega(x_\omega, z_\omega) = s$ and $d_\omega(z_\omega, y_\omega) = t$. $\square$

Lemma 9.5. *(See [KL97, Lemma 2.3].) For a sequence (X_n, x_n^0) of based $CAT(0)$-spaces and an ultrafilter ω consider the ultralimit*

$$(X_\omega, x_\omega^0) = \omega\text{-}\lim_n (X_n, x_n^0).$$

Let $Y_n \subset X_n$ be closed[1] convex subsets with $\omega\text{-lim}_n\, d(x_n^0, Y_n) = \infty$. Then

$$f := \omega\text{-}\lim_n (d_{Y_n} - d_{Y_n}(x_n^0))$$

is a Busemann function on X_ω, where d_{Y_n} is the distance function to Y_n.

Proof: For $x_n \in X_n - Y_n$ let $\rho_{x_n} : [0, d_{Y_n}(x_n)] \to X_n$ denote the shortest geodesic segment from x_n to Y_n. Let $proj_{Y_n}$ denote the nearest-point projection to Y_n. Choose a pair of points $x_\omega = (x_n), x_\omega' = (x_n')$ in X_ω. The function $t \mapsto d(\rho_{x_n}(t), \rho_{x_n'}(t))$ is monotonously decreasing. Then the ultralimit

$$\rho_{x_\omega} := \omega\text{-}\lim_n \rho_{x_n} : [0, \infty) \to X_\omega$$

is a geodesic ray which does not depend on the choice of the sequence (x_n) representing x_ω, and all rays ρ_{x_ω} are asymptotic to the same ideal point $\xi_\omega \in \partial_\infty X_\omega$. The triangle inequality implies that

$$d(\rho_{x_n}(t), x_n') \geq d(x_n', proj_{Y_n}(x_n)) - d_{Y_n}(\rho_{x_n}(t)) \geq t + (d_{Y_n}(x_n') - d_{Y_n}(x_n)).$$

Passing to the ultralimit yields

$$d(\rho_{x_\omega}(t), x_\omega') \geq t + (b(x_\omega') - b(x_\omega))$$

where $b := \omega\text{-lim}_n\, d_{Y_n}$. According to the previous inequality we have:

$$d(x_\omega', \rho_{x_\omega}(t)) - t \geq b(x_\omega') - b(x_\omega)$$

and sending t to infinity we get:

$$\beta_{\xi_\omega}(x_\omega') - \beta_{\xi_\omega}(x_\omega) \geq b(x_\omega') - b(x_\omega).$$

Since we may exchange the roles of x_ω and x_ω', the equality holds and we conclude that the function $\beta_{\xi_\omega} - b$ is constant on X_ω. $\square$

[1] This assumption could be dropped.

Exercise 9.6. *If T is a metric tree, $-\infty < a < b < \infty$ and $f : [a, b] \to T$ is a continuous embedding then the image of f is a geodesic segment in T. (Hint: use PL approximation of f to show that the image of f contains the geodesic segment connecting $f(a)$ to $f(b)$.)*

Corollary 9.7. *Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of geodesic spaces with upper curvature bounds κ_i tending to $-\infty$. Then for every non-principal ultrafilter ω each component of the ultralimit X_ω is a metric tree.*

Proof: Let Δ_i be a sequence of geodesic triangles in X_i which ω-converges to a triangle Δ_ω in X_ω. Let $\overline{\Delta}_i \subset \mathbb{H}^2(\kappa_i)$ be the comparison triangles for Δ_i, then the radius r_i of the inscribed circle in $\overline{\Delta}_i$ tends to zero as $i \to \infty$. Therefore the ω-limit of Δ_i's is a tripod. $\square$

Lemma 9.8. *Let X be a $CAT(-1)$–space, k, c be positive constants, then there is a function $\theta = \tau(k, c)$ such that for any (k, c)-quasi-isometric embedding $f : [a, b] \to X$ the Hausdorff distance between the image of f and the geodesic segment $[f(a)f(b)] \subset X$ is at most θ.*

Proof: Suppose that the assertion of lemma is false. Then there exists a sequence of (k, c)-quasi-isometric embeddings $f_n : [-n, n] \to X_n$ to $CAT(-1)$-spaces X_n such that

$$\lim_{n\to\infty} d_H(f([-n, n]), [f(-n), f(n)]) = \infty$$

where d_H is the Hausdorff distance in X_n. Let $d_n := d_H(f([-n, n]), [f(-n), f(n)])$. Pick points $t_n \in [-n, n]$ such that $|d(t_n, [f(-n), f(n)]) - d_n| \le 1$. Consider the sequence of pointed metric spaces $(\frac{1}{d_n}X_n, f_n(t_n))$, $(\frac{1}{d_n}[-n, n], t_n)$. It is clear that $\omega\text{-}\lim n/d_n > 1/k > 0$ (but this ultralimit could be infinite). Let $(X_\omega, x_\omega) = \omega\text{-}\lim(\frac{1}{d_n}X_n, f_n(t_n))$ and $(Y, y) := \omega\text{-}\lim(\frac{1}{d_n}[-n, n], t_n)$. The metric space Y is either a nondegenerate segment in $\mathbb{R}$ or a closed geodesic ray in $\mathbb{R}$ or the whole real line. Note that the Hausdorff distance between the image of f_n in $\frac{1}{d_n}X_n$ and $[f_n(-n), f_n(n)] \subset \frac{1}{d_n}X_n$ is at most $1 + 1/d_n$. Each map

$$f_n : \frac{1}{d_n}[-n, n] \to \frac{1}{d_n}X_n$$

is a $(k, c/n)$-quasi-isometric embedding. Therefore the ultralimit

$$f_\omega = \omega\text{-}\lim f_n : (Y, y) \to (X_\omega, x_\omega)$$

is a $(k, 0)$-quasi-isometric embedding, i.e. it is a k-bilipschitz map:

$$|t - t'|/k \le d(f_\omega(t), f_\omega(t')) \le k|t - t'|.$$

In particular this map is a continuous embedding. On the other hand, the sequence of geodesic segments $[f_n(-n), f_n(n)] \subset \frac{1}{d_n}X_n$ also ω-converges to a nondegenerate geodesic $\gamma \subset X_\omega$, this geodesic is either a finite geodesic segment or a geodesic ray or a complete geodesic. In any case the Hausdorff distance between the image L of f_ω and γ is exactly 1, it equals the distance between x_ω and γ which is realized as $d(x_\omega, z) = 1$, $z \in \gamma$. I will consider the case when γ is a complete geodesic, the other two cases are similar and are left to the reader. Then $Y = \mathbb{R}$ and by Exercise 9.6 the image L of the map f_ω is a complete geodesic in X_ω which is within Hausdorff distance 1 from the complete geodesic γ. This contradicts the fact that X_ω is a metric tree. $\square$

9.3. The asymptotic cone of a metric space

Let X be a metric space and ω be a non-principal ultrafilter on $\mathbb{N}$. The *asymptotic cone* $\mathrm{Cone}_\omega(X)$ of X is defined as the based ultralimit of rescaled copies of X:

$$\mathrm{Cone}_\omega(X) := X^0_\omega, \qquad \text{where} \quad (X^0_\omega, x^0_\omega) = \omega\text{-}\lim_i (\frac{1}{i} \cdot X, x^0).$$

The limit is independent of the chosen basepoint $x^0 \in X$. The discussion in the previous section implies:

Proposition 9.9. *1. $Cone_\omega(X \times Y) = Cone_\omega(X) \times Cone_\omega(Y)$.*

2. $Cone_\omega \mathbb{R}^n \cong \mathbb{R}^n$.

3. The asymptotic cone of a geodesic space is a geodesic space.

4. The asymptotic cone of a CAT(0)-space is CAT(0).

5. The asymptotic cone of a space with a negative upper curvature bound is a metric tree.

Remark 9.10. Suppose that X admits a cocompact discrete action by a group G of isometries. The problem of dependence of the topological type of $Cone_\omega X$ on the ultrafilter ω was open until recently counterexamples were constructed in [TV], [Du00]. However in the both examples the group G is not finitely presentable.

To get an idea of the size of the asymptotic cone, note that in the most interesting cases it is homogeneous. We call a metric space X *quasi-homogeneous* if $diam(X/\mathrm{Isom}(X))$ is finite.

Proposition 9.11. *Let X be a quasi-homogeneous metric space. Then for every non-principal ultrafilter ω the cone $Cone_\omega(X)$ is a homogeneous metric space.*

Proof: The group of sequences of isometries $\mathrm{Isom}(X)^{\mathbb{N}}$ acts transitively on the ultralimit

$$\omega\text{-}\lim_i (\frac{1}{i} \cdot X)$$

which contains $\mathrm{Cone}_\omega(X)$ as a component. $\square$

Lemma 9.12. *Let X be a quasi-homogeneous $CAT(-1)$–space with uncountable number of ideal boundary points. Then for every nonprincipal ultrafilter ω the asymptotic cone $Cone_\omega(X)$ is a tree with uncountable branching.*

Proof: Let $x^0 \in X$ be a basepoint and $y, z \in \partial_\infty X$. Denote by γ the geodesic in X with the ideal endpoints z, y. Then $\mathrm{Cone}_\omega([x^0, y))$ and $\mathrm{Cone}_\omega([x^0, z))$ are geodesic rays in $\mathrm{Cone}_\omega(X)$ emanating from x^0_ω. Their union is equal to the geodesic $\mathrm{Cone}_\omega \gamma$. This produces uncountably many rays in $\mathrm{Cone}_\omega(X)$ so that any two of them have precisely the basepoint in common. The homogeneity of $\mathrm{Cone}_\omega(X)$ implies the assertion. $\square$

Chapter 10

Introduction to Group Actions on Trees

10.1. Basic definitions and properties

Recall that a metric tree T is a metrically complete nonempty geodesic metric space where each geodesic triangle is a tripod. In particular, every two points in T are connected by unique geodesic segment. *Vertices* (or *branch-points*) of T are the points $x \in T$ such that there are at least three geodesic segments in T emanating from x whose interiors are disjoint. A *nondegenerate* tree is a tree which contains more than one point.

We shall consider isometric actions of groups G on trees T. The pair: a tree T and an action of G on T is called a *G-tree*. A G-tree is called *minimal* if there is no G–invariant subtree in T different from T. A G-tree is called *trivial* if G has a global fixed point in T (the action of G on T is called *trivial* in this case). The action is called *small* if the stabilizer of any nondegenerate arc is a virtually nilpotent subgroup of G. In this case we will say that the G-tree T is *small*. Two G-trees are said to be *isomorphic* if there is a G-equivariant isometry between these trees.

For every element $g \in G$ the action of G on T defines the *translation length* $\ell_T(g)$ as follows:

$$\ell_T(g) = \inf\{d(x, gx) : x \in T\} \tag{10.1}$$

Definition 10.1. An isometry g of T is called **hyperbolic** or **axial** if it has an invariant geodesic A (called the **axis** of g) in T, so that g acts on A as a translation, in which case

$$\ell_T(g) = \mathrm{Length}(A/\langle g\rangle).$$

An isometry g of T is called **elliptic** if it has a fixed point in T.

Note that each axial element g has unique axis; if $T' \subset T$ is a g-invariant subtree of T then g is axial if and only if $g|T'$ is axial.

Lemma 10.2. *(See [CM87].) Each isometry of T is either hyperbolic or elliptic.*

Proof: If $g \in \mathrm{Isom}(T)$ has a fixed point in T then there is nothing to prove. So, suppose that the action of g on T is free. Pick a point $x \in T$, let α be the geodesic segment in T which connects x and $g(x)$. Consider the segments $g\alpha$, $g^{-1}\alpha$. Let a denote the point of intersection between α, $g^{-1}\alpha$ which is the most distant from x; let $b = g(a)$ and $\beta \subset \alpha$ be the geodesic subsegment between a, b. It follows that $b \neq a$ since g acts freely. Then the $\langle g \rangle$–orbit A of β is an isometric copy of the real line in T and it is invariant under $\langle g \rangle$. Suppose that $z \neq A$, then

$$d(z, gz) = 2d(x, A) + d(a, b) \tag{10.2}$$

Therefore $d(a, b)$ realizes the infimum in the definition of $\ell_T(g)$. $\square$

Corollary 10.3. *Suppose that $g \in Isom(T)$ has zero translation length in T. Then g has a fixed point in T.*

Lemma 10.4. *(See [CM87]) Assume that the elements g, h, hg act on the tree T as elliptic isometries. Then the group generated by g and h has a global fixed point in T.*

Proof: Suppose that fixed–point sets F_g, F_h of g, h are disjoint, these are closed convex subsets in T. Pick points $x \in F_g, y \in F_h$ and connect them by geodesic segment $[xy]$. Let $\alpha = [pq] \subset [xy]$ be the smallest subsegment such that $p \in F_g, q \in F_h$. Then the path $(\alpha) * (h\alpha) = \omega$ is geodesic in T. The segments ω and $hg(\omega)$ meet only in a single point $h(p)$ which is one of the end–points of ω. Therefore the $\langle hg \rangle$–orbit of ω is a geodesic in T, which implies that hg has no fixed points in T. Contradiction. $\square$

Exercise 10.5. *Show that the Baumslag-Solitar group*

$$G = \langle a, b | ab^2a^{-1} = b^3 \rangle$$

is not residually finite (see [BS62]). Hint: show that the homomorphism $f : G \to G$, $f(a) = a, f(b) = b^2$ is onto but is not injective, its kernel contains the nontrivial element $k = (aba^{-1}b^{-1})^2 b^{-1}$. To prove that k is nontrivial it suffices to verify that the commutator $aba^{-1}b^{-1}$ does not fix a point in the tree corresponding to the decomposition of G as HNN-extension (use Lemma 10.4).

Corollary 10.6. *Suppose that G is a finitely generated and T is a non-trivial G-tree. Then G contains an axial element.*

Proof: Suppose that G consists only of elliptic elements. Then by the previous lemma, $F_g \cap F_h \neq \emptyset$ for each pair of elements $g, h \in G$. Let $S = \{g_\alpha, \alpha \in I\}$ denote a finite generating set of G. Choose points $x_{ij} \in F_{g_i} \cap F_{g_j}, 1 \leq i, j \leq 3$, for each a triple g_1, g_2, g_3 of elements of S. The geodesic triangle with the vertices x_{12}, x_{23}, x_{31} is a tripod. By convexity of the fixed-point sets, the central point of this tripod belongs to all three subsets F_{g_i}, $i = 1, 2, 3$. Continue inductively: suppose that each m-fold intersection of fixed-point sets of the generators is nonempty:

$$\bigcap_{\alpha \in J} F_{g_\alpha} \neq \emptyset$$

for every m-element index subset $J \subset I$. Take $m+1$ elements in S, $g_1, g_2, ..., g_{m+1}$. Pick the following points:

$$a \in F_{g_1} \cap ... \cap F_{g_m}, b \in F_{g_2} \cap ... \cap F_{g_{m+1}}, c \in F_{g_3} \cap ... \cap F_{g_{m+1}} \cap F_{g_1}.$$

Then the central point of the tripod with the vertices a, b, c is contained in each fixed-point set F_{g_i}, $i = 1, ..., m+1$. It follows that each $(m+1)$-fold intersections of fixed-point sets of elements of G is nonempty. By induction, G fixes a point in T. $\square$

Lemma 10.7. *Suppose that g is an isometry of a metric tree T such that there is a nondegenerate arc $J \subset T$ on which g induces a nontrivial orientation-preserving partial isometry $g : s \to s'$, $s, s' \subset J$. Then g is axial and its axis contains $s \cup s'$.*

Proof: Let m, m' be the mid-points of s and of $s' = g(s)$. Connect m, m' by the geodesic segment I, it is necessarily contained in J. Then our assumptions imply that $gI \cap I = \{m'\}$, $g^{-1}I \cap I = \{m\}$. Therefore the union

$$\bigcup_n g^n(I) = L$$

is a g-invariant geodesic in T. $\square$

Lemma 10.8. *Let T be a G-tree, $g, h \in G$ be axial elements whose axes have bounded intersection. Then for any pair of points $x \in A_h, y \in A_g$ there are numbers n, m such that $h^m g^n$ is axial and x, y belong to the axis of this element.*

Proof: It suffices to consider the case $x \notin A_h$, $y \notin A_g$. Let $A_h \cap [xy] = [x, q]$, $A_g \cap [xy] = [py]$. (Note that $p = q$ if $A_g \cap A_h$ is nonempty.) We can assume that q lies between $x, h(x)$ and p lies between $g^{-1}(y), y$ (otherwise we change g or/and h to its inverse). Then for sufficiently large n we have: the composition $g^n h^n$ induces an orientation-preserving partial isometry of the geodesic segment $[h^{-n}(p)g^n(q)]$. Hence by Lemma 10.7 the segment $[h^{-n}(p)g^n(q)]$ is contained in the axis of $g^n h^n$. On the other hand, $[xy] \subset [h^{-n}(p)g^n(q)]$ (see Figure 10.1) which concludes the proof. $\square$

Corollary 10.9. *If T is a G-tree then the union of axes of axial elements of G is connected.*

Definition 10.10. A nontrivial G-tree T is said to be **unipotent** if G fixes a point ξ in $\partial_\infty T$, i.e. there exists a geodesic ray $\rho : [0, \infty) \to T$ (representing ξ) such that for each $g \in G$ there is a constant $c \in \mathbb{R}$ so that $g \circ \rho(t) = \rho(t + c), t \gg 0$.

Exercise 10.11. *Suppose that a G-tree T is minimal, nontrivial and is not isometric to $\mathbb{R}$. Show that T is unipotent provided that for every pair of axial elements $g, h \in G$ the intersection $A_g \cap A_h$ is an infinite ray.*

This exercise implies that if a G-tree T is nontrivial, minimal, nonunipotent and is not isomorphic to $\mathbb{R}$, then there are axial elements $g, h \in G$ such that $A_g \cap A_h$ is bounded. Therefore there is $n > 0$ such that the axes of g and $h^n g h^{-n}$ are disjoint.

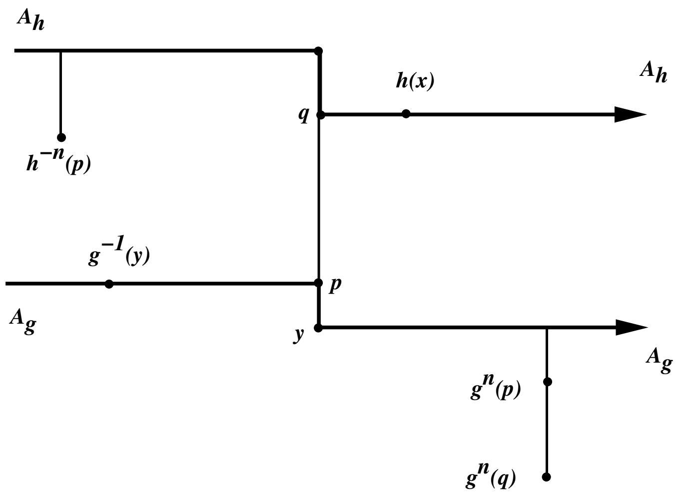

Figure 10.1:

Theorem 10.12. *Suppose that G is a countable group acting on a tree T. Then either (a) the action is trivial, or (b) the action is unipotent, or (c) G contains axial elements in which case T contains a unique minimal G-invariant subtree which is the union T_G of axes of axial elements of G.*

Proof: We first note that if G contains axial elements then the subset T_G is a G-invariant subtree. If in this case $S \subset T$ is another G-invariant subtree then $\partial_\infty S = \partial_\infty T_G$ which implies that $T_G \subset S$. Therefore, if G contains axial elements, then the union of axes T_G is the minimal G-invariant subtree in T. Thus, according to Corollary 10.6, we are done in the case if G is finitely generated or if G contains axial elements.

Suppose that G contains no axial isometries. Since G is a countable group, we can represent G as the union $G = \cup_n G_n$ of nested finitely generated subgroups $G_1 \subset G_2 \subset G_3 ....$ Each subgroup G_i fixes pointwise a closed convex nonempty subset $C_i \subset T$, $C_1 \supset C_2 \supset C_3 ....$ Pick a point $x_0 \in T$ and let $\gamma_i = [x_0 x_i]$ denote the shortest geodesic segment in T connecting x_0 to C_i. The reader will verify that the segments γ_i also form a nested family: $\gamma_1 \subset \gamma_2 \subset \gamma_3 ....$ Therefore (since T is complete) the union of these segments is either (1) a closed geodesic segment $[x_0 x]$, (2) or a half-open geodesic segment $[x_0 x)$, (3) or a complete geodesic ray $[x_0 \xi)$. Hence either the point $x \in T$ or the point $\xi \in \partial_\infty T$ is fixed by the each subgroup G_i. It follows that in the cases (1) and (2) the group G fixes a point in T (thus the action $G \curvearrowright T$ is trivial) and in the case (3) G fixes a point in $\partial_\infty T$ (thus the action is unipotent). $\square$

Notation 10.13. If G is a countable group and T is a G-tree which is nontrivial and non-unipotent then $T_G \subset T$ will denote the minimal G-invariant subtree in T, i.e. the union of axes of axial elements of G.

Below we give an example of unipotent nontrivial G-tree T so that G contains no axial elements. Take a simplicial tree T where each vertex has valence 3 and each edge has unit length. Pick a point $\xi \in \partial_\infty T$, a geodesic ray ρ asymptotic to ξ and a sequence of vertices $x_i \in \rho$ which converge to ξ. For each i choose an elliptic isometry $g_i : T \to T$ whose fixed-point set intersects ρ along the subray $[x_i\xi)$. Then the group $G \subset \mathrm{Isom}(T)$ generated by g_i's, $i = 1, 2, 3, \ldots$, contains only elliptic elements and acts nontrivially on T since

$$\bigcap_{i=1}^{\infty} \rho_i = \emptyset.$$

The action $G \curvearrowright T$ is unipotent.

We now modify the above example to get a minimal unipotent H-tree S so that H contains axial isometries and S is not isometric to $\mathbb{R}$. Namely, start with the same tree T as above and pick an axial isometry $h : T \to T$ with an axis A. There exists an elliptic isometry $g \in \mathrm{Isom}(T)$ which fixes a subray $R \subset A$ pointwise and sends $A - R$ to a distinct geodesic ray. Then take $h' := ghg^{-1}$. The axis A' of h' equals $g(A)$. Thus we let $H \subset \mathrm{Isom}(T)$ be the group generated by h, h'. To get a minimal action choose a minimal subtree $S = T_H \subset T$ which is invariant under H. Clearly the action of H on S is unipotent.

Lemma 10.14. *(Compare [Pau97].) Let T be a minimal nonunipotent G-tree. Then for every nondegenerate arc $[xy] \subset T$ there exists an axial element $f \in G$ whose axis contains $[xy]$.*

Proof: Suppose that the assertion is false. This excludes the cases when T is trivial or isometric to $\mathbb{R}$. Minimality of T implies that T is the union of axes of axial elements of G. Hence we can assume that the points x, y belong to the axes A_h, A_g of certain elements $h, g \in G$. Our assumption implies that $y \notin A_h, x \notin A_g$. I claim that we can choose h, g so that $A_g \cap A_h$ is bounded. If $A_g \cap A_h$ is unbounded choose $g_1 \in G$ whose axis is disjoint from A_x, A_y. Then the axis of $g_1^n g^m$ has bounded intersection with A_h and still contains x (see Corollary 10.8). Thus we will assume that $A_g \cap A_h$ is bounded. Applying Corollary 10.8 again we get an element $f = g^n h^m$ whose axis contains x, y. $\square$

Theorem 10.15. *(M. Culler, J. Morgan, [CM87].) Suppose that G is a finitely generated group and T, R are minimal non-unipotent G-trees, so that the associated length functions*

$$\ell_T : G \to \mathbb{R}, \quad \ell_R : G \to \mathbb{R}$$

coincide. Then there exists a G-equivariant isometry between T and R.

Remark 10.16. Consider the Baumslag-Solitar group $G = \langle a, b | ba^2b^{-1} = a\rangle$ and let T be the Bass-Serre tree associated with the HNN-decomposition of G. Then T is minimal and not isometric to $\mathbb{R}$. However the associated length function ℓ_T is the same as the length function for an isometric action of G on $\mathbb{R}$ which factors through $G \to \langle b\rangle$.

Note that even in the case of a finitely generated free group G it is not enough to verify equality of the length functions on a finite subset of

G, see [SV92]. However, if G is the fundamental group of a closed surface, a minimal *small* G-tree is uniquely determined by the translation lengths of a finite number of elements of G, see Corollary 11.32.

10.2. Actions on simplicial trees

The concept of *simplicial tree* is a refinement of metric tree: simplicial tree Γ is a 1-dimensional regular simply-connected CW-complex. The 0-cells of Γ are called *vertices* and 1-cells are called *edges*. Every simplicial tree Γ can be given structure of a metric tree T where each edge is given the unit length. Note however that the set of vertices of T is a priori different from the set of vertices of Γ.

A metric tree T is called *simplicial* if it is isometric to the metric tree obtained from a simplicial tree by the above procedure. Equivalently, the set V of vertices of T is discrete and components of $T - V$ have integer lengths. The procedure of conversion of simplicial to metric trees allows us to transfer notation and results from the theory of metric trees to the theory of simplicial trees. The main difference will be the notion of G-tree and isomorphism: the action of group G on simplicial tree must preserve the CW-complex structure, an isomorphism between simplicial G-trees is a G-equivariant isomorphism of CW-complexes.

Definition 10.17. (Morphisms between trees.) Let T, T' be G-trees and $f : T \to T'$ a continuous G-equivariant map. The map f is called a **morphism** if each geodesic segment $[a_0a_n] \subset T$ can be subdivided into subsegments

$$[a_0a_1], [a_1a_2], ..., [a_{n-1}a_n]$$

so that the restriction of f to each $[a_ia_{i+1}]$ is an isometry.

Lemma 10.18. *Suppose that T, T' are G-trees where T' is small and T is simplicial and satisfies the property:*

if e, e' are distinct edges incident to a common vertex v then the subgroup of G_v generated by the stabilizers $G_e, G_{e'}$ of e, e' is not virtually nilpotent.

Then for every morphism $f : T \to T'$ morphism between these G-trees and every $x, y \in T$ which belong to distinct edges we have: $f(x) \neq f(y)$.

Proof: Suppose that $x, y \in T$ belong to distinct edges and $f(x) = f(y)$. Connect x and y by a geodesic segment γ in T. The map f is a morphism, thus there exists a subdivision of γ into subsegments γ_i so that the restriction of f to each γ_i is injective. On the other hand, $f(x) = f(y)$, therefore we can find two distinct edges $e, e' \subset T$ which share a common vertex $v \in \gamma$ so that $k = f(e \cap \gamma) = f(e' \cap \gamma)$. It follows that the subgroups $G_e, G_{e'}$ fix k which consists of more than one point. On the other hand, the subgroup of G generated by $G_e, G_{e'}$ is not virtually nilpotent. This contradicts the assumption that the G-tree T' is small. $\square$

Corollary 10.19. *Suppose that the map f as above is an isometry on each edge. Then f is injective.*

A group G acts on a simplicial tree T *without inversions* if the G-stabilizer of each edge $e \subset T$ fixes this edge pointwise.

Suppose that T is a minimal nontrivial simplicial G-tree, where a group G acts without inversions so that T/G is a finite graph. According to the Bass-Serre theory (see [Ser80]) any such action of G on T corresponds to a *graph of groups decomposition* of G which can be thought as an iterated HNN extension and amalgamated free product decomposition. Below is a very brief description of this decomposition.

Suppose that we have a finite graph Γ where each edge is oriented. We assume that each vertex v of Γ is assigned a *vertex* group G_v and each edge e is assigned an *edge* group G_e. Each inclusion $v \hookrightarrow e$ of a vertex into edge (as the initial or the terminal vertex) corresponds to a monomorphism $h_{ev} : G_e \to G_v$ The collection

$$(\Gamma, \{G_e, G_v, h_{ev} : \quad \text{where } e, v \text{ are edges and vertices of } \Gamma\})$$

is called the *graph of groups* L. The fundamental group $\pi(L)$ is defined as follows.

There is a collection of injective homomorphisms $\psi_v : G_v \to \pi(L)$, $\psi_e : G_e \to \pi(L)$ which satisfies the following universal property:

Suppose that we are given a group H and a collection of homomorphisms $\phi_v : G_v \to H$, $\phi_e : G_e \to H$ so that

$$\phi_v \circ h_{ev} = \phi_e$$

for each inclusion $v \hookrightarrow e$. Then there is a homomorphism $\theta : \pi(L) \to H$ so that

$$\theta \circ \psi_v = \phi_v, \theta \circ \psi_e = \phi_e.$$

If G is a group isomorphic to $\pi(L)$ then we say that G has a *graph of groups* decomposition. According to the Bass-Serre theory of actions of groups on trees, each graph of groups decomposition of G gives rise to an action of G on a simplicial tree T so that each vertex stabilizer is conjugate to the image in G of one of the vertex groups G_v (which injects in G) and each edge stabilizer is also conjugate to the image of one of the vertex groups G_v (which injects in G). The graph Γ is the quotient T/G. The graph of groups decomposition is said to be *trivial* if the action of G on T has a global fixed point. Clearly each (nontrivial) graph of groups decomposition gives rise to a (nontrivial) decomposition of G as amalgamated free product or HNN-extension.

The concept of graph of groups appears naturally in relation to Seifert-Van Kampen theorem.

Definition 10.20. Suppose that we are given a finite graph Γ and each vertex v and edge e of Γ is assigned a connected CW-complex X_v, X_e (they are called the vertex and edge spaces) and each inclusion $v \hookrightarrow e$ (as initial or terminal vertex) corresponds to a closed cellular embedding $\epsilon_{ev} : X_e \hookrightarrow X_v$. We assume that the images of all these embeddings are disjoint. Glue the spaces X_v, X_e using these embeddings. The result is a **graph of spaces** Δ.

Each graph of spaces $(\Gamma, \{X_v, X_e, \epsilon_{ev}\})$ determines a graph of groups $(\Gamma, \{G_e, G_v, h_{ev})$ where G_v is the image of $\pi_1(X_v)$ in $\pi_1(X)$, G_e is the image of $\pi_1(X_e)$, the maps $G_e \to G_v$ are induced by the inclusions ϵ_{ev}.

Remark 10.21. Note that we do not assume that $\pi_1(X_v)$, $\pi_1(X_e)$ inject in $\pi_1(X)$.

The following is a version of Seifert- Van Kampen Theorem (cf. [Mas91, Theorem 2.2]):

Theorem 10.22. *The fundamental group of each graph of spaces is naturally isomorphic to the fundamental group of the corresponding graph of groups.*

Below we illustrate the above decomposition in the following special case:

- G is the fundamental group of a compact pared 3-manifold (M, P).
- The action on T is elliptic with respect to P, i.e. the fundamental group of each component of P has a fixed vertex on T.
- The action of G on T is small.

Our discussion follows more general arguments of [SW79]. Let $P = P_1 \cup \ldots \cup P_q$. The quotient $S = T/G$ be a finite graph. Given the action of G on the simplicial tree T we construct a complex K as follows. For each vertex v of S we take an aspherical CW complex K_v with the fundamental group G_v which is the stabilizer of $\tilde{v} \in T$ in G. Suppose that vertices $v, w \in T$ are connected by an edge e of T. In each G_v, G_w we have cyclic subgroups equal to the stabilizer G_e of $e \subset T$ in G. Choose the subcomplexes $K_{v,e}$, $K_{w,e}$ in K_v, K_w whose fundamental groups are copies of G_e (we can assume that these subcomplexes are simple loops). We can assume also that these subcomplexes are chosen so that each $\pi_1(P_j) \subset \pi_1(K)$ is conjugate to the fundamental group of a subcomplex $K(P_j) \subset K$ which is disjoint from any $K_{v,e}$. (Here we use ellipticity of the action on T.) Note that each G_e is either trivial or cyclic.

Connect $K_{v,e}$, $K_{w,e}$ by a copy of $K_{v,e} \times I$ according to the isomorphisms $\pi_1(K_{v,e}) \cong G_e \cong \pi_1(K_{w,e})$. We repeat this for each vertex and edge of S. The result is a complex K whose fundamental group is G. Now we construct a PL homotopy–equivalence $h : M \to K$ assuming that it is transversal to the union of the middle loops $L_e = K_{v,e} \times \{1/2\}$ of $K_{v,e} \times I$. Therefore the preimage of every L_e is a compact properly embedded 2-submanifold in M which is disjoint from P. The image of the fundamental group of every component of $h^{-1}(L_e)$ is conjugate into the corresponding subgroup G_e in G. Using the Loop Theorem we can deform h so that for each edge e:

- Every component of $h^{-1}(L_e)$ is incompressible.
- Each annulus in $h^{-1}(L_e)$ is essential.
- Each disk in $h^{-1}(L_e)$ is compressing.

- No distinct connected components of $h^{-1}(L_e)$ are properly isotopic.

This implies Theorems 1.32, 1.33.

Now we restrict to the case: $M = \Sigma \times I$, where Σ is a compact hyperbolic surface, $P = \partial S \times I$. Then each component of $h^{-1}(L_e)$ is an essential "vertical" annulus: $\gamma_j \times I$, where we can assume that each γ_j is a simple closed geodesic in Σ.

So, we get a decomposition of the surface Σ by a union σ of simple closed disjoint geodesics. Lift this decomposition in $\mathbb{H}^2$ and let Γ be the tree dual to the corresponding decomposition of the hyperbolic plane. Stabilizer of every vertex of Γ in G is a free non-Abelian group, which is the fundamental group of the corresponding component of $\Sigma - \sigma$, whose image under $h : M \to K$ is contained in the complement to the set of all middle loops in K. Therefore it is contained in the stabilizer of a vertex of T.

Stabilizer of each edge is a cyclic subgroup of G and for two distinct edges their stabilizers generate a free non-Abelian subgroup. Now we construct a G–equivariant map $q : \Gamma \to T$ as follows. Every vertex v of Γ is stabilized by a (non-Abelian) subgroup G_v of G, we shall send it to the fixed point of G_v on T (the last point exists by the argument above and is unique since the action of G on T is small). We extend this map to the set of edges of Γ by an affine map. The resulting map can be noninjective only if it folds the tree Γ at some vertex $v \in \Gamma$, i.e. for two adjacent segments e, d at v in Γ we have $f(e) = f(d)$. However Lemma 10.18 implies that f cannot fold. Therefore, T is isomorphic to $\Gamma \subset \mathbb{H}^2$.

The collection of geodesics $\sigma \subset \Sigma$ is said to be *dual* to the action of G on T. We therefore proved

Lemma 10.23. *Let T be a small simplicial G-tree which is elliptic relative to the collection of peripheral subgroups of G (i.e. G fixes a point x in T). Then there is a system of disjoint simple homotopically nontrivial loops $\alpha_1, ..., \alpha_n$ so that the simplicial G-tree Γ, dual to the decomposition of the universal cover of S by the inverse images of the loops α_i, is equivariantly isomorphic to the simplicial G-tree T.*

10.3. Limits of isometric actions on $CAT(0)$-spaces

There are several ways to compactify the space of (conjugacy classes of) discrete and faithful representations $\mathcal{D}(\Gamma, G)$ of a finitely generated group Γ by actions of Γ on trees. The first such compactification was introduced by Culler and Shalen [CS83] (in the case of $G = SL(2, \mathbb{C})$), later it was generalized by Morgan and Shalen [MS84] and by Morgan [Mor86] (in the case $G = \mathrm{Isom}(\mathbb{H}^n)$). Further, more general, geometric constructions of this compactification were found by Bestvina [Bes88] and Paulin [Pau88].

In this section, following [Chi91] and [KL95], we will give an interpretation of this compactification using the language of ultralimits.

Take a finitely generated group Γ with a finite generating set $\mathcal{G}$. Let X_i be a sequence of $CAT(0)$-spaces and $\rho_i : \Gamma \to \mathrm{Isom}(X_i)$ be a sequence of representations. For $x \in X_i$, let $D_i(x)$ denote the diameter of the set $\rho_i(\mathcal{G})(x)$. Set $D_i := \inf_{x\in X_i} D_i(x)$. We assume that the sequence (ρ_i) diverges in the sense that $\lim_{i\to\infty} D_i = \infty$. Choose points $x_i \in X_i$ such that $D_i(x_i) \le D_i + 1/i$. For any non-principal ultrafilter ω, there exists a natural isometric action ρ_ω of Γ on the ultralimit of rescaled spaces

$$(X_\omega, x_\omega) := \omega\text{-}\lim_i (D_i^{-1} \cdot X_i, x_i).$$

X_ω is a CAT(0)-space and the action ρ_ω has no global fixed point. If X_n are $CAT(-1)$-spaces, then X_ω is a metric Γ-tree.

10.4. Compactification of character varieties

Now we can discuss the relation between actions of groups on trees and sequences of discrete and faithful representations into $\mathrm{Isom}(\mathbb{H}^n)$. Suppose that Γ is a finitely generated group which is not almost Abelian, $[\rho_i] \in \mathcal{D}(\Gamma, \mathrm{Isom}(\mathbb{H}^n))$ is a sequence of discrete and faithful representations which is not relatively compact. For each ρ_i define the number D_i and a point $x_i \in \mathbb{H}^n$ as in §10.3, where $X_i = \mathbb{H}^n$ for all i.

Theorem 10.24. *(J. Morgan and P. Shalen [MS84], J. Morgan [Mor86], M. Bestvina [Bes88], F. Paulin [Pau88], [Pau89].) Under the above conditions there exists a subsequence ρ_{i_j} and a metric Γ-tree T such that the following is satisfied:*

1. *T is the minimal Γ-invariant subtree in the tree*

$$X_\omega = \omega\text{-}\lim_i (D_i^{-1} \cdot \mathbb{H}^n, x_i)$$

for some ultrafilter ω on $\mathbb{N}$.

2. $\lim_{j\to\infty} \ell(\rho_{i_j}(g))/D_{i_j} = \ell_T(g)$.

3. *The action of Γ on T is minimal, small and the tree T is nondegenerate.*

Proof: First of all, since the sequence $[\rho_i]$ is not relatively compact, it follows that

$$\overline{\lim_{i\to\infty}} D_i = \infty$$

(otherwise we conjugate ρ_i to ρ_i' so that the sequence of points x_i is relatively compact) and hence we can choose a subsequence $\{i_j\}$ so that $\lim_{j\to\infty} D_{i_j} = \infty$.

Choose an ultrafilter ω which contains the subsequence $\{i_j\}$. Then the ultralimit X_ω (as in *(1)*) is a Γ-tree. Let T be the minimal G-subtree in X_ω. We now show that the action of Γ on the tree X_ω satisfies

$$\lim_{j\to\infty} \ell(\rho_{i_j}(g))/D_{i_j} = \ell_{X_\omega}(g)$$

this will imply that the action of G on T satisfies *(3)*. Consider the case when $\rho_i(g)$ is loxodromic for ω-all i. Let L_i denote the axis of $\rho_i(g)$ and $proj_i : \mathbb{H}^n \to L_i$ be the nearest-point projection. There are two possible cases:

Case 1: $\omega\text{-lim}_i D_i^{-1} d(x_i, L_i) < \infty$. Then $\omega\text{-lim}_i L_i = L_\omega \subset X_\omega$ is a g-invariant geodesic and the assertion is obviously true.

Case 2: $\omega\text{-lim}_i D_i^{-1} d(x_i, L_i) = \infty$. If

$$\omega\text{-}\lim_i \ell(\rho_i(g))/D_i > 0$$

then the exponential contraction property of $proj_i$ (Lemma 3.6) implies that the segment $[x_i, \rho_i(g)(x_i)] \subset \mathbb{H}^n$ contains point y_i so that $d(y_i, L_i) \le const$ for ω-all i. Hence the triangle inequality implies:

$$d(x_\omega, gx_\omega) = \omega\text{-}\lim_i d(x_i, g(x_i))/D_i = \infty$$

which is impossible. Thus

$$\omega\text{-}\lim_i \ell(\rho_i(g))/D_i = 0$$

Let us prove that g acts as an elliptic element on T. The ultralimit of the rescaled distance functions $d(\bullet, L_i)/D_i$ is a Busemann function β on X_ω (Lemma 9.5), which must be invariant under the action of g. Let $\xi \in \partial_\infty X_\omega$ be the ideal boundary point corresponding to β, $g(\xi) = \xi$. If g is not elliptic, it has the axis A in T. Connect ξ to A by a geodesic ray R which intersects A only at the origin. Then, since g acts on A as a translation, we conclude that $gR \cap R = \emptyset$. Contradiction.

The cases when $\rho_i(g)$ are elliptic or parabolic for ω-all i are similar to the Case 2 above, which finishes the proof of *(2)*.

Proof of *(3)*. Suppose that Γ fixes a point $(y_i) = y_\omega \in X_\omega$. Then for each g in the finite generating set $\mathcal{G}$ we have:

$$0 = d(g(y_\omega), y_\omega) = \omega\text{-}\lim_i d(\rho_i(g)(y_i), y_i)/D_i$$

and $d(\rho_{i_j}(g)(y_{i_j}), y_{i_j}) = o(D_{i_j})$ which contradicts the choice of D_i in §10.3. Thus the Γ-tree X_ω is nontrivial and the tree T is nondegenerate. Finally we will prove that the Γ-tree X_ω is small. Let $I = [y_\omega z_\omega]$ be a nondegenerate arc in X_ω and $\Gamma_I \subset \Gamma$ is the subgroup stabilizing I pointwise. We will use the discrete and faithful representation $\rho_1 : \Gamma \hookrightarrow \mathrm{Isom}(\mathbb{H}^n)$ and regard Γ as a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^n)$. Then Γ_I is virtually nilpotent if and only if it is an elementary subgroup of $\mathrm{Isom}(\mathbb{H}^n)$. Hence we assume that Γ_I is nonelementary and thus contains a free subgroup generated by a pair of nonelliptic elements $g, h \in \Gamma_I \subset \mathrm{Isom}(\mathbb{H}^n)$. Choose $\epsilon > 0$ so that $\epsilon < \mu_n/28$, where μ_n is the Margulis constant for $\mathbb{H}^n$. Let S_4 denote the collection of words in the generators $g^{\pm 1}, h^{\pm 1}$ of the length ≤ 4.

Let $y_\omega = (y_i), z_\omega = z_i$. For each $f \in \Gamma_I$ and i define

$$\lambda_i(f) := \max_{u_i \in [y_i z_i]} d(\rho_i(f)u_i, u_i).$$

Then

$$0 < \omega\text{-}\lim_i d(y_i, z_i)/D_i < \infty$$

and for each $g \in \Gamma_I$ we have

$$0 = \omega\text{-}\lim_i \lambda_i(g)/D_i.$$

Lemma 3.10 implies that for ω-all i the segment $[AB] := [y_i z_i]$ contains a subinterval $[A'B']$ which satisfies the following:

- For each $f \in S_4$ the image $f([A'B'])$ is contained in the ϵ-neighborhood of $[AB]$ and $Proj(f(A'))$ is closer to A than the point $Proj(f(B'))$, where $Proj$ is the nearest-point projection to the complete geodesic through $[AB]$.
- $\omega\text{-}\lim d(A', B')/D_i > 0$.

Let $P, Q \in [A'B']$ be such that $d(P,Q) = d(A',B')/9 = d(A',P)/4 = d(Q,B')/4$. Then for all $f \in S_4$ and ω-all i we will have $\lambda_i(f) \leq d(P,Q)$. For each element $f \in S_4$ define the mapping

$$\bar{f} : [PQ] \to [A'B'], \bar{f}(x) = Proj(f(x)).$$

We will isometrically identify the segment $[A'B']$ with an interval in $\mathbb{R}$. We leave it to the reader to verify the following properties of the mapping $f \mapsto \bar{f}$:

1. For each $x \in [PQ]$, $d(f(x), \bar{f}(x)) \leq \epsilon$.
2. For each f the mapping $\bar{f}$ is a 2ϵ-approximate translation, i.e.
$$|d(x, \bar{f}(x)) - d(y, \bar{f}(y))| \leq 2\epsilon$$
for all $x, y \in [PQ]$.
3. The mapping $f \mapsto \bar{f}$ is a 2ϵ-approximate homomorphism: for each $f_1, f_2 \in S_4$ such that $f_1 f_2 \in S_4$, we have:
$$|\bar{f}_1 \bar{f}_2(x) - \overline{f_1 f_2}(x)| \leq 2\epsilon.$$

Let $f_1 := g^{\pm 1}, f_2 := h^{\pm 1}$. Thus for each $x \in [PQ]$ we have:

$$d([f_1, f_2](x), \overline{[f_1, f_2]}(x)) \leq 16\epsilon$$

$$|[\overline{f_1}, \overline{f_2}](x) - \overline{[f_1, f_2]}(x)| \leq 8\epsilon$$

$$|[\overline{f_1}, \overline{f_2}](x) - x| \leq 4\epsilon$$

and therefore

$$d([f_1, f_2](x), x) \leq 28\epsilon < \mu.$$

Kazhdan-Margulis theorem now implies that the commutators

$$[g, h], \quad [h^{-1}, g], \quad [h, g^{-1}]$$

belong to a common elementary subgroup of Γ_I. If one of these commutators is trivial then the group generated by g, h is abelian and we get a contradiction. Otherwise, the assumption that the group generated by g, h is free implies that all these commutators have infinite order and therefore are not elliptic; thus their fixed point sets in $\partial_\infty \mathbb{H}^n$ are equal:

$$F := Fix([g,h]) = Fix([h,g^{-1}]) = Fix([h^{-1},g]) \ .$$

Note that

$$g^{-1}[g,h]g = [h,g^{-1}], \quad h^{-1}[g,h]h = [h^{-1},g],$$

thus

$$g^{-1}(F) = h^{-1}(F) = F$$

which implies that the set F (which consists of one or two points) is invariant under both g and h, hence the group generated by g and h is elementary. Contradiction. $\square$

10.5. Proper actions

Suppose that T is a complete metric space, and G is a finitely generated group with a finite generating set $\mathcal{G} = \{g_1, ..., g_r\}$. Suppose that $G \curvearrowright T$ is an effective and isometric action. Given this action define the function $D : T \to \mathbb{R}$

$$D(x) = \max\{d(g_j x, x), j = 1, ..., r\}.$$

Pick a base-point $x_0 \in T$. The action of G on T is called *proper* if

$$\lim_{d(x,x_0)\to\infty} D(x, x_0) = \infty.$$

(Thus the action is proper if the function D is "proper", i.e. its sublevel sets are bounded.) Clearly properness of the action does not depend on the choice of the base-point and generators. If $G_0 < G$ and the action of G_0 on a G-tree T is proper, then the action of G is proper as well. A minimal action of G on T is called *axial* if $T \cong \mathbb{R}$. In this case G is almost Abelian and a subgroup of index 2 in G acts on T by translations.

Theorem 10.25. *Suppose that G is finitely generated and T is a nontrivial minimal G-tree. Then one of the following mutually exclusive cases occurs:*

- *The action of G is proper.*
- *The action of G is axial and G acts on $\mathbb{R}$ by translations.*
- *T is not a line and there exists an infinite ray $R \subset T$ so that for each element h of the commutator subgroup $[G,G]$ of G, there exists an infinite subray $R(h) \subset R$ which is pointwise fixed by h. (I.e. this action is* **unipotent***.)*

Proof: Since the action of G does not have a global fixed point, Lemma 10.4 implies that the group G contains a hyperbolic element g with the axis A. It follows from hyperbolicity of g that

$$d(gx, x) \to \infty \quad \text{as} \quad d(x, A) \to \infty.$$

Suppose that we have two hyperbolic elements $g_1, g_2 \in G$ whose axes intersect in a compact subset. Thus for any $t > 0$ the intersection

$$\{x : d(x, g_1(x)) \le t\} \cap \{x : d(x, g_2(x)) \le t\}$$

is bounded. Hence the action of $G_0 := \langle g_1, g_2 \rangle$ on T is proper, which implies that the action of G is also proper. Now assume that the action of G is not proper. We conclude that axes of any two hyperbolic elements must intersect at least by an infinite ray. Then there exists an infinite ray $R \subset T$ so that for any hyperbolic element $g \in G$ the intersection between R and the axis A_g is an infinite subray. Let $\xi \in \partial_\infty T$ denote the ideal point corresponding to R. Then ξ is fixed by all hyperbolic elements of G. Let Γ be the normal subgroup of G generated by all hyperbolic element. Note that the fixed-point set for the action of Γ on $\partial_\infty T$ is invariant under the action of G. If this set contains more than one point, then Γ has an invariant geodesic L in T, which connects two of these points and which is the axis of *all* hyperbolic elements of G. Thus L is G-invariant and $\mathbb{R} \cong L = T$, by minimality of T. This is the axial case, since the action is not proper, the group G must preserve the orientation on $\mathbb{R}$. In this case the assertion of Theorem is clearly valid.

Finally consider the case when Γ fixes exactly one point ξ on $\partial_\infty T$. The point ξ is also fixed by the whole group G. We conclude that for any element $h \in G$ there exists an infinite ray $R(h)$ terminating in ξ so that $R(h)$ is either pointwise fixed by h (if h is elliptic) or $R(h)$ is contained in the axis of h (if h is hyperbolic). It is clear that for each element $h \in [G, G]$ the ray $R(h)$ can be chosen so that $R(h)$ is pointwise fixed by h. $\square$

Corollary 10.26. *Suppose that under the conditions of Theorem 10.25 the action of G on T is small. Then either (a) the commutator subgroup of G is almost nilpotent, or*

(b) the action $G \curvearrowright T$ is proper.

Proof: Suppose that the action is not proper. The only case we need to consider is the case of *unipotent action*. Since the action of G is small, the commutator subgroup $[G, G]$ of G must be almost nilpotent. $\square$

Suppose that M is a closed Riemannian manifold with the fundamental group G and $G \curvearrowright T$ is a proper action on a $CAT(0)$-space. One of the important applications of properness of this action is existence of a G-equivariant harmonic map from the universal cover of M to T, see [KS97].

Chapter 11

Laminations, Foliations and Trees

In this chapter we discuss several essentially equivalent mathematical concepts:

- Geodesic currents with zero self-intersection.
- Measured foliations.
- Measured geodesic laminations.
- Trajectories of holomorphic quadratic differentials.
- Measured train tracks.
- Generalized interval exchange transformations.
- Small actions of surface groups on metric trees.

There is yet another point of view on measured geodesic laminations [Hat88] which will not be discussed here.

11.1. The Euclidean motivation

In this section we describe spaces of measured foliations and geodesic laminations on the 2-torus S, how they compactify the Teichmüller space of S and how elements of the mapping class group of S act on this compactification. It is helpful to keep in mind this description since it is the simplest version of what we will do in the case of hyperbolic surfaces.

Consider a flat torus S of unit area. Let $\mathcal{C}(S)$ denote the union of the empty set with the set of all (unoriented) closed geodesics in S. We will identify $\mathcal{C}(S)$ with the set of primitive elements of $(H_1(S, \mathbb{Z}))/\pm 1$ (where zero corresponds to the empty set). Define the *geometric intersection number* $i(\alpha, \beta)$ between elements of $\mathcal{C}(S)$ to be the number of points of transversal intersection. This is the same as the minimal number of points of intersection between loops α', β' in the homotopy classes of α, β. We have the intersection form Q

$$Q : H_1(S, \mathbb{R}) \times H_1(S, \mathbb{R}) \to \mathbb{R}.$$

The absolute value of the restriction of this form to $\mathcal{C}(S)$ gives the geometric intersection number between α and β: $|Q(\alpha, \beta)| = i(\alpha, \beta)$. Let $\alpha, \beta \in \mathcal{C}(S)$ be a pair of closed geodesics with intersection number 1. Then any element $\gamma \in H_1(S, \mathbb{Z})$ is uniquely determined by the pair $(Q(\gamma, \alpha), Q(\gamma, \beta))$. However, elements γ of $\mathcal{C}(S)$ are not uniquely determined by $f(\gamma) := (i(\gamma, \alpha), i(\gamma, \beta))$. The ambiguity is: $f(n\alpha + m\beta) = f(-n\alpha + m\beta)$. Thus there are at most two distinct elements of $\mathcal{C}(S)$ in $f^{-1}(z, w)$. To avoid this ambiguity we take into account intersection with the third geodesic δ corresponding to the homology class $\alpha + \beta$. Then

$$Q(n\alpha + m\beta, \alpha + \beta) = n + m, \quad Q(-n\alpha + m\beta, \alpha + \beta) = -n + m$$

and $i(n\alpha + m\beta, \delta) = i(-n\alpha + m\beta, \delta)$ if and only if either $n = 0$ or $m = 0$, which means that the triple

$$(i(\gamma, \alpha), i(\gamma, \beta), i(\gamma, \delta))$$

uniquely determines $\gamma \in \mathcal{C}(S)$.

The projectivized homology group $PH_1(S, \mathbb{R})$ is homeomorphic to the unit circle and the set:

$$\{a/|a| : a \in \mathcal{C}(S)\}$$

is dense in $PH_1(S, \mathbb{R})$.

Choose α, β as generators of $\pi_1(S)$. The *word norm* $|x|$ (i.e. the distance between 0 and $x \in \pi_1(S)$ with respect to the word metric) can be calculated geometrically as $i(x, \alpha) + i(x, \beta)$. The elements of the space $H_1(S, \mathbb{R})$ can be represented as measured oriented geodesic foliations. Here the word "measure" means "a transversal measure". For example, if a represents a simple closed geodesic of the word norm $|a|$, then it determines a foliation of S by geodesics parallel to a. Suppose that J is any finite geodesic arc in S transversal to a. Then we define $mes_a(J)$ as $Length(J) \cdot |a|$. To define the *transversal measure* on smooth arcs transversal to a we assume that the transversal measure does not change if we slide the end–points of J along the foliation and keep J transversal to the foliation a. The space of projective classes of *measured geodesic laminations* on S is the same thing as the space $PH_1(S, \mathbb{R})$ of (nonoriented) classes of geodesic foliations on S. The Teichmüller space $\mathcal{T}(S)$ of the torus S is naturally identified with the hyperbolic plane. Assuming that S is the origin in $\mathcal{T}(S)$ we conclude that $PH_1(S, \mathbb{R})$ serves as a natural compactification of $\mathcal{T}(S)$. Namely, suppose that we are given a direction $x \in PH_1(S, \mathbb{R})$. Think about x as a direction in the universal cover $\mathbb{R}^2$ of S. Then we renormalize the metric on $\mathbb{R}^2$ keeping the same action of the fundamental group so that we squeeze the metric in the direction x and stretch it in the orthogonal direction (to keep the area equal to 1). This sequence of metrics on the torus S has a *limit*: a degenerate bilinear form on $\mathbb{R}^2$ which is defined up to a multiple. The only information about this form is that it vanishes on tangent lines which are parallel to x. It corresponds to the projective class of a measured geodesic foliation on S.

Now we consider the classification of elements of infinite order in the classical modular group $SL(2,\mathbb{Z})$ (the group $PSL(2,\mathbb{Z})$ is the mapping class group of the torus). Each hyperbolic element $A \in SL(2,\mathbb{Z})$ has two fixed points in $PH_1(S,\mathbb{R})$: one repulsive $[A_-]$ and the other attractive $[A_+]$. Both are represented by eigenvectors of A. These eigenvectors give rise to a pair of irrational foliations on the torus S. We will identify $H_1(S,\mathbb{R})$ with the universal cover $\mathbb{R}^2$ of the torus S so that A is a linear transformation.

Given the the mapping A, the flat metric on the torus S can be chosen so that A is symmetric and thus repulsive and attractive directions are orthogonal. Stretching–squeezing the metric in these directions corresponds to moving along a geodesic in $\mathcal{T}(S)$. Suppose that $x \in H_1(S,\mathbb{Z})$. Then we can apply to x iterations $A^n(x)$. As $n \to +\infty$ the normalized sequence of vectors $x_n = A^n(x)/|A^n(x)|$ converges to $[A_+]$ and as $n \to -\infty$ the normalized sequence x_n converges to $[A_-]$.

We choose representatives A_+, A_- in the projective classes $[A_+], [A_-]$. Then $A(A_+) = \lambda_+ A_+$ and $A(A_-) = \lambda_- A_-$, where $\lambda_\pm$ are the eigenvalues of A. The foliations A_+ and A_- are "binding" the surface S: any nonzero foliation has nonzero intersection number either with A_+ or with A_- (the corresponding eigenvectors are linearly independent).

Diffeomorphisms of the torus S which possess such pair of attractive/repulsive foliations are called *Anosov*.

In contrast, if A is a parabolic element, then it has only one fixed point; therefore the corresponding foliation does not bind S. The element A is an iteration of Dehn twists along a simple loop a, the foliation $A_\pm$ has closed leaves which represent the homotopy class of a.

The goal of the rest of this chapter is to extend this picture to the case of hyperbolic surfaces. Such extension was implicit in earlier works of Dehn and Nielsen (see [Deh87]). In the modern time it was discovered and brought back to mathematics by Thurston [Thu81], one of whose motivation was to prove the hyperbolization theorem.

11.2. Geodesic currents

Let $\mathcal{M}_\infty$ denote the space of unordered pairs of distinct points on $\mathbb{S}^1$:

$$\mathcal{M}_\infty := \{(z,w) \in \mathbb{S}^1 \times \mathbb{S}^1 : z \neq w\}/(z,w) \sim (w,z).$$

Topologically $\mathcal{M}_\infty$ is just the Moebius band. We identify $\mathbb{S}^1$ with the ideal boundary of the hyperbolic plane $\mathbb{H}^2$. Then the group $PSL(2,\mathbb{R})$ acts on $\mathcal{M}_\infty$. Recall that a *Radon measure* on a space X is a measure which defines a continuous functional on the space of continuous functions with compact support (with the topology of uniform convergence). Fix a discrete torsion-free group $G \subset PSL(2,\mathbb{R})$ such that $\mathbb{H}^2/G = S$ is a hyperbolic surface. A *geodesic current* μ on S is a G-invariant Radon measure on $\mathcal{M}_\infty$. Let $\mathcal{GC}(S)$ denote the space of all geodesic currents. Clearly, sums of geodesic currents and their positive multiples again belong to $\mathcal{G}C(S)$. The space $\mathcal{G}C(S)$ of geodesic currents has a natural weak* topology of convergence on continuous functions. More on geodesic currents is contained in [Bon86], [Bon88].

At this point the definition of geodesic current may look abstract and useless, however later on we shall see how it corresponds to more geometric definitions of measured laminations and foliations.

The Basic Example. Suppose that $g \in G$ is a nontrivial element representing a simple closed geodesic γ on S (if S has boundary we assume that γ is non-peripheral). Take the G-orbit of the fixed-point set $Fix(g) \in \mathcal{M}_\infty$. This orbit forms a discrete subset E in $\mathcal{M}_\infty$ and we can choose the Dirac measure on this orbit. This is the simplest example of a geodesic current on S (which is the sum of δ-functions supported on the points of E).

Consider the square $\mathcal{M}^2_\infty := \mathcal{M}_\infty \times \mathcal{M}_\infty$, elements of this space correspond to pairs of geodesics in $\mathbb{H}^2$. In the space $\mathcal{M}^2_\infty$ take the open subset $\mathcal{I}M^2_\infty$ corresponding to pairs of geodesics which have transversal intersections in $\mathbb{H}^2$. The group G acts naturally on $\mathcal{I}M^2_\infty$ and it is easy to see that this action is properly discontinuous. Suppose now that μ, ν are geodesic currents in $\mathcal{G}C(S)$, the product of these measures defines an G-invariant measure $\mu \times \nu$ on the open subset $\mathcal{I}M^2_\infty$ of $\mathcal{M}^2_\infty$. Finally we project this product measure to $\mathcal{I}M^2_\infty/G$ and take the total mass of the quotient measure. The result is called the *intersection number* $i(\mu, \nu)$ between the currents μ, ν.

Theorem 11.1. *(F. Bonahon, [Bon86], [Bon88].) The function*

$$i : \mathcal{G}C(S) \times \mathcal{G}C(S) \to \mathbb{R}_+$$

is continuous and bilinear.

Exercise 11.2. *If μ, ν are geodesic currents corresponding to distinct simple closed geodesics α and β (as in the Basic Example) then $i(\mu, \nu)$ equals the number of points of intersection between α and β.*

11.3. Measured foliations on hyperbolic surfaces

Let S be a compact hyperbolic surface, possibly with nonempty geodesic boundary. (The discussion in the case of surfaces of finite area is similar, see [Gar87].) Roughly speaking, a *measured foliation* on S is a singular C^1–foliation with singularities similar to the singularities of holomorphic quadratic differentials, which is given a *transversal measure.* More precisely, suppose that $P = \{p_1, .., p_m\}$ is a finite subset of S, which is the *singular set* of the foliation. If S has nonempty boundary we require each boundary circle to contain at least one point of P. Pick a coordinate covering U_j on $S - P$ and assume that in each U_j we are given a real–valued C^1–function v_j such that $|dv_j| = |dv_i|$ on $U_j \cap U_i$. Assume also that for each singular point p_s there is a coordinate neighborhood V with complex coordinate z such that $|dv| = |Im(z^{k/2}dz)|$ for some positive integer k. Leaves of the foliations are the graphs immersed S in along which the functions v_j are constant. Then $|dv|$ is called a *measured quasifoliation.* If in addition each boundary circle of S is contained in a singular leaf, then $|dv|$ is called a *measured foliation.*

If J is an arc in S then the transversal measure (height) $h(J)$ of J with respect to the foliation $|dv|$, is the integral

$$\int_J |dv|.$$

The *height* $h_\gamma(|dv|)$ of a (free homotopy class) of a loop γ on S is the infimum of heights of loops in $[\gamma]$, this defines a map

$$h : (\gamma, |dv|) \mapsto h_\gamma(|dv|) \in \mathbb{R}.$$

We define *lines* in S and in the universal cover $\tilde{S}$ of S the same way we defined them for quadratic differentials (Definition 5.2): they are certain subsets of singular leaves which can be approximated by nonsingular leaves.

Exercise 11.3. *Let $L \subset \tilde{S}$ be a lift of a boundary loop of S to the universal cover $\tilde{S}$. Then L* **is not a line***.*

Alternatively one can describe a measured foliation as follows. Let λ be a foliation on S (which is smooth away from P) with the set of isolated singularities P. We require that near each point $p_j \in P$ the foliation λ looks like the image of the vertical foliation of $\mathbb{C}$ under the map $z \mapsto z^{2/k}$, the number k is called the degree of the singular point $p_j := 0 \in \mathbb{C}$. See Figure 11.1. We require each boundary circle of S to be contained in a singular leaf of λ.

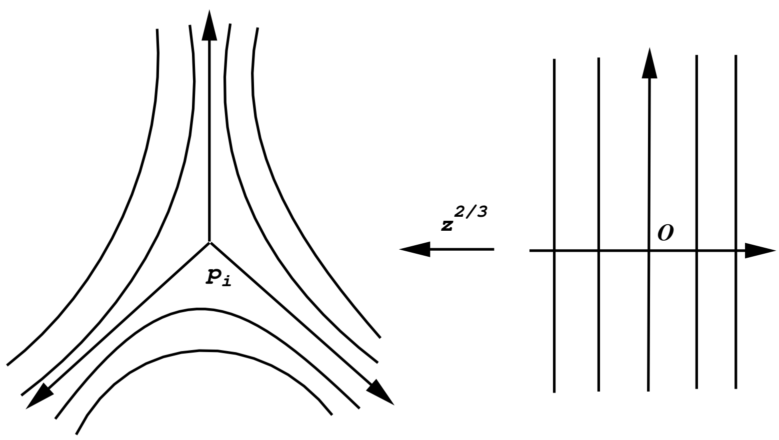

Figure 11.1: *Measured foliation near the critical point p_i where $k = 3$.*

Now suppose that $S - P$ is covered by charts so that images of leaves of the foliation are vertical lines in $\mathbb{R}^2 = \mathbb{C}$ and transition maps not only preserve the vertical directions but also are isometries in the horizontal direction (maybe orientation reversing). Then the measure of an arc J contained in the coordinate neighborhood is the Lebesgue measure of its projection to the horizontal direction. The measure is invariant under sliding the end-points of J along leaves of the foliation. This gives λ a transversal measure.

Definition 11.4. Critical leaves of the foliation λ are those which contain at least one singular point $p_j \in P$.

Basic examples of measured foliations are constructed from holomorphic quadratic differentials $\phi(z)dz^2$ on the surface S (if S is a Riemann surface without boundary) as follows:

$$|dv| = |Re(\sqrt{\phi(z)}dz)|.$$

This defines a map $q : \phi \mapsto |dv|$. Leaves of the foliation $|dv|$ are the vertical trajectories of this quadratic differential, i.e. the curves on which the quadratic differential has zero real part and positive imaginary part. Things are more complicated if we have a compact orientable surface $\dot{S}$ with boundary. Let S be the interior of $\dot{S}$, introduce a complex structure of finite type on S. Let ϕ be a holomorphic quadratic differential on S which has at worst simple poles at the punctures. Then as above we define measured foliation $\mathcal{F}$ on the surface S (which is the horizontal foliation of ϕ). Let Σ denote the conformal compactification of S (i.e. the Riemann surface S is obtained by removing a finite subset from Σ).

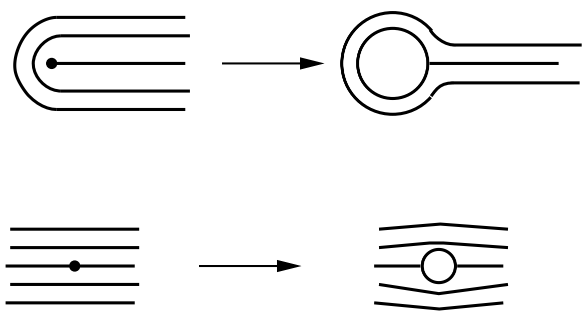

Figure 11.2: *Blow up of a puncture.*

Let p_j be a puncture of S (we identify p_j with $0 \in \mathbb{C}$) where

$$\phi(z) = z^{k_j}\psi(z), \quad k_j \geq -1$$

$\psi(z)$ is holomorphic at p_j and $\psi(p_j) \neq 0$. Then we blow up each puncture $p_j \in \Sigma$ into a circle γ_j. The result is diffeomorphic to the surface $\dot{S}$, the foliation $\mathcal{F}$ extends to $\dot{S}$ where we treat each circle γ_j as a part of a leaf. Since ϕ has at worst simple poles at the punctures, each γ_j contains at k_j+2 singular points. In the Figure 11.2 we describe the blow up if $k_j = -1, 0$.

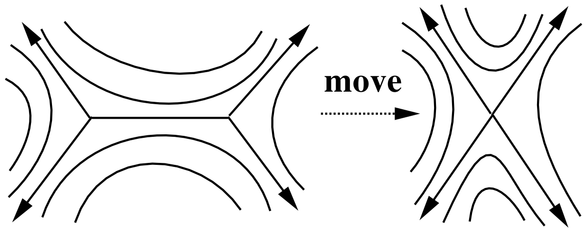

Figure 11.3: *Whitehead move.*

Two measured foliations $|du|, |dv|$ are called *measure–equivalent* if the heights of (the free homotopy classes) of all simple closed curves in S with respect to these foliations are equal, i.e.

$$h(\gamma, |du|) = h(\gamma, |dv|), \quad \text{for each loop } \gamma \text{ in } S.$$

Equivalently (if S is a closed surface) one can use the *Whitehead* equivalence relation on singular foliations by collapsing compact critical intervals to points (the *Whitehead move*) and taking isotopy of foliations. (See [HM79], [Gar87], [FLP79] for details.) Note that the Whitehead move maps leaves to leaves and lines to lines. It is clear that Whitehead moves preserve measure equivalence classes, the opposite direction is not obvious. In Figure 11.4 we give an example (taken from [HM79]) of a measured foliation which is not isotopic to the foliation of a holomorphic quadratic differential but is Whitehead equivalent to it.

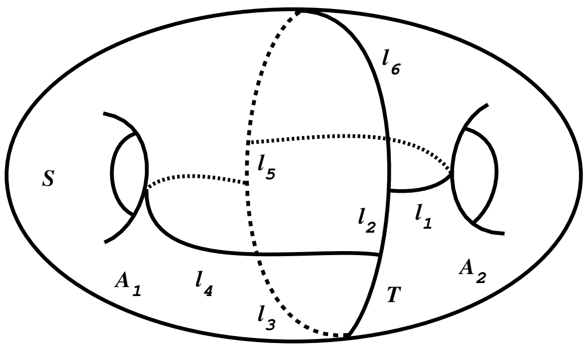

Figure 11.4: *Example of a measured foliations L on the surface S of genus 2 which is not isotopic to the horizontal foliation of a quadratic differential. The complement to the graph T in the surface S is the disjoint union of two annuli A_1, A_2. These annuli are foliated by horizontal loops. The transversal measure is given by the "height" of the transversal interval. Existence of a Euclidean metric with conical singularities corresponding to this foliation would force the equality $l_2 = l_5$. Collapsing the intervals l_2, l_5 to points makes L the horizontal foliation of a quadratic differential.*

Let $\mathcal{MF}(S)$ denote the space of all measure equivalence classes of measured foliations on the Riemann surface S. For convenience we include in $\mathcal{MF}(S)$ the empty set (with zero transversal measure). Note that if S is a triply punctured sphere then $\mathcal{MF}(S) = \{\emptyset\}$. We have a collection of height functions

$$h_\gamma = h(\gamma, \cdot) : \mathcal{MF}(S) \to \mathbb{R}_+.$$

The topology on $\mathcal{MF}(S)$ is the topology of pointwise convergence of functions h_γ on $\mathcal{C}(S)$, which is the space of homotopy classes of simple loops on S.

Theorem 11.5 below implies that there is a finite collection of simple loops $\gamma_1, \ldots., \gamma_k$ on S so that it is enough to verify convergence for the finite set of functions h_{γ_j}, $1 \le j \le k$.

Theorem 11.5. *(See [FLP79, Expose VI].) There exists a finite collection of simple loops $\Gamma := \{\gamma_1, ..., \gamma_k\}$ on S so that the height function $h : \mathcal{C}(S) \times \mathcal{MF}(S) \to \mathbb{R}$ is determined by its restriction to Γ. The space $\mathcal{MF}(S)$ is homeomorphic to $\mathbb{R}^{-3\chi(S)-n}$, where $\chi(S)$ is the Euler characteristic of S and n is the number of boundary circles.*

A measured foliation is called *uniquely ergodic* if for the corresponding singular foliation F on S, any two transversal measures are proportional. H. Masur has proved in [Mas82] that uniquely ergodic foliations form a set of full measure in $\mathcal{MF}(S)$.

Suppose now that S is a closed oriented surface and fix a complex structure on S. Let $Q(S)$ denote the space of all holomorphic quadratic differentials on S.

Theorem 11.6. *(J. Hubbard and H. Masur, see [HM79] and [Gar87].) Under the above conditions any measured foliation on S is equivalent to the foliation of vertical trajectories of a quadratic differential. Moreover the natural map*

$$q : Q(S) \to \mathcal{MF}(S)$$

is a homeomorphism.

Corollary 11.7. *Suppose that S is a compact hyperbolic surface with (possibly empty) geodesic boundary and $\mathcal{F}$ is a measured foliation on S. Embed isometrically the universal cover $\tilde{S}$ into $\mathbb{H}^2$ as a convex subset. Let $\tilde{\mathcal{F}}$ denote the lift of $\mathcal{F}$ to $\tilde{S}$. Then there is a pair (k, c) so that each line L in $\tilde{\mathcal{F}}$ is a (k, c)-quasi-geodesic in $\mathbb{H}^2$. In particular, L has exactly two accumulation points in $\partial_\infty \mathbb{H}^2$.*

Proof: (i) We first assume that S is a closed Riemann surface, then $\mathcal{F}$ is Whitehead-equivalent to a vertical foliation $\mathcal{F}'$ of a quadratic differential on S. Under the sequence of Whitehead moves relating $\mathcal{F}$ and $\mathcal{F}'$ the line L corresponds to a line L' of $\tilde{F}'$. Each Whitehead move is a homotopy-equivalence of S, hence it lifts to a quasi-isometry $W : \tilde{S} \to \tilde{S}$. Then our discussion in §5.3 implies that L' is a quasi-geodesic, hence L a quasi-geodesic as well.

(ii) If S is closed non-orientable, then the assertion follows by taking a 2-fold orientation covering over S.

(iii) Suppose that S has nonempty boundary. Take the *double* DS of the surface S along its boundary. The surface DS admits a canonical hyperbolic metric so that the isometry between the universal cover of DS and $\mathbb{H}^2$ extends the isometric embedding $\tilde{S} \hookrightarrow \mathbb{H}^2$. We double the measured foliation $\mathcal{F}$ as well and get a measured foliation $D\mathcal{F}$ on DS which contains $\mathcal{F}$ as a subfoliation. The line L is a line in the lift of $D\mathcal{F}$ to $\mathbb{H}^2$. Thus the assertion that L is a quasi-geodesic follows from the Part (ii) of the proof applied to the foliation $D\mathcal{F}$ in DS.

Recall that each quasi-geodesic in $\mathbb{H}^2$ has exactly two accumulation points in $\partial_\infty \mathbb{H}^2$ (see Corollary 3.44). □

Now, given a measured foliation $\mathcal{F}$ on S, we will construct a geodesic current on S. Let $\tilde{\mathcal{F}}$ be the lift of the foliation $\mathcal{F}$ to the hyperbolic plane. Let $E = E(\mathcal{F})$ denote the set of unordered pairs $(z, w) \in \mathcal{M}_\infty$ such that $z, w \in \partial_\infty \mathbb{H}^2$ are the end-points of a line $L_{z,w}$ of $\tilde{\mathcal{F}}$. (This line may be contained in a singular leaf and it can happen that several lines have the same end-points.) Let us verify that the set E is closed in $\mathcal{M}_\infty$. Suppose that L_{z_n,w_n} is a sequence of lines connecting the points $(z_n, w_n) \in E$ so that

$$(z, w) = \lim_{n\to\infty} (z_n, w_n).$$

Let γ_{z_n,w_n} denote the geodesic in $\mathbb{H}^2$ with the end-points z_n, w_n. Since the lines L_{z_n,w_n} are (k, c)-quasi-geodesics, the Hausdorff-distance between L_{z_n,w_n} and γ_{z_n,w_n} is bounded by a constant C independent on n. Hence convergence of γ_{z_n,w_n} to a geodesic in $\mathbb{H}^2$ (connecting z and w) implies that the sequence L_{z_n,w_n} also converges. The limit of lines in S is a line, thus the limit of the lines L_{z_n,w_n} in $\tilde{S}$ is again a line. The limiting line $L_{z,w}$ necessarily connects z and w (since $\gamma_{z,w}$ does and $d_H(\gamma_{z,w}, L_{z,w}) \le C$). This proves that E is closed. Note that if $L \subset \tilde{S}$ is a boundary arc of $\tilde{S}$ then L is not a line.

Now we construct a measure on E. For a subset $B \subset \mathcal{M}_\infty$ we shall denote by $L(B)$ the union of all lines in $\tilde{\mathcal{F}}$ which connect points of B. Suppose that K is a compact subset in $\mathcal{M}_\infty$ which corresponds to the product of two disjoint intervals in $\mathbb{S}^1$. Then there exists a compact $K' \subset \mathbb{H}^2$ such that all leaves of $\tilde{\mathcal{F}}$ connecting points of K intersect K'. Lines of $\tilde{\mathcal{F}}$ are quasigeodesics in the hyperbolic plane, therefore there exists a compact "product" neighborhood C of $(z, w) \in K$ such that $L(C) \cap K'$ does not contain any critical points. Thus all these lines in K' intersect orthogonally an interval $J \subset K'$ and the intersection $L(K) \cap J$ is closed. We declare $\mu_\infty(C)$ to be $\mu(J \cap L(C))$ where μ is the transversal measure to $\mathcal{F}$. Hence we defined the measure on compact subsets of $\mathcal{M}_\infty$ which are covered by products of small arcs in $\mathbb{S}^1$. There is a standard procedure of extension of this measure to a Borel measure on $\mathcal{M}_\infty$ which is essentially the same as the construction of the Lebesgue measure on $\mathbb{R}^n$ starting with rectangles (see for instance [Kol61]). The resulting measure μ_∞ is supported on E and is G-invariant.

The set E also has the following important property. Suppose that (x, y), $(z, w) \in E$, then the hyperbolic geodesics $\gamma_{x,y}$ and $\gamma_{z,w}$ connecting x to y and z to w do not intersect.

Remark 11.8. The reader will generalize the above discussion to the case of **measured quasifoliations**.

A measured foliation $\mathcal{F}$ on the surface S is called *rational* if it has only finitely many finite leaves connecting singular points and if all other leaves are closed. (See Figure 11.5.) Clearly this property is invariant under Whitehead moves. In the case of tori this class of foliations consists of rational foliations (where all leaves are closed).

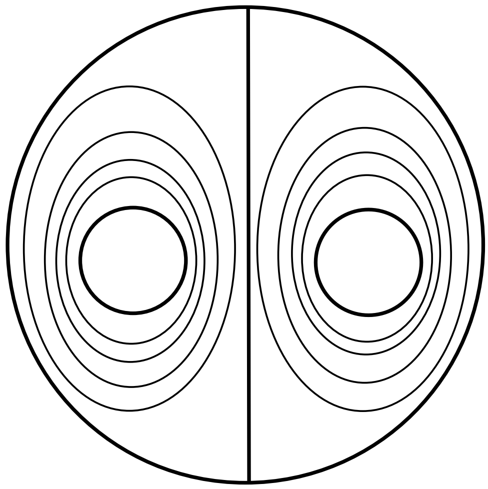

Figure 11.5: *Example of a rational foliation on the surface of genus 2: double this foliation on the pair of pants.*

Theorem 11.9. *(See [FLP79].) Rational foliations are dense in $\mathcal{MF}(S)$.*

It is easy to see that rational foliations on S are exactly those for which $E(\mathcal{F})$ is a discrete set. More generally, given a measured foliation $\mathcal{F}$ on S, the surface S admits a dynamical decomposition into a union of annuli A_i where each leaf of $\mathcal{F}$ is closed (these are called *rational components*) and subsurfaces B_i where each leaf of $\mathcal{F}$ is dense (these are called *minimal components*). The boundary of each rational and minimal component consists of a finite union of singular leaves of $\mathcal{F}$. We will prove this (in the context of measured geodesic laminations) in Lemma 11.21 and in Theorem 12.24 in the context of unions of bands.

11.4. Interval exchange transformations

In this section we describe a dynamical interpretation of measured foliations. Let $r = (r_1, r_2, ..., r_n)$ be a vector with positive coordinates and $\sigma \in S_n$ be a permutation on n symbols. Let $\chi : \{1, ..., n\} \to \{-1, 1\}$ be a map. We define the *generalized interval exchange transformation* $E_{r,\sigma,\chi}$ associated with the data (r, σ, χ) as follows:

Let $\rho = r_1 + ... + r_n$. Choose the points $x_0 = 0, ..., x_i = x_{i-1} + r_i, ..., x_n = x_{n-1} + r_n = \rho$ and the points $y_0 = 0, ..., y_i = y_{i-1} +$

$r_{\sigma(i)}, \dots, y_n = y_{n-1} + r_{\sigma(n)} = \rho$ in the interval $[0, \rho]$. Note that the intervals $[x_{i-1}, x_i]$ and $[y_{\sigma^{-1}(i)-1}, y_{\sigma^{-1}(i)}]$ have the same length. Let

$$\lambda_i : [x_{i-1}, x_i] \to [y_{\sigma^{-1}(i)-1}, y_{\sigma^{-1}(i)}]$$

be the isometry which is orientation-reserving if $\chi(i) = -1$ and is orientation-preserving otherwise. Then $E_{r,\sigma,\chi}$ is the collection of maps $\lambda_i, i = 1, ..., n$. If $\chi(i) = 1$ for each i then $E_{r,\sigma,\chi}$ is called the *interval exchange transformation.*

Apparently, interval exchange transformations were invented by V. Oseledeć in [Ose66][1] and were given the current name by M. Keane in [Kea75]. The reader can find a detailed discussion of the relation between the interval exchange transformations and the measured foliations in [Str84], [Mas82], here I will give only a sketch.

Let S be a compact hyperbolic surface and $\mathcal{F} = (F, \mu)$ be a measured foliation on S. Choose a closed interval $J \subset S$ transversal to F so that:

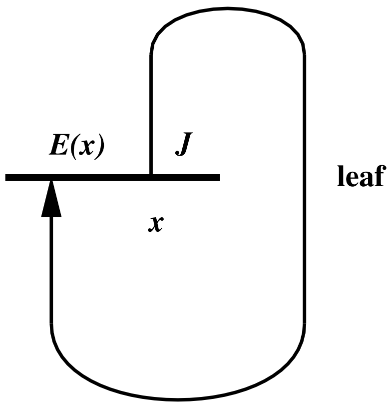

Figure 11.6: *The first return map $E(x)$.*

(a) If γ is an infinite ray in a leaf of F then $\gamma \cap J \neq \emptyset$.

(b) J does not contain singular points of F.

Existence of such interval J is clear in the case of a minimal foliation F, the general case is a bit more complicated and we will not discuss it here.

Fix a coorientation on J; this defines locally a direction on the trajectories of F emanating from the points of J. We call a point $x \in J$ *nonspecial* if the *positive* leaf of F emanating from x hits the interior of J in the positive direction before bumping into a singular point or an end-point of J. For each nonspecial point x we have a well-defined *first return* $E(x) \in int(J)$, which is the point of the first intersection of the positive leaf emanating from x with J so that the intersection is in the positive direction (see Figure 11.6).

There are only finitely many singular points of F and singular leaves leading to them from J; also there are only two end-points of J and only one direction leading to each of them. Suppose for example that two arcs

[1] At least this is the earliest reference I could find.

in the leaves of F (starting in J) before the first return enter the same singular point from the same direction. Then one of these arcs is contained in the other one, therefore at least one of them has the 1-st return to J before entering a singular point.

Thus the number of special points on J is finite. For each open maximal nonspecial interval J_i the map E is a Euclidean isometry $E_i : J_i \hookrightarrow J$ (which must preserve the orientation if S is orientable). The map E defined outside of the set of special points of J determines a generalized interval exchange map. In §12.15 we will describe how to recover the equivalence class of a measured foliation from the corresponding interval exchange transformation.

11.5. Train tracks

Let S be a compact hyperbolic surface with (possibly empty) geodesic boundary, T is a finite trivalent graph. The image τ of an embedding $T \to int(S)$ is called a *train track* in S if the following conditions are satisfied:

1. τ is C^1-smooth outside of its vertices.
2. Near each vertex the graph τ is C^1-equivalent to
$$\{(x,0), -\infty < x \leq 0\} \cup \{(x,x^2), x \geq 0\} \cup \{(x,-x^2), x \geq 0\} \quad (11.1)$$
(see Figure 11.7).
3. Suppose that C is a component of $cl(S-\tau)$, $D(C)$ is the double of C along C^1-smooth boundary edges of C (we exclude cusp points). The double $D(C)$ has negative Euler characteristic.

Vertices v_i of τ are called *switches* and edges b_j are called *branches*. Local branches near v_i are components of $\tau - v_i$ in the local model (11.1). Note that for each switch there are 3 local adjacent branches, two of them are *outgoing* (they correspond to quadratic components in (11.1)) and one is *incoming* (it corresponds to the linear component in (11.1)).

A *transversal measure* μ for τ is an assignment of nonnegative real numbers μ_j to branches b_j of τ. These numbers must satisfy the following *conservation law* in each vertex v of τ:

If b_1, b_2 are outgoing local branches at v and b_3 is the incoming branch then $\mu_1 + \mu_2 = \mu_3$.

The *support* of this measure is a subgraph of τ consisting of branches with nonzero transversal measure. The pair (τ, μ) is called *a measured* train track. Note that given τ, the space of measured train tracks (τ, μ) is a polyhedron. We will consider train tracks in S up to C^1-isotopy. A more general equivalence relation (which involves transformations similar to the Whitehead move) is required to establish equivalence between the categories of measured train tracks and measured laminations; we do not discuss it here. For a comprehensive discussion of measured train tracks see [PH92].

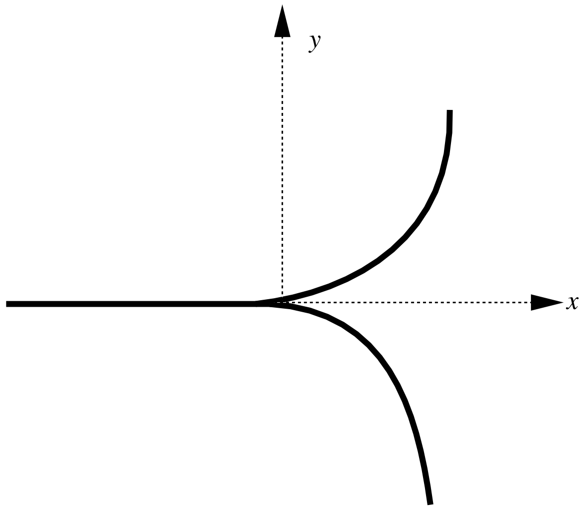

Figure 11.7: *Local picture of a train track.*

11.6. Measured geodesic laminations

Our discussion of geodesic laminations in this section mainly follows [CB88]. Let S be a compact hyperbolic surface, possibly with geodesic boundary, let $PT(S)$ denote the projectivized tangent bundle of S. A *geodesic lamination* L in S is a compact subset of $int(S)$ which is a union of complete disjoint geodesics which are called the *leaves* of L.

Remark 11.10. The reason to require the lamination to be disjoint from the boundary of S is to make the definition consistent with the definition of a measured foliation where we assume that each boundary loop is in a singular leaf. Similarly to the case of measured foliations we define a **geodesic quasilamination** by dropping the assumption that $L \subset int(S)$.

Example 11.11. A simple closed geodesic in $int(S)$ is a geodesic lamination.

Definition 11.12. A geodesic lamination L is called **maximal** if it a maximal geodesic lamination with respect to inclusion.

Consider the space $\mathcal{GL}(S)$ of all geodesic laminations on S. Since geodesic laminations are closed subsets of the metric space S, we use the Hausdorff distance to define the topology on $\mathcal{GL}(S)$. If S is closed then $\mathcal{GL}(S)$ is compact since S is compact. The case of surfaces with boundary is slightly more complicated since we assume that geodesic laminations on S are disjoint from the boundary of S.

Exercise 11.13. *Let γ be a boundary loop of S. Show that there is a constant $\epsilon > 0$ such that if a geodesic $\alpha \subset S$ contains a point x which is within the distance $\leq \epsilon$ from γ then:*

- *Either $\alpha = \gamma$.*

- *Or α crosses itself.*
- *Or some lifts of α, γ are asymptotic to each other.*

Thus there exists a compact subsurface $S_c \subset int(S)$ so that any geodesic lamination in S is contained in S_c.

Exercise 11.14. *Each geodesic lamination is contained in a maximal geodesic lamination.*

Exercise 11.15. *Suppose that L is a maximal geodesic lamination on S. Then the closure of each complementary region C in $S - L$ is either isometric to the ideal triangle in the hyperbolic plane or (in case C contains a boundary geodesic γ of S) the region C is isometric to a "crown", see Figure 11.8. In the latter case the interior of C is homeomorphic to the open annulus and has two boundary geodesics γ and α so that the opposite geodesic rays in α are asymptotic to each other.*

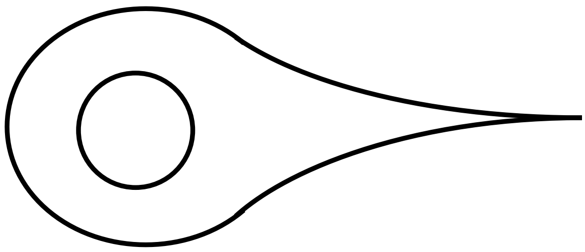

Figure 11.8: *The crown.*

Boundary leaves of L are those leaves which belong to the boundary of one of the complementary regions $S - L$. (Recall that the boundary geodesics of S do not belong to L.) Each point $x \in L$ belongs to a single leaf L_x of L. Therefore we have a well-defined map $\sigma : L \to PT(S)$. This map associates with the point $x \in L$ the tangent line to L_x at x. Using the fact that all geodesics in L are disjoint, it is easy to show that σ is a continuous map.

Lemma 11.16. *Each geodesic lamination L is a proper subset of S.*

Proof: The assertion is clear in the case of surfaces with boundary. Thus we assume that S is closed and L is a geodesic lamination on S which covers the full surface S. Then the bundle $PT(S)$ admits a globally defined continuous section σ. However the double cover of $PT(S)$ is the unit tangent bundle to S. Therefore the Euler number $e(PT(S)) = 2\chi(S)$ is nonzero. Contradiction. $\square$

Lemma 11.17. *Every geodesic lamination L on a compact hyperbolic surface S has empty interior.*

Proof: Let $\tilde{L}$ denote the lift of L to $\mathbb{H}^2$. Choose an orientation for leaves of $\tilde{L}$ which varies continuously from leaf to leaf. Fix a geodesic segment

J in $\mathbb{H}^2$, then the angles of intersection between the leaves of $\tilde{L}$ and J vary continuously. We assume that geodesics in $\tilde{L}$ are parameterized by the arc-length. Then for each point $x \in J$ we have a well-defined map $\epsilon(\bullet, t) : x \mapsto \gamma_x(t)$ where γ_x is the geodesic leaf of $\tilde{L}$ which crosses J at x. This map exponentially expands the distance and it is continuous since the section σ is continuous.

If J is entirely contained in $\tilde{L}$ and is transversal to one of the geodesics in $\tilde{L}$, then we apply to each point $x \in J$ the map $\epsilon(x, t)$. For large t the image of ϵ covers a fundamental domain of $F = \pi_1(S)$, which means that L covers the whole surface S. Contradiction. $\square$

A geodesic lamination L is called *minimal* if every ray of each leaf of L is dense in L.

Lemma 11.18. *(Properties of geodesic laminations.) Suppose that L is a geodesic lamination. Then:*

1. *The number of boundary leaves of L is finite.*
2. *The Lebesgue measure of L is equal to zero.*
3. *The intersection of L with any transversal geodesic segment is a totally disconnected subset of zero measure.*
4. *The union of all boundary leaves is dense in L.*

Proof: We prove this lemma in case of closed surfaces, the general case is proven by doubling the surface S along the boundary.

(1) The complement to L in S has finite hyperbolic area, thus it is the union of a finite number of hyperbolic surfaces with boundary which have finite area. They can have only finite number of boundary geodesics.

(2) Triangulate each complementary surface in $S - L$ using ideal triangles only. We get a new lamination L' which contains L. Extend L' in each ideal triangle to a singular foliation F with a single singular point inside of each triangle, see Figure 11.9.

The winding number of F around each singular point is equal to ± 1. Hence $-2\chi(S) = |e(PT(S))| \leq n$, where n is the number of singular points which is the same as the number of ideal triangles. The area of each ideal triangle is π. Therefore the area of $S - L$ is at least $n\pi \geq -2\pi\chi(S)$. However we know that $Area(S) = 2|\chi(S)|$; this implies that $n = -2\chi(S)$ and $Area(S) = Area(S - L)$. Thus $Area(L) = 0$.

The property (3) follows from (2). In any Cantor subset $C \subset [0, 1]$ the set of boundary points of complementary intervals is dense, this implies (4). $\square$

One can visualize neighborhoods of a geodesic lamination L using *fat train tracks*. Namely, if L is a geodesic lamination and $\epsilon > 0$ is sufficiently small, then the ϵ-neighborhood $Nbd_\epsilon(L)$ is a *fat train track* around L. Note that for a fixed ϵ we can find a very long simple closed geodesic L on S, so that the ϵ-fat train-track around L has large (non-Abelian) fundamental group. It is most convenient to see the structure of a geodesic lamination using the fundamental domain D of the group $F = \pi_1(S)$. Let $\tilde{L}$ denote the

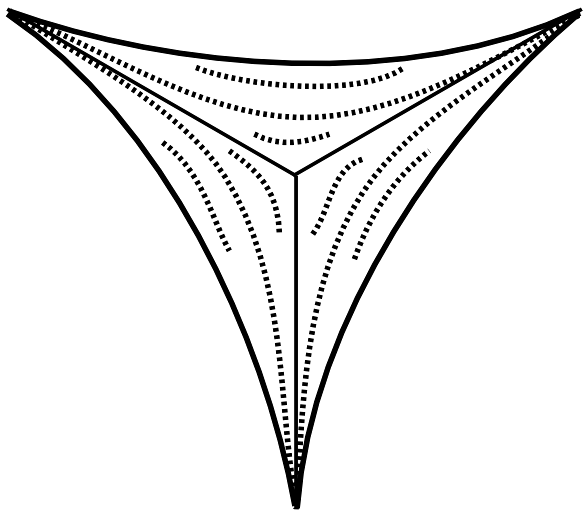

Figure 11.9: *Extension of the measured lamination.*

lift of L to the hyperbolic plane. Then $\tilde{L} \cap D$ is a disjoint union of geodesic segments. The ϵ-fat train track is the union of *rectangles* (also called the *flow boxes*). Inside of each rectangle the lamination can be completed to a vertical foliation (see Figure 11.10). For generic ϵ on the top (and bottom) of each flow box we have not more than two other flow boxes attached. The corresponding segment (top or bottom) is called a *switch*, if this number is exactly two. Suppose for concreteness than we have two rectangles R_1, R_2 attached to the top segment σ of the rectangle R_3. At the switch σ the rectangle R_3 is (locally) incoming and the rectangles R_1, R_2 are (locally) outgoing. To get the usual train tracks from fat ones we collapse each flow box along horizontal segments. The result is a finite graph in S. To get a train track structure for this graph we assign the *incoming/outgoing* labels to local branches emanating from vertices according to the labels on the original fat train track.

Note that compactness of $\mathcal{GL}(S)$ implies that there is a finite collection of *fat train tracks* $T_1, ..., T_k$ on S so that each geodesic lamination is contained in one of these tracks.

A *measured geodesic lamination* λ on S is a geodesic lamination L together with a *transversal measure* μ supported on L. The set L is called the *support* of λ. Here is one way to make this definition precise (see [PH92]). Given L we consider a collection $\mathcal{J}$ of all compact smooth 1-dimensional submanifolds in S with end-points in $S - L$ which are transversal to L. A *transversal measure* μ is a function

$$\mu : \mathcal{J} \to \mathbb{R}_+$$

which has the following properties:

- On each $J \in \mathcal{J}$ the restriction $\mu|_J$ is a (σ-additive) Borel measure.

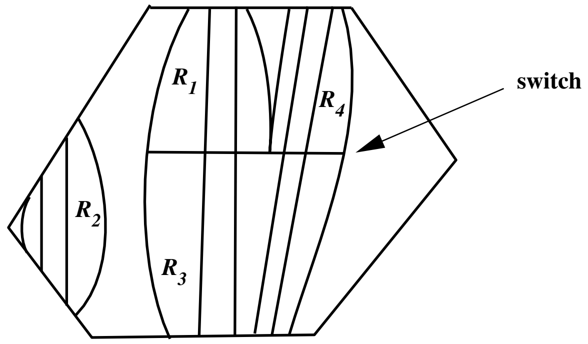

Figure 11.10: *Train tracks in the fundamental domain. Each track is the union of rectangles (*flow boxes*). Inside of each rectangle the lamination has especially simple* product *structure, where all tracks (leaves of the lamination) are almost parallel. Switch is the place where tracks in different boxes are starting to diverge. With respect to the product structure inside of each flow box, the transversal measure corresponds to a measure supported on a Cantor set which is the projection of the lamination on the bottom of the rectangle.*

- $\mu(J) = \mu(J')$ if $J, J' \in \mathcal{J}$ are isotopic through elements of $\mathcal{J}$.
- For each $J \in \mathcal{J}$, $\mu(J) > 0$ iff $J \cap L \neq \emptyset$.

Remark 11.19. Similarly one defines transversal measure for geodesic quasilaminations L by taking double DS of S across the boundary (do not double the quasilamination!). Then $L \subset DS$ is a lamination and we consider transversal measure for L on DS.

Here is another way to think about transversal measures. Cover the lamination L by a finite union of flow boxes forming a fat train track $\widehat{\tau}$. Inside of each flow box the lamination is the product of a Cantor set (or a finite set) and a vertical segment. The transversal measure corresponds to a Borel σ-additive measure whose support is this Cantor (or finite) set. Equivalently speaking, μ defines a Borel measure on any compact geodesic interval $J \subset S$ which is transversal to L. If J_t is a continuous family of such intervals so that end-points are contained in $S - L$, then the transversal measures on all J_t coincide.

After collapsing $\widehat{\tau}$ to a train track $\tau \subset S$, the transversal measure for L will produce a transversal measure for τ. (The conservation law follows from finite additivity of the transversal measure μ.) The advantage of train tracks (versus measured geodesic laminations) is that they are purely combinatorial objects (unlike laminations which are usually transcendental objects).

The space of measured geodesic laminations $\mathcal{ML}(S)$ on the surface S has a natural weak* topology given by evaluation of measures on various intervals in S. One can prove that this topology is equivalent to the topology defined via homotopy classes of simple loops on S similarly to the topology on $\mathcal{MF}(S)$. There is a natural map

$$\flat : \mathcal{ML}(S) \to \mathcal{GL}(S).$$

The problem however is that $\flat$ fails to be both injective and surjective. A measured geodesic lamination $(L, \mu) = \lambda$ is called *uniquely ergodic* if any transversal measure supported in L is proportional to μ. The simplest example of a measured geodesic lamination which is not uniquely ergodic is the one where L consists of two simple closed geodesics in S. Then the space of transversal measures for L can be naturally identified with

$$\mathbb{R}^2 \setminus (\mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}).$$

There are also minimal measured geodesic laminations which are not uniquely ergodic.

The mapping $\flat$ is also only upper semi-continuous: if (L_i, μ_i) is a sequence of measured geodesic laminations converging to (L, μ) as $i \to \infty$ then $L \subset L_\infty$ for each accumulation point L_∞ of $L_i \in \mathcal{GL}(S)$.

Lemma 11.20. *(1) Suppose that γ is a closed leaf of a measured geodesic lamination $\lambda = (L, \mu)$ on S. Then γ is isolated in L, i.e. for a small interval J transversal to γ, the intersection $J \cap L = J \cap \gamma$ is a single point.*

(2) If γ is not a closed leaf then γ is recurrent, i.e. it accumulates to itself in the both directions.

Proof: We will prove only (1), the reader should either derive the second assertion from Theorem 12.24 or read [CB88]. Suppose that γ is not isolated. Choose a lift $\tilde{\gamma}$ of γ to the universal cover $\mathbb{H}^2$. Let $\langle g \rangle$ denote the stabilizer of $\tilde{\gamma}$ in $\pi_1(S)$. There is a sequence of leaves l_n of $\tilde{L} - \tilde{\gamma}$ which is convergent to $\tilde{\gamma}$. First we suppose that end-points of all l_n are disjoint from the end-points of $\tilde{\gamma}$. If π_γ is the orthogonal projection to $\tilde{\gamma}$ then for large n the diameter of $\pi_\gamma(l_n)$ is greater than the translation length $\ell(g)$ of the hyperbolic element g. It means that $g(l_n) \cap l_n \neq \emptyset$ (see Figure 11.11).

This contradicts the property that distinct leaves of L are disjoint. Now assume that for large $n > n_0$ the end-point l_n^+ of every leaf l_n is the repulsive fixed point of g. Let J be a small transversal interval to $\tilde{\gamma}$. Then we can assume that the interval J intersects $\tilde{L}$ along the orbit $l_0 = l, l_1 = g(l), l_2 = g^2(l), ...$ The restriction of the transversal measure to J is atomic and, by invariance, the mass of each point $J \cap L_k$ is the same nonzero number. This however contradicts countable additivity of the transversal measure μ. $\square$

The above lemma gives examples of geodesic laminations L which do not admit any transversal measures supported on the whole L: take a simple closed geodesic $\gamma \subset S$ and choose an infinite geodesic α disjoint from γ which is asymptotic to γ in the both directions. Then any nonzero transversal measure whose support is contained in $L = \gamma \cup \alpha$ is actually supported only on γ. Lemma 11.20 also implies the following:

Lemma 11.21. *(Decomposition lemma.) Each measured geodesic lamination L is a disjoint union of a finite number of* **minimal** *sublaminations L_j.*

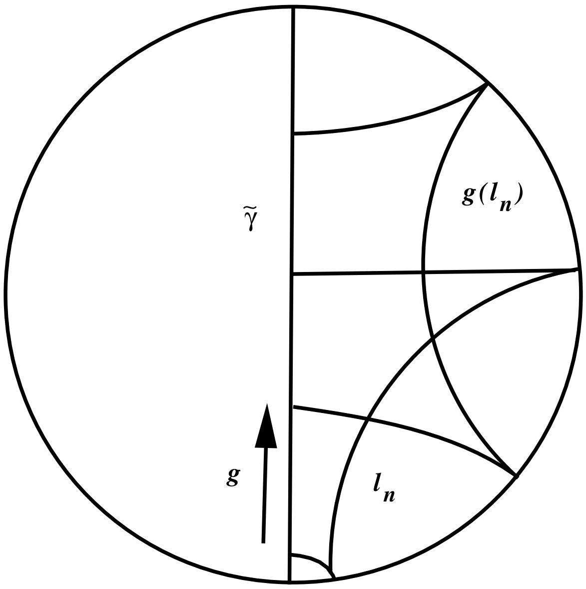

Figure 11.11:

Minimal components of L which are closed geodesics are called *periodic*, other components are called *recurrent*.

The concepts of geodesic lamination and measured geodesic lamination generalize to *blown up* geodesic lamination and *blown up* measured geodesic lamination: replace *geodesics* in the definition of geodesic lamination by *complete curves of constant*[2] *curvature which do not lift to metric circles in* $\mathbb{H}^2$.

11.7. Topology on measured laminations

Suppose that $(L,\mu) \in \mathcal{ML}(S)$ and $\widehat{\tau}$ is a fat train-track around L, $\widehat{\tau} = \cup_{j=1}^{n} B_j$ is the union of flow boxes. Then for each box B_j and horizontal interval $J_j \subset B_j$ (transversal to L) we have the number $\mu(J_j)$. If $(L_s,\mu_s) \in \mathcal{ML}(S)$ is convergent to (L,μ) then for large n the leaves of $L_s \cap B_j$ will be transversal to all J_j. Thus we will have:

$$\lim_{s\to\infty} \mu_s(J_j \cap L_s) = \mu(J_j) \tag{11.2}$$

One can prove the opposite as well. Namely, if L_s is Hausdorff convergent to L and the equality (11.2) above holds, then $(L_s,\mu_s) \to (L,\mu)$.

Definition 11.22. A pair of measured geodesic laminations α, β is **binding** the surface S if

$$i(\gamma,\alpha) + i(\gamma,\beta) > 0$$

for any nonzero geodesic current γ on S.

[2]The constant depends on the leaf.

A pair of simple closed geodesics α, β is *binding* S if and only if $S - (\alpha \cup \beta)$ consists only of simply-connected components and annular neighborhoods of boundary loops. Take a binding pair of measured geodesic laminations α, β. We will consider the sum $\gamma = \alpha + \beta$ of the corresponding *geodesic currents*. Then γ *binds* the surface S.

Lemma 11.23. *(F. Bonahon [Bon88].) If γ is a binding geodesic current then the set*

$$C = \{\lambda \in \mathcal{ML}(S) : i(\lambda, \gamma) \le 1\}$$

is compact.

Proof: The set C is clearly closed, so, by the definition of the weak* topology, it suffices to show that for every compact subset $K \subset \mathcal{M}_\infty$ the set $\{\mu(K), \lambda = (L, \mu) \in C\}$ is bounded in $\mathbb{R}$. Since the set K is compact and γ is binding, we can cover K by a finite number of open neighborhoods U_g of geodesics $g \in K$ so that for each g there exists a small open neighborhood U_h of a geodesic h (whose end-point set is contained in the support of γ) such that:

- Each pair of geodesics $g' \in U_g, h' \in U_h$ have nonempty transversal intersection.
- $\gamma(U_h) > 0$.
- $U_g \cap U_h$ project injectively to $\mathcal{M}^2_\infty / G$.

Thus $1 \ge i(\lambda, \gamma) \ge \gamma(U_h) \cdot \mu(U_g)$, which implies that $\mu(U_g) \le [\gamma(U_h)]^{-1}$. $\square$

11.8. From foliations to laminations

The concepts: *measured foliations* and *measured laminations* are essentially equivalent. Below we construct a natural homeomorphism between the spaces $\mathcal{MF}(S)$ and $\mathcal{ML}(S)$ (an alternative description of this homeomorphism is contained in [Lev83]). Suppose that $\mathcal{F}$ is a measured foliation in S with the lift $\tilde{\mathcal{F}}$ to the hyperbolic plane. Recall that in §11.3 we defined a closed subset $E = E(\mathcal{F}) \subset \mathcal{M}_\infty$ (which is the set of *end-points* of leaves of $\tilde{\mathcal{F}}$) and a G-invariant measure (geodesic current) μ_∞ supported on E.

Now connect all points (z, w) in $E = E(\mathcal{F})$ by hyperbolic geodesics $\gamma_{z,w}$. The union of these geodesics is a G-invariant closed subset $\tilde{L}$ in $\mathbb{H}^2$. No distinct geodesics in this subset intersect and if γ is a lift of a boundary geodesic of S then γ is disjoint from the closure of $\tilde{L}$.

Therefore $\tilde{L}$ is the lift to the universal cover $\tilde{S}$ of a geodesic lamination L in S. It remains to construct a transversal measure ν with the support L. To do this we *push* the measure μ_∞ back to the hyperbolic plane. Namely, let J be a compact geodesic arc in $\mathbb{H}^2$ which is transversal to $\tilde{L}$. Consider the union of all leaves of $\tilde{L}$ which intersect J. This union corresponds to a closed subset $\phi(J \cap \tilde{L}) \subset E(\mathcal{F}) \subset \mathcal{M}_\infty$. Moreover, there exists a homeomorphism $\phi : J \cap \tilde{L} \to \phi(J \cap \tilde{L})$. Then define the transversal measure ν on J to be the pull back of μ_∞ via ϕ from $E(\mathcal{F})$. It is easy to see that this defines a transversal measure for L.

11.9. From laminations to foliations

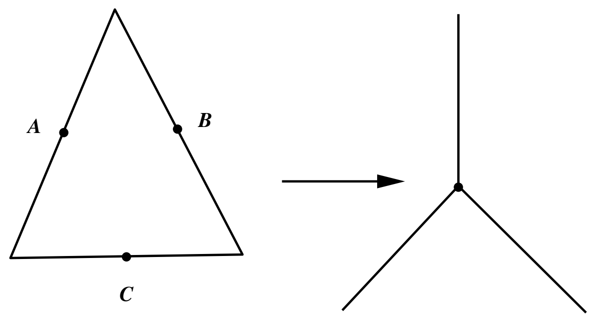

Figure 11.12: *Collapse of complementary triangles.*

We now go back from laminations to foliations. Let us first consider the case when $\lambda = (L, \mu)$ is a measured lamination on S such that L is a maximal geodesic lamination. Take a fat train-track neighborhood U of L. Inside of U the lamination λ has an obvious extension to a measured foliation $\mathcal{F}'$, namely each flow box $U_j = I_j \times I$ has a coordinate system so that $\lambda = C_j \times I$, where C_j is a Cantor set. Use a monotone continuous *Cantor function* $f_j : C_j \to I_j$ which is a homeomorphism everywhere except at boundary points of complementary intervals. We get a collection of continuous mappings $(f_j, id) : C_j \times I \to U_j$ which define a continuous map from L to a foliation of U by the *vertical segments* $\{t\} \times I \subset U_j$.

Take a complementary triangle $T \subset S - U$. Choose three points A, B, C on different sides of T which are not vertices. Then we collapse the triangle T in S to a tripod so that A, B, C are collapsed to a single vertex of the tripod. (See Figure 11.12.)

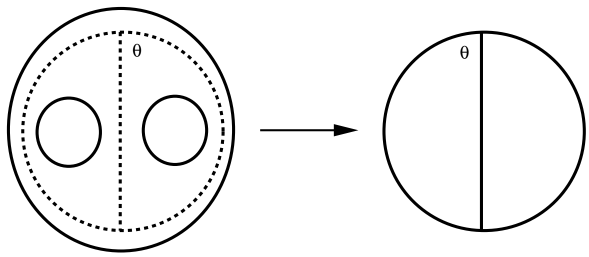

Figure 11.13: *Collapse of pairs of pants.*

The result has a natural structure of a singular foliation $\mathcal{F}$. We also get a natural map $L \to \mathcal{F}$, which is a homeomorphism everywhere except on the boundary leaves. Hence we can push-forward the transversal measure μ to $\mathcal{F}$ and get a measured foliation.

What to do in the case when L is not a maximal lamination? If $L = \emptyset$ we let $\mathcal{F} := 0$. Otherwise we do the following. First of all add disjoint simple closed geodesics $\gamma_i \subset int(S)$ to L so that each component of $S' = S - (\partial S \cup L \cup \cup_i \gamma_i)$ is either diffeomorphic to a disk or annulus or pair of pants. If C is a component of S' which is bounded by three closed geodesics we collapse C to the "θ-graph" θ embedded in C so that $C-\theta$ is the disjoint union of three annuli. See Figure 11.13.

In the case C has at least one ideal boundary vertex we add to C disjoint simple geodesics (asymptotic to the ideal vertices of C) to triangulate C into the union of ideal triangles and crowns. (Some of the crowns may be adjacent to the boundary loops of S.) Collapse each ideal triangle on ideal tripod as in the case of maximal laminations.

Then we collapse each crown onto the union of the boundary circle and the geodesic ray asymptotic to its vertex, see Figure 11.14.

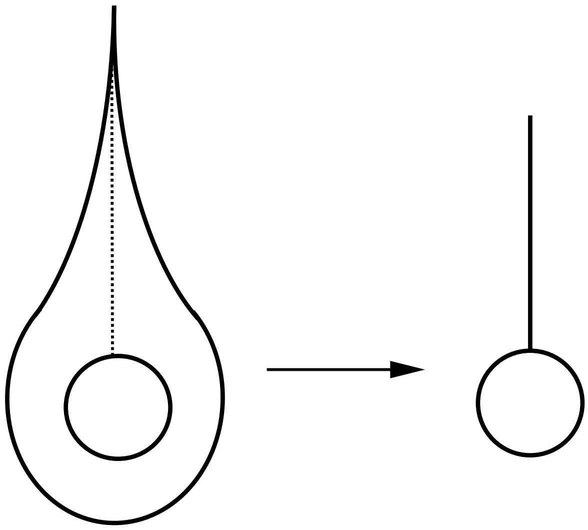

Figure 11.14: *Collapse of complementary crowns.*

Finally split S open along the isolated closed geodesics γ in L and insert foliated annuli $U(\gamma)$ with the Lebesgue transversal measure so that the total mass of this measure is $\mu(J)$, where J is a small transversal segment to γ. The result is a measured foliation $\mathcal{F}$.

The map $(L,\mu) \to \mathcal{F} \to E(\mathcal{F}) \in \mathcal{GC}(S)$ defines a homeomorphic embedding of $\mathcal{ML}(S)$ to the space of geodesic currents $\mathcal{GC}(S)$.

Remark 11.24. We leave it to the reader to generalize the above correspondence to a map from measured quasifoliations to measured quasilaminations. Note that in this case the correct notion of equivalence relation on the space of measured quasifoliations is the Whitehead equivalence.

11.10. Action of Mod_S on geodesic laminations

Recall that each element of the Teichmüller modular group Mod_S is represented by a quasiconformal diffeomorphism. Diffeomorphisms of S naturally act on $\mathcal{MF}(S)$ by *push-forward* of transversal measures. (This means that for any $\mathcal{F} \in \mathcal{MF}(S)$ and $h \in Diff(S)$ we let $h(\mathcal{F})$ be the image of the foliation $\mathcal{F}$ and to calculate the transversal measure of J with respect to $h(\mathcal{F})$ we take the transversal measure of $h^{-1}(J)$ with respect to $\mathcal{F}$.)

Using geodesic currents one can describe this action as follows. Let $S = \tilde{S}/\Gamma$ where $\tilde{S}$ is the universal cover of S which is realized as a convex domain in $\mathbb{H}^2$ and $\Gamma \subset PSL(2,\mathbb{R})$ is a discrete torsion-free subgroup. Let $E(\mathcal{F})$ be the geodesic current corresponding to $\mathcal{F}$, its support set is contained in the projection $\Lambda^2(\Gamma)$ of the square of the limit set $\Lambda(\Gamma)$ to $\mathcal{M}_\infty$. Any quasiconformal homeomorphism $h : S \to S$ lifts to a quasiconformal homeomorphism $\tilde{h} : \tilde{S} \to \tilde{S}$ which conjugates Γ to itself. Thus $\tilde{h}$ extends to the homeomorphism $h_\infty : \Lambda(\Gamma) \to \Lambda(\Gamma)$ (see Theorem 8.16). We get the "product" homeomorphism $(h_\infty, h_\infty) : \Lambda^2(\Gamma) \to \Lambda^2(\Gamma)$. Then $E(h(\mathcal{F})) = (h_\infty, h_\infty)(E(\mathcal{F}))$ where (h_∞, h_∞) acts as push-forward of measures on $\mathcal{M}_\infty$.

We conclude that the action of h depends only on the homotopy class of h. Therefore we get a well-defined action of the modular group Mod_S on $\mathcal{MF}(S)$. We leave it to the reader to verify that the group Mod_S acts by homeomorphisms on $\mathcal{MF}(S)$. (Use the embedding $\mathcal{F} \to E(\mathcal{F}) \in \mathcal{G}C(S)$ and the fact that elements of Mod_S act as homeomorphisms on the unit circle.)

11.11. The geometric intersection number

Let $\mathcal{C}(S)$ denote the set of geodesic laminations on S which are simple closed (nonperipheral) geodesics. We shall use the notation $\mathcal{MC}(S)$ for the collection of measured laminations whose support sets are the elements of $\mathcal{C}(S)$.

Theorem 11.25. *$\mathcal{MC}(S)$ is dense in $\mathcal{ML}(S)$.*

Proof: See [ECG87], [PH92]. □

Suppose that α, β are two simple closed geodesics on S. We assume that they are assigned the *counting* transversal measure, i.e. if $\gamma = \alpha$ or $\gamma = \beta$ and J is an arc in S with end-points in $S - \gamma$, then the transversal measure of J is the number of points of transversal intersection between J and γ. We define the *geometric intersection number* $i(\alpha, \beta)$ to be the number of points of transversal intersection between α and β. By linearity the function i extends to the function

$$i : \mathcal{MC}(S) \times \mathcal{MC}(S) \to \mathbb{R}.$$

Theorem 11.26. *(W. Thurston, F. Bonahon.) There exists a continuous extension of i from $\mathcal{MC}(S)$ to $\mathcal{ML}(S)$. This extension is obtained by restricting the intersection number i from $\mathcal{G}C(S)$ to $\mathcal{ML}(S)$ (see Theorem 11.1).*

Proof: See [Bon86], [PH92]. □

This extension is also called the *(geometric) intersection number* and is denoted by i.

11.12. From laminations to trees

Let S be a compact hyperbolic surface with (possibly empty) geodesic boundary, $\tilde{S} \subset \mathbb{H}^2$ is the universal cover of S and $G \cong \pi_1(S)$ is the group of covering transformations for $\tilde{S} \to S$. We first define the *blow up* of a measured geodesic lamination $\lambda = (L, \mu)$ (recall that L is the support set of μ).

Definition 11.27. For each isolated leaf γ of L we do the following. Take a small tubular neighborhood $U(\gamma)$ of γ so that $U(\gamma) \cap L = \gamma$ and foliate $U(\gamma)$ by closed curves at constant distance from γ. We give the foliated annulus $U(\gamma)$ the Lebesgue transversal measure so that the total mass of this measure is $\mu(J)$ (where J is sufficiently small arc transversal to γ). The resulting measured lamination $\lambda' = (L', \mu')$ of S (by curves of constant curvature) is called the **blow up** of λ.

Let $(\tilde{L}, \tilde{\mu}), (\tilde{L}', \tilde{\mu}')$ denote the lifts of λ, λ' to the universal cover $\tilde{S}$ of S. We introduce the following pseudo-metric $\tilde{d}_\lambda$ on $\tilde{S}$. Given any two points p, q we connect them by various piecewise-geodesic curves $\alpha(p,q)$ and minimize the transversal $\tilde{\mu}'$-measure of $\alpha(p,q)$. Let us verify that this minimum is always realized. We construct a metric ρ of nonpositive curvature on $\tilde{S}$ so that the complement to $\tilde{L}'$ is isometric to $\tilde{S} - \tilde{L}$, the restriction of ρ to the foliated strips of $\tilde{L}'$ is flat and the leaves of $\tilde{L}'$ are geodesics. Then geodesic segments of ρ give minimizing curves for the pseudo-distance $\tilde{d}_\lambda$.

Now we identify all pairs of points x, y in $\tilde{S}$ so that $\tilde{d}_\lambda(x,y) = 0$. The quotient is a metric space T_λ with the metric d_λ. It is easy to see that we identify two points if and only if :

- either they are in the same leaf of $\tilde{L}'$,
- or they are in the same complementary component $\tilde{S} - \tilde{L}'$.

The quotient topology coincides with the topology given by the metric d_λ. If J is an interval transversal to $\tilde{L}'$, then the projection $J \to T$ is given by a *Cantor* function $J \to T$ which is constant on all components of $J - \tilde{L}'$ and whose image is homeomorphic to J. It follows that each simple path in T can be lifted to an embedded piecewise-geodesic curve in $(\tilde{S}, \rho)$. Therefore T is a metric tree: each triangle $[abc]$ is a tripod (in the degenerate case it could be a segment), see Figure 11.15 where $[abc]$ is the projection of the triangle $[\tilde{a}\tilde{b}\tilde{c}] \subset \tilde{S}$.

The tree $T = T_\lambda$ is called the *dual tree to the lamination* λ. The action of the group $G = \pi_1(S)$ projects to an isometric action of G on the tree T. It is easy to see that this action is minimal: T is the union of axes of elements of G acting on T, the tree T is geodesically complete. Note that the action of each peripheral element of G on T is elliptic since boundary geodesics of S do not belong to L.

Definition 11.28. Let S be a compact surface with the fundamental group G. The action of G on a tree T is called **relatively elliptic** if each peripheral element of G has a fixed point in T. In this case the G-tree T is also called **relatively elliptic**.

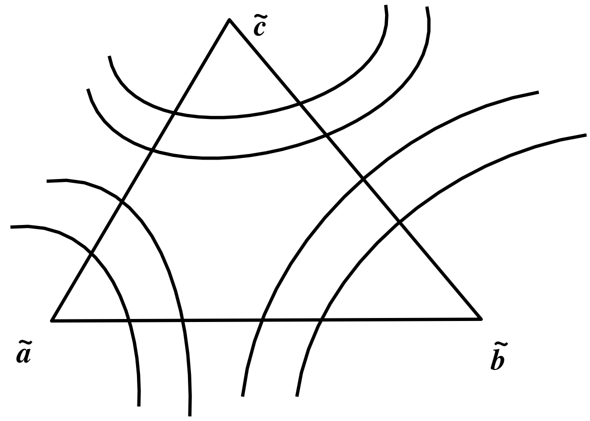

Figure 11.15: *Triangle in $\tilde{S}$ projects to a tripod in T.*

Lemma 11.29. *The action of G on T is small.*

Proof: Suppose that J is an interval in T which is fixed by a nontrivial subgroup $G' \subset G$. Then for generic $x \in J$ the preimage of x in $\tilde{S}$ is a single leaf of the lamination $\tilde{L}'$. This leaf is invariant under G'. Therefore G' is cyclic. $\square$

For example, take a pair α, β of simple closed geodesics on S which we regard as measured laminations using the Dirac transversal measure of the unit total mass. Let $g = g_\alpha$ denote an element of G corresponding to the loop α.

Exercise 11.30. *Under the above conditions $i(\alpha, \beta)$ equals the translation length $\ell_{T_\beta}(g)$ of the element g acting on the tree T_β.*

The following important theorem was proven by R. Skora in [Sko96], other proofs were given by J.-P. Otal [Ota96] and by B. Farb and M. Wolf [FW98]. Yet another proof is implicitly contained in the paper by F. Luo [Luo98] who gives a characterization of translation length functions $\ell_T : G \to \mathbb{R}$ for fundamental groups of hyperbolic surfaces which correspond to the trees T dual to measured laminations.

Theorem 11.31. *(Skora's Duality Theorem.) Suppose that T is a small minimal* **relatively elliptic** *G-tree (see Definition 11.28). Then there exists a measured geodesic lamination $\lambda \in \mathcal{ML}(S)$ so that its dual tree T_λ is equivariantly isometric to T.*

We will present a proof of Skora's Duality Theorem in the Chapter 12 as an application of the Rips' Theory (following [Bes97]).

Corollary 11.32. *There is a finite collection of elements $\gamma_1, ..., \gamma_k$ of G so that the equivariant isometry class of any small minimal relatively elliptic G-tree T is uniquely determined by the translation lengths $(\ell_T(\gamma_1), ..., \ell_T(\gamma_k))$.*

Let $\mathcal{T}ree(G)$ be the space of equivariant isometry classes of small minimal relatively elliptic actions of G on metric trees. Each element $T \in \mathcal{T}ree(G)$ corresponds to a *length* function $\ell_T : G \to \mathbb{R}_+$. We topologize $\mathcal{T}ree(G)$ using the product topology on the space of functions

$$G^{\mathbb{R}} = Maps(G, \mathbb{R}).$$

As we saw, it is enough to verify convergence on a finite number of elements of G. This implies

Theorem 11.33. *The correspondence $\mathcal{ML}(S) \ni \lambda \mapsto T_\lambda \in \mathcal{T}ree(G)$ is a homeomorphism.*

We define the projectivization $\mathcal{P}\mathcal{T}ree(G)$ of $\mathcal{T}ree(G)$ by identifying the nontrivial G-trees which are obtained from each other by rescaling the metric. Then $\mathcal{P}\mathcal{T}ree(G)$ is homeomorphic to a sphere.

11.13. An application of Skora's theorem

Let S be a compact hyperbolic surface with geodesic boundary, $G = \pi_1(S)$. Assume that we have a *relatively elliptic* action $G \curvearrowright T$ of G on a tree T (i.e. each element of G corresponding to a peripheral loop on S acts as an elliptic isometry of T). Suppose also that T is a small minimal G-tree. Recall that if α is a loop in S then g_α denotes the corresponding element of G. (Of course, g_α is well defined only up to conjugation and only after choice of an orientation on α. However since we are mainly interested in $\ell_T(g_\alpha)$ this does not matter.)

Theorem 11.34. *Consider the collection C_T of the conjugacy classes of all elements $g \in G - \{1\}$ which have fixed points in T. Then there exists a (possibly disconnected) subsurface $B \subset S$ with incompressible boundary such that $g \in C_T$ iff $g = g_\alpha$ for a loop α in B. The subsurface B is called the* **maximal elliptic subsurface** *in S.*

Proof: According to Skora's Theorem (theorem 11.31), there exists a measured geodesic lamination $\lambda = (L, \mu)$ on S so that $T = T_\lambda$. Let $\tilde{L}$ be the lift of L to the universal cover $\tilde{S} \subset \mathbb{H}^2$. It is clear that for every hyperbolic element $g \in G$, whose axis $A_g \subset \mathbb{H}^2$ intersects $\tilde{L}$ transversally, the projection of A_g to T_λ is a geodesic on which g acts as a translation. Thus, g acts on T as an elliptic element if and only if one of the following is satisfied:

- Either the axis $A_g \subset \mathbb{H}^2$ of g is a leaf of $\tilde{L}$.
- Or A_g is contained in a component on $\mathbb{H}^2 - \tilde{L}$.

Let $\lambda' = (L', \mu')$ denote the *blow up* of λ. In each non-contractible component C_j of $S - L'$ we take a compact subsurface B_j which is a deformation retract of C_j. The union of these subsurfaces with the collection of annuli $L' - L$ is B. $\square$

11.14. A characterization of aperiodic homeomorphisms

As we saw in the §11.1, each hyperbolic element α of the classical modular group $SL(2,\mathbb{Z})$ can be represented by an *Anosov* diffeomorphism A of the torus T^2. The diffeomorphism A has two invariant foliations: one stable (attractive) and the other unstable (repulsive). Thurston proved that a similar statement holds for homeomorphisms of hyperbolic surfaces. In this section we sketch a proof of Thurston's theorem following the approach of L. Bers [Ber78].

Let S be a complete oriented hyperbolic surface of finite area which has the genus g and m punctures. Take an *aperiodic* orientation-preserving homeomorphism $h : S \to S$ (see §1.9). Then h acts as an element h_* of the modular group Mod_S on the Teichmüller space $\mathcal{T}(S)$.

Theorem 11.35. *(L. Bers [Ber78].) The isometry h_* of the Teichmüller space has an invariant geodesic (with respect to the Teichmüller metric).*

Proof: I give a proof following [Ber78] and [Abi80]. Let $\ell(g)$ denote the *translation length* of the element $g \in Mod_S$ acting on $\mathcal{T}(S)$, i.e.

$$\ell(g) = \inf_{z \in \mathcal{T}(S)} d_{\mathcal{T}(S)}(z, g_*(z)).$$

The number of loops in a collection of pairwise disjoint simple closed geodesics in S is at most $N = 3g - 3 + m$. Our first goal is to show that $\ell(h)$ is realized in $\mathcal{T}(S)$. Let $r : \mathcal{T}(S) \to \mathcal{M}(S)$ denote the projection from the Teichmüller space to the moduli space of S. Choose the origin $z_0 \in \mathcal{T}(S)$ and let $\log(K) := 2d_{\mathcal{T}(S)}(z_0, h_*(z_0))$. Let $z_n \in \mathcal{T}(S)$ be a sequence so that

$$\lim_{n\to\infty} d_{\mathcal{T}(S)}(z_n, h_*(z_n)) = \ell(h).$$

If the projection $r(z_n)$ converges to $w \in \mathcal{M}(S)$ then there is a sequence $f_n \in Mod_S$ so that $x_\infty = \lim_{n\to\infty} f_n(z_n) \in r^{-1}(w)$ and thus $f_n \circ h_n \circ f_n^{-1}$ converges uniformly on compacts in $\mathcal{T}(S)$ to $g \in Mod_S$. Since the action of Mod_S on $\mathcal{T}(S)$ is discrete we conclude that $f_n \circ h_n \circ f_n^{-1} = g$ for large n. Moreover, $d_{\mathcal{T}(S)}(x_\infty, g_*(x_\infty)) = \ell(h) = \ell(g)$, which means that $\ell(h)$ is realized in $\mathcal{T}(S)$.

Thus we consider the case when $r(z_n)$ does not subconverge in $\mathcal{M}(S)$. Recall that $\mathcal{M}_\epsilon(S)$, which is the collection of complete hyperbolic structures of finite area on S where the length of the shortest closed geodesic is at least ϵ, is compact according to Mumford Compactness Theorem (theorem 5.9). Thus for large n the hyperbolic surface S_n (which corresponds to $r(z_n)$) contains a simple closed geodesic γ of the length $\epsilon < \mu_2/K^N$, where μ_2 is the Margulis constant for $\mathbb{H}^2$. If two closed geodesics on S_n have length $< \mu_2$ then they are disjoint (or equal). Recall that $d_{\mathcal{T}(S)}(z_n, h_*(z_n)) < \log(K)$, thus according to Theorem 8.57, for each $k \le N$ the length of the image of a closed geodesic γ_k^* representing $h^k(\gamma)$ in S_n is at most $K^k\epsilon < \mu_2$. Since h is aperiodic, the simple closed geodesics $\gamma, \gamma_1^*, ..., \gamma_N^*$ are disjoint. Contradiction.

Thus there is a point $O \in \mathcal{T}(S)$ so that $d(O, h_*(O)) = \ell(h)$ is minimal. If $h_*(O) = O$, then h acts as a conformal automorphism of O and hence $h_* : \pi_1(S) \to \pi_1(S)$ has finite order. This contradicts the assumption that h is aperiodic. Otherwise, $O \neq h_*(O)$ and the geodesic in $\mathcal{T}(S)$ through the points $O, h_*(O)$ is h-invariant. $\square$

Theorem 11.36. *(A. Marden, K. Strebel, [MS93].) Any aperiodic orientation-preserving homeomorphism $h : S \to S$ has* **unique** *invariant geodesic in $\mathcal{T}(S)$.*

Consider the point O in the invariant geodesic $L \subset \mathcal{T}(S)$ as the origin S of the Teichmüller space $\mathcal{T}(S)$. Assume $f : S \to S$ is the Teichmüller map homotopic to h. It follows that f^2 is also a Teichmüller map and $K(f^2) = K(f)^2$. According to the description of extremal quasiconformal maps (see §5.4), there exists a holomorphic quadratic differential ϕ on S such that the point (S, f) on L is given by the Beltrami differential

$$\mu = t\overline{\phi}/|\phi|, t \in \mathbb{R}.$$

Local calculations imply that the Beltrami differential of f^{-1} is given by

$$-t\overline{\phi}/|\phi|, t \in \mathbb{R}.$$

It follows that *final* and *terminal* quadratic differentials of the map f coincide (up to a positive multiple).

Recall what the above assertion means for the points $O, h_*(O)$. Let K be the coefficient of quasiconformality of f. Assume that the conformal structure on S is given by O. Then the quadratic differential ϕ defines a flat metric ρ_ϕ with conical singularities on S and a measured foliation $\mathcal{F}^-$ consisting of vertical trajectories of ϕ. The set of singular points $C(\phi)$ of this foliation is invariant under the solution f of the Beltrami equation

$$\overline{\partial}h/\partial h = t\overline{\phi}/|\phi|.$$

The restriction of the map h to $S - C(\phi)$ is affine with respect to the flat metric ρ_ϕ. It stretches the distance by $k = \sqrt{K} > 1$ in the horizontal direction and squeezes it by k^{-1} in the vertical direction. The surface S has also the second measurable foliation $\mathcal{F}^+$ given by the horizontal trajectories of ϕ (or equivalently by the vertical trajectories of the differential $i\phi$). The foliations $\mathcal{F}^+$ and $\mathcal{F}^-$ are transversal (in nonsingular points), have the same set of singular points and have no common leaves.

Definition 11.37. If $\mathcal{F}_1, \mathcal{F}_2$ are measured foliations on a surface S then they are called **transversal** if their singular sets are equal, $\mathcal{F}_1$ is transversal to $\mathcal{F}_2$ away from the singular set and near each singular point the pair $(\mathcal{F}_1, \mathcal{F}_2)$ looks like the pair $(\mathcal{F}^+, \mathcal{F}^-)$ above.

Considering $\mathcal{F}^\pm$ as measured foliations we conclude that:

$$h(\mathcal{F}^+) = k\mathcal{F}^+, \quad h(\mathcal{F}^-) = k^{-1}\mathcal{F}^-, \text{ where } k > 1 \tag{11.3}$$

(see §11.10 for the description of the action of Mod_S on measured foliations). If the surface S is not compact then the quadratic differential ϕ has at worst simple poles in the punctures of the surface S. Hence we can compactify the surface S by adding one circle for each puncture so that $\mathcal{F}^{\pm}$ extend to measured foliations $\bar{\mathcal{F}}^{\pm}$ on the resulting compact bounded surface $\overset{\bullet}{S}$.

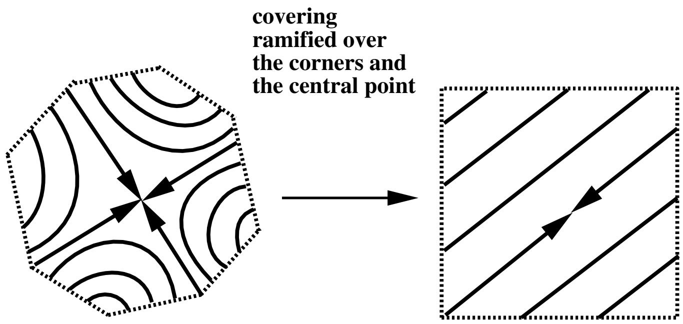

Figure 11.16: *Surface S of genus 2 is a 2-fold ramified covering over the torus with irrational foliation $\mathcal{F}$. The singular foliation $\tilde{\mathcal{F}}$ on S has 2 critical points, one of them is the center of the fundamental domain, the other one corresponds to the corner. The foliation $\mathcal{F}$ is the stable foliation of an Anosov diffeomorphism f of the torus. f lifts to a pseudo-Anosov homeomorphism of S with the stable foliation $\tilde{F}$.*

Let S be a compact hyperbolic surface (possibly with nonempty geodesic boundary).

Definition 11.38. A homeomorphism $h : S \to S$ is called **pseudo-Anosov** if there are two measured foliations $\mathcal{F}^+, \mathcal{F}^-$ such that the conditions (11.3) are satisfied, the foliations $\mathcal{F}^+, \mathcal{F}^-$ are transversal and h is a diffeomorphism outside of the singular set of these foliations. The foliation $\mathcal{F}^+$ is called **stable** and the foliation $\mathcal{F}^-$ is called **unstable** foliation of h. The number k is called the **stretch factor** of h.

Thus we have the following:

Theorem 11.39. *(W. Thurston.) Suppose that S is a compact oriented hyperbolic surface with geodesic boundary and $h : S \to S$ is an orientation-preserving aperiodic homeomorphism. Then h homotopic to a pseudo-Anosov homeomorphism.*

Remark 11.40. The assumptions that S is oriented and h is orientation-preserving are unnecessary, however we will not need the more general result.

Theorem 11.41. *(See [FLP79].) Stable and unstable foliations of any pseudo-Anosov homeomorphism are uniquely ergodic.*

There are numerous examples of pseudo-Anosov homeomorphisms of surfaces, in a sense any *generic* element of Mod_S is aperiodic. The simplest example is given by branched coverings over the 2-torus $S \to T$, where we lift an *Anosov* diffeomorphism $A : T \to T$ to a pseudo-Anosov homeomorphism of S, see Figure 11.16.

Lemma 11.42. *If $\mathcal{F} = F^+$ is stable foliation of a pseudo-Anosov homeomorphism $f : S \to S$ then no leaf of $\mathcal{F}$ connects singular points and no leaf of $\mathcal{F}$ connects a singular point to itself.*

Proof: Suppose that $I \subset S$ is an interval in a leaf of $\mathcal{F}$ which connects critical point x and y. Let $\mathcal{F}^- = (F, \mu)$ be the unstable foliation of f. Then $\mu(f(I)) = k\mu(I)$. However there exists $n > 0$ such that $f^n(I) = I$ since there are only finitely many singular points and leaves which connect them. Contradiction. $\square$

Theorem 11.43. *Any pseudo-Anosov homeomorphism $f : S \to S$ of compact hyperbolic surface with (possibly empty) geodesic boundary, is aperiodic. The stable/unstable foliations $\mathcal{F}^+, \mathcal{F}^-$ of f do not have closed leaves.*

Proof: Let $\mathcal{F}^+, \mathcal{F}^-$ be the stable/unstable pair of foliations of f with the stretch factor $k > 1$. Note that for any nonperipheral loop γ on S,

$$h_\gamma(f_*\mathcal{F}^+) = kh_\gamma(\mathcal{F}^+), \quad h_{f_*(\gamma)}(\mathcal{F}^+) = kh_\gamma(\mathcal{F}^+)$$

where $h_\gamma(\bullet)$ is the hight function. Thus $f_*^m(\gamma)$ is never homotopic to γ unless $m = 0$ or $h_\gamma(\mathcal{F}^+) = 0$. However if $h_\gamma(\mathcal{F}^+) = 0$ then γ is homotopic to a loop γ' in the foliation $\mathcal{F}^+$. Let $\gamma := \gamma'$. There are two possible cases.

Case A. The leaf γ is not critical. Then γ is transversal to the foliation $\mathcal{F}^-$. There are only finitely many maximal foliated annuli in the foliation $\mathcal{F}^+$, thus by taking if necessary sufficiently large power of f, we get: $f(\gamma)$ is a nonsingular leaf isotopic to γ through leaves of $\mathcal{F}^+$. Then $h_{f(\gamma)}(\mathcal{F}^-) = h_\gamma(\mathcal{F}^-) \neq 0$ since $\mathcal{F}^+$ (and in particular γ) is transversal to $\mathcal{F}^-$. This is impossible.

Case B. The loop γ is contained in a critical leaf L. Then L contains an arc which connects two singular points which contradicts the previous lemma. $\square$

Other classes of homeomorphisms.

It is clear from the proof of Theorem 11.35 that any orientation-preserving homeomorphism $h : S \to S$ is either *aperiodic*, or h^m is homotopic to the identity for certain $m \geq 1$ (i.e. h is homotopic to a *periodic* homeomorphism of S), or h preserves (up to isotopy) a collection of simple disjoint homotopically nontrivial nonperipheral loops in S, i.e. h is *reducible.* This classification of the elements of Mod_S is called *Nielsen-Thurston* classification.

The important example of a homeomorphism in the latter class is the *Dehn twist.* Let γ be a smooth simple nonperipheral homotopically nontrivial loop in S. Let U be a tubular neighborhood of γ in S. We will identify U with the annulus $A = \{z \in \mathbb{C} : 1 \leq |z| \leq 2\}$. Define the map $h : A \to A$ by the formula:

$$h(re^{2\pi i\theta}) = r \cdot e^{2\pi i(\theta + r)}.$$

Thus h is the identity on each boundary circle of A. It is clear however that h is not homotopic to the identity (rel. ∂A). Now extend h from $A = U$ to the rest of the surface S as the identity map. We get a map $D_\gamma : S \to S$ which is called the *Dehn twists* along γ. Any homeomorphism $f : S \to S$ which is homotopic to a Dehn twist is also called a Dehn twist.

Exercise 11.44. *Show that for each $n > 0$ the iterated Dehn twist $h^n : S \to S$ is not homotopic to the identity. Hint: describe algebraically the action of h^n on $\pi_1(S)$ using the decomposition of $\pi_1(S)$ in the amalgamated free product (or HNN extension) along the subgroup generated by $[\gamma]$.*

Exercise 11.45. *Let $h : S \to S$ be the composition of iterated Dehn twists along disjoint simple homotopically nontrivial nonperipheral loops. Suppose that $f : S \to S$ is a homeomorphism such that some iteration of f is homotopic to h. Show that the mapping torus of f is a 3-dimensional* **graph-manifold** *M, i.e. each component of the JSJ decomposition of M is a Seifert manifold.*

11.15. Dynamics of aperiodic homeomorphisms

In this section we discuss the action of aperiodic elements of the Teichmüller modular group on the space of measured geodesic laminations $\mathcal{ML}(S) \cong \mathcal{MF}(S)$ and on its projectivization.

We consider the following equivalence relation on the space $\mathcal{MF}(S)$:

$$\mathcal{F} \sim t\mathcal{F}, \quad t > 0.$$

Equivalently, we identify two quadratic differentials ϕ, ψ if $\psi = t^2\phi$, $t \in \mathbb{R} - \{0\}$. The quotient

$$\mathcal{PMF}(S) = (\mathcal{MF}(S) - 0)/\sim$$

is the space of projective classes of measured foliations. Similarly we define:

$$\mathcal{PML}(S) = (\mathcal{ML}(S) - 0)/\sim .$$

These spaces are homeomorphic to $\mathbb{S}^{-3\chi(S)-n-1}$, where n is the number of boundary circles. Here is a concrete way to embed $\mathcal{PML}(S)$ back to $\mathcal{ML}(S)$. Take a *binding pair* of simple closed geodesics α, β. We will consider $\alpha + \beta$ as a *geodesic current* using the counting transversal measures. According to Lemma 11.23 the set

$$\{\lambda \in \mathcal{ML}(S) : p(\lambda) = i(\lambda, \alpha + \beta) \leq 1\}$$

is compact.

Corollary 11.46. *$\mathcal{PML}(S)$ is compact.*

Thus the map $\mathcal{PML}(S) \ni [\lambda] \mapsto \lambda/p(\lambda)$ is a homeomorphism. When convenient we will think of projective classes $[\lambda]$ as elements of the "sphere" $\{\lambda \in \mathcal{ML}(S) : p(\lambda) = 1\}$. Recall that for each pseudo-Anosov homeomorphism $h : S \to S$ we have a pair of projectively invariant measured foliations $(\mathcal{F}^+, \mathcal{F}^-)$:

$$h(\mathcal{F}^+) = k\mathcal{F}^+, \quad h(\mathcal{F}^-) = k^{-1}\mathcal{F}^-, \text{ where } k > 1.$$

Let λ^+, λ^- be the corresponding measured geodesic laminations. Thus, h has a pair of distinct fixed points on $\mathcal{PML}(S)$.

Theorem 11.47. *(See [Iva92].) The sequence of homeomorphisms*

$$h^n : \mathcal{PML}(S) \to \mathcal{PML}(S), \quad n > 0$$

converges to the constant map $[\mu] \mapsto [\lambda^+]$ *uniformly on compacts in* $\mathcal{PML}(S) - [\lambda^-]$.

Thus pseudo-Anosov homeomorphisms act on $\mathcal{PML}(S)$ the same way hyperbolic elements of $\mathrm{Isom}(\mathbb{H}^n)$ act on the ideal boundary $\partial_\infty \mathbb{H}^n$. Note that the above theorem includes the statement that the foliations $\mathcal{F}^\pm$ are uniquely ergodic (see [FLP79]).

Exercise 11.48. *Derive Theorem 11.36 from Theorem 11.47.*

We conclude with the following description of the laminations $\lambda^\pm$ for a pseudo-Anosov homeomorphism h. Let γ be any simple closed geodesic on S, we regard γ as an element of $\mathcal{ML}(S)$. Consider the sequences

$$\lambda_n^+ := h^n(\gamma), \quad \lambda_n^- := h^{-n}(\gamma) \in \mathcal{ML}(S).$$

Then the projective classes $[\lambda_n^+], [\lambda_n^-]$ converge to $[\lambda^+], [\lambda^-]$. Fix any pair of binding simple closed geodesics α, β on S. The sequences of geodesic laminations $\lambda_n^\pm$ converge to the geodesic laminations $L^\pm$ underlying $\lambda^\pm$ in the Chabauty topology; the number of points of intersection between $L^\pm$ and at least one of the geodesics α, β (actually both) is infinite. Thus

$$\lim_{n\to\infty} i(\lambda_n^\pm, \alpha) + i(\lambda_n^\pm, \beta) = \infty.$$

After normalization each sequence

$$\lambda_n^\pm / p(\lambda_n^\pm)$$

converges to a measured lamination in the projective class $[\lambda^\pm]$.

Proposition 11.49. *The pair of measured laminations* λ^+, λ^- *is* **binding** *the surface* S.

Proof: Suppose that this pair is not binding and μ is a measured lamination such that

$$i(\mu, \lambda^+) + i(\mu, \lambda^-) = 0.$$

Recall that complementary regions $C^{\pm} = S - \lambda^{\pm}$ are simply-connected polygons of finite area. Let ℓ be a leaf of μ: if it belongs to λ^+, then transversality of the foliations $\mathcal{F}^+, \mathcal{F}^-$ on S implies that ℓ intersects λ^- transversally, hence $i(\mu, \lambda^-) > 0$. Similarly we conclude that ℓ is not contained in λ^-. Thus it has to be contained in one of the complementary regions. The leaf ℓ cannot be closed, thus μ contains a sequence of leaves $\ell_k \to \ell$ which must eventually intersect both $\partial C^+, \partial C^-$. Contradiction. $\square$

11.16. A compactification of the Teichmüller space

In this section I will describe "Morgan-Shalen" model [MS84] of Thurston's compactification of the Teichmüller space. The original Thurston's approach to this compactification is described in [FLP79]. Let S be a complete hyperbolic surface of finite area, $S = \mathbb{H}^2/G$, where G is a Fuchsian group. We consider the *real-analytic model*

$$\mathcal{T}_R(G) \subset Hom_a(G, PSL(2,\mathbb{R}))/PSL(2,\mathbb{R}) \subset \mathcal{R}_{par}(G, PSL(2,\mathbb{R}))$$

of the Teichmüller space $\mathcal{T}(S)$ of the surface of S as in §5.4. Note that according to Theorems 8.4 and 8.53, the space $\mathcal{T}_R(G)$ is a component $D_{par}(G, PSL(2,\mathbb{R}))$ of the character variety $\mathcal{R}_{par}(G, PSL(2,\mathbb{R})))$ which consists of conjugacy classes of discrete, faithful and *admissible* representations. The space of discrete, faithful, admissible representations of G to $PSL(2,\mathbb{R})$ is compactified by the space $\mathcal{P}Tree(G)$ of projective classes of small nontrivial minimal relatively elliptic actions of F on metric trees, where we use the translation length function to topologize the union. Recall that a sequence $[\rho_n] \in D_{par}(G, PSL(2,\mathbb{R}))$ converges to the projective class of a tree $[T] \in \mathcal{P}Tree(G)$ iff for each $g \in G$

$$\lim_{n\to\infty} c_n^{-1}\ell_{\rho_n}(g) = \ell_T(g)$$

for a certain sequence $c_n \in \mathbb{R}_+$ such that $\lim_{n\to\infty} c_n = \infty$.

According to Theorem 11.33 there is a homeomorphism $\mathcal{P}Tree(G) \to \mathcal{PML}(S)$. Thus we get a compactification of $\mathcal{T}(S)$ by adding $\mathcal{PML}(S)$. Let $\partial_\infty \mathcal{T}(S)$ denote the set of points in $\mathcal{PML}(S)$ which are limits of sequences in $\mathcal{T}(S)$.

Suppose that γ is a simple closed geodesic on S, λ is the corresponding element of $\mathcal{PML}(S)$. One can construct a sequence of points in $\mathcal{T}(S)$ which is convergent to λ as follows. Let D_γ be the Dehn twist around γ, we will view D_γ simultaneously as a homeomorphism $S \to S$ and as an element of the mapping class group Mod_S acting on the Teichmüller space $\mathcal{T}(S)$. Define the sequence $\psi_n = D_\gamma^n([S]) \in \mathcal{T}(S)$, where $[S]$ is the origin of $\mathcal{T}(S)$ represented by (S, id). The sequence ψ_n corresponds to the sequence of representations $\rho_n : G \to PSL(2,\mathbb{R})$ given by the formula:

$$\rho_n(g) = D_\gamma^n(g), \quad g \in G.$$

Theorem 11.50. *The sequence ψ_n converges to λ.*

Proof: Splitting S along γ defines a decomposition of $F = \pi_1(S)$ into an amalgamated free product $A *_C B$ (or HNN decomposition $A *_C$) with the amalgamation over $C = \langle\gamma\rangle$. Recall that the small G-tree T dual to the lamination $\gamma \subset S$, is projectively equivalent to the simplicial tree T of the decomposition $A *_C B$ (or $A *_C$), the quotient of T by G is a segment or a circle with only one edge. We will consider only the case of amalgamated free product, the case of HNN decomposition is left to the reader. Let a, b be vertices of T connected by an edge e so that a is stabilized by A and b is stabilized by B. The G-orbit of e is the whole tree T. Choose a finite generating set $\mathcal{G}$ for G and an ultrafilter ω. Consider the ultralimit

$$X_\omega := \omega\text{-}\lim_n c_n^{-1}(\mathbb{H}^2, x_n)$$

defined as in Theorem 10.24, where c_n is the minimal displacement of $\rho_n(G)$ (with respect to the generating set $\mathcal{G}$). Then X_ω is a nontrivial G-tree. Let Q be the minimal G-subtree in X_ω. Note that the action of the elements D_γ^n on $\mathcal{T}(S)$ does not change marked hyperbolic structures on the components of $S - \gamma$. Thus the groups A and B have global fixed points v, w in Q. Note that both v, w are fixed by the subgroup C. Scale the metric on T so that $d(a, b) = d(v, w)$. We then construct a morphism $f : T \to Q$ by mapping a to v, b to w and extending f to the rest of T using the action of G. The morphism f is necessarily an isomorphism (see Lemma 10.18).

Corollary 11.51. *The accumulation points of $\mathcal{T}(S)$ are dense in $\mathcal{PML}(S)$.*

Proof: This follows from the above theorem and the fact that projective classes of laminations supported on simple closed geodesics are dense in $\mathcal{PML}(S)$ (see Theorem 11.25). □

The union $\mathcal{T}(S) \cup \mathcal{PML}(S)$ is called *Thurston's compactification* of the Teichmüller space. It is known that $\mathcal{T}(S) \cup \mathcal{PML}(S)$ is homeomorphic to the d-dimensional closed ball, where d is the (real) dimension of $\mathcal{T}(S)$, see [FLP79].

Another approach to the description of $\mathcal{T}(S) \cup \mathcal{PML}(S)$ is due to Bonahon [Bon88], we describe it below (for simplicity we restrict to the case of closed hyperbolic surfaces). Bonahon defines the *Liouville* measure L on $\mathcal{M}_\infty$ as follows. Consider the product

$$[a, b] \times [c, d] \subset \mathbb{S}^1 \times \mathbb{S}^1 - Diag$$

of two circular arcs. Then the measure of its projection to $\mathcal{M}_\infty$ is

$$L([a, b] \times [c, d]/\sim) := \left| \log \left| \frac{(a-c)(b-d)}{(a-d)(b-c)} \right| \right| .$$

Since Moebius transformations preserve the cross-ratio, this measure is invariant under $\mathrm{Isom}(\mathbb{H}^2)$. Fix the origin $S = \mathbb{H}^2/F$ in the Teichmüller space $\mathcal{T}(S)$ which we now identify with the space of marked hyperbolic structures on S. Then for each point $q \in [X, f] \in \mathcal{T}(S)$ we get an isomorphism

$\rho : F \to F'$ to a Fuchsian subgroup $F' \subset PSL(2,\mathbb{R})$. This isomorphism is induced by a homeomorphism $h : \mathbb{S}^1 \to \mathbb{S}^1$. Let L_q be the pull-back of the measure L via h. Thus we get an embedding

$$\iota : \mathcal{T}(S) \hookrightarrow \mathcal{GC}(S), \quad q \mapsto L_q.$$

Bonahon proves that for every L_q the self-intersection number of L_q is $i(L_q, L_q) = \pi^2|\chi(S)|$. Recall that $\mathcal{ML}(S)$ is the subset of $\mathcal{G}C(S)$ which is defined by

$$\{\alpha : i(\alpha, \alpha) = 0\}$$

Each point $q \in \mathcal{T}(S)$ defines the length function $\ell_q : \mathcal{C}(S) \to \mathbb{R}$ as follows: if a representation ρ corresponds to q, then $\ell_q(\gamma) := \ell(\rho(\gamma))$ is the translation length. The function ℓ_q extends continuously to $\ell_q : \mathcal{GC}(S) \to \mathbb{R}$ which coincides with $i(L_q, \bullet)$. In particular, $\ell_q(\gamma) = 0$ iff $\gamma = \emptyset$.

If α, β is a binding pair of simple closed geodesics on S then

$$\lim_{j\to\infty} L_{q_j}/p(L_{q_j}) = \lambda \in \mathcal{ML}(S)$$

(where the limit is understood as the limit of sequences of geodesic currents) if and only if the sequence of marked hyperbolic structures q_j converges to the projective class $[\lambda]$ in $\mathcal{T}(S) \cup \mathcal{PML}(S)$ (with respect to the equivariant Gromov-Hausdorff topology). One way to see it is the following. Let L_∞ denote the limit of $L_{q_j}/p(L_{q_j})$ (which exists up to a subsequence). Since $i(L_{q_j}, L_{q_j}) = const$ and the intersection number is a continuous function, $i(L_\infty, L_\infty) = 0$ and hence $L_\infty \in \mathcal{ML}(S)$. Each geodesic current σ is determined by the length function $i(\sigma, \bullet) : \mathcal{C}(S) \to \mathbb{R}$. Then Theorem 10.24 implies that $L_\infty = \lambda$.

Chapter 12

The Rips' Theory

Consider a pared 3-manifold (M, P) with incompressible boundary $\partial_0 M$; let $W \subset M$ be the window of (M, P) and let $G = \pi_1(M)$. We call an action $G \curvearrowright T$ on a metric tree *relatively elliptic* (with respect to P) if the fundamental group of each component of the parabolic locus P has a global fixed point in T. The following theorem is crucial for the proof of the Hyperbolization Theorem as presented in this book:

Theorem 12.1. *(J. Morgan and P. Shalen [MS88b].) Suppose that T is a small minimal relatively elliptic G-tree. Then the fundamental group of each component M_j of $M - W$ has a global fixed point in T.*

The original proof of this theorem was based on a detailed study of measured surface laminations in 3-manifolds which give *resolutions* to the action $G \curvearrowright T$. These laminations were deformed to incompressible tori and essential annuli and thus related to the JSJ decomposition of the manifold M. The Rips' theory greatly generalizes the original result of Morgan and Shalen and deals with actions of general finitely presentable groups on trees (these groups a priori have nothing to do with the 3-manifolds). I will use the *Rips theory* to prove theorem 12.1. Our description of the *Rips' theory* follows [BF95]. The reader can also find discussion of the Rips' theory in the papers of Paulin [Pau97], Gaboriau, Levitt and Paulin [GLP94], [GLP95] and Guirardel [Gui98].

Roughly speaking the *Rips' theory* is a device which for the given input:

A stable nontrivial action of a finitely presentable group G on a metric tree T.

produces the output:

A graph of groups decomposition of G and "simple" actions of the vertex groups on trees.

The definition of *stable* action is in the following section. Properties of the graph of groups decomposition of G are described in §12.13 (Theorem 12.72, Corollary 12.73), where it also becomes clear what the "simple" actions of the vertex groups on trees are. In §12.17 we derive Theorem 12.1 from Corollary 12.73.

12.1. Stable trees

Recall that if G is a group acting on a space X and $Y \subset X$, then $Fix_G(Y)$ denotes the subgroup of G whose elements fix Y pointwise.

Definition 12.2. (Stable subtrees) Consider a G-tree T and a nondegenerate subtree $S \subset T$. The subtree S is called **stable** if for each nondegenerate subtree $S' \subset S$ we have: $Fix_G(S') = Fix_G(S)$. Note that neither S nor S' have to be G-invariant. If S is not stable then any subtree $S' \subset S$ such that $Fix_G(S') \neq Fix_G(S)$ is called a **destabilizing** subtree.

Example 12.3. If $Fix_G(\alpha) = \{1\}$ for every nondegenerate segment $\alpha \subset S$, then S is stable.

Example 12.4. Suppose that S_1, S_2 are stable subtrees in T and $S_1 \cap S_2$ is nondegenerate. Then $S := S_1 \cup S_2$ is also a stable subtree in T. Indeed, let $S' \subset S$ be a destabilizing subtree, let $g \in G$ be an element in $Fix_G(S') - Fix_G(S)$. Without loss of generality we can assume $g \notin Fix_G(S_1)$. Since S_1 is stable we conclude that $S_1' := S_1 \cap S'$ is either empty or is a single point. Hence $S' \subset S_2$ and stability of S_2 implies that $g|S_2 = id$. Therefore $g \in Fix_G(S_1 \cap S_2) - Fix_G(S_1)$ and by stability of S_1 we get: $S_1 \cap S_2$ is either empty or a single point which contradicts our assumptions.

Clearly if $S_1 \subset S_2$ are nondegenerate trees and S_2 is stable then S_1 is also stable. Example 12.4 implies that each stable subtree $S \subset T$ is contained in a **unique** maximal stable subtree M. In particular, if two distinct maximal stable subtrees intersect then the intersection is degenerate.

Exercise 12.5. *Construct example of a G-tree T with two distinct maximal stable subtrees $M_1, M_2 \subset T$.*

Definition 12.6. A nontrivial action of a group G on a tree T is called **stable** if every nondegenerate subtree $S \subset T$ contains a stable subtree.

For instance, consider a simplicial tree T (where each edge has unit length) and let G be a subgroup in the group of automorphisms of T. Then the action of G on T is stable. Indeed, if S is any nondegenerate subtree of T then S contains a proper nondegenerate subsegment α of an edge of T. This segment α is a stable subtree: if $g \in G$ fixes a nondegenerate subsegment $\alpha' \subset \alpha$ then g fixes α as well.

To get a simple example of *unstable* action consider $T := \mathrm{Cone}_\omega(\mathbb{H}^2)$, $G := \mathrm{Isom}(T)$. Then for each nondegenerate arc $\alpha \subset T$ there is an element $g \in G$ and a subarc $\alpha' \subset \alpha$ such that $g|\alpha \neq id$, $g|\alpha' = id$.

Exercise 12.7. *Construct an example of a finitely generated group G and an unstable action of G on a tree. (See page 278.)*

Proposition 12.8. *Let T be a nontrivial G-tree. Then each of the following conditions implies stability of the action of G on T:*

1. *The set of arc stabilizers satisfies the ascending chain condition, i.e. if $G_1 \subset G_2 \subset G_3 \subset \ldots$ is a chain of arc stabilizers then $G_i = G_{i+1}$ for all but finitely many i.*

2. *For each nondegenerate arc $\alpha \subset T$ we have: $Fix_G(\alpha)$ is trivial.*

3. *The action of G is small and G is the fundamental group of a compact Haken manifold or G is a discrete group of isometries of a complete simply-connected manifold X whose sectional curvature is pinched between two negative constants (for instance $X = \mathbb{H}^n$).*

4. *If $s_1 \supset s_2 \supset ...$ is a sequence of arcs in T converging to a point then $Fix_G(s_i) = Fix_G(s_{i+1})$ for all but finitely many i.*

Proof: (1) Let $S \subset T$ be a subtree, then S contains a subarc J such that $Fix_G(J) = Fix_G(J')$ for any subarc $J' \subset J$. Hence J is a stable subtree in S.

(2) Obvious.

(3)Recall that every finitely generated virtually nilpotent group satisfies the ascending chain condition for arbitrary chains of subgroups (Theorem 4.2). Since the action of G is small, arc stabilizers are virtually nilpotent subgroups in G. If G is the fundamental group of a Haken manifold then each virtually nilpotent subgroup of G is finitely generated (see Theorem 1.56). In the second case Bowditch [Bow93a] proves that point $p \in \partial_\infty X$ the stabilizer of p in $Isom(X)$ is a virtually nilpotent finitely generated group. In both cases, if $N_1 \subset N_2 \subset ...$ is a chain of virtually nilpotent subgroups in G, then $N_i = N_{i+1}$ for all sufficiently large i and we can apply (1).

(4) This assertion is analogous to (1) and we leave it to the reader. □

The above proposition explains the name *stability*: every sequence of stabilizers will eventually stabilize. Note however that smallness of the action alone does not imply that the action is stable: Dunwoody [Dun97] constructs an example of a finitely generated group G and small but unstable action of G on a tree T.

12.2. Unions of bands and band complexes

A general G-tree T is a complicated dynamical system, the goal of this section is to describe the corresponding *finite* combinatorial object which encodes the information about the action of G on T. Roughly speaking *a union of bands* is a 2-dimensional complex with *singular one-dimensional foliation* that is given a transversally invariant measure. Generic leaves of this foliation do not have to be 1-manifolds, they are 1-dimensional graphs.

We start with a *real graph*, i.e. a finite *simplicial* [1] 1-complex Γ where each edge is identified with an interval in $\mathbb{R}$. We orient each edge of Γ.

Remark 12.9. This definition does not require existence of a global metric on Γ whose restriction to the edges e of Γ is consistent with the embeddings $e \hookrightarrow \mathbb{R}$. For instance, the triangle with the sides of the length $1, 1$ and 3 is an example of a real graph. However we do have a metric on each edge of Γ. Given these metrics we construct the induced path-metric on Γ.

[1] It will become clear in the Definition 12.14 that we cannot work with general finite graphs.

Recall that I denotes the unit interval $[0,1]$. A *band* B is the product $b \times I$ where b is a compact nondegenerate segment of $\mathbb{R}$. The segment b will be identified with $b \times \{0\}$, which is called the *bottom* of B. The product $b \times \{1\}$ is called the *top* of B. We will use the name *bases* for the top and bottom segments of B. There is a canonical reflection

$$\delta_B : B \to B, \quad \delta_B(s,t) = (s, 1-t).$$

If S is a subset of one of the bases of B then $\delta_B(S)$ is called the *dual* of S.

Subsets of $b \times \{t\}$ (for $t \in I$) and $\{s\} \times I$ (where $s \in b$) will be called *vertical* and *horizontal* subsets of B respectively. We will think of B as a fiber bundle over b with the *vertical fibers* $\{s\} \times I$. Take a finite collection $B_i := b_i \times I$ $(i = 1, ...n)$ of bands and isometric embeddings

$$f_i, f_i' : b_i \times \{0\}, \quad b_i \times \{1\} \to \Gamma, \quad i = 1, ...n,$$

whose images are contained in the (closed) edges of Γ. However we do not require the images of these maps to be equal to the edges of Γ.

Definition 12.10. A **union of bands** is the quotient space Y of the disjoint union

$$\Gamma \sqcup B_1 \sqcup ... \sqcup B_n = \Gamma \sqcup B$$

where we identify points of bases with their images in Γ. We will identify each band with its image in Y. We require each point of Γ to belong to the image of at least one base.

Define the *bases* in Y as the images of bases of B_j in Y. We retain the notation b for the bases in Y. Define the *boundary* ∂Y as the union of the images of the vertical boundary segments of the bands B_i in Y.

Remark 12.11. The union of bands Y is not just a topological space but we retain the information about the geometry of Γ, bands, attaching maps and the product decomposition of bands.

Each union of bands Y is foliated, the *leaves* of Y are the equivalence classes generated by the equivalence relation:

$$x \sim y \quad \text{if} \quad x, y \quad \text{belong to a vertical fiber of a band.}$$

Thus each leaf is a union of vertical fibers in bands. Each leaf has a natural topology: a subset is open iff its intersection with every vertical fiber of every band is open. Hence every leaf is homeomorphic to a locally finite graph.

The reader can think of Y as a foliated space with a **flat connection** (given by the vertical segments) and a metric structure transversal to the foliation.

Definition 12.12. A subset of Y is called **horizontal** if it is contained in Γ or is contained in a single band and is horizontal there.

Figure 12.1 gives an example of a union of bands. In this case the union of bands is a surface with boundary and the foliation is a foliation by leaves of a holomorphic quadratic differential on this surface. Generic leaf of this foliation is a 1-manifold, however there are several leaves which branch, classically they are called *singular* (or *critical*) leaves. This union of bands has three singular points p, q, r, each singular point is located on the boundary of the surface.

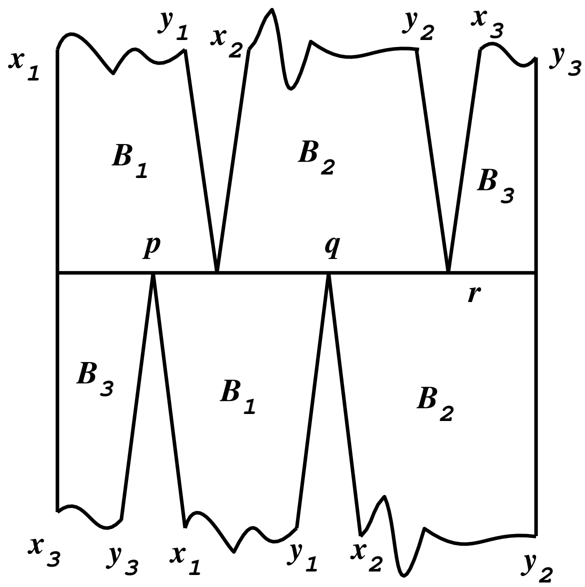

Figure 12.1: *Example of a union of bands. Identify the curves with the labels $x_i y_i$ to get the band B_i, $i = 1, 2, 3$.*

Since for general unions of bands we cannot expect generic leaves to 1-manifolds, the definition of *singular leaves* has to be modified. One way to do it is to declare a leaf to be nonsingular iff it is disjoint from the vertical boundary fibers of every band, this is the approach of [GLP94]. It is clear however that this definition is not equivalent to the classical in the case of the foliation of a compact annulus by the concentric circles: the boundary leaves are obviously noncritical. We will give another definition:

Definition 12.13. A leaf L is called **nonsingular** (or **regular**) if it can be exhausted by compact subgraphs C_j so that the following is satisfied:

Let C_j^o be the interior of C_j in L. Then there is an open neighborhood $U \subset Y$ of C_j^o such that U is naturally homeomorphic to an interval bundle over C_j^o where the interval is either open or half-open. This interval bundle has a flat connection whose leaves are contained in the leaves of Y and the fibers of the interval bundle are horizontal in Y.

Otherwise L is called **singular**.

In the next section we will give an alternative definition in terms of *pushing* in Y.

Definition 12.14. (Measures of complexity of unions of bands.) Define the **height** $\eta(Y)$ of a union of bands Y to be the total number of bands in

Y. For each base b of a band B in Y define the **weight** $w(b)$ as follows:

- If B is an annulus (i.e. $f_{dual(b)} = f_b \circ \delta_b$) we let $w(b) = 0$.
- If B is a Moebius band we let $w(b) = 1/2$.
- Otherwise $w(b) = 1$.

Define the **weight** $w(z)$ of a point $z \in \Gamma$ to be the sum of weights of bases containing z. A **block** γ in Y is the closure of a component of the union of interiors of bases of Y.[2] The **complexity** $\sigma(\gamma)$ of a block γ equals

$$Complexity(\gamma) = \sigma(\gamma) = \max\{0, -2 + \sum_{b \text{ is a base in } \gamma} w(b)\}.$$

The complexity $\sigma(Y)$ is the sum of complexities of the blocks in Y.

It may seem odd that we subtract 2 in the definition of complexity σ, however it is crucial for proving that complexity does not increase under the Processes I and II of the Rips machine.

Exercise 12.15. *List all blocks of zero complexity which do not contain bases of zero weight.*

12.3. Pushing

Let $S, S' \subset Y$ be horizontal subsets and p be a path in a leaf of Y.

Definition 12.16. The set S **pushes into** the set S' along the path p if there is a homotopy $H : S \times I \to Y$ so that:

- $H(x, 0) = x$ for each $x \in S$, $H(S, 1) \subset S'$.
- $H(S, t)$ is horizontal for each $t \in I$.
- $H(z \times I)$ is contained in a leaf of Y for each $z \in S$.
- There is a point $z_0 \in S$ such that $H(z_0, t) = p(t)$, $t \in I$.

We say that S **pushes onto** S' along the path p if it pushes into S' along the path p and $H(S, 1) = S'$. The homotopy H is called **pushing**.

Given a point $z \in Y$ consider the maximal connected horizontal subset $h(z) \subset Y$ containing z. Clearly $h(z)$ is a finite graph which is a horizontal segment of a band if $z \notin \Gamma$. The finite graph $h(z)$ has a natural path-metric induces from the metrics on the edges of Γ and the metrics on the horizontal slices of bands. Given $\epsilon > 0$ we define the *horizontal neighborhoods* $N(z, \epsilon)$ of points $z \in Y$ as closed ϵ-neighborhoods of z in $h(z)$.

Given $z \in Y$ define $P(z)$ as the collection of points $y \in Y$ so that a horizontal neighborhood of z pushes *onto* a horizontal neighborhood of y along a path

$$p : I \to \text{ a leaf of } Y \text{ which connects } z \text{ to } y.$$

Thus $y \in P(z)$ if and only if $z \in P(y)$.

[2] Since Γ is a simplicial complex it follows that each block is an interval.

Remark 12.17. Note that our definition of $P(z)$ is slightly different from Bestvina and Feighn's [BF95].

Definition 12.18. A subset S of Y is called **pushing-saturated** if given every leaf L of Y, a path $p : I \to L$ (so that $p(0) = z$) and $\epsilon > 0$ such that $N(z, \epsilon)$ pushes along p, it follows that $p(1) \in S$.

Note that $P(z)$ is the *minimal* pushing-saturated subset of Y containing z. In other words, $S \subset Y$ is pushing-saturated iff $S \supset P(z)$ for each $z \in S$.

Exercise 12.19. *Show that the closure and the accumulation set of each pushing-saturated set are again pushing-saturated. Unions and intersections of pushing-saturated sets are pushing-saturated.*

Exercise 12.20. *A leaf is singular iff it contains a proper pushing-saturated subset. Equivalently, a leaf L is non-singular iff $L = P(z)$ for each $z \in L$.*

Definition 12.21. A union of bands is **minimal** if every pushing-saturated subset of Y is dense in Y and is **simplicial** if every leaf of Y is compact.

At the first glance this definition does not look very natural: why do we define minimality using pushing-saturated sets instead of simply saying that every leaf is dense? The main reason for that is the way the Rips Machine works. The Machine operates not with the leaves L of Y but rather with unions of neighborhoods $U_j = U(C_j^o)$ of finite subgraphs C_j^o which exhaust L as in the Definition 12.13. One of the features of the Machine is step-by-step elimination of certain neighborhoods U_j from the union of bands. Such neighborhoods U_j exist only for pushing-saturated subsets and not for arbitrary leaves.

Definition 12.22. Suppose that W is a compact proper and pushing-saturated subset of a leaf L of Y. Then W is called a **fault**.

It is clear that each leaf of Y is pushing-saturated. Below we give a *necessary condition* for a leaf L of Y to contain a proper pushing-saturated subset (see Figure 12.2):

(a) Either the leaf L contains a *local cut point*, i.e. there is a point $z \in L \cap \Gamma$ which locally separates the union of Γ and those bands that contain z.

(b) Or the leaf L contains a point $z \in L \cap \Gamma$ such that: z is an end-point of a base b and an interior point of a base b' (such point z will be called a *dead end*).

Point z as in (a) or (b) is called a *critical point* of L. Note however that existence of a critical point is only necessary but not sufficient condition for L to be singular.

Exercise 12.23. *Prove that:*

- *Each singular leaf contains only finitely many critical points.*

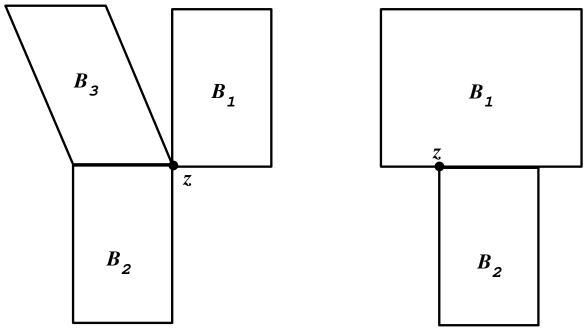

Figure 12.2: *Critical points of leaves.*

- *If $L' \subset L$ is a proper pushing-saturated subset of L then the frontier of L' in L consists of critical points. In particular, L contains only finitely many pushing-saturated subsets and Y contains only finitely many singular leaves.*

12.4. Transversal measure on union of bands

Let Y be a union of bands, the goal of this section is to define a *transversal measure* on Y (with respect to the foliation of Y into vertical leaves). Let $g : I \to B$ be a continuous map of the unit interval into a band $B = b \times I$. Then we say that g is *transversal* to the vertical foliation of B if the composition $\pi \circ g$ is injective, where $\pi : B \to b$ is the projection to the first factor. If g is transversal then we define the *transversal measure* $\mu(g)$ as the length of the subinterval $\pi(g(I)) \subset b$. Suppose now that $g : I \to Y$ is a continuous map and we have a finite subdivision of I into subintervals I_j so that for each j the image $g(I_j)$ is contained in a single band $B_j \subset Y$. Then we say that g is transversal to the vertical foliation of Y if each restriction $g|I_j$ is transversal to the vertical foliation of B_j. For transversal paths g we define the *transversal measure* $\mu(g)$ of g as the sum

$$\sum_j \mu(g|I_j).$$

This definition extends to continuous paths g such that I has a finite subdivision into subintervals I_j so that for each j either $g|I_j$ is transversal to the vertical foliation of B_j or $g(I_j)$ is contained in a leaf of B_j; in the latter case we let $\mu(g|I_j) := 0$.

12.5. Dynamical decomposition of unions of bands

The following theorem was proven by H. Imanishi [Ima79] in the context of singular foliations on manifolds (which have Morse-type singularities). Our discussion follows [GLP94].

Theorem 12.24. *(Dynamical decomposition of unions of bands.) Each union of bands Y contains only finitely many faults. Suppose that Y has no faults at all. Then Y splits into disjoint union of components each of which is either minimal or simplicial. Each simplicial component is an interval bundle over a leaf in that component.*

Proof: The first assertion follows from the Exercise 12.23. Consider a finite regular leaf L. Thus (by the definition of a regular leaf) all nearby leaves are also finite and regular, let $U(L)$ be the maximal connected neighborhood of L which is the union of finite regular leaves. Clearly $U(L)$ is an interval bundle over L. The frontier of $U(L)$ in Y consists of finite singular leaves. Since each finite singular leaf contains a *fault*, $U(L)$ is a component of Y. By compactness there are only finitely many such components, let U denote their union.

Pick a component M of $Y - U$. Our goal is to show that M is minimal, i.e. each pushing-saturated subset L of M is dense in M. Without loss of generality we may assume that L is contained in a single infinite leaf.

Step 1:

Proposition 12.25. *There is a number $\delta > 0$ (depending on L) such that for every base b, every component in $b - cl(L)$ has length at least δ. In particular, no infinite leaf closure in M intersects a base in a Cantor set.*

Proof: Without loss of generality we can assume that $L \cap b$ is an infinite set. Choose δ as follows: if x is an end-point of a base c then the distance between x and $int(c) \cap L$ is either zero or at least δ.

Suppose that $J = [xu]$ is a component of $cl(L) \cap b$ which has length $\lambda < \delta$. Since the accumulation set of a pushing-saturated set is also pushing-saturated we conclude that for each end-point x of J the set $P(x)$ has infinite length and intersects b infinitely many times. Thus we can find two distinct points $y, z \in P(x) \cap b$ which satisfy:

- $|y - z| < \lambda$;
- a nondegenerate subinterval $[xw]$ of J pushes[3] into the interval $[yz]$ along a vertical path p connecting x to y.

Define $x' \in [xu]$ by $|x - x'| = |y - z| < \lambda$. Note that $x' \notin cl(L)$, hence the interval $[xx']$ does not push into $[yz]$ along p. The path p is a finite composition of vertical paths:

$$p_1 * .... * p_k$$

each contained in a single band. Hence we can find $j \geq 1$ so that $[xx']$ pushes along $p'' := p_1 * ... * p_{j-1}$ and does not push along $p' := p_1 * ... * p_j$. (If $j = 1$ then we set p'' be the constant path.) Let $x'' \in (xx']$ be the point closest to x such that $[xx'']$ does not push along p'. Then:

[3] Up to this point all our arguments were valid for leaves (instead of pushing-saturated sets). However to get the interval $[xw]$ we have to use *pushing*.

- The point $p''(x', 1) \in \Gamma$ is not in $cl(L)$.
- $p''(x', 1)$ is an end-point of one of the bases c (otherwise we can continue pushing along p').
- $0 < d(c, L) = \lambda < \delta$, since the whole open interval $(p''(x, 1), p''(x', 1))$ is the image of (xx') under pushing along p' and thus is disjoint from $cl(L)$.

This contradicts our choice of δ and hence proves Proposition 12.25. □

Step 2: Consider the intersection $b \cap cl(L)$ of a base $b \subset M \cap \Gamma$. Without loss of generality we may again assume that this intersection is infinite. Pick a pushing-saturated subset $K = P(x)$ for a point x of the frontier of $b \cap cl(L)$ in b. We may choose x an interior point of b, let $J \subset b$ denote an interval with end-point x such that $int(J) \cap cl(L) = \emptyset$. Then K has infinite length, let $\delta = \delta(K)$ be given by the Proposition 12.25. Choose $y, z \in K$ such that:

- $0 < |y - z| < \delta$.
- If p is a path in K connecting x to y then p pushes a nondegenerate subinterval $[xx'] \subset J$ into the interval $[yz]$.

Then $[y, z]$ is contained in $cl(K) \subset cl(L)$ (by Proposition 12.25). On the other hand, an interval $[xx']$ pushes into $[yz]$ and $(xx'] \cap cl(L) = \emptyset$. Contradiction. □

Corollary 12.26. *Y is minimal if and only if each leaf of Y is dense (in Y) and Y contains no compact proper pushing saturated subsets.*

12.6. Band complexes

Note that each union of bands is homotopy equivalent to a finite graph. Thus unions of bands clearly do not suffice to describe actions of groups on trees (unless these groups are free groups). To get more complicated finitely presented groups we have to add 2-cells to the unions of bands.

Definition 12.27. A **band complex** X **based** on a union of bands Y is a finite 2-dimensional CW-complex obtained by taking Y and attaching 0, 1 and 2-cells as follows:

- The 1-cells of X intersect Y in a subset of Γ.
- Each component of the intersection of Γ and a 2-cell of X is a point.
- Each component of the intersection of a band of Y and a 2-cell of X is vertical.

We modify the definition of the boundary ∂Y of Y as the union of ∂Y with the 1-cells in X whose interiors are disjoint from any band.

We define the *leaves* in a band complex X (based on a union of bands Y) as the leaves in Y. Similarly, X is called *minimal* (resp. *simplicial*) if Y

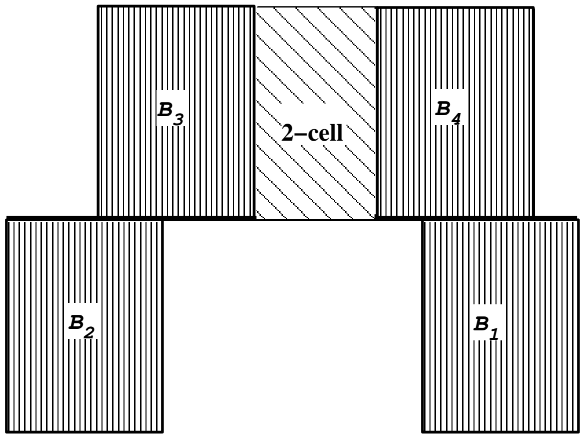

Figure 12.3: *A band complex.*

is *minimal* (res. *simplicial*). The *complexity* $\sigma(X)$ and the *height* $\eta(X)$ are defined as $\sigma(Y)$ and $\eta(Y)$ respectively. Define the transversal measure on X as follows.

A path $g : I \to X$ is said to be *transversal* to the vertical foliation of X if I can be subdivided into finite number of subintervals I_j so that for each j either:

(a) $g|I_j : I_j \to Y$ is transversal to the vertical foliation of Y (see §12.4) in which case we let $\mu(g|I_j)$ be the transversal measure of $g|I_j : I_j \to Y$, as defined in §12.4.

(b) Or $g_j(I_j)$ is contained in the closure of $X - Y$, in which case $\mu(g|I_j) := 0$.

For transversal paths g we define the transversal measure $\mu(g)$ as the sum

$$\sum_j \mu(g|I_j).$$

It is clear that if $g : I \to X$ is a path such that $g(0) = x_0, g(1) = x_1$ then we can homotope g (rel. $\{0, 1\}$) to a path transversal to the vertical foliation of X.

Definition 12.28. Let $\tilde{X} \to X$ be a cover of X (usually but not always it will be the universal cover). The transversal measure lifts from X to $\tilde{X}$. A **generalized leaf** in $\tilde{X}$ is an equivalence class for the equivalence relation: $x \sim x'$ iff there is a path in $\tilde{X}$ connecting x to x' whose transversal measure equals zero. Then the **leaf space** $\tilde{X}/\sim$ of $\tilde{X}$ is the quotient of $\tilde{X}$ by the above equivalence relation. The leaf space is equipped with the pseudometric

$$d([x], [x']) := \inf\{\mu(g) : g \text{ is a path in } X \text{ connecting } x \text{ to } x'\}.$$

We will say that the pair $(X, \tilde{X})$ is **regular** if the leaf space (T_X, d) is a metric tree.

Example 12.29. (Band complexes corresponding to the interval exchange transformations.) Let $E_{r,\sigma,\chi}$ be a **generalized interval exchange transformation** (defined in Section 11.4).

We let Γ be the real graph $[0,\rho]$. Let $B_i := [x_{i-1}, x_i] \times I$ be the bands, $i = 1, ..., n$, define the attaching maps for these bands as $id = f_i : b_i = [x_{i-1}, x_i] \to [x_{i-1}, x_i] \subset \Gamma$ and $f_i' = \lambda_i : b_i \times \{1\} \to \Gamma$. As the result we get a union of bands Y associated with the interval exchange transformation $E_{r,\sigma,\chi}$. It is clear that the total space of Y is homeomorphic to a surface with boundary S. We let X be the band complex obtained by attaching a 2-cell to each boundary component of S so that each attaching map is a homeomorphism. (More generally, one can attach cells via maps of degree ≥ 1, in which case the fundamental group of the resulting band complex is the fundamental group of a 2-dimensional orbifold.)

Below we use the interval exchange transformations to give two examples of band complexes: one where d is a metric and the second where d is identically zero, but $\tilde{X}/\sim$ is not a single point.

Example 12.30. A band complex X so that the pseudometric d associated to the universal cover of X is a metric and $(\tilde{X}/\sim, d)$ is a metric tree.

Start with a closed hyperbolic surface S and a singular measured foliation $\mathcal{F}$ on S associated with a holomorphic quadratic differential on S. Then using the construction of interval exchange transformation in §11.4, we get a union of bands Y obtained by splitting S along several arcs contained in the leaves of $\mathcal{F}$. We recover S from Y by attaching 2-cells along loops in Y corresponding to these arcs. We get a band complex X. Let $\tilde{X}$ be the universal cover of X. Then $(\tilde{X}/\sim, d)$ is isometric to the tree $T_{\mathcal{F}}$ dual to the lift of $\mathcal{F}$ to $\tilde{X}$. $\square$

Example 12.31. A band complex X so that the pseudometric d associated to the universal cover of X is identically zero.

Let T^2 be a flat 2-torus, let $\mathcal{F}$ be a transversally measured geodesic foliation on T^2 so that each leaf of $\mathcal{F}$ is dense. Let $\tau : T^2 \to T^2$ be an orientation-preserving involution which fixes four points. Clearly $\mathcal{F}$ is invariant under τ. The quotient $T^2/\langle\tau\rangle$ is the 2-sphere $\mathbb{S}^2$ and the foliation $\mathcal{F}$ projects to a measured foliation $\bar{\mathcal{F}}$ on $\mathbb{S}^2$ with four points removed (these are the projections of the points fixed by τ). We next associate an interval exchange transformation to the foliation $\bar{\mathcal{F}}$.

We describe the quotient foliation $\bar{\mathcal{F}}$ in Figure 12.4: the torus T^2 is obtained by identifying sides of the fundamental parallelogram R; the foliation $\mathcal{F}$ corresponds to the vertical geodesic foliation on R. The orbifold $T^2/\langle\tau\rangle$ is obtained by making identifications on sides of the triangle Δ, which is "one half" of R (see Figure 12.4 where we identify the pairs of oriented edges labeled α, β, γ on the boundary of R to get the quotient sphere $\mathbb{S}^2$).

To obtain X we split $\mathbb{S}^2$ open along seven disjoint segments each of which is contained in a leaf of $\bar{\mathcal{F}}$ and attach 2-cells to each boundary circle of the resulting surface Σ by homeomorphic maps of the circles.

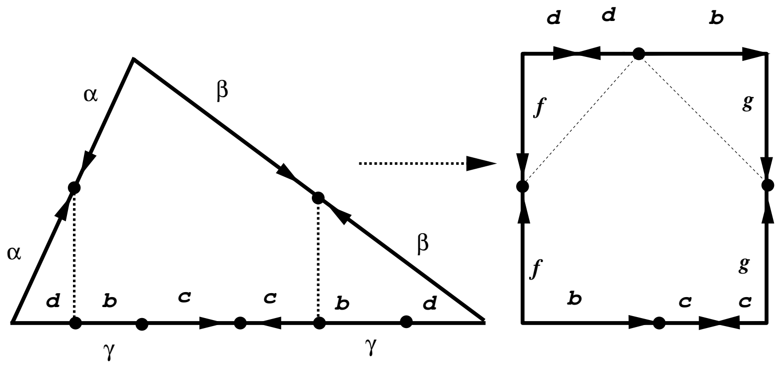

Figure 12.4: *A band complex.*

Since each leaf of $\mathcal{F}$ is dense in T^2, the same is true for $\bar{\mathcal{F}}$. The sphere X is simply connected, hence the universal cover $\tilde{X}$ is the same as X itself. We conclude that the **leaf space** $\tilde{X}/\sim$ is not Hausdorff (each point is dense) and the pseudometric d on $\tilde{X}/\sim$ is identically zero. On the other hand, $\tilde{X}/\sim$ consists of continuum of points. $\square$

Let X be a band complex, if $\tilde{X}$ is a cover of X then we shall use the notation T_X for the quotient $\tilde{X}/\sim$ (we will suppress the choice of $\tilde{X}$). Let $q : \tilde{X} \to T_X$ denote the projection.

Definition 12.32. Let G be a finitely presented group acting on a tree T. A **resolution** for this action is a G-equivariant map $\rho : \tilde{X} \to T$, where $\tilde{X}$ is the cover of a band complex X such that $\pi_1(X)/\pi_1(\tilde{X}) = G$ and:

- For each generalized leaf $L \subset X$ and each component $\tilde{L}_0$ of its lift to $\tilde{X}$ the image $\rho(\tilde{L}_0)$ is a point.
- For each base $b \subset Y \subset X$ and each component $\tilde{b}_0$ of its lift to $\tilde{X}$ one can subdivide $\tilde{b}_0$ into finitely many subintervals I_j so that $\rho|I_j$ is an isometric embedding for each j.

The band complex X is said to be **resolving** the tree T. The resolution ρ is called **exact** if for each point $t \in T$ the preimage $\rho^{-1}(t)$ is path connected.

Clearly, if $(X, \tilde{X})$ gives an exact resolution of T then $(X, \tilde{X})$ is necessarily regular and the generalized leaves of X are path connected. Below we construct resolutions which are adapted to a collection of finitely generated subgroups F_i of G:

Theorem 12.33. *Suppose that G is a finitely presentable group, T is a G-tree and $F_1, ..., F_s$ are finitely generated subgroups of G each of which fixes a point in T (s can be equal to zero). Then T has a resolving regular band complex X such that $\tilde{X}$ is the universal cover of X and for each F_j there exists a connected subgraph $D_j \subset \partial Y$ such that F_j is contained in the image of $\pi_1(D_j)$ in G.*

Proof: Let $\langle g_1, ..., g_n | R_1, ..., R_k \rangle$ be a finite presentation for the group G where the list of generators includes all generators of the groups $F_1, ..., F_s$. Let $Z = \{z_1, ..., z_s\} \subset T$ be a finite collection of fixed points of the groups $F_1, .., F_s$. Pick a point $x \in T$ and consider a finite subset $S \subset T$ which consists of:

- $\{x\} \cup Z$ and the images of $\{x\} \cup Z$ under $g_j^{\pm 1}$, $j = 1, ..., n$.
- The images of x under all subwords in the relations R_i, $i = 1, ..., k$.

Let K be any finite subtree in T which contains the set S, for instance we can take the convex hull of S as the subtree K. Consider the collection of maps

$$\phi_j : T_j := K \cap g_j^{-1} K \to T'_j := g_j K \cap K$$

which are the restrictions of the actions of g_j, $j = 1, ..., n$. The subtrees T_j cover K (because of the first condition on S). We introduce structure of a simplicial complex on K as follows: the set of vertices consists of $\{x\} \cup Z$ and of the vertices of K. Each edge of the resulting simplicial complex has length induced from K. Finally, K becomes a *connected* real graph Γ after we orient each edge.

Next we construct the corresponding band complex Y. For each $j = 1, ..., n$ we do the following. For each each edge of Γ choose maximal (with respect to the inclusion) arcs b so that:

- $b \subset T_j$.
- $\phi_j(int(b))$ is disjoint from $Vertices(\Gamma)$.

We will say that the arcs b *correspond* to j. Note that the arcs b (corresponding to different indices j) can overlap, there are only finitely many arcs b and they cover the whole graph Γ. The arcs b (and their images in K under ϕ_j's) will be the bases of the union of bands Y. Then for each $j = 1, ..., n$ and each base b *corresponding* to j, we attach to Γ the band $b \times I = B$, where the attaching map on the bottom b is the identity and the attaching map on the top is $\phi_j \circ \delta_B$. This defines Y. We construct the band complex X as follows:

Step 1. First of all, for each j and for each pair of distinct bands $B = b \times I, B' = b' \times I$ (whose bases $b = [w, y], b' = [w, y']$ *correspond* to j) we attach the disk $D^2 \cong I \times I$ so that:

- The vertical sides of D are identified with the vertical sides $w \times I \subset B$ and $w \times I \subset B'$ of the bands B, B'.
- The horizontal edges of D are mapped to the points w and $\delta_B(w) = \phi_j(w)$ respectively.

Let X' be the CW complex resulting from this procedure, $\pi_1(X')$ is isomorphic to the free group on n generators, these generators correspond to the segments

$$w_j \times I \subset b_j \times I$$

where we choose one edge b_j and one end-point w_j of b_j for each pair j.

Step 2. Each relation R_i is represented by a vertical loop r_i in Y which is based at the vertex x (by the second condition on the finite set $S \subset T$). Attach to X' a 2-disk along r_i. Repeat this for each $i = 1, ..., k$.

Note that each subgroup F_j of G is generated by closed vertical boundary loops in Y which are based at the vertex z_j. Then the subgraph D_j in the assertion of Theorem is the union of these loops.

Let $\tilde{X}$ be the universal cover of the band complex X. Cut X open along the top bases $dual(b)$. We get a simply-connected complex X^0 equal to K with 2-cells and bands B attached along the bottom bases b. We retain the notation X^0 for a lift of X^0 to $\tilde{X}$. Then X^0 is a fundamental domain for the action of G on $\tilde{X}$. There is a canonical mapping $\rho : X^0 \to T$ which is the identity embedding $K \hookrightarrow T$ on K and is constant on each vertical segment and on each 2-cell. Since $X^0 \subset \tilde{X}$ is a fundamental domain for the action of G, the map ρ extends to a G-equivariant map $\rho : \tilde{X} \to T$ which is a resolution for the G-tree T. We leave it to the reader to verify that the pair $(X, \tilde{X})$ is regular. □

Remark 12.34. In the above theorem the map $\rho : T_X \to T$ factors through $q : \tilde{X} \to T_X$ and becomes a morphism $f : T_X \to T$ of G-trees. The projection of $K \subset X^0 \subset \tilde{X}$ to T_X is an isometric embedding as well as the map $f : q(K) \to K \subset T$.

We will say that resolving complex X satisfying the property:

> for each F_j there exists a connected subgraph $D_j \subset \partial Y$ such that F_j is contained in the image in G of $\pi_1(D_j)$, $j = 1, ..., s$,

is a *relative resolving complex* with respect to the collection of subgroups $\{F_j\}$.

Definition 12.35. A stable G-tree T is called **pure** if it admits a **pure resolving band complex** X, i.e. X is minimal, Y is connected and $\pi_1(Y) \to \pi_1(X)$ is onto.

Note that the resolving band complex X constructed in Theorem 12.33 is always such that Y is connected and $\pi_1(Y) \to \pi_1(X)$ is onto. However in general the complex X is not minimal.

12.7. Holonomy of vertical paths in X

Pick a band $B = b \times I$ in a union of bands Y, let $b \subset \gamma, dual(b) \subset \alpha$ where γ, α are edges of Γ. Then B determines an isometry $h_B : \gamma \to \alpha$ whose restriction to b equals $\delta_B : b \to dual(b)$. If we interchange the top and the bottom of the band B we get the inverse isometry $h_B^{-1} : \alpha \to \gamma$. We will say that h_B is the *holonomy* of any vertical path in B which connects b to $dual(b)$, we also call h_B the holonomy of B. We say that B is *orientation-preserving* (resp. *orientation-reversing*) if its holonomy is orientation-preserving (resp. *orientation-reversing*). Suppose that we have a cellular path $p : [0, 1] \to L$ where L is a leaf in Y, $x = p(0) \in b$, $y = p(1) \in c$

where b, c are bases in Y and a horizontal neighborhood of x in b pushes along p into a horizontal neighborhood of y in c. Then (by subdividing p appropriately) we represent p as a composition $p_1 * ... * p_n$ of paths each of which is contained in a single band B_j. Hence we can define the *holonomy* $h_p : b \to c$ of p as the composition of holonomies of the paths p_j. It is clear that the holonomy of p depends only on the *combinatorics* of p, i.e. the sequence of bands $(B_1, ..., B_n)$.

We say that p is *orientation-preserving* (resp. *orientation-reversing*) if h_p preserves (res. reverses) the orientation. In general this property depends on the choices of orientation on edges β and γ of Γ containing b and c, however if $\beta = \gamma$ then it is independent on the orientation of edges of Γ.

If b, c are contained in a single edge $\beta = \gamma$ of Γ and p is orientation-preserving we define the *translation length* $\ell(p)$ of p as the translation length of the isometry $h_p : \gamma \to \gamma$.

Here is an interpretation of these notions in terms of actions on trees. Suppose that X is a band complex based on Y which resolves an action of $G = \pi_1(X)$ on a tree T. We assume also that under this resolution each edge γ of Γ embeds isometrically into T and that path p is such that b, c are contained in a single edge γ. Define the *closing up* $\widehat{p}$ of the path p as the composition $\lambda * p$ where λ is a path in γ connecting $y = p(1)$ to $x = p(0)$.

Then $\widehat{p}$ determines an element g of $G = \pi_1(X, x)$ which depends only on p. The choice of a lift of x to $\tilde{X}$ defines isometric embeddings $\iota : \gamma \hookrightarrow T, \iota : b \hookrightarrow T$. It is clear that:

- p is orientation-preserving if and only if the partially defined isometry
$$g : \iota(\gamma) \to \iota(\gamma)$$
is orientation-preserving.
- If $\ell(p)$ is defined then it is equal to the translation length of this partially defined isometry.

In particular, if the translation length of p is different from zero then g is a nontrivial element of G. Moreover, by Lemma 10.7, if $\ell(p)$ is different from zero then g is an axial isometry of T, the axis of g contains $\iota(b)$ and the translation length $\ell_T(g)$ of g equals $\ell(p)$.

12.8. Kazhdan-Margulis theorem for actions on trees

It is clear that direct generalization of Kazhdan-Margulis theorem fails for group actions on trees, for instance stabilizer of a point can be arbitrarily large, etc. Nevertheless it turns out that some residual form of Kazhdan-Margulis lemma still holds for stable actions on trees which admit resolution by minimal band complexes (see Theorem 12.39 and Corollary 12.40 below).

Definition 12.36. Let J be a nondegenerate subarc of an edge of Γ with a base-point z_0 in the interior of J. A loop α in X based at z_0 is said to be J-**short** if it is a composition $p_1 * \lambda * p_2$ where p_1, p_2 are paths in the interior of J and λ is a path in a **nonsingular** leaf of Y.

In this section we assume that X is a band complex which has no *faults*. In the following two propositions we assume that $C(Y)$ is a *minimal* component of Y. We let $C(\Gamma) := \Gamma \cap C(Y)$.

Proposition 12.37. *For each nondegenerate horizontal segment $J \subset C(Y)$ and each $z \in C(Y)$ there is $\epsilon > 0$ such that $N(z, \epsilon)$ pushes into J.*

Proof: Directly follows from minimality of $C(Y)$. $\square$

Proposition 12.38. *Let J be a nondegenerate subarc of an edge of Γ with a base-point z_0 in the interior of J. Then the image of $\pi_1(C(Y))$ in $\pi_1(X)$ is generated by J-short loops.*[4]

Proof: Recall that Y has only finitely many singular leaves. Let γ be a loop in $C(Y)$ based at z_0. Then γ is homotopic to a composition of paths

$$p_1 * \lambda_1 * p_2 * \lambda_2 * ... * \lambda_k * p_k \tag{12.1}$$

where each $p_j : I \to C(\Gamma)$ is an injective path in an edge of Γ and each path λ_i is contained in a *nonsingular* leaf. Using the previous proposition subdivide each segment $p_j(I)$ into subintervals $p_j(I_m)$ so that:

- Each $p_j(I_m)$ pushes onto a (parameterized) segment J_{jm} in J via homotopy $H_{jm} : I_m \times [0,1] \to X$, $I_m = [t_m, t_{m+1}]$.
- The end-points $p_j(I_m)^{\pm}$ of each $p_j(I_m)$ belong to nonsingular leaves.

Let $\sigma^+_{im}, \sigma^-_{im}$ be paths in the (nonsingular!) leaves of Y which connect $p_j(I_m)^+$ to the initial point of J_{jm} and the terminal point of J_{jm} to $p_j(I_m)^-$:

$$\sigma^+_{im}(\tau) = H_{jm}(t_m, \tau), \tau \in [0,1],$$

$$\sigma^-_{im}(\tau) = H_{jm}(t_m, 1-\tau), \tau \in [0,1].$$

See Figure 12.5. Now in the composition of paths (12.1) we replace each path p_j, $j = 2, ..., k-1$, by the composition of paths of the form

$$\sigma^+_{jm} * J_{jm} * \sigma^-_{jm}$$

where each $\sigma^{\pm}_{jm}$ is contained in a nonsingular leaf. This gives the decomposition of γ as a product of J-short loops. $\square$

Recall that if T is an H-tree then T_H denotes the unique minimal H-invariant subtree in T or is a fixed point of H in T (if H has any) or the empty set if H has no minimal invariant subtrees in T. The latter does not occur if H is finitely generated.

[4]The number of generators can be infinite.

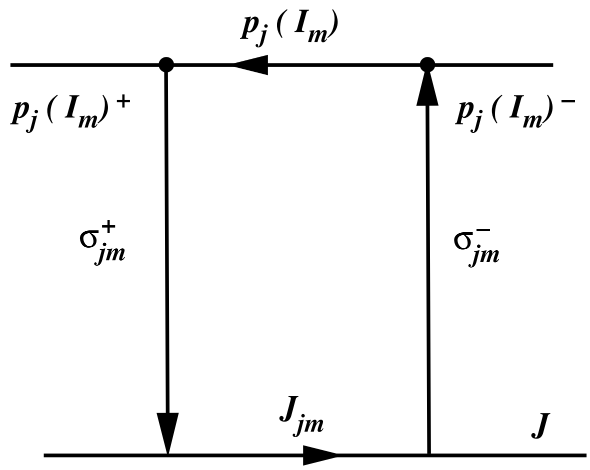

Figure 12.5: *Construction of short loops.*

Theorem 12.39. *Suppose that X is a (regular!) band complex resolving a stable G-tree T. Let $C(Y) \subset Y \subset X$ be a minimal component of Y and let H denote the image of $\pi_1(C(Y))$ in $G = \pi_1(X)$. If $h \in H$ is an element fixing an arc of T then h lies in the kernel of the action of H on T_H.*

Proof: We note that H is finitely generated, hence T_H is either a single point (in which case there is nothing to prove) or a nondegenerate tree. (Actually, because $C(Y)$ is a minimal component it follows that H does not fix a point in T.) Choose a subarc $J \subset C(Y)$ of an edge of Γ so that J maps via the resolution $\rho : \tilde{X} \to T$ to an arc $\rho(J)$ in a maximal stable subtree $S \subset T_H$. Let $\mathcal{G}_J$ be the set of generators of G constructed in the Proposition 12.38. For each generator $g \in \mathcal{G}_J$ we have:

$$g(\rho(J)) \cap \rho(J) \quad \text{is a nondegenerate arc.}$$

Since S is a maximal stable subtree, our discussion in §12.1 implies that $g(S) = S$. However T_H is minimal with respect to the action of H, hence $S = T_H$. Stability implies that if $h \in H$ fixes an arc in S it fixes the whole S. Theorem follows. □

Corollary 12.40. *Under the assumptions of Theorem 12.39 suppose that α, β are elements of H acting as axial isometries of T. If the length of $L := Axis(\alpha) \cap Axis(\beta)$ is larger than the sum of the translation lengths of α and β then $Axis(\alpha) = Axis(\beta)$.*

Proof: Consider the commutator $\gamma = [\alpha, \beta]$. Let J be a subsegment of L adjacent to one of the end-points so that

$$length(J) = length(L) - \ell_T(\alpha) - \ell_T(\beta) > 0.$$

Our assumptions imply that the arc J is fixed by the element γ. Hence γ acts trivially on the tree T_H. Thus the restrictions of α, β to T_H commute. The tree T_H contains the axes of α and β, therefore $Axis(\alpha) = Axis(\beta)$. □

Remark 12.41. This corollary is similar to the behavior of axes of hyperbolic elements g, h in discrete groups of isometries of hyperbolic spaces (which follows from the Kazhdan-Margulis theorem): if axes A_g, A_h of g, h are sufficiently close to each other along subintervals of A_g, A_h whose lengths are sufficiently long (with respect to the translation lengths of g, h), then $A_g = A_h$. (Compare the proof of Theorem 10.24.)

The following is a construction (due to F. Paulin) of a band complex which resolves unstable action on a tree.

Example 12.42. Unstable action of a finitely generated group on a tree.

Let T^2 be the 2-torus with measured geodesic foliation $\mathcal{F}$ where each leaf is dense. Then the dual tree to this foliation is the line $\mathbb{R}$ with a nondiscrete isometric action of $\mathbb{Z}^2$. Cut T^2 open along an arc θ which is contained in a leaf of the foliation and represent the result Y as a union of two bands based on graph Γ which is a single segment. Attach a 2-cell to ∂Y to recover the torus T^2; the result is a band complex Z.

Pick disjoint geodesic segments b, b' in Γ which have the same transversal measure. Attach to Z the foliated band B with the bottom b and the top b' so that the width of B equals the transversal measure of b. We get a measured foliation on the resulting band complex X. We leave it to the reader to check that $(X, \tilde{X})$ is regular (where $\tilde{X}$ is the universal cover of X), that X has no faults and is minimal. Let T be the quotient tree $\tilde{X}/\sim$, $G := \pi_1(X)$. Since each leaf of the foliation $\mathcal{F}$ is dense, there exists an arc α contained in a leaf of $\mathcal{F}$ which connects a point $x \in b$ to a point $y' \in b'$. Let β be a vertical arc in B which connects y' to a point $y \in b$. Finally, let $\gamma \subset b$ be arc connecting y to x. Let $g \in G$ be the element which corresponds to the composition of paths $\alpha * \beta * \gamma$. We will use x as the base-point for $\pi_1(X)$. The tree T contains a geodesic L_1 which is the image of the quotient of the universal cover of T^2 by the lift of $\mathcal{F}$. This geodesic is stabilized by $G_1 := \mathbb{Z}^2 = \pi_1(T^2, x)$. Let $L_2 := g(L_1)$, it is stabilized by $G_2 := gG_1g^{-1}$. Then $L_2 \cap L_1$ is a finite arc $\tilde{b}$ (a copy of b). The group G_1 contains elements h with arbitrarily small translation lengths. On the other hand, the intersection of the axes of h and ghg^{-1} is $L_1 \cap L_2 = \tilde{b}$. Therefore Corollary 12.40 implies that the action of G on T is unstable.

12.9. The moves

In this section we describe a collection of moves that transform one band complex X to another band complex X'. These moves have the property that:

- $\pi_1(X) \cong \pi_1(X')$.
- If X resolves a G-tree T then X' also resolves T.
- If the resolution X of T was exact then the new resolution X' is exact as well.

- If we are given a finite collection of subgroups $F_i \subset G$ as in Theorem 12.33 and X is a relative resolving complex with respect to this collection of subgroups then X' also is.

Move M0: "Add a 2-cell". Add a 2-cell D to X along a loop λ in the union

$$Y \cup \ \text{1-skeleton of} \ \ X$$

so that λ meets Y in a union of finitely many vertical fibers and λ is nulhomotopic in X.

Move M1: "Add an annulus". Let J be a subarc of an edge of Γ. Attach to X a weight zero band B with the base J and a 2-cell D along the loop that is formed by one of the vertical fibers of B.

Move M2: "Subdivide a band". Let $B = [p, q] \times I$ be a band of Y. Pick a point x in the open interval (p, q). We obtain Y' by replacing B with the two bands $B_1 = [p, x] \times I$ and $B_2 = [x, q] \times I$ by splitting B open along the vertical segment $x \times I$. As we result we get two copies of is vertical segment σ, σ'. Attach a 2-cell D along the loop $\sigma \cup \sigma'$. The cell D is called a *subdivision cell.*

Move M3: "Split a point". Suppose that z is a point of Γ that does not meet the interior of any base, however z is a local cut-point and belongs to at least one base. Split Γ open at z, the resulting graph Γ' has a collection of vertices $Z := \{z_1, ..., z_s\}$ corresponding to z. Add to Γ' a cone $Cone(Z)$ over Z (this cone will be a part of $X' - Y'$). Attach bands and cells to $\Gamma' \cup Cone(Z)$ as before to get X'.

Lemma 12.43. *By applying a finite sequence of the moves M2 and M3 to a band complex X we may arrange that X has no faults.*

Proof: We recall that according to our discussion in §12.3 the band complex X has only finitely many faults in X, each of which is a finite subgraph in a singular leaf, the frontier of this subgraph consists of *critical points*. We split Y open along each fault using the moves M2 and M3. Recall that there are two types of critical points in the singular leaves of Y (*local cut points* and *dead ends*), the Move M3 eliminates the local cut-points, the Move M2 eliminates the dead ends. These moves also do not introduce new critical points (on the frontiers of faults). Thus using the Moves M2 and M3 we eliminate all critical points (on the frontiers of faults), as the result we get a band complex without faults. □

We can assume that the boundary of each 2-cell in X is subdivided so that if the interior of an edge of the subdivision meets Y then this edge is contained in Y.

Theorem 12.44. *Let G be a finitely presentable group and $G \curvearrowright T$ be a stable action. Suppose that the band complex X has no faults and resolves T. Then G has a graph of groups decomposition such that:*

- *The action on T of each vertex group of the decomposition is either* **trivial** *or* **pure**.

- *The action on T of each edge group is* **trivial**. *Moreover, each such group is contained in the image in G of the fundamental group of a leaf of X.*

- *If we are given a finite collection of subgroups $F_i \subset G$ as in Theorem 12.33 and X is a relative resolving complex with respect to this collection of subgroups, then each F_i is contained in the image of one of the vertex groups of the decomposition of G (up to conjugation).*

Proof: By Theorem 12.24, each component $C(Y)$ of Y is either minimal or simplicial, in the latter case $\pi_1(C(Y))$ fixes a point in T. It is clear that $cl(X - Y)$ has a natural CW-complex structure. We give X the structure of a graph of spaces as follows:

(a) The vertex spaces are the components of Y and of $cl(X - Y)$.

(b) The edge spaces are the components of intersection of $cl(X - Y)$ and Y. Note that each edge space is contained in a leaf of Y.

Remark 12.45. This decomposition of X does not satisfy one of the properties of a graph of spaces (Definition 10.20): the images of the edge-spaces can intersect. To resolve this problem we use (as the vertex-spaces) instead of the components of $X - Y$ their deformation retracts and (as the edge-spaces) instead of the components Σ of the intersection $Y \cap cl(X - Y)$, the appropriate neighborhoods of Σ in X which admit deformation retractions to Σ. The details of this are left to the reader.

If a vertex space X_v is a simplicial component of Y or a component of $cl(X-Y)$ then the action of $\pi_1(X_v)$ on T has a global fixed point. Otherwise X_v is a minimal component of Y. Hence in this case the action of $\pi_1(X_v)$ on T is pure. $\square$

Note that vertex and edge groups in this graph of groups decomposition of G can be non-isomorphic to the fundamental groups of vertex and edge spaces, they are merely the quotients of such groups. Hence, a priori vertex and edge groups can fail to be finitely presentable. (We choose the coverings $\tilde{X}_v \to X_v$ using the restrictions of the universal covering $\tilde{X} \to X$ to the inverse images of X_v in $\tilde{X}$.) At this stage we also cannot control the group-theoretic structure of the edge groups except for the edge groups incident to the vertices corresponding to simplicial components of Y.

Move M4: "Slide". Suppose that we have a pair of distinct bands $B = b \times I$ and $C = c \times I$ in a band complex X. Assume that $f_b(b) \supset f_c(c)$. We change the union of bands Y to Y' by changing the attaching map f_c to

$$f_c' := f_{dual(b)} \circ \delta_B \circ f_b^{-1} \circ f_c.$$

Informally, we "slide" the base c from b to $dual(b)$ along the band B so that each point of c "slides" along a vertical segment of B, as the result the band C in X transforms to a band C' in X', see Figure 12.6.

We say that b is the **carrier** of the slide and c is the **carried** base.

Remark 12.46. Note that the move M4 can possibly increase the complexity. Here are the possibilities how it may happen:

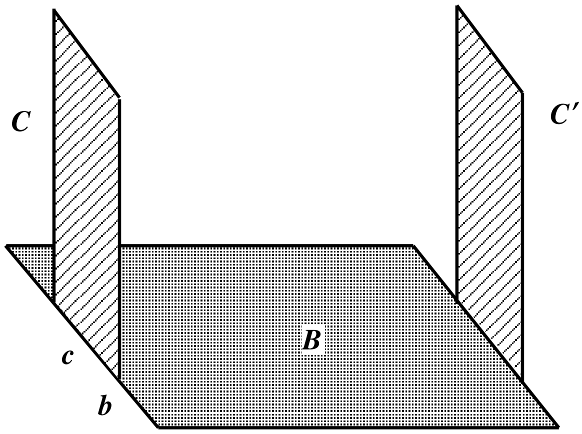

Figure 12.6: *Slide.*

(1) The carried base c has weight zero.

(2) The carried base c has weight $1/2$.

(3) The carried base c has weight 1 and is contained in a block β of zero complexity, which means that β contains only the bases b and c whose weights are equal to 1. See Figure 12.7. (The complexity will not change if $weight(b) = 1/2$.)

To avoid the increase of complexity in the first two cases we should slide along B both bases of the band C in a single move.

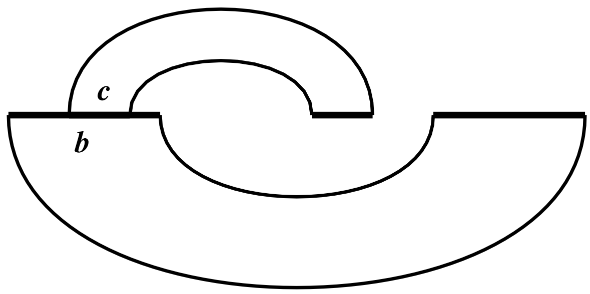

Figure 12.7: *Two bands of the unit weight.*

Lemma 12.47. *Suppose that we have a nondegenerate segment $s \subset \Gamma \subset X$ such that s pushes onto a segment $s' \subset \Gamma$. Then there is a finite sequence of moves M1 and M4 which transforms X to a band complex X' whose underlying union of bands Y' equals $Y \cup A'$ where A' is a band with the bases s, s'.*

Proof: First we attach a band A of zero weight to s via the move M1. Now keep one of the bases a of A attached to s and realize the pushing of s onto s' by a sequence of slides of $a' = dual(a)$ that terminates when we slide a' onto s'. $\square$

The following two lemmas are used to describe the next (and the last) Move M5 in presence of weight $1/2$ bases.

Lemma 12.48. *Let $B = b \times I$ be a band with weight$(b) = 1/2$. Then there is a sequence of moves that replaces B by a pair of bands $B' = b' \times I$ and $A = a \times I$, where weight$(b') = 1$, weight$(a) = 0$ and $b' = a$ (i.e. the images of these bases in Y coincide), $b' \cap dual(b')$ is a single point.*

Proof: Subdivide B along the midline into two weight 1 bands (Move M2), then slide one of these bands over the other one (Move M4). $\square$

Lemma 12.49. *Let $B = b \times I$ be a band with weight$(b) = 1/2$, $b = [u, v]$, let m denote the midpoint of b. Pick a point $x \in (m, v)$ and denote by $y \in (u, m)$ the point dual to x. Then there is a sequence of moves which replaces B by weight $1/2$ band $B' = b' \times I$, $b' = [x, y]$, a weight zero band $D = d \times I$ with $d = [u, y]$ and a weight 1 band $E = e \times I$ with $e = [x, v]$ and dual$(e) = d$.*

Proof: First subdivide along the vertical loop in B that contains x, y. Then slide the band D along the band E (note that they share the common base e). See Figure 12.8. $\square$

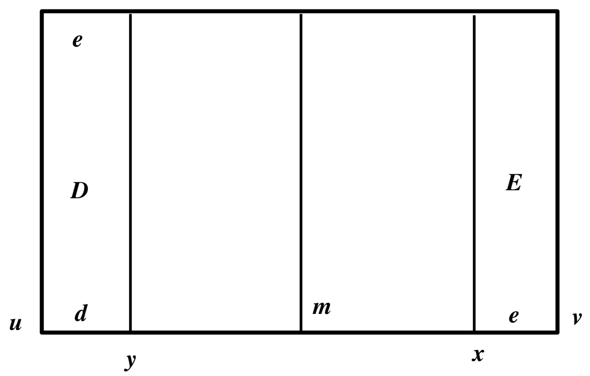

Figure 12.8: *Subdivision of a Moebius band.*

Definition 12.50. A subarc c of a base b is **free** if either:

1. b has weight 1 and the interior of c does not meet any other base of positive weight.

2. b has weight $1/2$, the interior of c does not contain the midpoint of b and the bands $b, dual(b)$ are the only positive weight bases that meet the interior of c.

Below we describe the Move M5, I recommend the reader to assume first that the free arc c of a base b does not meet interiors of any bases (except b) and that $weight(b) = 1$. See Figure 12.10.

Move M5: "Collapse from a free subarc". Let $B = b \times I$ be a band in Y and $c = [p, q]$ be a free subarc of b.

(1) We first use the Move M2 to subdivide all zero weight bands meeting c until for every zero weight base d that meets the interior of c we have: $d \subset c$. This step creates subdivision disks. Let X_1 be the resulting band complex.

(2) Next, use the Move M4 to slide over B (into $dual(c)$) all weight zero bases d contained in c. As the result the subdivision disks become annuli (called *subdivision annuli*). [5] Now c does not meet interior of any base but b (except if $weight(b) = 1/2$ then c also meets $dual(b)$). Let X_2 denote the resulting band complex.

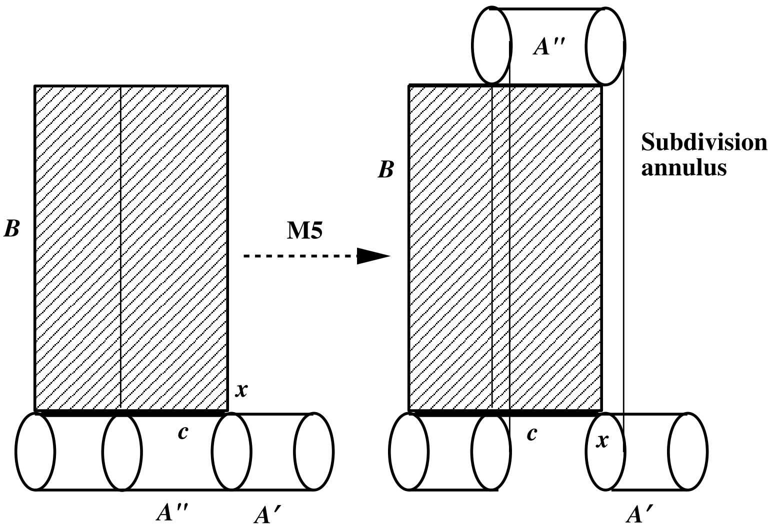

Figure 12.9: *Creation of the subdivision annuli.*

(3a) If $weight(b) = 1$ then collapse $c \times I$ to $dual(c) \cup \partial c \times I$. In the special case when one of the end-points x of c is also an end-point of b and the open segment $x \times (0,1)$ does not meet any cells in X_2, we eliminate $x \times (0,1)$. We modify the graph Γ by removing the interior of c. The result is the band complex X'.

There are two other cases when $weight(b) = 1/2$ and c is contained in one half of b.

(3b) Suppose that $weight(b) = 1/2$ and one of the end-points x of c is the mid-point of b. Then apply Lemma 12.48 to the band B transforming it to an annulus A and a weight 1 band B' so that the interior of c is disjoint from A. Now c is a free arc in a weight 1 base of B' and we can apply (3a).

---

[5]Here is a description of how these annuli look like. On the Step (1) we split open a weight zero band A into two bands, say A', A'', where we suppose for simplicity that A' is disjoint from the interior of c and the base of A'' is contained in c. As the result we get two vertical circles: one in A', the second is in A''. These circles are identified in X and share a common end-point x of c; these loops are homotopic in X_1, the homotopy is given by a subdivision disk. After the sliding (Move M4) the circles become disjoint, however they still have to be homotopic: the homotopy is given by a subdivision annulus that contains the vertical interval $x \times I$ of B. See Figure 12.9.

(3c) Suppose that (3b) does not occur, let x be the end-point of c which is the closest to the midpoint m of b (and $m \neq x$). Then, similarly to the case (3b), apply Lemma 12.49 and reduce the problem to the case when c is a free arc in a weight 1 base.

This finishes description of the Move M5.

Remark 12.51. (1) Suppose that q is the end-point of c that belongs to the interior of b. Then the open vertical segment $q \times (0,1) \subset X'$ meets $cl(X' - Y')$ only along the subdivision annuli.

(2) If $D_j \subset \partial Y$ is a connected subgraph which carries the subgroup F_j and D'_j is the corresponding subgraph in $\partial Y'$ then the intersection $D'_j \cap q \times (0,1)$ is empty. See Figure 12.10.

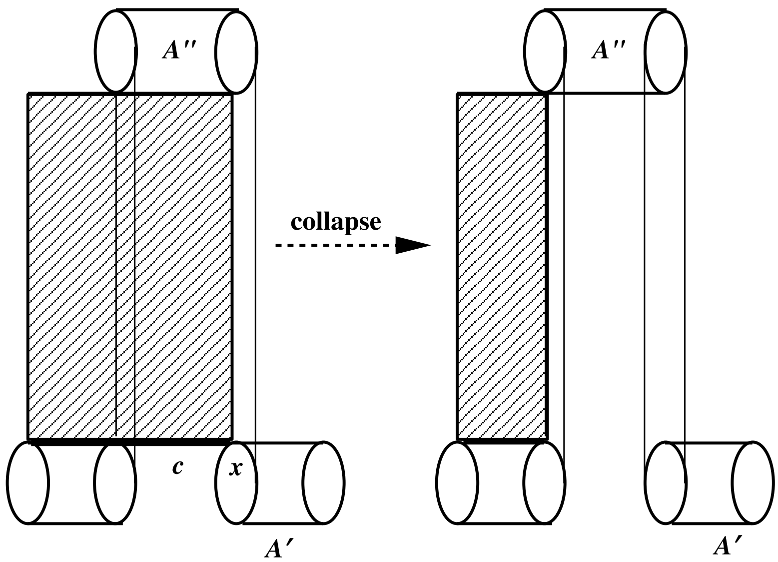

Figure 12.10: *Collapse in the Move M5.*

12.10. Preliminary motions of the Rips machine

We start with the action of a finitely presentable group G on a tree T, in the relative case we are given a finite collection of finitely generated subgroups F_i each of which fixes a point in T. Then we construct a (regular) band complex X which resolves this action, the map $\rho : \tilde{X} \to T$ from the universal cover of X to T is as in Theorem 12.33.

We now use the Moves M2 and M3 to transform X to another band complex (which we still denote X) so that the new band complex X satisfies the following properties:

(A1) The real graph Γ is the disjoint union of its edges. Each edge is a block.

(A2) The union of bands is a disjoint union of components each of which is either minimal or simplicial and each simplicial component is a trivial I-bundle.

(A3) If $B = b \times I$ and $A = a \times I$ are weight $1/2$ bands such that a, b share midpoints then $B = C$ (this is achieved via the Move M2).

(A4) The resolution $\rho : \tilde{X} \to T$ embeds lifts of bases into T, where $\tilde{X}$ is the universal cover of X.

(A5) The complex X has no faults.

12.11. The machine

Definition 12.52. A sequence $B_1, ..., B_n$ of weight 1 bands forms a **long band** B provided that, after exchanging if necessary tops and bottoms of some B_j's, the following is satisfied:

1. The top of B_j is contained in the bottom of B_{j+1} for $j = 1, 2, ..., n-1$.
2. $B_i \cap B_j = \emptyset$ if $|i - j| > 1$.
3. $B_1 \cup .... \cup B_n$ intersects other bands of positive weight in a subset of the union of the bottom b of B_1 and the top b' of B_n.

See Figure 12.11.

We will refer to the bottom of B_1 and the top of B_n are the bottom and top of the long band B. We let δ denote the composition $\delta_{B_n} \circ ... \circ \delta_{B_1}$.

Suppose that $b = [x, y]$ is a base of weight $1/2$ with the midpoint m. Then the *halves* of b are the intervals $[x, m]$ and $[m, y]$. We will refer to these halves as *half-bases.*

Definition 12.53. (Isolated bases.) A weight 1 base is **isolated** if its interior does not meet any other base of positive weight. A half h of a base b (of the weight $1/2$) is **isolated** if the interior of h meets no positive weight bases other than $b, dual(b)$.

A weight 1 base b is **semi-isolated** if a punctured neighborhood in b of one of its endpoints p meets no other positive weight bases. A half h of a weight $1/2$ base b is **semi-isolated** if a punctured neighborhood of an end-point p of h meets no positive weight bases other than $b, dual(b)$. In these cases we say that b and h are semi-isolated at p.

Remark 12.54. Note that if there is isolated half then we get contradiction with the minimality assumption, the same is in the case when we have semi-isolated half which is semi-isolated at the mid-point p of the corresponding base. The pushing-saturated non-dense subset is the mid-loop of the Moebius band through p.

Below is the list of rules (R0)–(R4) which describe the Process I as applied to a **minimal** component $C(Y)$ of a union of bands Y (satisfying properties (A1)–(A5)), the Process I transforms X to a band complex X':

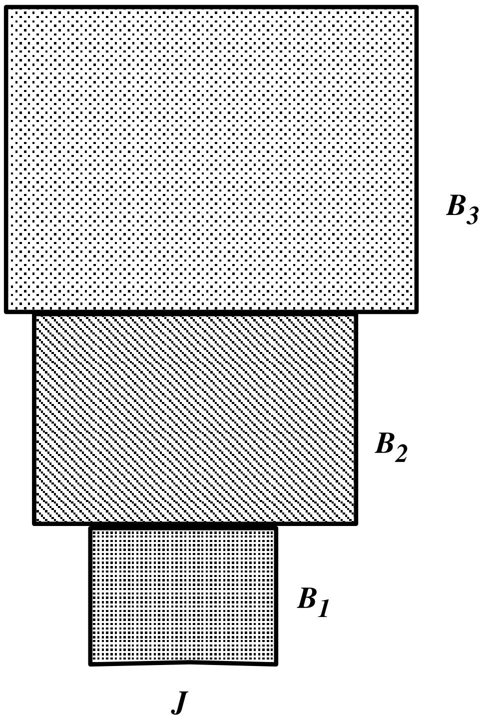

Figure 12.11: *A long band.*

(R0) If there is no free subarc of a base b of $C(Y)$, then we go to the Process II. Otherwise we find a maximal (with respect to inclusion) free subarc J of a base b of $C(Y)$. Use the Move M5 to collapse from J. It could be that there are several such subarcs J, here is how we choose one of them:

(R1) If there is an isolated (half-) base c take $J = c$ (if there are several choices pick at random).

(R2) Suppose there are no isolated (half)-bases but there is a semi-isolated (half-) base c which is semi-isolated at end-point p. Then choose a maximal free subarc $J \subset c$ that contains p.

(R3) If there are no isolated or semi-isolated (half-) bases then choose J at random.

(R4) Treat the *long bands* as units. Namely, if we have a long band $(B_1, ..., B_n)$ and we use J in the bottom of B_1 then we perform n consecutive collapses, i.e. we collapse from J then from the bottom of B_2, etc, and finally from the bottom of B_n. These n operations are treated as a single move producing X'.

Remark 12.55. Let J be a free arc in the bottom of a long band B and $\delta(J)$ is defined as in Definition 12.52. Then:

- The arc $\delta(J)$ contains no free subarcs. Indeed, otherwise we would have vertical leaves in B terminating at $\delta(J)$ which are not dense in $C(Y)$, this contradicts minimality of $C(Y)$.

- If $p \in int(\delta(J))$ is an end-point of a base, then there is a base c distinct from the top of B which contains p. We verify this is the case when top of B_i is the bottom of B_{i+1} for all i. Connect p by a vertical segment λ to the point $q = \delta(p) \in J$. If the top of B is the only base containing p then the point p is a **dead end** and the segment λ is a fault in the leaf through p which is a contradiction.

Note that the Process I does not change the number of components of Y.

Lemma 12.56. *Suppose that X' is produced from X by Process I. Then*

$$Complexity(X') \leq Complexity(X).$$

Proof: We will consider the "generic" case when (R3) is applied, the rest is left as an exercise to the reader. Let B be the (long) band whose bottom base b contains J and γ be the block containing b. Then after collapse from J the block γ splits into two or three components depending on the location of $\delta_B(J)$. We consider the case when γ splits into two components γ', γ'' of nonzero complexity and $\delta_B(J)$ is contained in a block $\alpha \neq \gamma$ so that $Complexity(\alpha)$ is positive. Let α' be obtained from α as the result of collapse from J (note that α' is still a single block by the Remark 12.55). Then after collapse from J we have:

$$Complexity(\gamma') + Complexity(\gamma'') =$$
$$-3 + \sum_{c \text{ is a base in } \gamma} w(c) = Complexity(\gamma) - 1$$

and

$$Complexity(\alpha') \leq Complexity(\alpha) + 1$$

Therefore

$$Complexity(\gamma') + Complexity(\gamma'') + Complexity(\alpha') \leq$$
$$Complexity(\gamma) + Complexity(\alpha)$$

As far as other blocks are concerned, their complexity does not change. □

Remark 12.57. There is a special case when B is a long band and the Move M5 splits each intermediate base $b_i, 1 < i < n$, into two components. However each of these b_i forms a block of zero complexity and splitting into two pieces does not change this fact.

The following proposition describes the result of application of the Process I.

Proposition 12.58. *Suppose that we have an infinite sequence $X_0, X_i, X_2 ...$ of band complexes so that X_{i+1} is obtained from X_i via the Process I for each $i > 0$. Then the following hold:*

1. *The choice of J as in (R1) occurs only finitely many times.*
2. *The number of weight $1/2$ bases does not increase.*

3. *There is a number N so that all bands in each X_i can be organized into at most N long bands.*

4. *For any n there is a segment of length n contained in some leaf so that this segment is eventually collapsed.*

5. *The choice of J as in (R3) occurs infinitely many times.*

Proof: (1) Collapse from an isolated base J will eliminate one block and decrease the complexity of the block α containing the base $dual(J)$, unless α contains only two bases one of which contains $dual(J)$ in which case B is a part of a *long band* and the rule (R4) is applied. In any case the complexity of $C(Y)$ will decrease.

(2) Obvious.

(3) Using Lemma 12.56 we may assume that the complexity of X_i is a constant σ for all i. Suppose that the number of long bands of X_{i+1} is larger than the number of long bands in X_i. This can happen only if the rule (R3) is applied, i.e. there are no (semi-) isolated bases and half-bases. Hence each long band contains a base in a block of positive complexity, therefore the number of long bands in X_i is bounded by 3σ and the number of long bands can increase by at most 1 in X_{i+1}. We conclude that in each X_j the number of long bands is at most

$$3\sigma + \text{ the number of long bands in } X_0.$$

(4) Define inductively the functions $N(k,x), N(k), k \in \mathbb{N}$ where x belongs to a free subarc $J(x)$ of $X_{N(k)}$ that is eventually collapsed under the Process I.

Remark 12.59. Note that it is a priori possible that certain free arc in a base of X_0 is never collapsed.

Set $N(0,x) := 0, N(0) := 0$. Suppose that $N(k,x), N(k)$ are defined. Let $N(k+1,x)$ be the least integer such that the arc $J(x)$ containing x is collapsed at the stage $N(k+1,x)$. Note that for the fixed k the function $N(k+1,x)$ is constant on the interior of each free arc in $X_{N(k)}$. Let

$$N(k+1) := \max_x N(k+1,x).$$

Clearly $\lim_{k\to\infty} N(k) = \infty$. Each point $x \in X_0$ that is collapsed at the stage $N(n)$ belongs to a vertical segment of length n in X_0 that is totally collapsed after the stage $N(n)$ of the Process I.

(5) Recall that $\eta(X_i)$ (the *height* of X_i) is the total number of bands in X_i (here we *do not* count long bands as a single band). Suppose that only collapses of the type (R2) occur. This implies that the height $\eta(X_i) = \eta$ does not depend on i. Our goal is to show that these assumptions imply that the Process I will terminate after finitely many steps.

By (4) for each $n > 0$ there is an (oriented) vertical segment λ of length n which is collapsed after the stage $N(n)$ of the Process I. Choose $n > 4\eta$. Then the segment λ intersects a band $B = b \times I$ in X in at least five distinct vertical segments λ_j. Among these five segments we choose three segments $\lambda_1, \lambda_2, \lambda_3$ which have the following property:

Orient λ_j using orientation of λ and order these segments so that λ_1 is the first and λ_3 is the last with respect to the linear order on λ. Let x_j denote the first point of λ_j that belongs to b. Then we require the paths $s_1 = [x_1, x_2]$ and $s_2 = [x_2, x_3]$ to be orientation-preserving (see §12.7).

Then the segments λ_j are separated by the distance $\theta > 0$ which *does not depend* on B and λ but depends only on the original complex of bands X.[6]

At least one of these segments (say λ_2) is separated from the vertical boundary of B by the two other segments λ_1, λ_3. Hence the distance from λ_2 to the both vertical boundary segments of B is at least θ. By our assumption the segment λ is collapsed after the stage $N(n)$ of the Process I and only the rule (R2) is applied, thus after the stage $N(n)$ the length of at least one base (namely b) had decreased by at least θ. We apply this argument inductively to conclude that between the stages $N(kn)$, $N((k+1)n)$ the length of at least one base had decreased by at least θ. This implies that eventually all bases will be eliminated and the Process I will terminate after finitely many stages. Contradiction. $\square$

Now we describe the **Process II.** Let $C(Y)$ be a minimal component of Y. We suppose that the Process I does not apply to $C(Y)$. Hence:

(*) The weight of each point $z \in C(\Gamma) := \Gamma \cap C(Y)$ is at least 2.

(Otherwise z would be contained in a free arc.) We identify the components of $C(\Gamma)$ with disjoint closed intervals in $\mathbb{R}$. This gives $C(\Gamma)$ a linear order. Let F be the first component of $C(\Gamma)$ and x be the initial point of F. Let b denote the longest base of positive weight containing x (take b to have weight 1 if such base exists, if there are several such bases then choose one at random).

Operation 1. Slide from b all those positive weight bases c contained in b (except of $b, dual(b)$) whose midpoint is moved away from x as the result of the slide. The base b is the *carrier* of each of these slides. We will say that the base c was *carried* during this operation.

Operation 2. Collapse from the maximal free initial segment J of b. Note that the segment J is necessarily nondegenerate.

Definition 12.60. The bands $b \times I$ and $c \times I$ are said to have **participated** in the Process II.

The **Process II** consists of consecutive application of the Operations 1 and 2. Let X' be the result of application of the Process II. We adopt the following convention about labeling the bases of positive weight in X and X':

(i) If $c \subset b$ is a base which participated in the Process II, then we retain the name c for the base in X' obtained from c via sliding along B.

[6] The holonomy transformations $g_1, g_2 \in G$ determined by the segments s_1, s_2 belong to a finite subset of G. The number θ is the minimal translation length of these transformations.

(ii) We retain the name b for the base in X' obtained from b by collapsing the segment J.

With this convention it makes sense to talk about *dynamics* of bases of X under the sequence of iterations of the Process II, for instance we talk about the *limiting position* and the *limiting length* of a base d of X.

Proposition 12.61. *If X' results from X via the Process II then:*

1. *$Complexity(X') \leq Complexity(X)$.*

2. *If $Complexity(X') = Complexity(X)$ then (*) holds for X'.*

Proof: (1) According to the Remark 12.46, the slides in the Operation 1 do not increase the complexity, except possibly the last slide where we slide from a block of zero complexity (case (3) in Remark 12.46). Let β denote the block in X containing b. If the last slide from b increases complexity then after all these slides only the base b of the block β remains and Operation 2 eliminates the block β completely thus decreasing the complexity.[7] The Operation 2 clearly cannot increase the complexity.

(2) Since (*) is satisfied by X there are the following reasons it might fail for X':

(a) There is a base c that after sliding along B becomes a base c' of weight 0 (this decreases the total weight of bases containing points of $int(c')$). However in this case $Complexity(X') < Complexity(X)$.

(b) Similarly a base c might become a base of weight $1/2$, this again leads to decrease of the complexity by $1/2$.

(c) Let J denote the maximal initial free segment of b that is collapsed by the Operation 2. It might happen that $w(z) \leq 2.5$ for some $z \in \delta_B(J)$ after the Operation 1 and the Operation 2 reduces the weight of z by 1. However this would imply that the base b of X contains a free subarc, in which case we have to apply the Process I instead of the Process II to the complex X. $\square$

If the complexity decreases, we terminate the Process II and continue with the Process I.

Proposition 12.62. *Consider a sequence $X_0, X_1, \ldots$ of band complexes so that for each $i \geq 0$ the complex X_{i+1} is obtained from X_i via the Process II. Then one of the following holds:*

1. *The sequence terminates after finitely many steps.*

2. *(The **surface case**.) For all sufficiently large i and each point of $\Gamma_i = \Gamma(X_i)$, which is not an end-point of a base, we have: $w(z) = 2$.*

3. *(The **toral case**.) (a) Every base has a limiting position in Γ. (b) Some base b is a carrier and is carried infinitely often. (c) The length of the base b does not converge to zero. (d) The bases $b, dual(b)$ have the same limiting positions in $\mathbb{R}$.*

[7] There is a special case when β has only two bases. These bases must be equal since X has no free arcs. We leave this case to the reader.

Proof: Suppose that neither (1) nor (2) occur, thus we have the *toral* case. Then the complexity of this sequence of band complexes is constant.

(a) Every base b has a limiting position since on each step we have: either b is carried, thus it moves to the right and its length remains constant, or b is the carrier, in which case the right end-point of b does not move and the length of b does not increase. This Proves (a). Without loss of generality we may assume that:

(i) If a band participates in the Process II it does so infinitely often.

(ii) Weights of bases do not change as the result of the Process II.

Remark 12.63. It is a priori possible that during the Operation 1 we have a band $c \times I$ of weight 1 which is carried and after that becomes a band of weight zero or $1/2$ (an annulus or a Moebius band). However this would lead to decrease of complexity which we assume is not the case.

Recall that we identify Γ with a disjoint union of intervals in $\mathbb{R}$. Let D_i denote the total measure of the subset of points in Γ_i between the initial point and the minimal point of Γ that lies in a band that never participates. Define

$$\tau_i = \sum \{mes(b) \cdot weight(b) | b \text{ is a base of } X_i \text{ which eventually participates}\}$$

$$excess(X_i) := \tau_i - 2D_i \ .$$

Lemma 12.64. *$excess(X_i) > 0$ and $excess(X_i) = excess(X_{i+1})$.*

Proof: The first assertion follows from the fact that the weight of each point is at least 2 and that the *surface* case (2) does not occur.

To verify the second assertion we analyze the Operations 1 and 2. As the result of the Operation 2 (when we collapse the maximal free initial subinterval J contained in the carrier d) both $2D_i$ and τ_i decrease by $2 \cdot mes(J) \cdot weight(d)$, hence the *excess* does not change. During the Operation 1 we only slide bands over each other and this does not change their weights (see the Assumption (ii) above), hence τ_i and D_i do not change as well. □

It follows from the above lemma that there is a base b of positive weight that participates infinitely often and its length does not converge to zero. We claim that the base b has to be carried infinitely often. Suppose not. Then the base $dual(b)$ is also carried only finitely many times. Thus without loss of generality we can assume that each time the base b (and $dual(b)$) *participates* in the Process II it is the carrier. Under these assumptions the right end-points of b and $dual(b)$ and the translation length ν of $B = b \times I$ (if B is orientation-preserving) remain (eventually) constant for all members of the sequence X_i. There are several cases to consider:

(i) b and $dual(b)$ have disjoint interiors and the corresponding band $B = b \times I$ is orientation preserving. Then each base can be carried by b only finitely many times, hence the Process II eventually terminates, which is a contradiction.

(ii) Suppose that B is orientation-reversing. Then a base c can be carried by b at most once and we get the same conclusion as in (i).

(iii) Suppose that the band B has weight 1 and is orientation-preserving. Then each base c which is carried from b moves away (to the right) from its initial point by the translation length $\nu > 0$ of b. If b is a carrier infinitely many times then some base c moves to the right by ν infinitely many times which is impossible.

This proves (b) and (c). Finally we consider (d). If the base b under consideration has the weight $1/2$ there is nothing to prove. Suppose $weight(b) = 1$ and $q > 0$ is the distance between the mid-points of $b, dual(b)$. Each time the base b is the carrier, a base carried by b moves the distance q to the right. Hence the distance q must tend to zero as we proceed with the Process II. $\square$

12.12. The machine output

Recall that we start with a finitely presentable group G acting on a tree T and a finite collection of finitely generated subgroups $F_j \subset G$ each of which fixes a point in T. We construct a band complex X which resolves an action of $G = \pi_1(X)$ on tree T (relative to the collection of subgroups F_j) as in Theorem 12.33.

Definition 12.65. If eventually only the Process I is applied to $C(Y)$ then $C(Y)$ is said to have the **thin type**[8]. In the case when eventually only the Process II is applied and $excess = 0$ then we say that $C(Y)$ has the **surface type**. Otherwise the Process II is applied and *excess* is a positive constant in which case $C(Y)$ has the **toral type**.

We let $\overline{Y}$ denote the union of bands obtained from Y by omitting all zero weight bands. Given a component $C(Y)$ of Y we shall use the following notation:

Set $\overline{C(Y)} := C(Y) \cap \overline{Y}$. We let ι denote the homomorphism $\iota : \pi_1(C(Y)) \to G$ induced by the inclusion, $H := \iota(\pi_1(C(Y)))$, let N be the normal subgroup of H normally generated by the images of the fundamental groups of annuli in the closure of $C(Y) - \overline{C(Y)}$ and let $\overline{H} := H/N$.

Now we start case-by-case discussion of the Machine output.

(i) Output of the Process II: surface type case.

Recall that a group is *cyclic* if it is 1-generated, this includes the trivial group, $\mathbb{Z}/n$ and $\mathbb{Z}$.

Proposition 12.66. *Suppose that $C(Y)$ is a component of Y which has the* **surface type**. *Then we have:*

1. *For each leaf L of $C(Y)$ the image of $\pi_1(L)$ in $G = \pi_1(X)$ admits a map onto a cyclic group Δ so that the kernel is contained in the kernel of the action of H on T_H.*
2. *Suppose that X is pure (thus $C(Y) = Y$, G is the image of $\pi_1(Y)$ in $\pi_1(X)$) and has surface type, then there is a short exact sequence:*

$$1 \to Ker(G \curvearrowright T) \to G \to \pi_1(O) \to 1$$

[8] In this case the widths of bands in $C(Y)$ converge to zero, thus the name.

where O is a cone type 2-dimensional orbifold. Each of the subgroups F_j of G maps into either peripheral or torsion subgroup of $\pi_1(O)$.

Proof: Without loss of generality we can assume that only the Process II can be applied to $C(Y)$ and $excess(C(Y)) = 0$. If $C(Y)$ has any bases of the weight $1/2$ then we apply Lemma 12.48 to the corresponding Moebius bands and transform each of these bases to a pair of bases of weight 1 (with disjoint interiors) and a pair of equal bases of zero weight (corresponding to an annulus). This way we eliminate all Moebius bands and still retain $w(z) = 2$ for each point $z \in C(\Gamma)$ that is not an end-point of a base. Now $\overline{C(Y)}$ is obtained by attaching bands to a collection of intervals in $\mathbb{R}$ so that: each point is either an end-point of a base (recall that this cannot be a *local cut-point*) or it belongs to exactly two bases. This implies that $\overline{C(Y)}$ is a surface (with boundary) S. Each nonsingular leaf of $\overline{C(Y)}$ is homeomorphic to $\mathbb{R}$. Recall that no singular leaf contains a proper compact pushing-saturated subset, hence each singular leaf of $\overline{C(Y)}$ is a boundary loop of S with finite (necessarily nonzero) number of rays attached. Hence each singular leaf in $\overline{C(Y)}$ has infinite cyclic fundamental group. Let L be a closed vertical loop in an annular band in $C(Y)$. Then the cyclic subgroup Δ of G (which is image of $\pi_1(L)$) fixes an arc in the tree T, hence Δ lies in the kernel of the action of H on the tree T_H (see Theorem 12.39). In general, the fundamental group of a leaf L of $C(Y)$ has a normal subgroup $N \cap \pi_1(L)$ (which is contained in the kernel of the action on T_H), the quotient group is cyclic (it corresponds to $L \cap \overline{C(Y)}$). This proves the first assertion of the proposition. Now consider the second assertion. We have a short exact sequence

$$1 \to N \to H \to \overline{H} \to 1.$$

Clearly $N = Ker(G \curvearrowright T)$. The group $\overline{H}$ is the fundamental group of $\overline{Y}$ with 2-cells attached along boundary components, hence $\overline{H}$ is isomorphic to the fundamental group of a cone type 2-dimensional orbifold O. (The nonempty singular locus appears when the degree of one of the attaching map is larger than 1.) This proves the Proposition in the absolute case (when we are not given a collection of finitely generated subgroups F_i fixing points in T). Each of the subgroups F_i is conjugate into the fundamental group of one of the *boundary leaves* of Y (see Theorem 12.33). Thus F_i projects either to a peripheral or to a torsion subgroup of $\overline{H}$. $\square$

(ii) Output of the Process II: toral case. We suppose that $C(Y)$ is a toral component of $Y \subset X$ and $X = X_1, X_2, \dots$ is a sequence of band complexes which is obtained from X via the Process II applied to $C(Y)$. By Proposition 12.62 we may assume that for each i the component $C(Y_i)$ has only one block. Let J denote the block of $C(Y)$. Each band of Y_i determines a partial isometry of J, extend this isometry to the isometry of $\mathbb{R}$ and let $\mathcal{B}(Y_i) \subset \mathrm{Isom}(\mathbb{R})$ denote the group generated by these isometries. Clearly this group is finitely generated and it is easy to see that $\mathcal{B} = \mathcal{B}(Y) = \mathcal{B}(Y_i)$ for all i. The group $\mathcal{B}$ is called the *Bass* group of $C(Y)$. Then each element of $\mathcal{B}$ is either a translation or a symmetry.

Lemma 12.67. *Let $\mathcal{B} \subset Isom(\mathbb{R})$ be a finitely generated subgroup.*

1. *Suppose that we have a sequence of translations* $\tau_j : x \mapsto x + t_j$, $j = 1, 2, ...$, *in* $\mathcal{B} - \{1\}$ *such that* $\lim_{j\to\infty} t_j = 0$. *Then this sequence contains translations* τ_i, τ_j *such that* $t_i/t_j \notin \mathbb{Q}$.

2. *Suppose that we have a sequence of distinct symmetries* $\sigma_1, \sigma_2, ...$ *in* $\mathcal{B}$ *such that* $\lim_j \sigma_j = \sigma_0$. *Then this sequence contains elements* $\sigma_i, \sigma_j, \sigma_k$ *such that the ratio of the translation lengths of* $\sigma_i\sigma_j$ *and* $\sigma_i\sigma_k$ *is irrational.*

Proof: We prove the first assertion of Lemma, the second is similar. Consider the subgroup $\mathcal{B}'$ of $\mathcal{B}$ generated by $\tau_1, \tau_2, ...$ Since $\mathcal{B}$ is a finitely generated virtually Abelian group, the subgroup $\mathcal{B}'$ is also finitely generated. Moreover, there is a number N so that the elements $\tau_1, ..., \tau_N$ generate $\mathcal{B}'$. The subgroup $\mathcal{B}' \subset \mathrm{Isom}(\mathbb{R})$ is not discrete. Hence there are two generators τ_i, τ_j of $\mathcal{B}'$ such that t_i/t_j is irrational. $\square$

Lemma 12.68. *We can apply the Moves M0–M4 and transform* X *to a band complex* X' *so that:*

There are two bands $A = a \times I, A' = a \times I$ *in* X' *which are orientation–preserving, they determine partial isometries* α, α' *of* J *so that the domain of the composition* $\alpha\alpha'\alpha^{-1}(\alpha')^{-1}$ *is a nondegenerate segment and the ratio of the translation lengths* $\ell(\alpha)/\ell(\alpha')$ *is irrational.*

Proof: Recall that according to Proposition 12.62 there is a base b in $C(Y)$ which is a carrier and is carried infinitely often in the sequence X_j. Let b_i denote the base in X_i corresponding to b. The bases b_i converge to a certain nondegenerate segment. By Proposition 12.62 we can choose a subsequence in X_i so that the sequence of isometries $h(b_i) \in \mathcal{B}$ of J determined by b_i is convergent. (Recall that $\mathcal{B}$ is the Bass group). The limit is the identity in the orientation-preserving case. In particular, all b_i are either orientation-preserving or reversing. Now we apply Lemma 12.67 to the sequence $h(b_i) \in \mathcal{B}$.

We first consider the orientation-preserving case. Then there are bases b_i, b_j so that the ratio of the translation lengths of $h(b_i), h(b_j)$ is irrational. We use Lemma 12.47 to attach a band $A = b_i \times I$ to $b_i, dual(b_i)$ and a band $A' = b_j \times I$ to $b_j, dual(b_j)$. The partial isometries α, α' are obtained by the restriction of $h(b_i), h(b_j)$ to b_i, b_j. Recall that

$$\lim_{i\to\infty} b_i = \lim_{j\to\infty} b_j, \quad \lim_{i\to\infty} h(b_i) = \lim_{j\to\infty} h(b_j) = id.$$

Hence, if i, j are sufficiently large, the composition $\alpha\alpha'\alpha^{-1}(\alpha')^{-1}$ of partial isometries has nondegenerate domain. This concludes the proof in the orientation-preserving case.

Suppose now that all the bases b_i are orientation-reversing. Then similarly to the above argument we find b_i, b_j, b_k $(i < j < k)$ so that the ratio of the translation length of $h(b_i)h(b_j)$ and $h(b_i)h(b_k)$ is irrational. Recall that the sequence of bases b_s is convergent to a certain nondegenerate segment in J. Therefore we can choose i so large that $b_{ij} := b_i \cap b_j, b_{ik} := b_i \cap b_k$ are nondegenerate segments. Consider the sub-bands $B_{ij} = b_{ij} \times I \subset b_j \times I = B_j$

and $B_{ik} = b_{ik} \times I \subset b_k \times I = B_k$. We subdivide the bands B_j, B_k (the Move M2) so that the bands B_{ij}, B_{ik} become bands in the new band complex X'. Now we slide the bases b_{ij}, b_{ik} from b_i (the Move M4). This produces bands B'_{ij}, B'_{ik} such that translation lengths of $h(B'_{ij}), h(B'_{ik})$ are equal to the translation length of $h(b_i)h(b_j)$ and $h(b_i)h(b_k)$. We let $A := B'_{ij}$, $A' := B'_{ik}$. To get nondegenerate domain for the composition $\alpha\alpha'\alpha^{-1}(\alpha')^{-1}$ we again choose sufficiently large i. $\square$

Proposition 12.69. *(1) The group H has an invariant line in T and H fits into a short exact sequence*

$$1 \to Ker(H \curvearrowright T_H) \to H \to \mathcal{A} \to 1$$

where $\mathcal{A}$ contains a free Abelian subgroup of the index 2. (2) Let λ be a leaf in $C(Y)$. Then the image π of $\pi_1(\lambda)$ in H fits into a short exact sequence

$$1 \to K \to \pi \to \Phi \to 1$$

where Φ is a group of order ≤ 2 and K fixes an arc in T.

Proof: We first prove (1). Let A, A' be the bands constructed in the previous lemma. Let α, α' denote the elements of H corresponding to A, A'; let $\rho(J)$ be the image of J in T so that α, α' are partial orientation-preserving isometries of $\rho(J)$. Then the commutator $[\alpha, \alpha']$ fixes a nondegenerate subarc $I \subset \rho(J)$. Therefore, by Theorem 12.39, the commutator $[\alpha, \alpha']$ fixes T_H pointwise. We conclude that α, α' act as translations on T_H with the same axis L. By Lemma 12.68, the ratio of the translation lengths of α, α' is irrational. Therefore the subgroup F generated by α, α' contains nontrivial translations with arbitrarily small translation lengths. By Proposition 12.38 the group H is generated by J-short elements, let γ be one of the generators. We will show that L is invariant under γ. Let $s \subset \rho(I)$ denote a closed subinterval such that $s' := \gamma(s) \subset \rho(J)$.

Case I: the partial isometry of J determined by γ is orientation-preserving. By Lemma 10.7 the isometry $\gamma : T \to T$ is axial and its axis contains $s \cup s'$. Let ϵ denote the diameter of the subset $s \cup s'$ in $\rho(J)$. Choose an element w in $F - \{1\}$ whose translation length is so small that $\ell(w) + \ell(\gamma) < \epsilon$. Then Corollary 12.40 implies that the axes of γ and w coincide which concludes the proof in the Case I.

Case II: the partial isometry of J determined by γ is orientation-reversing. Let x, x' denote the mid-points of s, s'. Let t' denote the subinterval of s' centered at x' whose length is the half of the length of s. We again choose $w \in F - \{1\}$ whose translation length is less than the length of t'. Then $w' := \gamma w \gamma^{-1}$ determines a partial orientation-preserving isometry of $\rho(J)$ which maps t' into s'. Again, by Corollary 12.40, the elements w, w' have the same axis L. This implies that $\gamma(L) = L$.

This finishes the proof of (1). To prove (2) note that π fixes a point in L. Let π^0 be the subgroup of index ≤ 2 in π which consists of elements preserving orientation on L. Then π^0 acts trivially on L which concludes the proof. $\square$

(iii) Output of the Process I: thin case.

Suppose that some component $C(Y)$ of Y has the *thin type.* Then we have an infinite sequence of band complexes $X_0, X_1, ...$, where for each i the complex X_{i+1} is obtained from X_i via the Process I applied to the minimal component $C(Y)$. Let $\mathcal{A}_n$ denote the collection of *subdivision annuli* in X_n (see definition of the Move M5).

Proposition 12.70. *Under the above conditions the group G splits over a subgroup that fixes an arc in T. The subgroups F_i are conjugate into vertex subgroups of this decomposition of G. Suppose that $C(Y)$ has no zero weight bands, then the above decomposition is a free product.*

Proof: It follows from the Proposition 12.58 (assertions 4 and 5) and Remark 12.51 that eventually there is a band $B_n = b_n \times I$ of weight 1 in $C(Y_n) \subset X_n$ that meets the closure of $X_n - (Y_n \cup \mathcal{A}_n)$ only in a subset of $b_n \cup dual(b_n)$. For any $t > 0$ there is a certain $k \geq 1$ so that the intersection $B_n \cap Y_{n+k}$ consists of at least t components.

The band B_n is formed as the result of a collapse when the rule (R3) is applied.

Remark 12.71. Let D_{jn} be the subgraph in ∂Y_n which carries the subgroup F_j in H_j (in the relative case). The Remark 12.51 (part (2)) implies that $D_{jn} \cap B_n$ is contained in $\Gamma(X_n)$.

Consider the horizontal segment $h_n := b_n \times \{1/2\}$ in B_n. It intersects certain subdivision annuli $A_{1,n}, ..., A_{s,n} \subset \mathcal{A}_n$ only in the end-points of h_n. Recall that $A_{j,n}$ does not have to be an embedded annulus in X_n, what we actually have are continuous maps

$$\alpha_{j,n} : \mathbb{S}^1 \times [0,1] \to A_{j,n} \subset X_n.$$

Let $\pi : \mathbb{S}^1 \times [0,1] \to [0,1]$ denote the projection. For each j we have a finite set

$$P_{j,n} := \alpha_{j,n}^{-1}(\partial h_n)$$

where distinct points have distinct projections to $[0,1]$. Thus we get a disjoint collection of "horizontal" circles $C_{j,n} := \pi^{-1}(\pi(P_{j,n}))$ in $\mathbb{S}^1 \times [0,1]$. We retain the notation $C_{j,n}$ for the (homeomorphic) image of $C_{j,n}$ in X_n and let

$$C_n := C_{1,n} \cup ... \cup C_{s,n} \subset X_n.$$

Then $\Lambda_n := h_n \cup C_n$ is a finite graph (two bouquets of circles connected by a segment) which locally separates the complex X_n.

Then X_n is a graph of spaces with the edge space Λ_n and the vertex spaces the compactified components of $X_n - \Lambda_n$. This defines a splitting of the fundamental group G of X_n. Every subgroup F_j in G is conjugate into one of the vertex subgroups of this decomposition of G, see Remark 12.71. Our next goal is to show that this splitting is nontrivial.

First of all, the generators of the fundamental groups of the circles $C_{j,n}$ are in the kernel of the action of $H = \pi_1(C(Y))$ on T_H (see Theorem 12.39). Thus the image of $\pi_1(\Lambda_n)$ in G fixes T_H and hence fixes an arc of T.

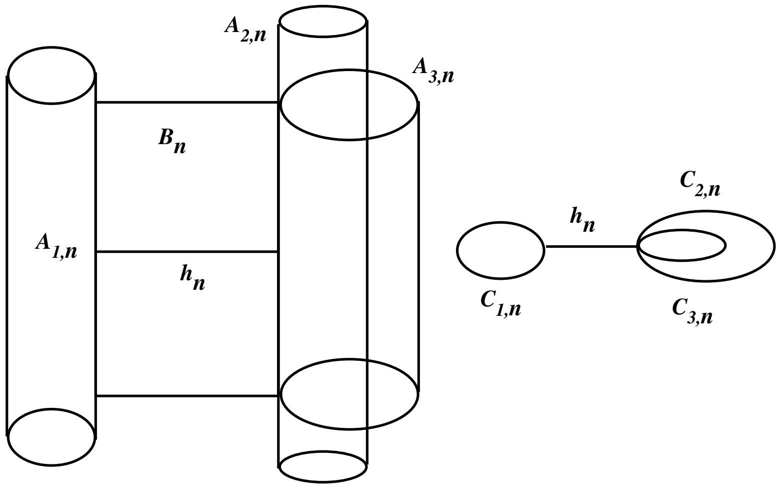

Figure 12.12: *Graphs corresponding to the subdivision annuli.*

If Λ_n does not separate X_n then G splits as an HNN-extension and we are done. Thus we assume that Λ_n separates X_n in two components X'_n and X''_n, in particular $b \cap dual(b) = \emptyset$. Since B_n is eventually cut into arbitrary many bands and the number of long bands is bounded from above, there are paths $p' : I \to X'_n, p'' : I \to X''_n$ and long bands B'_n, B''_n so that:

(a) $p'(I), p''(I)$ are contained in (nonsingular) leaves L', L''.

(b) $L' \subset B'_n, L'' \subset B''_n$.

(c) Each intersection $p'(I) \cap B_n \subset b$, $p''(I) \cap B_n \subset dual(b)$ consists of two distinct points.

Therefore both paths p', p'' have nontrivial holonomy. Hence each vertex subgroup $\pi_1(X'_n)$ and $\pi_1(X''_n)$ of the decomposition of G acts on T as a nontrivial group, therefore each vertex subgroup is nontrivial. On the other hand, the edge group (which is the image of $\pi_1(\Lambda_n)$ in G) fixes T, hence it is a proper subgroup of G. This implies that the splitting of G is nontrivial and we are done. $\square$

12.13. Proof of the decomposition theorem

In this section we assume that G is a finitely presentable group with a stable and minimal action on a tree T. Let $F_1, ..., F_n$ be a collection of finitely generated subgroups of G each of which fixes a point in T.

Theorem 12.72. *Under the above conditions the following is satisfied:*

- *If the action of G on T is not pure then G splits (as amalgamated free product or HNN extension) over a subgroup E so that E fits into a short exact sequence*

$$1 \to K_E \to E \to C \to 1$$

where K_E fixes an arc in T and C is either finite or cyclic. The subgroup E fixes a point in T. Each subgroup F_j is conjugate into one of the vertex subgroups of the decomposition of G.

- *If the action is pure then G belongs to one of the following types:*

  (a) Surface type:

$$1 \to K \to G \to \pi_1(O) \to 1$$

  where K is the kernel of the action of G on T and O is a 2-dimensional cone-type orbifold. Each subgroup F_j maps into a torsion or peripheral subgroup of $\pi_1(O)$.

  (b) Toral (axial) type: T is a line and

$$1 \to K \to G \to A \to 1$$

  where K is the kernel of the action of G on T and A is a subgroup of Isom$(\mathbb{R})$. Each subgroup F_j maps into a torsion subgroup of A.

  (c) Thin type: G splits over a subgroup that fixes an arc of T. Each subgroup F_j belongs to one of the vertex subgroups of the decomposition of G.

Proof: We start by constructing a band complex X as in Theorem 12.33 and then transform it to a band complex X' which has no faults (see Lemma 12.43). The we apply the Rips Machine to each minimal component $C(Y')$ of the union of bands $Y' \subset X'$.

If a component $C(Y')$ has thin type then we proceed as in Proposition 12.70: the image of $\pi_1(C(Y))$ in G admits the required splitting. This splitting is given by decomposition of the band complex X_n (obtained by applying the Process I inductively to the component $C(Y')$) as in Proposition 12.70, this decomposition yields the required splitting of the group G.

Hence in what follows we can assume that no component of Y' has thin type. If a component $C(Y')$ is of *surface* or *axial* type then:

For each leaf $L \subset C(Y)$ the image π of $\pi_1(L)$ in G fits into a short exact sequence

$$1 \to K_L \to \pi \to C \to 1$$

where K_L fixes an arc in T and C a cyclic group (see Propositions 12.66, 12.69). Thus if X' is pure then the assertion directly follows from Propositions 12.66, 12.69.

If X is not pure then G has a graph of groups decomposition which is given by a *graph of spaces* decomposition of X (see Theorem 12.44 and Remark 12.45). Suppose that the decomposition is nontrivial. We already know that each edge group E of this decomposition has a fixed point on T. Each edge group E is contained in the image of the fundamental group of a leaf L of a component $C(Y')$. If this component is minimal (i.e. has surface or axial type) then E satisfies the assertion of Theorem by Propositions 12.66, 12.69. Each simplicial component $C(Y')$ is an I-bundle over a leaf.

By passing to a finite cover $\widetilde{C(Y')}$ over $C(Y)$ we can assume that $\widetilde{C(Y')}$ is the trivial interval bundle over a leaf. Hence, if $D \subset C(Y')$ is an *edge-subset* (of the *graph of spaces* decomposition of X), then its fundamental group is a finite extension of a group which fixes an arc in T. $\square$

Corollary 12.73. *Suppose that the action of G is stable and arc stabilizers are virtually solvable. Then one of the following holds:*

(a) If the action of G **is not pure** *or* **is pure and thin** *then G splits over a subgroup E so that E is a finitely generated virtually solvable group which fixes a point in T. Each subgroup F_j is conjugate into one of the vertex subgroups of the decomposition of G.*

(b) The **surface** *case:*

$$1 \to \quad \textit{a virtually solvable group} \quad \to G \to \pi_1(O) \to 1$$

where O is a 2-dimensional cone-type orbifold. Each subgroup F_j maps into a torsion or peripheral subgroup of $\pi_1(O)$.

(c) The **toral** *case: G is virtually solvable.*

12.14. Proof of Skora's duality theorem

In this section we start proof of Skora's duality theorem which will occupy this and the next two sections.

We suppose that S is either a compact hyperbolic surface with geodesic boundary or a flat annulus, $\tilde{S}$ is the universal cover of S, $G = \pi_1(S)$. Let T be a small *trivial* simplicial G-tree Let $\psi : \tilde{S} \to T$ be a G-equivariant map which maps each boundary geodesic L to a fixed point t_L of the stabilizer G_L of L in G. Let $T' \subset T$ be the subtree which is the convex hull of $\psi(\partial\tilde{S})$.

Proposition 12.74. *There is an extension $\phi : \tilde{S} \to T'$ of the map ψ and (in case S is hyperbolic) a finite measured (quasi)lamination $(\mathcal{L}, \mu)$ with the blow up $(\mathcal{L}', \mu')$, so that the fibers of ϕ are the complementary regions of $\tilde{\mathcal{L}}'$ and the leaves of $\tilde{\mathcal{L}}'$. Here $\tilde{\mathcal{L}}'$ is the lift of $\mathcal{L}'$ to $\tilde{S}$. In the case S is the annulus, the same conclusion holds for a nonsingular measured foliation $(\mathcal{F}, \mu)$ on S: the fibers of ϕ are the leaves of the lift of $\mathcal{F}$ to $\tilde{S}$. In both cases the mapping ϕ projects to an isometry from the tree dual to the (quasi)lamination $(\mathcal{L}, \mu)$ to T.*

Remark 12.75. Note that in case ψ is a constant mapping we get $(\mathcal{L}, \mu) = 0$.

Proof: We will consider the case when S is hyperbolic leaving the annular case to the reader. Since the action of G on T is small, the point x is the unique fixed point for the action of G on T. For each boundary loop α choose a component $\tilde{\alpha} \subset \partial\tilde{S}$, let g_α denote a generator for the stabilizer of $\tilde{\alpha}$ in G, then the point $x_\alpha := \psi(\tilde{\alpha}) \in T$ is fixed by g_α. Let $[x x_\alpha]$ denote the geodesic segment in T, it is clear that this (possibly degenerate) segment is fixed by g_α pointwise. Then for $\alpha \neq \beta$ the intersection $[x_\alpha x] \cap [x_\beta x]$ is the point x (since otherwise the free nonabelian group generated by g_α, g_β fixes a nondegenerate segment in T which contradicts smallness of the action of G on T). If S is a hyperbolic surface define measured geodesic lamination $(\mathcal{L}, \mu)$ so that $\mathcal{L} \subset \partial S$ consists of the union of those $\alpha \subset \partial S$ for which $r_\alpha = d(x, x_\alpha) > 0$. For each $\alpha \subset \mathcal{L}$ we let the transversal measure be the transversal Dirac measure multiplied by r_α. Then $(\mathcal{L}', \mu')$ is the union of foliated annuli A_α of the transversal mass r_α. We define the mapping ϕ on $\tilde{S} - \tilde{\mathcal{L}}'$ to be the constant mapping equal to x. For the component $\tilde{A}_\alpha$ of the inverse image of A_α in $\tilde{S}$ (which contains $\tilde{\alpha}$) we let

$$\pi_{\tilde{A}_\alpha} : \tilde{A}_\alpha \to \tilde{A}_\alpha / \sim$$

be the projection along the leaves. The transversal measure projects to quotient so that it becomes a segment of the length r_α. We let $\tau_\alpha : \tilde{A}_\alpha / \sim \to [x_\alpha x]$ be the isometric mapping which sends the boundary leaf of $\tilde{S}$ to the point x_α. We let ϕ be the composition

$$\tau_\alpha \circ \pi_{\tilde{A}_\alpha}.$$

We extend the mapping ϕ equivariantly to other lifts of A_α. It is clear that the resulting mapping satisfies assertion of the proposition. $\square$

12.15. Geometric actions on trees

Definition 12.76. Let T be a G-tree where G is a finitely presentable group. The action of G on T is called **geometric** if it admits an **exact** resolution via the universal cover of a band complex. Otherwise the action is called **nongeometric**.

We first consider this notion in the case of **simplicial** trees T. It is clear that the action of G on T is geometric iff all vertex and edge stabilizers are finitely generated groups. This immediately gives us an example of a *nongeometric* action. Namely, let S be a closed hyperbolic surface $\phi : \pi_1(S) = G \to \mathbb{Z}$ be a nontrivial homomorphism. Clearly the kernel N of ϕ is infinitely generated. Then G is an HNN-extension of N which gives us an action of G on the simplicial tree $\mathbb{R}$ (vertices are the integers), the action of G on $\mathbb{R}$ factors through $\mathbb{Z}$ acting on $\mathbb{R}$ by integer translations. Vertex and edge stabilizer subgroups of this action are infinitely generated (they are all equal to N). Hence this is a nongeometric action.

There are more complicated examples of nongeometric *free* actions of groups on trees (when G is a finitely generated free group), we refer to [GL95] for details.

In this section we prove Skora's duality theorem (Theorem 11.31) under the extra assumption that the group action is geometric:

Theorem 12.77. *Let S be a compact hyperbolic surface with the fundamental group G and the boundary components $\alpha_1, ..., \alpha_m$, $F_i := \iota_{i*}(\pi_1(\alpha_i))$, $i = 1, ..., m$, where $\iota_i : \alpha_i \hookrightarrow S$ is the inclusion. Consider a small minimal geometric action of G on a tree T such that each subgroup F_i fixes a point in T ($i = 1, ..., m$). Then there is a measured foliation $\mathcal{G}$ on S such that the dual G-tree $T_{\mathcal{G}}$ is isomorphic to the G-tree T.*

Proof: Let X be a band complex whose universal cover gives an exact resolution to the G-tree. Apply the Preliminary Motions of the Rips Machine to the band complex X (see §12.10), we retain the name X for the resulting band complex. Let $Y \subset X$ be the union of bands in X. We apply the results of Sections 12.12, 12.13 to X. Recall that each component $C(Y)$ of $Y \subset X$ is either minimal or $\pi_1(C(Y))$ fixes a point in T. In each case $C(Y)$ corresponds to a vertex group in a graph of groups decomposition of G (relative for $\{F_1, ..., F_m\}$) and the incident edge groups are cyclic (since the G-tree T is small). If G_v is a component of the graph of spaces decomposition of X corresponding to a component of $X - Y$ then G_v fixes a point in T and the incident edge groups are edge subgroups of $\pi_1(C(Y))$ for some $C(Y)$. Thus all edge groups in the graph of groups decomposition of G are cyclic.

Let $C(Y)$ be a minimal component of Y and $B \subset C(Y)$ a weight zero band. Then the fundamental group $\pi_1(B)$ of the annulus corresponding to B acts trivially on the tree T_H where H is the image of the fundamental group of $C(Y)$ in G (see Theorem 12.39). Thus, since the action of G on T is small, it follows that the image of $\pi_1(B)$ in G is trivial. Let b be a base of such band B and $r : X \to cl(X - B)$ the retraction whose restriction to the band B is the map $(x, y) \mapsto x$. Therefore the map r induces an isomorphism of fundamental groups and we can eliminate all the weight zero bands from the minimal components of Y (Move M0) without changing the dual tree. We retain the name X for the resulting complex.

Lemma 12.78. *Each minimal component of* Y *has surface type.*

Proof: (1) Suppose a minimal component $C(Y)$ has thin type. Then by Proposition 12.70 the group G splits as a nontrivial free product and this splitting is relative to the collection of the subgroups $F_i, i = 1, ..., m$. This contradicts the assumption that $F_1, ..., F_m$ represent the fundamental groups of all boundary circles of S.

(2) Suppose a minimal component $C(Y)$ has toral type. According to Lemma 12.68 and Proposition 12.69, the group H, which is the image of $\pi_1(C(Y))$ in G, has an invariant geodesic line $L \subset T$ and the action of G on L has nondiscrete Bass group. Therefore H contains an Abelian subgroup of rank 2, which contradicts the assumption that G is the fundamental group of a hyperbolic surface. □

Let $C(Y)$ be a minimal component of Y. We know that $C(Y)$ is homeomorphic to a compact surface. The measured foliation on Y restricts to a measured foliation on this subsurface. Our next goal is to understand how 2-cells Δ^2 of X are attached to the boundary loops of $C(Y)$. Without loss of generality we can assume that every attaching map $\partial\Delta^2 \to \partial C(Y)$ has nonzero degree.

Suppose that λ is a boundary loop of $C(Y)$ which is nul-homotopic in X. Then λ necessarily splits X into several connected components (otherwise G would split into a free product relative to the collection $F_1, ..., F_m$). Let Z be a component of $cl(X - \lambda)$ which does not contain $C(Y)$. Therefore the image of $\pi_1(Z)$ in G is trivial, thus Z does not contain any minimal components of Y and we replace Z by a disk attached along λ without changing the fundamental group of X and the dual tree. We do it with respect to each loop λ as above, we retain the notation X for the resulting complex.

Lemma 12.79. *Let λ be a boundary loop of $C(Y)$ such that there is a 2-cell in X attached along λ. Then the attaching map has degree ± 1.*

Proof: For each boundary loop λ of $C(Y)$ which has a 2-cell attached to it we pick a generator γ of $\pi_1(\lambda)$ and let m_λ denote the degree of the attaching map. We retain the name γ for the image of γ in $\pi_1(Y)$. Consider the kernel K of the map $\pi_1(C(Y)) \to \pi_1(X) = G$. The subgroup K is normally generated by the elements γ^{m_λ}. If at least one $m(\lambda)$ is different from ± 1 then the image H of $\pi_1(C(Y))$ in G is the fundamental group of a 2-dimensional orbifold of cone type with nonempty singular locus. Thus either H has torsion or it is a trivial group. Neither case is possible since G is torsion-free and minimality of $C(Y)$ implies that $\pi_1(C(Y))$ does not act on T as the trivial group. $\square$

Lemma 12.80. *Let λ be a boundary loop of $C(Y)$ such that there is a 2-cell in X attached along λ. Then λ contains at least two singular points of the measured foliation on $C(Y)$.*

Proof: Since λ is a boundary loop and the component $C(Y)$ is minimal, it follows that λ contains at least one singular point x. The point x is necessarily contained in the interior of a base b of a band B in Y. If x is the only singular point on λ then there is a concatenation of bands[9] $B_1 = b_1 \times I, B_2 = b_2 \times I, ..., B_s = b_s \times I$ so that:

- The bottom of B_1 is a base $b_1 \subset \Gamma$ which has x as an end-point; the top of B_s is a base $dual(b_s) \subset \Gamma$ which has x as an end-point.
- $(b_1 \cup dual(b_s)) \cap b$ forms a neighborhood of x in b.
- $\lambda - \cup_{i=1}^{s} b_i$ is the disjoint union of open segments $(x_i, x_{i+1}) \subset \partial B_i$, $x_1 = x$.

Thus we can find a vertical path α in $B_1 \cup ... \cup B_s$ near λ which starts in b_1 and ends in $dual(b_s)$. Since the loop λ is nul-homotopic in X, the path α lifts to a path $\tilde{\alpha}$ in the universal cover $\tilde{X}$ so that the end-points of $\tilde{\alpha}$ are contained in a lift $\tilde{b}$ of the base b. This implies that the resolution $\rho : \tilde{X} \to T$ is not injective on $\tilde{b}$ which contradicts the assumption (A4) (see §12.10). $\square$

Let $C_i(Y)$ be a minimal component of T, define $C_i(X)$ by adjoining to $C_i(Y)$ all 2-cells from X which are attached along the boundary circles of $C_i(Y)$. Note that $\pi_1(C_i(X))$ injects into G. Let Σ_i be the surface obtained from $C_i(X)$ by collapsing each 2-cell (that is not in Y) to a point. The singular measured foliation on $C_i(Y)$ projects to a singular measured foliation on the surface Σ_i (since every boundary circle of $C_i(Y)$ which is

[9]We may have to exchange the tops and the bottoms of some of the bands.

nul-homotopic in X contains at least 2 singular points). This measured foliation corresponds to a measured geodesic lamination on Σ_i (for a hyperbolic structure with geodesic boundary on Σ_i).

Remark 12.81. This would conclude the proof of Theorem 12.77 in the case when the action of G on T is pure. The main objective of the rest of the argument is to combine the **pure** and **simplicial** components of X into a measured lamination on a compact surface so that the **pure** components correspond to the **minimal** regions and the **simplicial** components correspond to the **rational** regions of the corresponding measured foliation.

We replace each $C_i(X)$ in X by the corresponding surface Σ_i. Our next goal is to make analogous replacement for components of $cl(X - \cup_i \Sigma_i)$. The decomposition of X into the union of components of $\cup_i \Sigma_i$ and components of $cl(X - \cup_i \Sigma_i)$ corresponds to a graph of groups decomposition of the group G where the edge subgroups are infinite cyclic. Therefore Lemma 10.23 implies that we can realize this decomposition of G by a geometric decomposition of the surface S along a collection of disjoint loops, none of which is nul-homotopic (however some of these curves could be homotopic to each other). We let S_i denote the components of this decomposition of S. We will color the components S_i in two colors: black if they correspond to $\Sigma_i \subset X$ and white if they do not. Clearly no two components of the same color are adjacent to each other.

For each surface $\Sigma_i \subset X$ the peripheral subgroups of $\pi_1(\Sigma_i)$ correspond to the edge subgroups incident to the group $\pi_1(\Sigma_i)$ in the graph of groups decomposition of $G = \pi_1(X)$. Let S_i denote the subsurface of S whose fundamental group corresponds to $\pi_1(\Sigma_i)$ under the isomorphism $\pi_1(S) \to \pi_1(X)$. Then the isomorphism $\pi_1(S_i) \to \pi_1(\Sigma_i)$ preserves the peripheral structure (since it is given by the incident edge subgroups), therefore it is induced by a homeomorphism $S_i \to \Sigma_i$. We pull-back the measured foliation from Σ_i to S_i. Let $\tilde{S}_i$ be a lift of S_i to the universal cover $\tilde{S}$ of S. The quotient $\tilde{S}_i/\sim$ by the lift of the (blown up) measured lamination from S_i, is a metric tree acted upon by $\pi_1(S_i)$. This tree has canonical injective equivariant map into the tree T which is given by the restriction of the resolution $\rho : \tilde{X} \to T$ to the appropriate lift of Σ_i.

We now consider a *white* component S_j in the decomposition of S. Let $\tilde{S}_j$ be a lift of S_j to $\tilde{S}$. Then we have an equivariant map

$$\psi : \partial \tilde{S}_j \to T$$

which is given by restriction of the maps of adjacent *black* components of $\tilde{S}$ to their boundary arcs. Using Proposition 12.74 we construct an equivariant extension ϕ of this map to $\tilde{S}_j$ so that the fibers are the leaves and complementary components of a measured lamination (or foliation if S_j is an annulus) on S_j. The image of ϕ is contained in the convex hull of $\phi(\partial \tilde{S}_j)$. This implies that $\phi(\tilde{S}_j) \subset T_{Z_j}$ where T_{Z_j} is the dual tree to the foliation of the component $Z_j \subset X$ corresponding to S_j.

We let $\mathcal{L}$ be the measured lamination on the surface S which is the union of measured laminations of *black* and *white* components. The foliation $\mathcal{L}$ has the properties:

- There is a map $\phi : \tilde{S} \to T$ which induces a representation $p_* : \pi_1(S) \to \mathrm{Isom}(T)$ so that $p_* = \iota \circ q_*$ where ι is the isomorphism $\pi_1(S) \to \pi_1(X)$.

- The fibers of ϕ are the leaves of and complementary regions for the measured lamination $\tilde{\mathcal{L}}$, which is the lift of $\mathcal{L}$ to $\tilde{X}$.

This concludes the proof of Theorem 12.77. $\square$

12.16. Nongeometric actions on trees

The main objective of this section is to prove Skora's theorem in the general case.

Definition 12.82. Suppose that we have a collection of G-trees $T, T_n, n \in \mathbb{Z}_+$. Then the G-trees T_n converge **strongly** to the G-tree T if the following is satisfied:

- There is a sequence of morphisms $f_n : T_n \to T, f_{n,k} : T_n \to T_k, k \geq n$.

- $f_n = f_k \circ f_{n,k}$.

- For each n and each pair of points $x, y \in T_n$ there is $N = N(n, x, y)$ so that for all $k \geq N$ we have:

$$d(f_{n,k}(x), f_{n,k}(y)) = d(f_n(x), f_n(y)) \ .$$

Lemma 12.83. *If the G-tree T is small and T_n converge strongly to T then for sufficiently large n each G-tree T_n is small.*

Proof: The proof is obvious and is left to the reader. $\square$

Theorem 12.84. *(G. Levitt, F. Paulin [LP97].) Suppose that we have a finitely presentable group G and a G-tree T (which is relatively elliptic with respect to a finite collection of finitely generated subgroups F_i). Then:*

(1) T is a strong limit of minimal geometric actions of G on trees T_n (relatively elliptic with respect to the collection of subgroups F_i).

(2) If T is small then T_n are also small for all sufficiently large n.

(3) If T is minimal then T_n are also minimal for all sufficiently large n.

(4) If X_n is the band complex which gives an exact resolution of the G-tree T_n then there exists a continuous foliation-preserving map $\phi_{n,n+i} : X_n \to X_{n+i}$ which induces an isomorphism of the fundamental groups.

Proof: We apply Theorem 12.33 to construct a band complex X and (relative with respect to the collection of subgroups F_j) resolution $\rho : \tilde{X} \to T$ of the action of G on T, where $\tilde{X}$ is the universal cover of X. Recall that in the construction of X we are free to choose a finite subtree $K \subset T$ containing a certain finite set. We choose an increasing sequence K_n of finite subtrees which exhaust the tree T. According to the construction in Theorem 12.33, the tree K_n is the graph on which the band complex X_n is based. Moreover, if $\tilde{K}_n$ is a component of the lift of K_n to $\tilde{X}_n$, then the resolution $\rho_n : \tilde{X}_n \to T$ embeds $\tilde{K}_n$ isometrically into T.

By construction, each group F_i fixes a point in T_n. Now we need to verify that minimality of T implies minimality of T_n. The G-orbit of the subset $\tilde{K}_1 \subset T_n$ is the whole tree T_n. Thus it suffices to show that (for sufficiently large n) each point of $\tilde{K}_1$ lies in the axis of an axial element of G acting on T_n. Since the G-tree T is minimal it follows that each point of $K_1 \subset T$ is contained in the axis A_g of an axial isometry g. Let $D_g = \Phi_g \cup g(\Phi_g) \subset A_g$, where Φ_g is a fundamental domain for the action of $\langle g \rangle$ on A_g. We choose K_n such that

$$D_{g_i} \subset K_n, i = 1, ..., s, \quad \text{where} \quad K_1 \subset \cup_{i=1}^{s} D_{g_i}.$$

Lemma 10.7 implies that each element $g_i, i = 1, ..., s$, acts as an axial isometry on T_n and the union of the axes of g_i $(i = 1, ..., s)$ contains the image of $\tilde{K}_1$ in T_n. Therefore the G-tree T_n is minimal.

The isometric embeddings $\tilde{K}_n \hookrightarrow T_{n+i}$, $\tilde{K}_n \hookrightarrow T$ determine a collection of morphisms

$$f_{n,n+i} : T_n \to T_{n+i}, \quad f_n : T_n \to T$$

as follows. The inclusion $K_n \hookrightarrow K_{n+i}$ extends to a continuous foliation-preserving map $\phi_{n,n+i} : X_n \to X_{n+i}$ which induces an isomorphism of the fundamental groups. The map $f_{n,n+i}$ is the projection of the lift of $\phi_{n,n+i}$ to the quotient trees. (According to the construction in Theorem 12.33 each band in X_n is also a band in X_{n+i}.) We conclude that the action of $\pi_1(S)$ on T is the strong limit of geometric actions as required by Theorem 12.84. $\square$

Proposition 12.85. *Suppose that G is the fundamental group of a compact surface and T is a small minimal relatively elliptic G-tree. Then the morphisms $f_{n,k}$ (which appear in Theorem 12.84) are isomorphisms for all sufficiently large n.*

Proof: Firstly, strong convergence implies that for each element $g \in G$ there is N so that for all $n \geq N, k \geq n$ we have the equality of the translation lengths:

$$\ell_{T_n}(g) = \ell_{T_k}(g) = \ell_T(g).$$

By Corollary 11.32 there is a finite collection $g_1, ...g_m$ of elements of G so that the equivalence class of each measured foliation $\mathcal{F}$ is uniquely determined by the translation lengths

$$(\ell_{T_{\mathcal{F}}}(g_1), ..., \ell_{T_{\mathcal{F}}}(g_m)).$$

It follows that (for sufficiently large n) all the G-trees T_n are isomorphic[10] for sufficiently large n and they are dual to the same measured geodesic lamination $\mathcal{L}$ on the hyperbolic surface S. Now suppose that

$$f := f_{n,k} : T_n = T_{\mathcal{L}} \to T_k = T_{\mathcal{L}}$$

[10] A priori such isomorphism is not given by the maps $f_{n,k}$.

is a nontrivial folding, i.e. there is an arc $[x, y] \subset T_n$ so that $f(x) = f(y)$. Let $\tilde{\mathcal{L}}$ be a measured geodesic lamination in $\mathbb{H}^2$ which is a lift of $\mathcal{L}$ to the universal cover of S. We lift $[x, y]$ to a hyperbolic geodesic segment $[\tilde{x}, \tilde{y}]$ in $\mathbb{H}^2$ so that the the projection

$$[\tilde{x}, \tilde{y}] \to [x, y]$$

is an isomorphism with respect to the transversal measure induced by $\mathcal{L}$ (on $[\tilde{x}, \tilde{y}]$) and the Lebesgue measure induced on $[x, y]$ by the metric on T. By making appropriate choice of x, y we may assume that $\tilde{x}, \tilde{y}$ do not belong to $\tilde{\mathcal{L}}$. Let $\tilde{\mathcal{L}}_1$ be the sublamination in $\tilde{\mathcal{L}}$ which consists of geodesics which intersect $[\tilde{x}, \tilde{y}]$ nontrivially. The infinite hyperbolic geodesic which contains $[\tilde{x}, \tilde{y}]$ can be approximated by axes of hyperbolic elements of G (see Lemma 3.24), hence there exists a hyperbolic element $g \in G$ with the axis A so that the transversal measure of the intersection $A \cap \tilde{\mathcal{L}}_1$ is the same as the transversal measure of the intersection $[\tilde{x}, \tilde{y}] \cap \tilde{\mathcal{L}}$. Recall that $f(x) = f(y) \in T_k$, hence the translation length of g in T_k is strictly smaller than the translation length of g in T_n. Contradiction. $\square$

Remark 12.86. Alternatively one can argue as follows (see [Bes97]). Let $\mathcal{L}_n$ be the measured geodesic lamination on S which is dual to T_n. Let $\tilde{S}$ denote the universal cover of S and $\tilde{\mathcal{L}}_n$ the lift of $\mathcal{L}$ to $\tilde{S}$. Use the part (4) of Theorem 12.84 to show that there is a G-equivariant map $\phi_{n,k} : \tilde{S} \to \tilde{S}$ which maps leaves of $\tilde{\mathcal{L}}_n$ to leaves of $\tilde{\mathcal{L}}_k$ and complementary regions to complementary regions. The map $\phi_{n,k}$ projects to the morphism $f_{n,k} : T_n \to T_k$. Show that $f_{n,k}$ is injective on each arc $\alpha \subset T_n$ which does not contain any vertices of T_n. If $f_{n,k}$ is a nontrivial folding then $\phi_{n,k}$ identifies certain leaves of $\tilde{\mathcal{L}}_n$. Thus the extension of $\phi_{n,k}$ to $\partial_\infty \tilde{S}$ is not injective, which contradicts the fact that $\phi_{n,k}$ is a quasi-isometry.

Now we can finish the proof of Skora's theorem (Theorem 11.31) in the general case. Consider an action of $G = \pi_1(S)$ on a tree T which is small, minimal and relatively elliptic. This action is a strong limit of geometric actions on G on the trees T_n. Proposition 12.85 implies that for each pair of points $x, y \in T_n$ and for all sufficiently large n, k, we have:

$$d(x, y) = d(f_{n,k}(x), f_{n,k}(y)) = d(f_n(x), f_n(y)).$$

This implies that $f_n : T_n \to T$ is an isomorphism for all sufficiently large n. Therefore T is dual to a measured foliation on the surface S. $\square$

12.17. Compactness of representation varieties

Proof of Theorem 12.1.

Suppose that (M, P) is a compact orientable pared Haken 3-manifold, which has incompressible boundary surface $\partial_0 M$.

Let W denote the window in (M, P). Note that each almost solvable subgroup of $\pi_1(M) = \Gamma$ is either trivial, or $\mathbb{Z}$, or a peripheral subgroup $\mathbb{Z}^2$.

Consider an isometric, minimal, small, nontrivial, relatively elliptic (with respect to P) action of Γ on a tree T. Such action is necessarily stable by Proposition 12.8. Thus we can apply Corollary 12.73.

Step 1. Consider the cases (b) and (c) of the surface and axial actions $\Gamma \curvearrowright T$. Then the group Γ is either a surface group and for each component $P_i \subset P$, $\pi_1(P_i)$ is a peripheral subgroup of Γ, or Γ is Abelian. In the both cases the manifold M is homeomorphic to an interval bundle over a compact surface Σ so that the parabolic locus P is the I-bundle over $\partial\Sigma$. Thus M itself is the window and Theorem 12.1 trivially follows.

Step 2. Now we consider the case (a): a nontrivial decomposition of $\pi_1(M)$. Since M is a pared manifold, each edge subgroup C in this decomposition is either trivial or infinite cyclic. If C is trivial, it follows that $\partial M - P$ is compressible (Theorem 1.32) which contradicts our assumptions. Thus C is infinite cyclic and according to Theorem 1.33 there is an essential annulus $A \subset M$, which is disjoint from P. Then A (up to isotopy) is contained in the window of M and we split M open along A. The resulting manifold M' consists either of one or two components, its boundary contains two annuli A_1, A_2 corresponding to A. Note that each $\pi_1(A_j)$ fixes a point in T. We let $P' := P \cup A_1 \cup A_2$ be the new designated parabolic locus for M'. It is easy to see that (M', P') is again a pared manifold, so we repeat this procedure inductively. The process of decomposition of (M, P) along essential annuli will terminate after finitely many steps when we get a manifold M_k (the Euler characteristic to ∂M is increasing on each step). Note that on each step we split only window components. Thus, each component of $(M, P) - W$ is a component of the complement to the window V of M_k. Let $Q \subset \partial M_k$ be the *designated parabolic locus* that we get in the process (it includes P and the annuli that we have created after splitting our manifolds).

Step 3. Let N be a component of M_k. We need to prove that the fundamental group of each component of $N - V$ fixes a point in T. If $\pi_1(N)$ has a global fixed point in T then we are done. So, suppose that we have a nontrivial action of $\pi_1(N)$ on a minimal subtree $T' \subset T$ which is $\pi_1(N)$-invariant. Then this action is of the *surface type*. However we have already considered this case in **Step 1**, when we concluded that N itself is the window in $(N, Q \cap N)$. This concludes the proof of Theorem 12.1. $\square$

Remark 12.87. Instead of the above induction arguments one can also use the approximation theorem of V. Guirardel [Gui98], who proved that small stable actions of a finitely presentable group G on a tree T could be approximated by simplicial actions of G on T_n so that if $g_1, ..., g_k \in G$ fix points in T, then these elements also fix points on T_n for large n.

Corollary 12.88. *Suppose that (M, P) is a pared acylindrical 3-manifold with the fundamental group G, T is a small G-tree so that the action $G \curvearrowright T$ is relatively elliptic (with respect to P). Then G has a global fixed point in T.*

Proof: Directly follows from Theorem 12.1 since $W = \emptyset$. $\square$

Corollary 12.89. *Suppose that (M, P) is a acylindrical pared 3-manifold with the fundamental group G, Γ is a group which contains G as a subgroup of finite index, $\Gamma \curvearrowright T$ is a small action of Γ on a metric tree T so that the restriction of this action to G is relatively elliptic. Then Γ has a global fixed point in T.*

Proof: Corollary 12.88 implies that G fixes a point in T. Without loss of generality we assume that G is normal in Γ, $F := \Gamma/G$. Let $x \in T$ be a point fixed by G. Then Γ-orbit of x is a finite set which is fixed by G pointwise. However, since the action of G is small, the group G has exactly one fixed point in T, this implies that x is fixed by Γ. $\square$

Exercise 12.90. *Show that if $\Gamma \curvearrowright T$ is an isometric action on a tree and $G \subset \Gamma$ is a finite index subgroup which fixes a point in T, then Γ also fixes a point in T.*

The following *Relative Compactness Theorem* was first formulated by W. Thurston [Thu86a], a "tree-theoretic" proof of this theorem was presented by J. Morgan and P. Shalen in the series of papers [MS84, MS88b].

Theorem 12.91. *(Relative Compactness Theorem.) Suppose that (M, P) is an acylindrical pared 3-manifold. Then the relative space of discrete and faithful representations*

$$\mathcal{D}(\pi_1(M), \pi_1(P); PSL(2, \mathbb{C}))$$

is compact.

Proof: Suppose not, then (according to Theorem 10.24) there is a small, minimal nontrivial relatively elliptic action of G on a tree T. However Corollary 12.88 implies that G fixes a point in T. Contradiction. $\square$

Similarly we get:

Theorem 12.92. *Suppose that (M, P) is an acylindrical pared 3-manifold, Γ is a group which contains $G = \pi_1(M)$ as a subgroup of finite index. Then*

$$\mathcal{D}(\Gamma, \pi_1(P); PSL(2, \mathbb{C}))$$

is compact.

Below we describe another application of Theorem 12.1 which will be used in §17.1. Let (M, P) be a connected orientable pared manifold. We will assume that $\chi(M) < 0$. Consider an isometric, minimal, small, relatively elliptic action of G on a tree T.

Theorem 12.93. *Suppose that for each component $\Sigma_j \subset \partial_0 M$ the group $\pi_1(\Sigma_j)$ fixes a point in T. Then G fixes a point in T as well.*

Proof: Consider the simplicial tree J dual to the JSJ-decomposition of M. If v is a vertex of J then G_v will denote its stabilizer in G and M_v the corresponding component of M.

Proposition 12.94. *For each M_v which is distinct from the solid torus, each component of $\partial_0 M_v = \partial M_v \cap \partial_0 M$ is not an annulus.*

Proof: The assertion is clear if M_v is an interval bundle over a surface S_v since in this case $\partial_0 M_v$ is a 2-fold cover of S_v. Suppose that M_v is not an interval bundle. Let A be an annular component of $\partial_0 M_v$. Then each boundary circle of A is adjacent to an annulus which either is a component of P or is a component of $\partial_1 M_v = \partial M_v - \partial_0 M_v$. If both circles are adjacent to the components of P then the definition of pared manifold implies that M_v is the solid torus. Similarly, if both adjacent annuli are in $\partial_1 M_v$ then (unless M_v is a solid torus) we can isotope A within M_v (so that ∂A slides over the adjacent annuli from $\partial_1 M_v$) to an essential annulus A' properly embedded in $M - P$. This contradicts the assumption that M_v is not an interval bundle. The remaining case (when one adjacent annulus is in P and the other is in $\partial_1 M_v$) is similar and is left to the reader. $\square$

Let $G' \subset G$ be the subgroup generated by those vertex subgroups G_v that are not cyclic. It is clear that G' is normal in G and that G' is not virtually Abelian. Thus G' can have at most one fixed point in T (since the tree T is small). If such point exist it is also fixed by the whole group G and we are done. Thus it suffices to show that G' fixes a point in T. If $F \subset J$ is a closed finite subtree we let G_F denote the subgroup of G' generated by noncyclic subgroups $G_v, v \in F$. Since the subgroups G_F exhaust the group G', it is enough to show that each G_F fixes a point in T. We argue by induction on the number of vertices in F. We will avoid one-point subtrees F so that $F = \{v\}$ and G_v is cyclic. Note that if G_v is cyclic then for each edge $[vw] \subset J$, the group G_w is not cyclic.

(1) If $F = \{v\}$ then the assertion follows from Theorem 12.1.

(2) Suppose the assertion holds for all subtrees F' with less than d vertices. Let $F \subset J$ be a finite subtree which is obtained from F' by adding an extra edge $[wv] = e$, $v \notin F'$. If G_v is cyclic then $G_F = G_{F'}$ and there is nothing to prove. Suppose that G_v is not cyclic. If G_w is not cyclic either then (according to Proposition 12.94) there is a component $\Sigma_j \subset \partial_0 M$ so that both surfaces $\partial_0 M_v \cap \Sigma_j$ and $\partial_0 M_w \cap \Sigma_j$ have negative Euler characteristic. Thus $Fix_T(\pi_1(M_v)) = Fix_T(\pi_1(\Sigma_j)) = Fix_T(\pi_1(M_w))$ is the same single point p as the one fixed by $G_{F'}$. Thus p is fixed by G_F. Assume that G_w is cyclic. Then F' contains an edge $[uw]$ such that G_u is not cyclic and by repeating the above arguments we conclude that G_u, G_v have the same (unique) fixed point. Thus the same point is fixed by G_F. $\square$

Corollary 12.95. *Suppose that (M, P) is a pared manifold with the fundamental group Γ. Then for each n the following restriction mapping is proper:*

$$D(\Gamma, \pi_1(P); SO(n,1)) \to \prod_{i=1}^{k} D(\pi_1(S_i), SO(n,1))$$

(here S_i's are the components of $\partial_0 M$). In other words, if $[\rho_j] \in D(\Gamma, \pi_1(P); SO(n,1))$ are conjugacy classes of representations which sub-

converge (up to conjugation) on the fundamental group of each surface S_i*, then the sequence* $[\rho_j]$ *subconverges as well.*

Chapter 13

Brooks' Theorem and Circle Packings

Which geometrically finite subgroups G of $\mathrm{Isom}(\mathbb{H}^3)$ are contained in lattices Γ in $\mathrm{Isom}(\mathbb{H}^3)$? It is clear that there are only countably many lattices Γ and there is a continuum of geometrically finite subgroups $G \subset \mathrm{Isom}(\mathbb{H}^3)$. In this chapter I present a theorem of R. Brooks which asserts that in some sense geometrically finite subgroups $G \subset \mathrm{Isom}(\mathbb{H}^3)$ which could be extended to lattices form a dense countable set. Below is an outline of the proof.

Suppose that $\mathcal{P}$ is a finite collection of closed round disks in the 2-sphere $\mathbb{S}^2$ so that the interiors of distinct disks are disjoint, however the boundary circles of these disks could be tangent. Such collection of circles is a *partial packing* of $\mathbb{S}^2$. A partial packing is a *packing* if each complementary component T to the union of disks in $\mathcal{P}$ is an *ideal triangle*, i.e. the boundary of T consists of three circular arcs. Given a partial packing $\mathcal{P}$ one may ask if it is possible to extend $\mathcal{P}$ to a packing $\mathcal{Q}$ by adding more round disks to the complementary components of $\cup_{D\in\mathcal{P}} D$. One can easily add more disks to $\mathcal{P}$ and get a partial packing $\mathcal{P}'$ where each complementary component is an ideal quadrilateral, however in general we may be unable to extend $\mathcal{P}'$ further, to a finite packing. We could keep adding more and more disks to each complementary quadrilateral

$$R_i \subset \mathbb{S}^2 - \bigcup_{D\in\mathcal{P}'} D$$

without reducing the number of the complementary quadrilaterals. By adding *infinite number* of round disks we could find an infinite packing $\mathcal{P}_\infty$ with a finite number of accumulation points of the disks: one for each complementary quadrilateral. Brooks' idea is to associate to $\mathcal{P}'$ a vector $r = (r_1, ..., r_n)$ in $\mathbb{R}^n_+$, where n is the number of complementary ideal quadrilaterals so that the continuous fraction expansion to r_i describes the combinatorics of the disks in $\mathcal{P}_\infty \cap R_i$. If the number r is rational then $\mathcal{P}_\infty$ is actually finite and we have succeeded. Rational vectors are dense in $\mathbb{R}^n_+$, so by making an arbitrarily small perturbation of $\mathcal{P}$ we get a partial packing $\mathcal{P}_\epsilon$ which extends to a finite packing $\mathcal{Q}_\epsilon$.

What does it have to do with Kleinian groups and lattices? If $\mathcal{Q}$ is a packing then for each complementary ideal triangle T_i we can add a round disk D_i containing T_i so that the boundary of D_i contains the vertices of T_i. As the result we get a finite collection $\mathcal{R}$ of round disks which cover $\mathbb{S}^2$ so that each point is covered by the interiors of at most two disks and the boundary circles of these disks are either disjoint, or tangent, or orthogonal. Such collection of round disks is a *pattern* of round disks which covers $\mathbb{S}^2$. Thus the group Γ generated by reflections in the boundary circles of the disks in this patters is a lattice in $\mathrm{Isom}(\mathbb{H}^3)$; the points of quadruple intersection of the circles are the fixed points of parabolic subgroups of Γ. To apply this construction to the problem about embedding geometrically finite groups G into lattices one has to generalize the above construction to patterns of round disks in the quotient surface $S(G)$. If one can find a pattern of disks which covers $S(G)$ then we can lift boundary circles of these disks to $\Omega(G)$ and take the group generated by G and all the reflections in the lifted circles. The result is a lattice Γ containing G. As before, there is no reason for the existence of a pattern $\mathcal{P}$ which covers $S(G)$. However after an arbitrarily small perturbation of G we get another geometrically finite group G_ϵ for which such pattern exists.

What we will actually need (and prove) is a *relative version* of the above construction, where we are already given a partial packing of $S(G)$ and we would like to extend it to a pattern covering $S(G)$ (after replacing G by a nearby geometrically finite group G_ϵ).

13.1. Orbifolds and patterns of circles

In this section I describe a procedure of constructing hyperbolic orbifolds and discrete subgroups of $\mathrm{Isom}(\mathbb{H}^3)$ associated with *patterns of round discs* on Riemann surfaces. This construction will be used repeatedly in this chapter and in the chapter 19.

Suppose that $G \subset \mathrm{Isom}(\mathbb{H}^3)$ is a geometrically finite torsion-free Kleinian group, $S(G) = \Omega(G)/G$ is the quotient surface. We define the *round disks* in $S(G)$ as follows:

(a) We count as the round disks the homeomorphic projections of compact round disks from $\Omega(G) \subset \widehat{\mathbb{C}}$ to $S(G)$. (Here and in what follows we treat half-planes in $\widehat{\mathbb{C}}$ as a special case of round disks.)

(b) The second class of *round disks* consists of closed punctured disks D in $S(G)$, where the puncture corresponds to a cusp and the disk D is covered by a round disk $\Delta - \{p\} \subset \Omega(G)$, so that Δ is precisely invariant under its parabolic stabilizer A in G and p is the fixed point of A.

Suppose that $\mathcal{P} = D_1 \cup ... \cup D_r$ is a finite collection of closed round disks in $S(G)$ which satisfy the following properties:

- Two distinct disks D_i, D_j either have disjoint interiors or their boundaries are orthogonal.
- No point in $S(G)$ is covered by interiors of more than two disks in $\mathcal{P}$.

If these conditions are satisfied we will say that $\mathcal{P}$ is a *pattern of round disks* in $S(G)$. We will say that $\mathcal{P}$ is a *partial packing* of S if every two distinct *disks* in $\mathcal{P}$ have disjoint interiors.

Each disk $D_i \in \mathcal{P}$ is the projection of a round disk $\tilde{D}_i \subset \widehat{\mathbb{C}}$. For each disk $\tilde{D}_i$ the convex hull $C(\tilde{D}_i) \subset \mathbb{H}^3$ is precisely invariant under the stabilizer of $\tilde{D}_i$ in the group G. Let $C(D_i)$ denote the projection of $C(\tilde{D}_i)$ to $M(G)$. Let $C(\mathcal{P}) := \cup_{i=1}^r C(D_i)$ and $M_{\mathcal{P}} := M(G) - int(C(\mathcal{P}))$. The boundary of the manifold $M_{\mathcal{P}}$ has a natural polyhedral piecewise-geodesic structure (faces are pieces of $\partial C(D_i)$'s, etc.). Thus the polyhedral structure on $\partial M_{\mathcal{P}}$ gives $M_{\mathcal{P}}$ the structure of a locally reflective all right orbifold O (see §6.3). Our goal is to give this orbifold a canonical complete hyperbolic structure and describe the holonomy group of this structure.

Let $G_{\mathcal{P}} \subset \mathrm{Isom}(\mathbb{H}^3)$ be the group generated by G and by the reflections in *all* lifts of the hyperplanes $\partial C(D_i), i = 1, ..., r$, into $\mathbb{H}^3$. This group is called the *group associated with the pattern* $\mathcal{P}$.

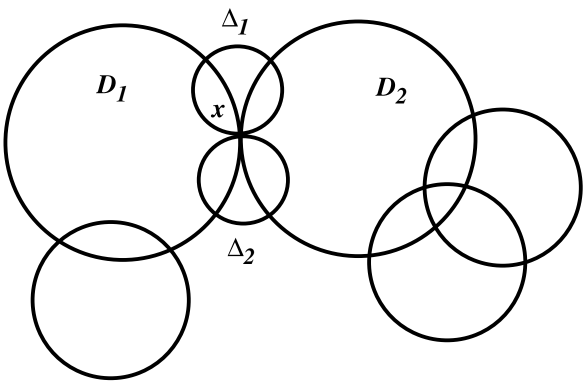

Figure 13.1: Case (a).

Theorem 13.1. *The group $G_{\mathcal{P}}$ is geometrically finite and the orbifold $\mathbb{H}^3/G_{\mathcal{P}}$ is naturally diffeomorphic to O.*

Proof: I will present a proof in the case when the manifold $M(G)$ has no cusps, the general case is similar and is left to the reader. Consider the convex domain $\Phi \subset \mathbb{H}^3$ which is the complement to G-orbit of the union of open hyperbolic half-spaces $intC(\tilde{D}_i), i = 1, ..., r$. This is a locally finite G-invariant polyhedron so that $\Phi/G_{\mathcal{P}} = M_{\mathcal{P}}$. Thus, according to Theorem 4.50, Φ is a fundamental domain for the group R generated by reflections in faces of Φ. Locally, the polyhedron Φ has the same structure as the orbifold O.

The quotient orbifold $\mathbb{H}^3/R$ is naturally identified with Φ (where the orbifold structure is given by lift from O). The group G acts on Φ as the group of covering transformations with the quotient $M_{\mathcal{P}}$. So, it is left to prove that the group $G_{\mathcal{P}}$ generated by R and G is geometrically finite. To show this we have to find a collection of cuspidal neighborhoods of cusps of the orbifold $\dot{O} = (\mathbb{H}^3 \cup \Omega(G_{\mathcal{P}}))/G_{\mathcal{P}}$ so that its complement is compact.

Consider a regular neighborhood $\mathcal{N} \subset \overset{\bullet}{M}(G)$ of $C(\mathcal{P})$. Then the manifold $\overset{\bullet}{M}(G) - \mathcal{N}$ is naturally homeomorphic to $\overset{\bullet}{M}(G)$. The frontier of $\mathcal{N}$ in $\overset{\bullet}{M}(G)$ is disjoint from the thin part of the orbifold O. Recall that the group G was convex-cocompact to begin with. Thus the $\overset{\bullet}{M}(G) - \mathcal{N}$ is compact.

Now we need to understand the cusps which are contained in the orbifold $\mathcal{N}$. Some of these cusps correspond to the *isolated ideal vertices* of $\mathcal{N}$. Namely, the intersection of the polyhedron $\mathcal{N}$ with $\partial \overset{\bullet}{M}$ can contain isolated points x. Four faces of $\overset{\bullet}{O}$ meet at such points. These points correspond to cusps of the virtual rank 2.

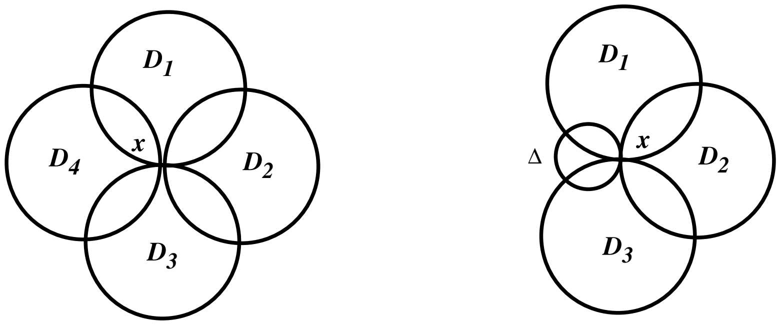

Figure 13.2: Cases (b) and (c).

The second class of cusps corresponds to the ideal vertices of the complement $\partial \overset{\bullet}{M} - \mathcal{P}$. We call them *nonisolated ideal vertices* x of $\mathcal{N}$. At such points two disks from the pattern $\mathcal{P}$ are tangent.

We have the following combinatorial possibilities for x. The first two are the cases of nonisolated ideal vertices.

(a) x belongs to exactly two circles $\partial D_1, \partial D_2$ which are tangent at x. Then we can find two compact round disks $\Delta_1, \Delta_2 \subset S(G)$ which are tangent to each other at the point x and which are orthogonal to $\partial D_1, \partial D_2$. (Figure 13.1.) The union of the convex hulls of Δ_1, Δ_2 in $\mathbb{H}^3$ with a horoball with center at x forms a cuspidal neighborhood of the cusp of O corresponding to the point x.

(b) The next case is when the point x belongs to three circles $\partial D_1, \partial D_2, \partial D_3$ so that $\partial D_1, \partial D_3$ are mutually tangent and ∂D_3 is orthogonal to both of them. In this case we take a small round disk Δ which is tangent to ∂D_2 at the point x, orthogonal of $\partial D_1, \partial D_3$ and is disjoint from all other disks in $\mathcal{P}$. The union of Δ with a horoball with center at x forms a cuspidal neighborhood of the cusp of O corresponding to the point x. (Figure 13.2.)

(c) Case of an isolated vertex x. In this case a horoball with center at x is a cuspidal neighborhood for the point x. (In Figure 13.2 we describe the case when x is a common point of four circles.)

After removing these cuspidal neighborhoods we get a compact orbifold. Thus $G_{\mathcal{P}}$ is geometrically finite. $\square$

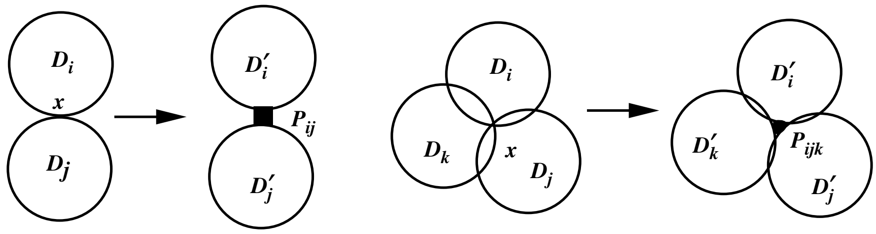

Figure 13.3: Shrink circles. Blow up the point x into a triangle or a square.

We use our analysis of the cusps in O to describe a compactification of O to an orbifold with boundary. Recall that no point in $S(G)$ is covered by interiors of more than two disks in $\mathcal{P}$. Thus, given the configuration $\mathcal{P}$ we can construct a graph $\mathcal{K} \subset S(G)$ as follows.

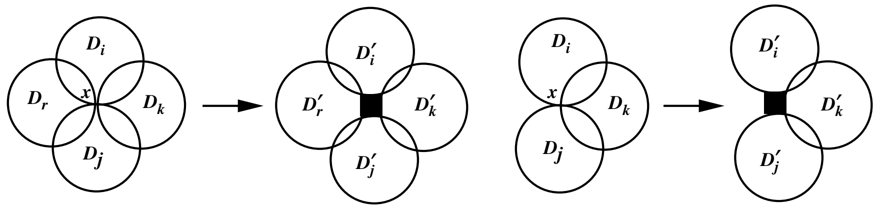

Figure 13.4: Step 1: Shrink circles, blow up the point x to a square.

Step 1. We start by decreasing slightly all disks in the family $\mathcal{P}$ so that each new disk D'_i is disjoint from ∂D_i. This procedure preserves the combinatorics of $\mathcal{P}$ except in the points x where either:

1. Two disks D_i, D_j are tangent: the new disks D'_i, D'_j become disjoint near x, instead of x we insert a 2-edged square orbifold P_{ij}, called a *black box.*

2. Two disks D_i, D_j are tangent and a disk D_k is orthogonal to both of them at the point x; instead of x we insert another *black box*, a 3-edged square orbifold P_{ijk}.

3. Four circles $\partial D_i, \partial D_j, \partial D_k, \partial D_r$ have the common point x, pairs of disks $\partial D_i, \partial D_j$ and $\partial D_k, \partial D_r$ are tangent and D_i is orthogonal to D_k; instead of x we insert a mirror square orbifold P_{ijkr}, called a *black box* as well.

See Figure 13.4.

Step 2. Finally we collapse each bigon $D'_i \cap D'_j$ to a segment $e_{ij} \subset D'_i \cap D'_j$ which connects the vertices of this bigon (see Figure 13.5). What remains of the disks D'_j we now call F_j, they are the *faces* of the orbifold $\widehat{O}'$.

As the result of this 2-step procedure we get a graph $\mathcal{K} \subset S(G)$, some of its complementary components are marked as *faces* F_i and some

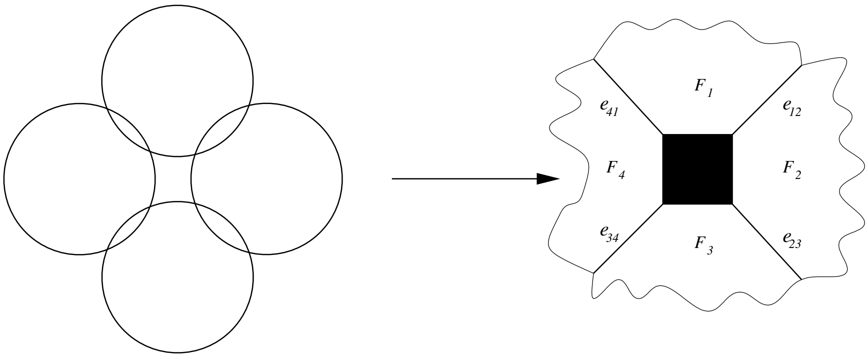

Figure 13.5: Step 2: Collapse bigons to segments.

as *black boxes*; the embedding $\mathcal{K} \hookrightarrow S(G)$ is well-defined up to isotopy. The orbifold $\widehat{O}'$ that we constructed has the underlying set $\overset{\bullet}{M}(G)$, the singular locus is the union of faces $F_j, 1 \leq j \leq r$, and the orbifold structure is given by assigning the angle $\pi/2$ to each edge of $\mathcal{K}$ which is obtained by collapsing a bigon. The orbifold $\widehat{O}'$ is obtained by removing the open cuspidal neighborhoods of those cusps of the orbifold O which belong to $\mathcal{N}$. To get the actual compactification $\widehat{O}$ of the orbifold O we also have to chop off cusps which are outside of $\mathcal{N}$.

We note that we get a pared orbifold $(\widehat{O}, P)$ by collecting together "black boxes" and tori and annuli corresponding to the cusps which are outside of $\mathcal{N}$.

13.2. Brooks' Theorem

Recall that for a geometrically finite Kleinian group $G \subset \mathrm{Isom}(\mathbb{H}^3)$ we have the Teichmüller space $\mathcal{T}(G)$ which consists of conjugacy classes of discrete embeddings $\rho : G \hookrightarrow \mathrm{Isom}(\mathbb{H}^3)$ that are induced by quasiconformal homeomorphisms. Let d_T denote the Teichmüller metric on $\mathcal{T}(G)$. Suppose that G is torsion-free and $\mathcal{P}$ is a *partial packing* of the surface $S(G)$. Let $G_{\mathcal{P}}$ be the discrete subgroup of $\mathrm{Isom}(\mathbb{H}^3)$ associated with $\mathcal{P}$ and $[\rho] \in \mathcal{T}(G_{\mathcal{P}})$. Let $G^\epsilon = \rho(G)$. Then the images of reflections in $G_{\mathcal{P}}$ under ρ are again reflections hence the partial packing $\mathcal{P}$ corresponds to a partial packing $\mathcal{P}^\epsilon = \mathcal{P}$ in $S(G^\epsilon)$ so that $G^\epsilon_{\mathcal{P}^\epsilon} = \rho(G_{\mathcal{P}})$.

Theorem 13.2. *(Brooks' Deformation Theorem, [Bro86].)*
Suppose that $G \subset \mathrm{Isom}(\mathbb{H}^3)$ is a geometrically finite torsion-free Kleinian group and $\mathcal{P}$ is a partial packing of $S(G)$. Then for any $\epsilon > 0$ there exists a homomorphism $\rho : G \to G^\epsilon \subset \mathrm{Isom}(\mathbb{H}^3)$ so that:

- $[\rho] \in \mathcal{T}(G)$, $d_T([id], [\rho]) \leq \epsilon$.
- *The partial packing $\mathcal{P}^\epsilon$ in $S(G^\epsilon)$ corresponding to $\mathcal{P}$ extends to a pattern of disks $\mathcal{R}$ so that the associated discrete subgroup $\Gamma = G^\epsilon_{\mathcal{R}}$ (which contains G^ϵ) is a lattice in $\mathrm{Isom}(\mathbb{H}^3)$.*

- *The convex hull* $C\Lambda(G^\epsilon)$ *is precisely invariant under the subgroup* G^ϵ *of* Γ.

- *The orbifold* $\mathbb{H}^3/\Gamma$ *is* **all right bipolar** *orbifold (see Definition 6.28).*

The proof of this theorem occupies the rest of this section. The main tool in the proof is a study of circle packings on the quotient surface $S(G)$ of the Kleinian group G, it was influenced by Thurston's generalization of Andreev's theorem. Brooks' paper [Bro86] was generalized by Bowers and Stephenson in [BS92]. Ever since appearence of Thurston's notes [Thu81, Chapter 13] and his subsequent influential paper [Thu86b] on applications of circle packings to the complex analysis, the study of circle packings bacame a small industry. We refer the reader to [RS87, MR90, He90, HR93, HS93, dV91, Du97].

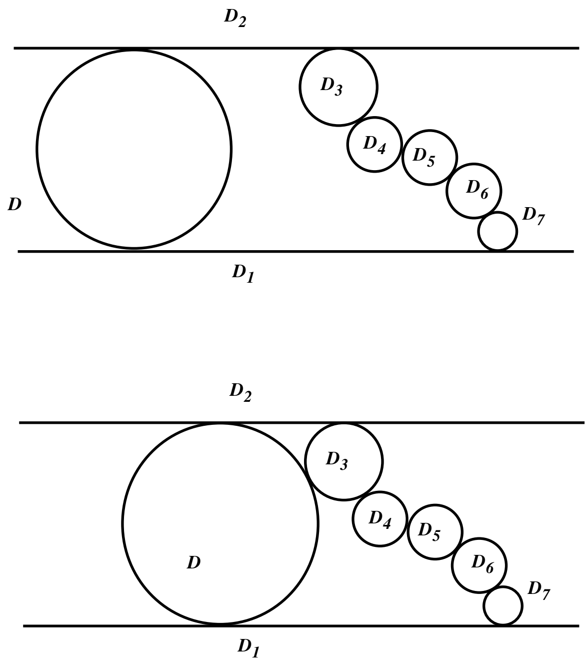

Figure 13.6:

13.3. A packing invariant of patterns of disks

Suppose that $D_1, ..., D_n$ $(n \geq 3)$ is a cyclically ordered collection of closed round disks in $\widehat{\mathbb{C}}$ so that the disks D_i, D_j are tangent if $i = j \pm 1$ and disjoint otherwise. The complement $\widehat{\mathbb{C}} - (D_1 \cup ... \cup D_n)$ consists of two components. Each component C of $\widehat{\mathbb{C}} - int(D_1 \cup ... \cup D_n)$ is called an *ideal circular polygon.* The vertices of C are points of tangency between the disks which belong to $cl(C)$.

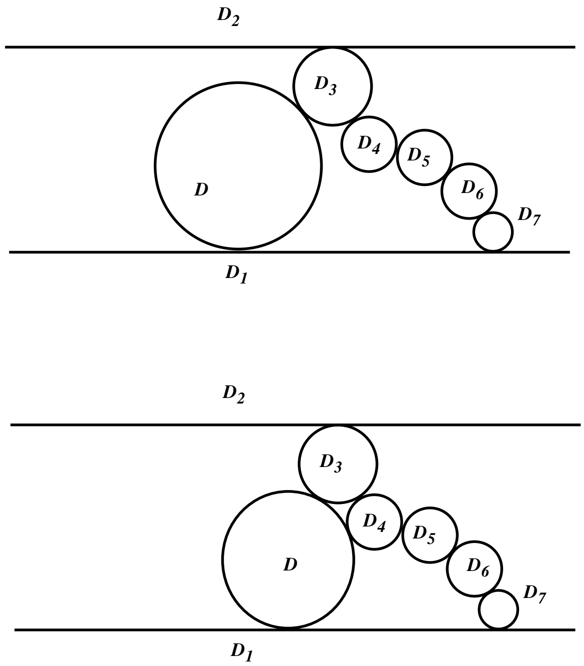

Figure 13.7:

Lemma 13.3. *Suppose that C is an ideal circular polygon in $\widehat{\mathbb{C}}$. Then there exists a finite collection of disks $D_{n+1}, ..., D_{n+N}$ such that each component of*

$$C - (D_{n+1} \cup ... \cup D_{n+N})$$

is either an ideal triangle or ideal rectangle.

Proof: Assume that the number of ideal vertices of C is at least 5. We start with an ideal vertex v of C formed by the disks D_1, D_2 and inscribe in this corner of C a round disk D which is tangent to D_1, D_2 and is disjoint form all other disks D_j. Then start moving D away from v keeping it tangent to D_1, D_2 until D will touch another disk D_i. If D_i is disjoint from D_1 and D_2 then we stop. Otherwise assume that (say) $D_i = D_3$ (see Figure 13.6). In this case we continue moving the disk D so that $D \subset C$ and D is tangent to D_2 and D_3. We stop when D will touch another disk D_j. It is easy to see that the disk D constructed this way separates the domain C into 3 ideal polygons C_1, C_2, C_3. One of them (bounded by boundary arcs of D, D_1, D_2, D_3) is an ideal rectangle. Direct computation shows that the number of ideal vertices of each polygon C_j is less than n (see Figure 13.7). The same is true if we stop after the disk D touches D_3 which is disjoint from D_1, D_2. Let $D_{n+1} := D$. Thus, arguing by induction, we construct the family of disks $D_{n+1}, ..., D_{n+N}$. $\square$

Remark 13.4. The above lemma has a straightforward generalization to the case of circular polygons in $\widehat{\mathbb{C}}$ whose inner angles are less than π.

Note that in general we cannot find a collection of round disks in C so that all complementary regions are ideal triangles. To do this we have to deform the original polygon C.

In what follows we will need several elementary facts about the *continuous fractions*, the proofs of these facts are left as exercises to the reader.

Let $(n_1, n_2, n_3, \dots)$ be a (possibly finite) sequence of integers, where $n_1 \geq 0$ and $n_i > 0$ for $i > 1$. Define the sequence of rational numbers:

$$q_m := n_1 + \cfrac{1}{n_2 + \frac{1}{n_3 + \dots \frac{1}{n_m}}}.$$

This sequence is called a *continuous fraction*. The sequence q_m has the following properties:

(1) q_m converges to a nonnegative real number r as $m \to \infty$, the usual notation for r is

$$r = n_1 + \cfrac{1}{n_2 + \frac{1}{n_3 + \dots}}.$$

The sequence q_m is called the *continuous fraction expansion* for r.

(2) Each nonnegative real number admits a continuous fraction expansion and this expansion is unique.

We now return to the discussion of ideal polygons. Consider an ideal circular quadrilateral $Q \subset \widehat{\mathbb{C}}$, let H be the group generated by the reflections in circular arcs which are the edges of Q. Then the fundamental domain for the action of H on $\widehat{\mathbb{C}}$ consists of the union of the interior quadrilateral Q and the exterior quadrilateral $\bar{Q}$ (see Figure 13.8), hence H is quasiconformally conjugate to a Fuchsian group (which is generated by reflections in four circles which are orthogonal to a straight line in $\widehat{\mathbb{C}}$). We conclude that the group H is quasifuchsian.

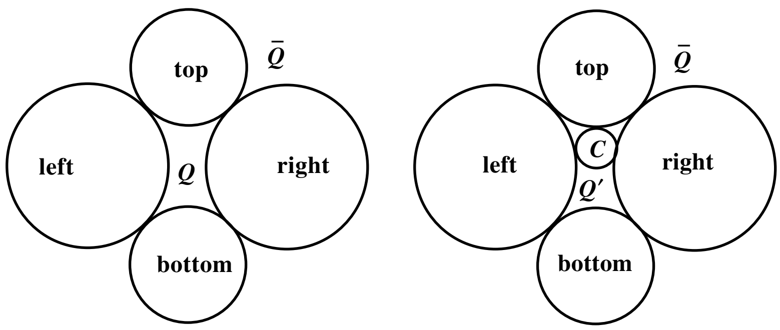

Figure 13.8: *Q is the interior quadrilateral and $\bar{Q}$ is the exterior quadrilateral. The circle C is vertical.*

We label the sides of Q as: top, right, bottom, left, so that this labeling is consistent with the natural orientation of ∂Q, otherwise the labeling has nothing to do with any particular coordinate system in $\widehat{\mathbb{C}}$ (see Figure 13.8).

Note that we can always inscribe into Q a round circle C which is tangent either to the top, bottom and left sides of Q (following [Bro85] we call it a *horizontal* circle) or to the top, left and right sides (we call it a *vertical* circle). If this circle is horizontal and vertical simultaneously, then we call this circle *neutral*. If C is not neutral then it splits Q into two ideal triangles and one ideal quadrilateral Q'. Three sides of Q' already have labels coming from the original quadrilateral Q, there is a unique way to label the remaining side so that Q' gets a top-right-bottom-left labeling. If C is a neutral circle then we stop, otherwise we inscribe a horizontal or a vertical round circle into Q'. Then we repeat the process ...

As the result we get a finite or infinite *circle packing* of Q. Each point of Q either belongs to a closed disk bounded by one of the circles of this packing or to an ideal triangle bounded by arcs of these circles, or it is an accumulation point of the circles. Note that since the spherical diameter of the sequence of quadrilaterals tends to zero, there is only one point of accumulation of circles. This point could possibly belong to one of the circles in the packing. Let $\mathcal{P} := \mathcal{P}(Q)$ denote the resulting packing. Define a number $r(Q) \in (0, \infty)$, which is called *the packing invariant* of Q, as follows.

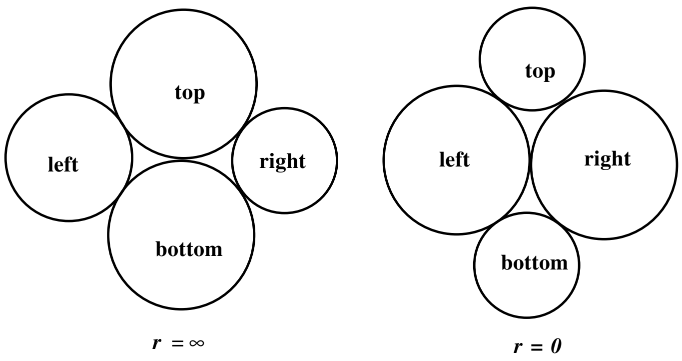

Figure 13.9: *Degenerate quadrilaterals.*

Let n_1 be the number of horizontal circles in $\mathcal{P}$ that we can inscribe until we will need a vertical circle. If one of these circles is neutral then we count it as a horizontal circle and let $r(Q) := n_1$. Otherwise let n_2 be the number of vertical circles $\in \mathcal{P}$ that we can inscribe until we again needs a horizontal one. As before, if one of these circles is neutral, then we count it as vertical and let

$$r(Q) := n_1 + \frac{1}{n_2} \; .$$

Otherwise we continue this process. In the limit we get a real number $0 < r(Q) < \infty$, whose continued fraction expansion is:

$$r(Q) := n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \dots}} \; .$$

The number $r(Q)$ is called the *packing invariant* of Q.

Now we generalize this invariant to the cases of *degenerate* quadrilaterals Q. Namely, we allow certain disks D_i, D_{i+2} be tangent inside the polygon

Q: if the top and bottom sides of Q are tangent then we let $r(Q) = \infty$, if left and right sides are tangent then we let $r(Q) = 0$. See Figure 13.9.

Consider the configuration space $\mathcal{C}(4)$ of labeled ideal quadrilaterals in $\widehat{\mathbb{C}}$. We assume that the circles containing boundary arcs of quadrilaterals $Q \in \mathcal{C}(4)$ do not degenerate to points, but we allow degeneration of quadrilaterals as on Figure 13.9: either top and bottom sides or left and right sides could be tangent. The space $\mathcal{C}(4)$ has a natural topology which is induced from the product topology on the space of quadruples of round circles in $\widehat{\mathbb{C}}$. The packing invariant defines a function $r : \mathcal{C}(4) \to [0, \infty]$.

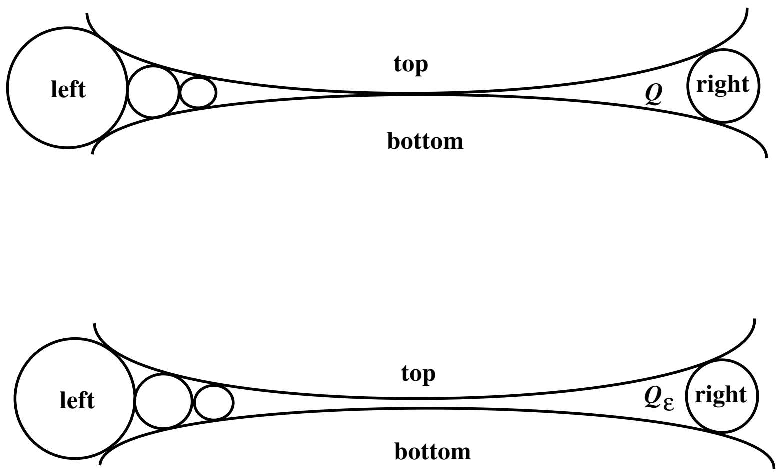

Figure 13.10: *Packing invariant for degenerate quadrilaterals.*

Lemma 13.5. *The function r is continuous.*

Proof: We first consider the cases of degenerate quadrilaterals. Suppose that Q is a degenerate quadrilateral where top and bottom sides are tangent. Then we can inscribe infinitely many horizontal circles C_j into the "left" half of the quadrilateral Q. Now we "regenerate" Q to a quadrilateral Q_ϵ, where ϵ is the smallest distance between the top and bottom sides of Q_ϵ (see Figure 13.10). It follows from elementary calculations that one can perturb $\approx const/\epsilon$ of the circles C_j so that they become horizontal circles inscribed into Q_ϵ. Here $const$ is a constant which depends only on the geometry of the original configuration. Thus

$$\lim_{\epsilon \to 0} r(Q_\epsilon) = \infty = r(Q).$$

Similar argument works in the case when the left and right sides of Q are tangent.

Now we can consider the generic case. Suppose that Q is a nondegenerate quadrilateral with an infinite circle packing $\mathcal{P}$, let $r := r(Q)$. Suppose that $\delta > 0$, $N = [1/\delta]$ (the greatest integer not exceeding $1/\delta$). Then we can find a neighborhood U of Q in $\mathcal{C}(4)$ such that any $Q_\epsilon \in U$ have the same first N numbers $n_1, ..., n_N$ describing combinatorics of circle packing of the quadrilateral Q_ϵ. Thus $|r(Q_\epsilon) - r(Q)| \leq const/N$. We conclude that the function r is continuous in such points Q. The last case we have to consider is when the circle packing of Q is finite. Let C_k denote the last (neutral) circle in $\mathcal{P}(Q)$, where the numbers $n_1, ..., n_k$ describe combinatorics of the circle packing of Q. Small perturbations of Q can move C_k to either a horizontal or a vertical circle (depending on a sequence Q_ϵ in $\mathcal{C}(4)$ convergent to Q). Let us assume that C_k becomes horizontal after this perturbation. Then we get quadrilaterals $Q_{k,\epsilon} \subset Q_\epsilon$, where C_k corresponds to the top or the bottom side of $Q_{k,\epsilon}$ and $\epsilon \to 0$ is the distance between the top and the bottom sides of $Q_{k,\epsilon}$. As $\epsilon \to 0$ the quadrilaterals $Q_{k,\epsilon}$ degenerate to a quadrilateral Q_k where the top and bottom sides of Q_k are tangent. Thus we are in the situation when we can apply our analysis of continuity of the function r at degenerate quadrilaterals. Namely,

$$r(Q_\epsilon) \approx n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \cdots \frac{1}{n_k + \epsilon/const}}} \approx r(Q).$$

The case when C_k becomes vertical in $Q_{k,\epsilon}$ is similar. $\square$

Consider a configuration C of four tangent round disks with the labels as in Figure 13.8. They bound two ideal quadrilaterals : Q and $\bar{Q}$. These quadrilaterals have packing invariants $r(Q)$ and $r(\bar{Q})$.

Proposition 13.6. *The pair $(r(Q), r(\bar{Q})) \in \mathbb{R}_+ \times \mathbb{R}_+$ uniquely determines the configuration C up to a Moebius transformation $\gamma : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.*

Proof: I will reduce this Proposition to a rigidity theorem of O. Schramm [Sch91]. Consider the circle packing of the Riemann sphere $\widehat{\mathbb{C}}$ given by $C \cup \mathcal{P}(Q) \cup \mathcal{P}(\bar{Q})$. At most two points in $\widehat{\mathbb{C}}$ are the accumulation points for these packings, thus at most two points in $\widehat{\mathbb{C}}$ are not in the union of corresponding round disks and ideal triangles. (In the terminology of [Sch91], the complement to the *carrier* of $C \cup \mathcal{P}(Q) \cup \mathcal{P}(\bar{Q})$ consists of at most two points.) If we have another configuration C' with the same pair of packing invariants $(r(\mathcal{P}(Q')), r(\mathcal{P}(\bar{Q}')))$ then the packings $\mathcal{P}(Q) \cup \mathcal{P}(\bar{Q})$ and $\mathcal{P}(Q') \cup \mathcal{P}(\bar{Q}')$ are combinatorially equivalent (by the uniqueness property of the continuous fraction expansion) and therefore they are Moebius-equivalent according to [Sch91]. Thus there is a Moebius transformation which carries C to C'. $\square$

13.4. A packing invariant of Kleinian groups

We associate a discrete group of Moebius transformations Γ_Q to the packing $\mathcal{P} = \mathcal{P}(Q)$ of each ideal quadrilateral Q as follows. Let C denote the configuration of four round circles which contain the edges of Q. For each ideal triangle T_j formed by the arcs of circles in $\mathcal{P}$ we add the extra circle $C(T_j)$ which contains the vertices of this triangle. As the result we get a pattern of circles $\mathcal{D}$ which consists of the circles in C, $\mathcal{P}$ and of $C(T_j)$, $j = 1, 2, ...$ Each pair of circles in $\mathcal{D}$ is either tangent, orthogonal or disjoint. Then take the group Γ_Q generated by the reflections in all circles of $\mathcal{D}$. Similarly to

§13.1, the convex polyhedron $\Phi \subset \mathbb{H}^3$ bounded by the convex hulls of the circles in $\mathcal{D}$ is a fundamental domain for the group Γ_Q. The quadrilateral $\bar{Q} = int(\partial_\infty \Phi)$ serves as a fundamental domain for the action of Γ_Q on $\widehat{\mathbb{C}}$.

Remark 13.7. Note that $\partial_\infty \Phi - \bar{Q}$ consists of a countable number of ideal vertices.

Take two configurations C, C' of four tangent round circles in $\mathbb{C}$ as above, let Q, Q' be the (bounded) ideal quadrilaterals that C and C' bound.

Proposition 13.8. *Suppose that $r(Q) = r(Q')$. Then there exists a quasiconformal homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which carries Q to Q', $\mathcal{P}(Q)$ to $\mathcal{P}(Q')$ and $\mathcal{D}$ to $\mathcal{D}'$.*

Proof: Our proof again follows [Bro85, Lemma 4.1], however in the final part of the proof I will use Theorem 12.92 to complete the argument. Let $R_a \subset \mathbb{C}$ denote the Euclidean rectangle with the vertices $\{0, a, a+i, i\}$. This rectangle has the natural top-right-left-bottom labeling of the sides. The labeled quadrilateral $\bar{Q}$ is nondegenerate, thus (for a certain $a > 0$) there is a Riemann mapping $\phi : \bar{Q} \to R_a$, whose extension to the boundary sends vertices to vertices and preserves labels. The number a is the *conformal modulus* of $\bar{Q}$. For any rectangle R_b define the affine map $A_b : R_a \to R_b$ by the formula

$$A_b : z \mapsto Re(z)b/a + iIm(z).$$

Then we get a Beltrami differential μ_b on $\bar{Q}$ given by

$$\phi^*(\frac{\overline{\partial} A_b}{\partial A_b} d\bar{z}/dz).$$

Recall that $\bar{Q}$ is a fundamental domain for the action of Γ_Q on $\widehat{\mathbb{C}}$. Disperse the Beltrami differential μ_b from $\bar{Q}$ to $\Omega(\Gamma_Q)$ via the action of Γ_Q and extend it by zero to the limit set. We retain the notation μ_b for the resulting Beltrami differential . As in the proof of the Bers' Theorem (Theorem 8.40), we let h_b be the normalized solution of the Beltrami equation

$$\mu_b \partial h_b = \overline{\partial} h_b.$$

The quasiconformal map h_b conjugates Γ_Q to another discrete group of Moebius transformations which we call Γ_b. Note that circles in $\mathcal{D}(Q)$ are fixed-point sets of involutions in Γ_Q. Thus h_b carries circles from $\mathcal{D} := \mathcal{D}(Q)$ to round circles in $\mathcal{D}_b$ and preserves the combinatorics of the circle packing. Hence, we get a 1-parameter family $\mathcal{D}_b$ of quasiconformally equivalent circle packings so that the conformal modulus of the quadrilateral $\bar{Q}_b$ equals b. The parameter b varies from 0 to ∞. Let $\rho_b : \Gamma_Q \to \Gamma_b$ denote the embedding of Γ_Q into $\mathrm{Isom}(\mathbb{H}^3)$ induced by h_b.

Lemma 13.9. *$\bar{Q}_b$ converges to degenerate quadrilaterals in $\mathcal{C}(4)$ as $b \to 0$ and $b \to \infty$.*

Proof: Take the first k circles $C_1, ..., C_k$ in $\mathcal{P}(Q)$ among which there is at least one horizontal and one vertical circle (one of them could be neutral).

We will consider the case when none of these circles is neutral, the remaining case is done similarly. The finite partial packing $\widehat{\mathcal{P}}^k := C \cup C_1 \cup ... \cup C_k$ has two complementary quadrilaterals : $\bar{Q}$ and Q_k. Let Γ_k be the subgroup of Γ_Q generated by reflections in the circles of $\widehat{\mathcal{P}}^k$.

By our choice of the number k, no pair of opposite sides of $\bar{Q}$ corresponds to the circles which give a pair of sides of Q_k (see Figure 13.11).

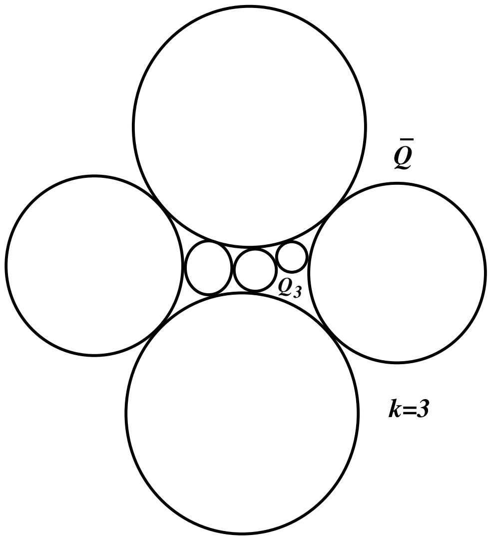

Figure 13.11: *Partial packing.*

Thus the hyperbolic orbifold $O := \dot{M}(\Gamma_k)$ is acylindrical and we can apply Theorem 12.92 to conclude that the sequence of discrete and faithful representations ρ_b subconverges to a representation ρ_0 as $b \to 0$ and subconverges to ρ_∞ as $b \to \infty$. Hence we get convergence of $\bar{Q}_b$ in $\mathcal{C}(4)$ as well. Clearly each of these limits is a degenerate quadrilateral since the conformal modulus depends continuously on $\bar{Q}_b \in \mathcal{C}(4)$. $\square$

Using Lemmas 13.5 and 13.9 we conclude that $r(\bar{Q}_b)$ varies continuously as a function of b from 0 to $+\infty$. Hence we can find a value of b such that $r(\bar{Q}_b) = r(\bar{Q}')$. Lemma 13.6 implies that the configurations C_b and C' are Moebius-equivalent. By construction there exists a quasiconformal homeomorphism of $\widehat{\mathbb{C}}$ which carries $\mathcal{D}$ to $\mathcal{D}_b$. This proves Proposition 13.8. $\square$

Suppose now that G is a geometrically finite discrete subgroup of $\mathrm{Isom}(\mathbb{H}^3)$ so that:

- A fundamental domain for the action of G in $\widehat{\mathbb{C}}$ consists of a finite union of ideal (nondegenerate) quadrilaterals $Q_1, ..., Q_m$.
- The group G contains reflections in the edges of these quadrilaterals.

Choose a top-right-bottom-left labeling for edges of the quadrilaterals $Q_1, ..., Q_m$. Let $[\rho] \in \mathcal{T}(G)$, $\rho(G) = G^\epsilon$ and ρ is induced by a quasiconformal mapping $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Then G^ϵ again has a fundamental domain

$Q_1^\epsilon \cup ... \cup Q_m^\epsilon$ (which is the image of $Q_1 \cup ... \cup Q_m$ under f) and the mapping f labels the edges of these quadrilaterals . Each of the labeled quadrilaterals Q_j^ϵ has the packing invariant $r(Q_j^\epsilon)$. Hence we get a function:

$$F : \mathcal{T}(G) \to (\mathbb{R}_+)^m, \quad F(\rho) = (r(Q_1^\epsilon), ..., r(Q_m^\epsilon)).$$

Definition 13.10. $F(\rho)$ is the **packing invariant** of $[\rho] \in \mathcal{T}(G)$.

Theorem 13.11. *(R. Brooks.) The function F is continuous and injective.*

Proof: The quadrilaterals Q_j vary continuously with the representation ρ, thus continuity of F follows from Lemma 13.5. The nontrivial part of the theorem is injectivity of F. Suppose that that $[\rho^\epsilon], [\rho^\delta] \in \mathcal{T}(G)$, $G^\epsilon := \rho^\epsilon(G), G^\delta := \rho^\delta(G)$, so that $F(\rho^\epsilon) = F(\rho^\delta)$. According to Proposition 13.8 there are quasiconformal homeomorphisms $\phi_j : Q_j^\epsilon \to Q_j^\delta$ which carry the packing $\mathcal{P}(Q_j^\epsilon)$ to $\mathcal{P}(Q_j^\delta)$ and the packing $\mathcal{D}^\epsilon$ to $\mathcal{D}^\delta$. The union of quadrilaterals Q_j^ϵ (resp. Q_j^δ) is a fundamental domain for the action of G^ϵ on $\widehat{\mathbb{C}}$ (resp. the action of G^δ on $\widehat{\mathbb{C}}$). Using the actions of the groups G^ϵ, G^δ we extend the homeomorphisms ϕ_j to an equivariant quasiconformal homeomorphism $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. As before we extend the groups G^ϵ, G^δ by adding the reflections in the circles of $\mathcal{D}(Q_j^\epsilon)$, $\mathcal{D}(Q_j^\delta)$, $j = 1, ..., m$. The resulting groups $\Gamma^\epsilon, \Gamma^\delta$ are discrete, their limit sets equal to $\widehat{\mathbb{C}}$. Recall that the groups G^ϵ, G^δ are geometrically finite and each packing $\mathcal{P}(Q_j^\epsilon)$, $\mathcal{P}(Q_j^\delta)$ has only one accumulation point in $\widehat{\mathbb{C}}$. Thus the groups $\Gamma^\epsilon, \Gamma^\delta$ have convex fundamental polyhedra with only countably many ideal vertices. Hence the action of Γ^ϵ on its limit set is recurrent (see Theorem 8.39). The quasiconformal homeomorphism h conjugates Γ^ϵ to Γ^δ and we apply Sullivan Rigidity Theorem 8.37 to conclude that h is Moebius. Therefore $[\rho^\delta] = [\rho^\epsilon]$ define the same point in $\mathcal{T}(G)$. $\square$

Corollary 13.12. *The mapping F is a local homeomorphism.*

Proof: Let Q_j be one of the ideal quadrilaterals in the fundamental domain of G. The subgroup G_j generated by the reflections in the edges of Q_j is the stabilizer of a component of $\Omega(G)$ which contains Q_j. The group G_j is quasifuchsian. Hence, according to § 8.15, the Teichmüller space $\mathcal{T}(G)$ is homeomorphic to $\mathbb{R}^m$. Thus F is an injective continuous map $\mathbb{R}^m \to \mathbb{R}^m$ and has to be a local homeomorphism. $\square$

Remark 13.13. Actually the mapping F is a global homeomorphism.

13.5. Proof of the Brooks' theorem

Finally we can finish the proof of Theorem 13.2. We first have to extend the partial packing $\mathcal{P}$ in $S(G)$ to a finite pattern $\mathcal{Q}$ of round disks in $S(G)$ (in the sense of § 13.1) so that the complementary regions to the union of disks in this pattern are ideal triangles and quadrilaterals (Steps 1–3 below).

Step 1. We first take care of the cusps in the surface $S(G)$. If $\mathcal{P}$ already contains a *round* disk which is a punctured neighborhood of a puncture p

in $S(G)$ then we ignore this cusp. For each remaining puncture p we add to $\mathcal{P}$ a *round disk* D in $S(G)$ which is a cusp neighborhood of p in $S(G)$. We assume that the union of $\mathcal{P}$ with the new *round disks* is still a partial packing. We retain the notation $\mathcal{P}$ for this partial packing. For a particular disk D which is a neighborhood of a puncture we take an appropriate conjugation of the group G in $PSL(2,\mathbb{C})$ so that that D is covered by an upper half-plane $\tilde{D} = \{z : Im(z) > t\} \subset \Omega(G_0)$. The stabilizer of $\tilde{D}$ in G_0 is a cyclic parabolic group A generated by the horizontal translation $\alpha : z \mapsto z+1$. The group A has the fundamental domain

$$\Psi := \{z : 0 < Re(z) < 1\}.$$

We inscribe into Ψ the round disk $\tilde{\Delta} := \{z : |z - it + 1/2| \leq 1/2\}$. See Figure 13.12.

By choosing t sufficiently large we guarantee that the interior of $\tilde{\Delta}$ projects homeomorphically into $S(G_0)$. Let Δ denote the projection. This way we choose pairs of round disks D_j, Δ_j for each cusp P_j in $S(G_0)$ so that the disks corresponding to the distinct cusps are disjoint and that each Δ_j does not intersect any disk from $\mathcal{P}$ except D_j. We assign the *white* label to the disks Δ_j and the *red* label to the disks in $\mathcal{P}$. Let $\mathcal{Q}_1$ denote the pattern of disks constructed on this step.

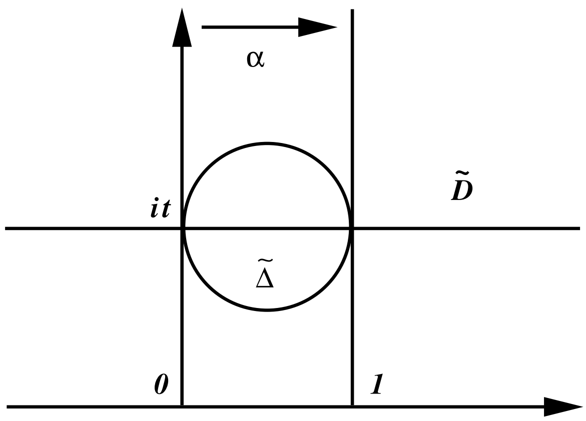

Figure 13.12: *Filling in cusps.*

Step 2. We use Lemma 13.3 to construct a collection of round disks in

$$S^*(G_0) := S(G) - \cup_{B_j \in \mathcal{Q}_1} B_j$$

which have disjoint interiors and so that each complementary region is either an ideal triangle T_k or an ideal quadrilateral. More precisely, first subdivide the surface $S^*(G)$ into cells whose edges are circular arcs so that each edge is either disjoint from $\partial S^*(G)$ or is orthogonal to it. Then cover the 1-skeleton T^1 of this cell complex (excluding $\partial S^*(G)$) by finitely many closed round disks so that each disk and edge in $T^1 - \partial S^*(G)$ are either orthogonal or disjoint and each edge in $\partial S^*(G)$ is either disjoint or tangent to a disk. Two disks in the covering are either disjoint, or equal or tangent to each other. The complementary regions to the above disks are simply-connected, we lift them to $\widehat{\mathbb{C}}$ and apply Lemma 13.3. The result is the pattern $\mathcal{Q}_2$ of round disks in $S(G)$.

Step 3. Let $Q_1, ..., Q_m$ be the complete list of complementary ideal quadrilaterals to $\mathcal{Q}_2$. For each complementary ideal triangle T_k to $\mathcal{Q}_2$, we choose a round disk in $S(G)$ whose boundary contains the vertices of T_k. After we add these disks to the pattern $\mathcal{Q}_2$ we get the required pattern $\mathcal{Q}$ of round disks in $S(G)$. We assign the *red* color to the disks bounded by these circles.

Step 4. Let $G_{\mathcal{Q}} \subset \mathrm{Isom}(\mathbb{H}^3)$ be the discrete subgroup associated with the pattern $\mathcal{Q}$ (as in §13.1).[1] The group $G_{\mathcal{Q}}$ satisfies the conditions of Theorem 13.11. We retain the notation $Q_1, ..., Q_m$ for the lifts of the quadrilaterals from $S(G)$ to the ideal quadrilaterals in the domain $\Omega(G)$. It follows from the description of fundamental polyhedron of the reflection group in §13.1, that $Q_1 \cup ... \cup Q_m$ is a fundamental domain for the action of $G_{\mathcal{Q}}$ on $\widehat{\mathbb{C}}$. Rational vectors are dense in $\mathbb{R}^m$, thus by Corollary 13.12 there is a dense subset $E \subset \mathcal{T}(G_{\mathcal{Q}})$ so that for each $[\rho] \in E$ the vector $F(\rho)$ has only rational components. Here the function F is the *packing invariant* of ρ. For given $\epsilon > 0$ we take $[\rho]$ such that $d_T([\rho], [id]) \leq \epsilon$.

Step 5. Let $[\rho] \in E$, $G^\epsilon = \rho(G)$, then the pattern $\mathcal{Q}$ corresponds to a pattern $\mathcal{Q}_\rho$ in $S(G^\epsilon)$. Each complementary quadrilateral Q_j^ϵ to $\mathcal{Q}_\rho$ admits a *finite* circle packing $\mathcal{P}(Q_j)$. We assign the white color to the corresponding disks. Finally, as in the Step 3, we "fill in" the complementary ideal triangles by round disks which will be labeled *red*. The result is the pattern of disks $\mathcal{R}$ in $S(G^\epsilon)$ whose disks completely cover this surface.

The associated group $\Gamma = G^\epsilon_{\mathcal{R}}$ is geometrically finite and has empty domain of discontinuity. Hence Γ is a lattice in $\mathrm{Isom}(\mathbb{H}^3)$. This proves the first two assertions of Theorem 13.2.

The rest of the assertions of Theorem 13.2 are easy to verify. Note that (to get Γ) we added to the group G^ϵ the reflections in the hyperbolic hyperplanes which are disjoint from the convex hull of $\Lambda(G^\epsilon)$. Thus there is a fundamental domain of the action of G^ϵ on $C\Lambda(G^\epsilon)$ which is contained inside of a fundamental domain of the lattice Γ_ϵ. This implies that $C\Lambda(G^\epsilon)$ is precisely invariant under G^ϵ. By the construction, the orbifold $\mathbb{H}^3/\Gamma$ is locally reflective. Our construction yields a white-red labeling of the hyperplanes which are the fixed point sets of the reflections in Γ, so that the hyperplanes with the same colors are disjoint and the hyperplanes with distinct colors are either disjoint or orthogonal in $\mathbb{H}^3$. The action of Γ preserves the labeling which implies that the orbifold $\mathbb{H}^3/\Gamma$ is *all right bipolar*.
$\square$

[1] Recall that we lift all round circles of the pattern to $\Omega(G)$ and add reflections in these circles to the group G.

Chapter 14

Pleated Surfaces and Ends of Hyperbolic Manifolds

14.1. Singular hyperbolic metrics

Consider a smooth connected closed surface $\hat{S}$ and a finite subset $P = \{p_1, ..., p_m\}$ in $\hat{S}$, so that the punctured surface $S := \hat{S} - P$ has hyperbolic type. Pick a complete hyperbolic structure on S. Let $V \subset S$ be a finite subset. A *"triangulation"* T of S based at $V \cup P$ is a finite collection Δ of smooth compact disjoint arcs in $\hat{S}$ with end-points in $V \cup P$ so that:

1. No two arcs are isotopic in $\hat{S}$ (mod $V \cup P$).

2. No arc is isotopic to a point in $\hat{S}$ (mod $V \cup P$).

3. This system is a maximal system of arcs with the properties (1), (2).

Note that such "triangulation" does not have to be a triangulation in the usual sense. In our applications, V will be either a single vertex or an empty set. The points in V are called *vertices* and the points in P are called the *ideal vertices* of the triangulation. The *edges* of the triangulation are the arcs in Δ. We define the *triangles* of the triangulation to be metric completions of the components of $S - \Delta$. Maximality of T implies that each triangle τ has three edges and three vertices (some of which may be ideal). The interior of each edge of τ maps bijectively to an edge of T, however distinct edges may have equal images.

Now suppose that each triangle τ of the triangulation T is given a complete hyperbolic metric h_τ of finite area and constant curvature -1 (this metric usually has nothing to do with the initial hyperbolic metric on S). We assume that the edges of each triangle are totally geodesic; note that the ideal vertices of τ correspond to the ideal points with respect to h_τ. We also assume that for every two triangles adjacent along an edge e, the length of e is the same in the both triangles (this length may be infinite). This collection of Riemannian metrics determines a path-metric

σ on the whole surface S. We call such metric a *singular hyperbolic metric* if the following properties are satisfied:

(A1) The total angle around each vertex in V is at least 2π.

(A2) The path-metric on S is complete.

Hence the universal cover $X := \tilde{S}$ of each singular hyperbolic surface is a $CAT(-1)$-space (see §3.2). There is a classification of isometries of X which is similar to the classification of isometries of $\mathbb{H}^n$. Any nontrivial isometry γ of X:

- Either has a fixed point in X (the elliptic case).
- Or γ has a single fixed point at infinity $\xi = \xi_\gamma \in \partial_\infty X$ and the Busemann function β_ξ is γ-invariant (the parabolic case).
- Or γ has a unique invariant geodesic $A_\gamma \subset X$ (the loxodromic case).

Suppose that $\gamma \in \mathrm{Isom}(X)$ is either loxodromic or parabolic. Consider the set

$$X_\epsilon := \{x \in X : \exists n \neq 0 | d(\gamma^n(x), x) \le \epsilon\}.$$

Then convexity of the distance function in X implies that:

- If γ is loxodromic then X_ϵ is the union of the shortest geodesic segments between the points of ∂X_ϵ and the geodesic A_γ.
- If γ is parabolic then X_ϵ is the union of geodesic rays emanating from the points of ∂X_ϵ and asymptotic to ξ_γ.

In any case X_ϵ is contractible.

Let $\tilde{S}$ denote the universal cover of $S - V$ and let $\tilde{T}$ denote the lift of T to S. For each singular hyperbolic surface there is a locally isometric developing map $d : S - V \to \mathbb{H}^2$ and a monodromy representation $\rho : \pi_1(S - V) \to \mathrm{Isom}(\mathbb{H}^2)$. To define d we first choose an isometric embedding $d|\tau$ into $\mathbb{H}^2$ of a triangle τ of the triangulation $\tilde{T}$. If τ' is an adjacent triangle which shares an edge e with τ then the restriction $d|e$ uniquely extends to an isometric embedding $d|\tau' : \tau' \to \mathbb{H}^2$ so that the coorientation on e determined by $d|\tau$ agrees with the coorientation determined by $d|\tau'$. Continue this process inductively. The resulting map d is equivariant with respect a representation $\rho : \pi_1(S-V) \to \mathrm{Isom}(\mathbb{H}^2)$. Note that if $v \in V$ then the monodromy around v is an elliptic isometry (or the identity) which is conjugate to $z \mapsto e^{i\theta} z$ where $\theta = \theta_v$ is the total angle around v with respect to the singular hyperbolic metric σ.

Exercise 14.1. *Show that completeness of (S, σ) is equivalent to the fact that the monodromy around each puncture $p \in P$ is parabolic.*

There is a combinatorial analogue of the Gauss-Bonnet formula for closed surfaces with singular hyperbolic metrics, it is proven exactly the same way as the ordinary Gauss-Bonnet formula:

$$Area(S, \sigma) = -2\pi\chi(S) + \sum_{v \in V} (2\pi - \theta_v) \le -2\pi\chi(S).$$

Suppose that M is a complete hyperbolic 3-manifold and S is a surface with a singular hyperbolic structure. Then a continuous map $f : S \to M$ is called a *singular pleated map*, if:

- The restriction of f to each triangle in the completed triangulation of S is locally a totally-geodesic isometric embedding.
- f is proper on each triangle.

We will retain the name *singular pleated map* for the lift of f to the universal cover $\tilde{f} : \tilde{S} \to \mathbb{H}^3$. We call the image of f a *singular hyperbolic pleated surface* . All singular hyperbolic surfaces (S, σ) that we shall consider satisfy the following extra axiom:

(A3) The surface (S, σ) admits a singular pleated map into a complete hyperbolic 3-manifold which induces a monomorphism of the fundamental groups.

Therefore, Kazhdan-Margulis' Theorem in $\mathbb{H}^3$ implies the following property:

Property 14.2. Suppose that (S, σ) satisfies the axioms (A1), (A2), (A3). Then for every loops α, β in S each having the length less than μ_3, which intersect in a point $x \in S$, α, β generate a cyclic subgroup in $\pi_1(S, x)$.

Similarly to the case of usual hyperbolic manifolds, it is true that for every $\mu_3 > \epsilon > 0$ and singular hyperbolic surface S satisfying the Axiom (A3), the ϵ-thin part of S consists of the union of compact topological annuli and punctured topological disks. Let S^0_ϵ denote the complement to the union of punctured disks which are in the ϵ-thin part of S_ϵ.

Given a constant $\epsilon < \mu_3$ and a singular hyperbolic surface S we collapse each component of $S_{(0,\epsilon]}$ to a point, the quotient $\nu_\epsilon(S)$ has a natural path metric. By repeating arguments of the proof of Mumford compactness theorem (see for instance [Abi80]) one proves:

Theorem 14.3. *(Compare [Bon86].) There is an universal function $\varphi(x, y)$ such that: For any singular hyperbolic surface S (satisfying the axioms (A1), (A2), (A3)) and any two points $a, b \in S_{[\epsilon,\infty)}$, the distance $d(\nu_\epsilon(a), \nu_\epsilon(b))$ in $\nu_\epsilon(S)$ is at most $\varphi(\chi(S), \epsilon)$.*

14.2. Existence theorem for singular pleated maps

Suppose that $\rho : \pi_1(S) \to \mathrm{Isom}(\mathbb{H}^3)$ is a homomorphism and S is a singular hyperbolic surface. Recall this homomorphism is called *relative parabolic* if the images of peripheral elements of $\pi_1(S)$ are parabolic isometries. An element $\gamma \in \pi_1(S)$ is called an *accidental parabolic element* for ρ if γ is not peripheral but $\rho(\gamma)$ is parabolic.

Theorem 14.4. *(Existence theorem for singular pleated maps.) Suppose that $S = \hat{S} - P$ is a surface of hyperbolic type (i.e. $\chi(S) < 0$) and $\rho : F :=$*

$\pi_1(S) \to \mathrm{Isom}(\mathbb{H}^3)$ is a nonelementary relative parabolic representation. Then there exists a singular hyperbolic structure on S and a ρ-equivariant singular pleated map $f : \tilde{S} \to \mathbb{H}^3$.

Proof: We will not prove this theorem here. Its proof is similar to the one presented below (under extra assumptions), but relies upon the following:

Proposition 14.5. *(Compare [Ka95b], [GKMar].) (a) Suppose that the surface S has genus > 0. Then there exists a simple loop $\gamma \subset S$ such that $\rho(\gamma)$ is a loxodromic element of $PSL(2, \mathbb{C})$.*

(b) Suppose that S is a sphere with punctures P. Then there exists a triangulation T of S (based on P) so that for each edge e of this triangulation and the regular neighborhood U of $cl_{\hat{S}}(e)$ in $\hat{S}$, the image $\rho(\pi_1(U \cap S))$ is a nonelementary subgroup of $\mathrm{Isom}(\mathbb{H}^3)$.

Instead we prove more restrictive theorem which will suffice here:

Theorem 14.6. *Suppose that ρ is discrete, faithful and has no accidental parabolic elements (i.e. the image of each nonperipheral loop is loxodromic). Then there exists a singular hyperbolic structure on S and a ρ-equivariant singular pleated map $f : \tilde{S} \to \mathbb{H}^3$.*

Proof: **Case 1.** The surface S is compact. Thus there is a simple, homotopically nontrivial and nonperipheral loop γ on S.

Choose a point $z \in \gamma$. Then construct a triangulation of S which has a single vertex $z \in V$ and γ is an edge of this triangulation. This triangulation lifts to a connected fundamental domain D for the action of $F = \pi_1(S)$ in $\tilde{S}$. Denote the vertices of D by $z_1, ...z_n$, they are preimages of z. Let e denote an edge of D emanating from z_1 which covers the loop γ. Then an element $g_2 \in F$ corresponding to γ sends z_1 to another vertex z_2 of e. Let $w_1 = f(z_1)$ be any point on the axis of the element $\rho(g_2) \in PSL(2, \mathbb{C})$, then we set $f(z_2) := \rho(g_2)(w_1)$ and extend the map f to the set of vertices of D in ρ–equivariant way. We assume that the image under f of any edge of the triangulation of D is a geodesic segment connecting corresponding points in $\mathbb{H}^3$. Extend f to each triangle of D by a homeomorphism whose image is the ideal triangle spanned by the images of the edges of D. Then construct a ρ-equivariant extension of the map f to the whole universal cover $\tilde{S}$. The pull-back of the hyperbolic metric from $\mathbb{H}^3$ defines a singular metric on $\tilde{S}$ which is F–invariant. What is left to check is that this singular metric is hyperbolic. Completeness of this metric is clear since S is compact. The image of the universal cover of the loop γ under f is a geodesic in $\mathbb{H}^3$. Therefore, the sum of angles of triangles adjacent to γ from the "left" is at least π, and the sum of angles of triangles adjacent to γ from the "right" is also at least π.

Case 2. The surface S is not compact. Then we let $V = \emptyset$ and triangulate the surface S using the ideal vertices (the points in P) only. We let $S = \mathbb{H}^2/F$. The representation ρ is faithful and type-preserving. Thus there is a natural ρ-equivariant bijection $\beta = \beta_\rho$ between the set $\tilde{P}_F$ of parabolic fixed points of the group F and the set $\tilde{P}_{\rho(F)}$ of parabolic fixed points of

$\rho(F)$. The hyperbolic plane is tessellated by ideal triangles with the vertices in $\tilde{P}_F$ which are lifts of the triangles in our triangulation of S. If $[ab]$ is an edge of this tessellation we take a homeomorphic map $\tilde{f} : [ab] \to [\beta(a)\beta(b)]$, it is clear that this map can be chosen ρ-equivariant. For each ideal triangle $[abc]$ of the tessellation of $\mathbb{H}^2$ we extend (in the ρ-equivariant manner) the map $\tilde{f}$ from the boundary of $[abc]$ to its interior so that the image of the resulting homeomorphism is the ideal triangle $[\beta(a)\beta(b)\beta(c)] \subset \mathbb{H}^3$. We let $\tilde{f} : [abc] \to [\beta(a)\beta(b)\beta(c)]$. This yields an equivariant pleated map $\tilde{f} : \mathbb{H}^2 \to \mathbb{H}^3$. Now define the singular hyperbolic structure on S by pull-back of the hyperbolic structure σ on $\mathbb{H}^3$ via $\tilde{f}$. Note that monodromy of σ around each puncture is parabolic since ρ maps parabolic elements to parabolic. Hence σ is complete. $\square$

14.3. Compactness theorem for pleated maps

Fix a complete hyperbolic 3-manifold M, a compact $B \subset M$ and a hyperbolic surface S of finite area. Recall that for a singular hyperbolic structure σ on S and $\mu \leq \mu_3$, S^0_μ denotes the complement to the union of punctured disk components of the μ-thin part of (S, σ). Note that S^0_μ is connected.

Theorem 14.7. *(Thurston's Compactness Theorem for Pleated Maps.) There exists a function $\theta(M, B, \chi(S), \mu)$ which has the following property. For any singular hyperbolic structure σ on S and any π_1-injective singular pleated map $f : (S, \sigma) \to M$ whose image intersects B and $f_* : \pi_1(S) \to \pi_1(M)$ is type-preserving, the diameter of $(S, \sigma)^0_\mu$ is at most $\theta(M, B, \chi(S), \mu)$.*

Proof: Let $\mu = \mu_3$, let $M_{[\mu/2,\infty)}$, $M_{(0,\mu/2]}$ be the $\mu/2$–thick and $\mu/2$–thin parts of M. We have the lower bound $\mu/2$ on the minimal distance between different thin components of M. Let C denote the collection of cusps in $M_{(0,\mu/2]}$. Collapse each thin component of M to a point, let $\nu : M \to \nu(M)$ denote the quotient map; the quotient $\nu(M)$ has a natural induced path metric. Then for each r, the r-neighborhood of $\nu(B)$ can contain not more than $j(r)$ elements of $\nu(M_{(0,\mu/2]})$, where $j(r) = j_{M,B}(r)$ is a function which depends only on geometry of M and location of B. The singular pleated map f does not increase the distance, therefore $f(S_{(0,\mu/2]}) \subset M_{(0,\mu/2]}$. This means that f projects to a distance-nonincreasing map $g : \nu_\epsilon(S) \to \nu(M)$ where $\epsilon = \mu/2$. According to Theorem 14.3, the diameter of $\nu_\epsilon(S)$ is at most $\varphi(\chi(S), \epsilon)$. Let $\{t_1, ..., t_s\}$ denote the set of those points in $\nu(M_{(0,\mu/2]} - C)$ which are within the distance at most $\varphi(\chi(S), \epsilon)$ from $\nu(B)$, the number s is at most $j(\varphi(g, \epsilon))$. These points correspond to solid tori in $M_{(0,\epsilon]}$. Hence the injectivity radius over $\nu^{-1}(\{t_1, ..., t_r\})$ is at least $\eta = \eta(M, B)$. Therefore the injectivity radius on $S - f^{-1}(C)$ is at least η. On the other hand, all components of $f^{-1}(C) \cap (S, \sigma)^0_\mu$ are homotopically trivial in S. This implies that the injectivity radius of $(S, \sigma)^0_\mu$ is least η. Thus, the diameter of (S, σ) is at most $\varphi(\chi(S), \eta) = \theta(M, B, \chi(S), \mu)$. $\square$

Corollary 14.8. *(W. Thurston.) Up to the precomposition with homeomorphisms of S there are only finitely many homotopy classes of maps from S into M which satisfy the hypothesis of Theorem 14.7.*

Proof: I will consider only the case of compact surfaces of the genus g, the general case is similar and is left to the reader. Fix a system of canonical generators $\{\gamma_1, ..., \gamma_{2g}\}$ of $F = \pi_1(S, v)$, where the base-point v is in V. Let $f_n : (S, \sigma_n) \to M$ be a sequence of maps as in Theorem 14.7. Since the diameter of (S, σ_n) is uniformly bounded, we may precompose f_n with a self–homeomorphisms of S to get generators g_j such that that the length of each based loop $f_n(g_j)$ in M is at most $2\theta(M, B, \chi(S), \mu)$. We may assume that $f_n(v) \in B$ for each n. Therefore the corresponding sequence of representations $\rho_n : F \to G \subset PSL(2, \mathbb{C})$ is relatively compact. Discreteness of G implies that this sequence contains only a finite number of different representations. $\square$

Corollary 14.9. *(W. Thurston.) Let M be a closed hyperbolic 3-manifold, $F \cong \pi_1(S)$ is the fundamental group of a closed hyperbolic surface. Then there is only a finite number of copies of F embedded in $\Gamma = \pi_1(M)$ up to conjugation in $\pi_1(M) = G$.*

Proof: Take the compact $B = M$. Each embedding $\rho_n : F \to G$ is induced by a singular pleated map $f_n : S \to M$ whose image definitely intersects B. Then finiteness of the number conjugacy classes of images $\rho_n(F)$ follows from Corollary 14.8. $\square$

This corollary admits further generalizations: one can relax the assumption that F is the fundamental group of a hyperbolic surface and Γ is the fundamental group of a hyperbolic manifold, see [Del95], [Sel97].

14.4. Geometrically tame ends of hyperbolic manifolds

In this section we study the structure of those ends of hyperbolic 3-manifolds with finitely generated fundamental groups, which are not geometrically finite. See §4.23 for the definitions and notation related to the ends of hyperbolic manifolds. Suppose that M is a complete hyperbolic 3-manifold with finitely generated fundamental group, $M^0 := M^0_\epsilon$ is the complement to the union of open Margulis cusps in M, $M^c = M^c_\epsilon$ is a relative compact core of M^0. We will consider only those manifolds M which satisfy the following condition of F. Bonahon:

Condition B. The surface $\partial_0 M^c = \partial M^c - \partial M^0$ is incompressible. Equivalently, for every nontrivial decomposition of $G = \pi_1(M)$ into a free product $A * B$ there is a parabolic subgroup in G which is not conjugate into any of the free factors.

An end E of M^0 is *geometrically finite* iff $E \cap CM(G)$ is compact, where $CM(G)$ is the convex core of $M = M(G)$. Another way to express it is to say that the subset $E \subset M^0$ is precompact in $M \cup \Omega(G)/G$. Note that geometric finiteness is independent on the choice of compact core M^c.

Remark 14.10. If we reduce the Margulis constant ϵ to $\epsilon' < \epsilon$ then there is 1-1 correspondence between the ends of M^0_ϵ and of $M^0_{\epsilon'}$: each end E' of $M^0_{\epsilon'}$ contains exactly one end of M^0_ϵ. We will retain the names of the ends in this case.

Recall that $P(E)$ denotes the union of E with those Margulis cusps which are adjacent to E. Let $S := int(\partial_0 E)$.

Definition 14.11. Suppose that $E \subset M^0_\epsilon - M^c_\epsilon$ is an end of M, $\gamma_n \subset E$ is a sequence of closed geodesics such that for each compact $K \subset E$ only finite number of members of this sequence intersect K. Then the sequence $\{\gamma_n\}$ is said to **exit** the end E. Similarly, a sequence of singular pleated maps $f_n : S \to P(E)$ **exits** E if $f_n(S) \cap E$ eventually leaves any compact subset of E. A geodesic ray $\rho : [0, \infty) \to E$ **exits** E if

$$\lim_{t\to\infty} d(\rho(t), S) = \infty.$$

Lemma 14.12. *Assume that there is a sequence of closed geodesics $\gamma_n \subset E$ which exits the end E. Then E is geometrically infinite.*

Proof: Suppose that E is geometrically finite. Then $E \cap CM(G)$ is compact and for all large $n \gg 1$ we have: $\gamma_n \cap CM(G) = \emptyset$. However this means that the lifts $\tilde{\gamma}_n \subset \mathbb{H}^3$ are not contained in the convex hull of the limit set. This contradicts to the property that the end–points of all geodesics $\tilde{\gamma}_n$ are limit points of G. $\square$

Theorem 14.13. *(F. Bonahon, [Bon86].) Suppose that the end E is not geometrically finite. Then (after reducing μ if necessary) there exists a sequence of closed geodesics $\gamma_n \subset E$ which exits the end E.*

Proof: For simplicity I will restrict our discussion to the case when $E = P(E)$, i.e. E is bounded by a closed surface S, see [Bon86] for the general case. Let $S_n \subset E$ be a surface homologous to $S := \partial E$ so that $d(S_n, S) \geq n$. Let E_n be the unbounded component of $E - S_n$. Consider the convex hull of S_n in E. Since E is not geometrically finite, the convex hull is unbounded. The convex hull of S_n is the union of convex hulls of finite subsets of S_n. However, for each finite subset $X \subset S_n$ the convex hull C_X is within the distance at most 2 from the union of geodesics in C_X connecting points of X. Therefore, there is a sequence of geodesic segments $s_k = [x_k, y_k] \subset E_n$ whose end–points belong to S_n so that $length(s_k) \to \infty$ as $k \to \infty$. We have:

$$d(y_k, x_k) \leq Const_n = diam(S_n).$$

Let h_k denote the shortest geodesic segment connecting x_k to y_k, then

$$length(h_k) \leq Const_n.$$

Therefore for large k the loops $q_k := s_k \cup h_k$ are homotopically nontrivial. For each n choose one of the loops q_n so that $\lim_n length(q_n) = \infty$ and

q_n are homotopically nontrivial. Let g_n denote the elements of G whose conjugacy classes represent q_n.

Up to a subsequence there are two possible cases:

(a) $\ell(g_n) \geq \delta > 0$.

(b) $\ell(g_n) \to 0$

(recall that ℓ is the translational length of the isometries of $\mathbb{H}^3$).

In the Case (a) we apply Lemma 4.59 and Lemma 4.60 to conclude that the closed geodesic $\gamma_n \subset M$ freely homotopic to q_n is within $(-\log(\delta/4)+5)$-neighborhood of q_n. Since $d(q_n, S) \to \infty$ we conclude that the sequence of geodesics γ_n exits the end E.

Consider (b). Let 2δ be less than the injectivity radius in the 1-neighborhood of the surface S. Then δ-Margulis tube (or cusp) $U_n \subset M$ corresponding to the element g_n is disjoint from S. On the other hand, it contains a point within the distance $\leq -\log(\delta/2) + 5$ from the loop q_n (Lemma 4.59). This means that all tubes U_n are contained in E. Since by our assumption the end E contains no cusps, we conclude that E contains a sequence of closed geodesics γ_n whose length tends to zero. Thus $\{\gamma_n\}$ exits the end E. $\square$

Definition 14.14. Suppose that E is an end of M and there is a sequence c_n of simple homotopically nontrivial loops in $\partial_0 E$ which corresponds to a sequence of closed geodesics γ_n in M, which exits the end E. Then the end E is called **geometrically tame**. If all ends of the manifold $M = \mathbb{H}^3/G$ are geometrically tame then the group G is said to be geometrically tame.

We will use the following deep theorem of F. Bonahon [Bon86].

Theorem 14.15. *(Bonahon's Tameness Theorem.) All geometrically infinite ends of manifolds M satisfying the condition* **B** *are geometrically tame.*

Let E be an end of M, $S = int(\partial_0 E)$; we will identify S with a punctured surface of finite type.

Theorem 14.16. *(F. Bonahon [Bon86].) Suppose E is a geometrically tame end as above. Then there exists a sequence of singular pleated maps $f_n : int(S) \to P(E)$ which are properly homotopic to the embedding $S \hookrightarrow E$ and such that the sequence $\{f_n(intS)\}$ exits E.*

Proof: Using the same arguments as in Theorem 14.6 we construct a sequence of pleated maps f_n so that $f_n(c_n) = \gamma_n$ (where c_n are simple loops in S). Suppose that there exists a compact K in M which intersects each surface in the sequence $f_n(intS)$. Then, according to Theorem 14.7, the diameter of $f_n(S^0_\epsilon)$ is bounded from above by some constant T independent on n. Therefore $f_n(S^0_\epsilon)$ is contained in the T-neighborhood of K. However this contradicts the assumption that the sequence of geodesics $\{\gamma_n\}$ in $f_n(intS)$ exits E. $\square$

The special case when the conditions of this theorem are satisfied, is when M is homotopy–equivalent to a closed hyperbolic surface. An end E of the manifold M^0 is *topologically tame* if it admits compactification to a manifold with boundary.

As a corollary of Theorem 14.16, Bonahon in [Bon86] proves

Theorem 14.17. *All geometrically tame ends of hyperbolic manifolds satisfying the condition* **(B)**, *are topologically tame. Thus, all such hyperbolic manifolds admit compactifications to manifolds with boundary.*

Sketch of the proof: Let $\{f_n : int(S) \to P(E)\}$ be a sequence of π_1-injective pleated maps which exits the end E. This yields a sequence of π_1-injective maps of the compact surface, $h_n : S \to E$, so that the sequence $h_n(S)$ exits E and $f_n(\partial S) \subset \partial E - \partial_0 E$. Then use Theorem 1.27 to find π_1-injective embeddings h'_n within uniformly bounded distance from h_n. Thus $\{h'_n\}$ also exits E. On the other hand, the images $S_n := h'_n(S)$ are incompressible surfaces properly homotopic to the identity embedding $S \hookrightarrow E$. After passing to a subsequence if necessary we can assume that $S_i \cap S_j = \emptyset$ for $i \neq j$ and no surface S_k separates S_i from S_{i+1}. We let $S_1 := S$. Therefore the region $E_n \subset E$ bounded by S_{n+1} and S_n is a Haken manifold so that the pair $(E_n, \partial E_n \cap P)$ is homotopy-equivalent to $(S \times I, \partial S \times I)$. Theorem 1.29 implies that $(E_n, \partial E_n \cap P)$ is homeomorphic to to $(S \times I, \partial S \times I)$. Thus the union

$$E = E_1 \cup E_2 \cup \ldots \cup E_n \cup \ldots$$

is homeomorphic to $S \times \mathbb{R}_+$ and the end E is topologically tame. $\square$

Conjecture 14.18. *(Marden's Topological Tameness Conjecture, [Mar74].) If M is a complete hyperbolic 3-manifold with finitely generated fundamental group $G \subset PSL(2, \mathbb{C})$ then M is topologically tame.*

A certain progress towards this conjecture was achieved in [CM96] and [Ohs97] where it was proven under the assumption that G appears as a certain limit of geometrically tame groups.

Theorem 14.19. *(D. Canary [Can93].) If the hyperbolic 3-manifold $M = M(G)$ is topologically tame then the group G satisfies the Ahlfors' Measure Zero Conjecture (Conjecture 4.135).*

Canary's proof follows the ideas of Thurston who proved Ahlfors' Conjecture in [Thu81] for discrete groups $\Gamma \subset \mathrm{Isom}(\mathbb{H}^3)$ satisfying the Condition **(B)** and assuming that all ends of $\mathbb{H}^3/\Gamma$ are geometrically tame. Even if $\Lambda(G) = \mathbb{S}^2$, Canary (following Thurston's arguments) proves that the action of G on $\Lambda(G)$ is ergodic. It is therefore conjectured that each finitely generated discrete subgroup of $\mathrm{Isom}(\mathbb{H}^3)$ acts ergodically on its limit set. Note that there are examples of finitely generated (but not finitely presentable!) discrete subgroups of $PU(2,1)$ which do not act ergodically on their limit sets (one such example is described in [Ka98b]).

Proposition 14.20. *Let M be a complete hyperbolic manifold with finitely generated fundamental group, E is an end of M. Then there exists a geodesic ray $\rho \subset E$ which exits the end E.*

Proof: Let $x_n \in E$ be a sequence of points which exits the end E. Let $[x_n y_n]$ denote the shortest geodesic segment between x_n and the surface $\partial_0 E$ which bounds the end E. Up to a subsequence, the sequence of points y_n is a convergent to a point y in the compact surface $\partial_0 E$ and there is a limit of segments $[x_n y_n]$ in M which is an infinite geodesic ray ρ in M. We claim that the ray ρ exits the end E. Suppose that ρ intersects a compact $B \subset M$ infinitely many times, i.e. there is a sequence of points $z_k \in \rho \cap B$ such that the distance between y and z_k along ρ tends to infinity. Then for each z_k we can find a point $w_k \in [x_n y_n]$ such that $d(z_k, w_n) \leq 1$ and $\lim_{n\to\infty} length[w_n y_n] = \infty$. However $d_M(w_n, y_n) \leq dist_H(B, \partial_0 E) + 1/2$ is bounded from above. This contradicts the assumption that the segment $[x_n y_n]$ is the shortest between x_n and $\partial_0 E$. $\square$

14.5. Ending laminations

Let G be a finitely generated Kleinian group satisfying Bonahon's condition (B). Let $\Sigma_i = \partial_0(E_i)$ be the incompressible surfaces in $M = M(G)$ corresponding to the geometrically infinite ends E_i of M. According to Bonahon's Tameness Theorem, for each end E_i there is a sequence of simple loops $\gamma_n \subset \Sigma_i$ homotopic to closed geodesics $\gamma_n^* \subset M$ so that the sequence $\{\gamma_n^*\}$ *exits* the end E_i. Choose a hyperbolic metric with geodesic boundary on Σ_i. Consider the projection $\{[\gamma_n]\}$ of the sequence of geodesic currents $\{\gamma_n\}$ to $\mathcal{PML}(\Sigma_i)$. Since $\mathcal{PML}(\Sigma_i)$ is compact (see Corollary 11.46), the sequence $\{[\gamma_n]\}$ has a convergent subsequence.

Theorem 14.21. *Let $[\lambda], [\mu]$ be projective classes of measured laminations on Σ_i which are limits of subsequences in $\{[\gamma_n]\}$. Then*

- *λ, μ are maximal and connected.*
- *$i(\lambda, \mu) = 0$.*

Thus the support sets of λ, μ are equal.

Proof: See [Bon86]. $\square$

The key idea of Bonahon's paper is that instead of working with a sequence of simple loops $\gamma_n \subset \Sigma_i$ if suffices to consider *any* sequence of loops γ_n so that the sequence $\{\gamma_n^*\}$ *exits* the end E_i. Then Bonahon proves that any limiting geodesic current $[\lambda]$ for the sequence of projectivized geodesic currents $\{[\gamma_n]\}$ satisfies:

$$i(\lambda, \lambda) = 0$$

which implies that λ is a measured geodesic lamination. Then, λ is the limit of a sequence of geodesic currents α_n corresponding to simple closed loops on S and the sequence $\{\alpha_n^*\}$ necessarily exits the end E_i. This implies that E_i is tame.

Definition 14.22. The geodesic lamination L equal to the support set of some (any) limit of a subsequence in $\{[\gamma_n]\}$ is called the **ending lamination** of the end E_i and is denoted $\epsilon(E_i)$.

Conjecture 14.23. *(Thurston's Ending Lamination Conjecture.) The isometry class of the complete hyperbolic manifold $M = M(G)$ is uniquely determined by the topology of M, the collection of ending laminations of* *the geometrically infinite ends of M and by the conformal structure of $\Omega(G)/G$. More precisely, suppose that G, G' are discrete finitely generated subgroups of $PSL(2,\mathbb{C})$ satisfying Bonahon's condition* **(B)**. *Let $\phi : G \to G'$ be a type preserving isomorphism which is induced by a homeomorphism $f : M(G) \to M(G')$ so that:*

- *For each component $S_i \subset \partial \overset{\bullet}{M}(G)$ the homeomorphism $f : S_i \to S_i' \subset \partial \overset{\bullet}{M}(G')$ is homotopic to a conformal mapping.*
- *For each geometrically infinite end E_j of $\overset{\bullet}{M}(G)^0$ bounded by an incompressible subsurface $\Sigma_j \subset M(G)^0$, the end E_j' of $M(G')^0$ corresponding to E_j under f, is bounded by an incompressible subsurface $\Sigma_j' = f(\Sigma_j) \subset M(G')^0$ and $f : \Sigma_j \to \Sigma_j'$ carries the ending geodesic lamination $\epsilon(E_j)$ to $\epsilon(E_j')$.*

Then $f : M(G) \to M(G')$ is homotopic to an isometry.

A partial progress towards this conjecture was achieved by Y. Minsky (see [Min94b], [Min94a]) who proved it in the case when the injectivity radius of $M(G)$ is bounded from below and G is a surface group and in [Min99] where Minsky proves the ending lamination conjecture for the groups which are isomorphic to the fundamental group of a punctured torus via a type-preserving isomorphism. The results of [Min94a] were extended by M. Mitra [Mit98] and E. Klarreich [Kla97] who treated the case of freely indecomposable groups G so that $M(G)$ has injectivity radius bounded from below.

14.6. Infinite coverings of hyperbolic manifolds

Theorem 14.24. *(Thurston's Covering Theorem, W. Thurston [Thu81], K.-I. Ohshika [Ohs92], D. Canary [Can96].) Let $p : N \to M$ be a covering between complete hyperbolic manifolds, where $\pi_1(N)$ is finitely generated, nonelementary and N satisfies Bonahon's Condition* **(B)**. *Suppose that E is a geometrically tame end of N. Then the following alternative holds:*

(a) Either M is finitely covered by a manifold M' fibered over a circle and the group $\pi_1(N) \cap \pi_1(M')$ is a surface group corresponding to the fundamental group of the fiber.

(b) Or the restriction of p to E has finite multiplicity.

Remark 14.25. D. Canary [Can96] proved Theorem 14.24 under more general conditions: instead of Bonahon's condition **(B)** it is enough to assume that the end E is topologically tame.

Proof: Clearly it suffices to prove this theorem under the assumption that $\pi_1(N) \cong F$ is the fundamental group of an oriented complete hyperbolic surface $\dot{S}$ of finite area, and this isomorphism preserves the type of elements. (The surface $\dot{S}$ is naturally homeomorphic to the interior of a component of

$\partial_0 N^c$ which bounds E.) The surface $\dot{S}$ is distinct from the triply punctured sphere (since the end is geometrically finite in such case, see Lemma 4.34). Suppose that $f_n : (\dot{S}, \sigma_n) \to P(E)$ is a sequence of singular pleated maps which exits E, σ_n are singular hyperbolic structures on $\dot{S}$. We will identify F with the subgroup of G. Up to a subsequence there are two possible cases:

Case 1. The images $p(f_n(\dot{S}))$ intersect a compact $K \subset M$ for all n.

Case 2. The singular pleated surfaces $p(f_n(\dot{S}))$ eventually leave every compact subset of M.

In the 1-st case, according to Corollary 14.8, after taking a subsequence if necessary, we have a sequence of admissible automorphisms $\alpha_n : F \to F$ and elements $\gamma_n \in \Gamma$ so that the sequence of representations $\rho_n := ad(\gamma_n) \circ \alpha_n : F \hookrightarrow \Gamma$ consists of the identically equal representations of F.

The sequence of outer automorphisms $[\alpha_n] \in Mod_{\dot{S}}$ cannot be finite since the singular pleated surfaces $f_n(\dot{S})$ exit the end E and hence for each nonperipheral element $g \in F$ the length of the corresponding closed geodesic in $(\dot{S}, \sigma_n)$ tends to ∞ as $n \to \infty$. Since $\gamma_n F \gamma_n^{-1} = F$, the elements γ_n belong to the normalizer $\mathcal{N}(F)$ of F in Γ. The projection of the sequence $\{\gamma_n\}$ to $\mathcal{N}(F)/F \subset Mod_{\dot{S}}$ is a infinite, since $[\alpha_n] \in Mod_{\dot{S}}$ is not subconvergent. Therefore there is an element γ in $\mathcal{N}(F)$ such that $\gamma^n \notin F$ for any $n \in \mathbb{Z} - \{0\}$ (see Corollary 5.14). Hence we get a subgroup $\Gamma' := F \rtimes \langle \gamma \rangle \subset \Gamma$ which is isomorphic to the fundamental group of the mapping torus M_τ of a homeomorphism of $\tau : \dot{S} \to \dot{S}$ induced by $ad(\gamma) : F \to F$. Note that the boundary of M_τ has zero Euler characteristic, therefore Γ' is a lattice in $\mathrm{Isom}(\mathbb{H}^3)$. Hence $|G : \Gamma| < \infty$. This finishes the proof in the 1-st case.

Case 2. Before giving a proof in this case I will describe an example which illustrates the situation.

Remark 14.26. Actually, the general case could be reduced to the particular example that we consider below.

Suppose that $M^c = M^c_\epsilon$ is a (relative) Scott compact core of M^0_ϵ, $p(S) \cong S$ is a boundary component of $\partial_0 M^c$, $\pi_1(M) = \Gamma = F * \mathbb{Z}$, $F \cong \pi_1(S)$. Let E' denote the end of M bounded by S. Then our conclusion would be that $p(E) = E'$ and the restriction of p to E is a one-to-one. Assume that this is false. The manifold N has two ends E and D, so we have to assume that D is the end covering E'. Let g_1 denote a generator of the free factor $\mathbb{Z}$ in Γ. We will use the notation: $g_0 = 1, g_1^2 = g_2 \in \Gamma$ and $F_j = g_j F g_j^{-1}$, $j = 0, 1, 2$. So we can take a covering M' over M whose fundamental group is $F_0 * F_1 * F_2$ and M' has at least three ends (covering E') which correspond to the subgroups F_j. We denote these ends by E', E_1, E_2. We retain the notation M for this covering, Γ for its fundamental groups as well as for the covering $p : N \to M'$ induced by the inclusion $F_0 \hookrightarrow \Gamma$. Let $\{f_n : \dot{S} \to P(E)\}$ be a sequence of pleated surfaces which exits the end E. (See Figure 14.1.)

We have two ends E_1, E_2 of M which are bounded by surfaces $S_j \subset M^c$ that are not homotopic to S. Then there exist complete (distance-minimizing) geodesic rays $\ell_j \subset E_j$, $j = 1, 2$ which exit these ends. Lift these rays to the rays R_j in N via the covering p. Then R_j must exit either the end E or the end D of N. However, R_j's cannot exit the end

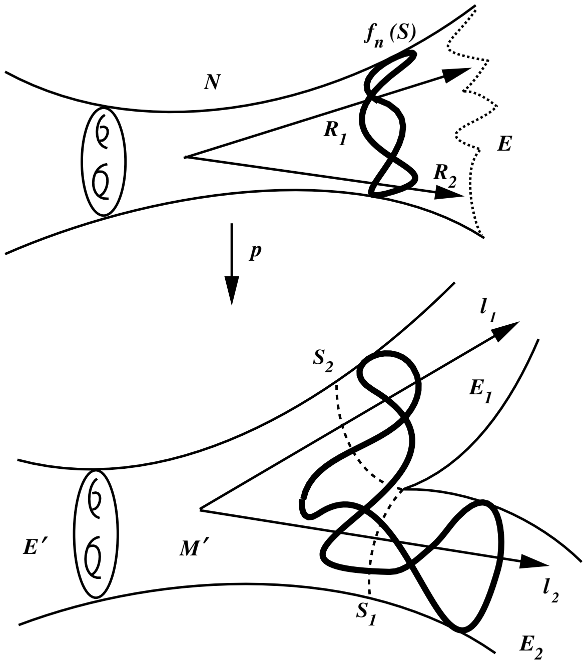

Figure 14.1: *A singular pleated surface tries to exit two ends simultaneously.*

D since $p(R_j) = \ell_j$ does not intersect the surface S. Thus each R_j exits the end E. This means that there are unbounded sequences of points $x_n \in R_1 \cap f_n(\dot{S}), y_n \in R_2 \cap f_n(\dot{S})$. Projections $p(x_n), p(y_n)$ of these sequences exit the ends E_1, E_2 respectively. However the sequence of singular pleated surfaces $pf_n(\dot{S})$ contains both $p(x_n), p(y_n)$, so it oscillates between two different ends E_1, E_2. This means that for each n the intersection $p(f_n(S)) \cap M^c$ is nonempty. Now Theorem 14.7 implies that $p(f_n(\dot{S})) \cap M_\epsilon^0$ has uniformly bounded diameter and is contained in a compact subset of M. This contradicts the assumption that the sequences x_n, y_n in $p(f_n(\dot{S}))$ exit the ends of M. This contradiction proves that $p(E) = E'$.

Now we consider the general situation. Our arguments follow [Ohs92]. Let p_E denote the restriction of the covering p to the end E of N_ϵ^0 which is bounded by the compact surface $S \subset \partial_0 E$. As before let $\dot{S}$ denote the interior of S.

Lemma 14.27. $p_E^{-1}(p(S)) \subset E$ *is compact.*

Proof: Pick a sequence $f_n : \dot{S} \to E$ of singular pleated surfaces which exits the end E. Suppose that the inverse image $p_E^{-1}(p(S)) \subset E$ is not compact. Since $p_E^{-1}(p(S))$ is properly embedded in N_ϵ^0, it intersects S in a finite collection of loops and arcs. Thus there are two possible cases.

(a) It may happen that $p_E^{-1}(p(S))$ contains an unbounded connected component S_1. In this case S_1 intersects all the singular pleated surfaces $f_n(\dot{S})$ which exit the end E. Hence $p(f_n(S))$ intersect the compact set $p(S)$, which is impossible.

(b) Thus we assume that $p_E^{-1}(p(S))$ contains infinitely many compact components S_n disjoint from S. Each of them is properly immersed in E. Fix an orientation on S. This determines orientation on each S_n. Each oriented surface S_n determines a relative homology class $[S_n] \in H_2(E, \partial_P E; \mathbb{Z})$. Note that the immersed surfaces S_n are π_1-injective, thus the images of their fundamental groups have finite index in $\pi_1(N)$. Hence all these relative homology classes are nontrivial and they are integer multiples of $[S]$. After taking a subsequence we can assume that $[S_n] = k_n[S], k_n \cdot k_m > 0$; it is enough to consider the case $k_n > 0$. Then there is a relative 3-chain $h_1 \in C_3(E, \partial_P E; \mathbb{Z})$ such that $\partial h_1 = k_1[S_1] - k_2[S_2]$. Note that images of the classes $p_*[S_n] \in H_2(M, \partial_P M; \mathbb{Z})$ are equal to $k_n[p(S)]$, therefore $0 = \partial p_\#(h_1) \in C_2(M, \partial_P M; \mathbb{Z})$. Thus $p_\#(h_1)$ is a relative 3-cycle which is nontrivial since p is a local diffeomorphism. This implies that M^0_ϵ is compact, which again contradicts the assumption that we are in the *Case 2*. $\square$

Clearly the above Lemma is valid for any choice of the surface S bounding the end E in the product manifold N^0_ϵ. We conclude that the map $p_E : E \to M$ is proper. Let $\tilde{S}$ be one of the lifts of the surface $S \subset N$ to the universal covering space $\tilde{M} = \tilde{N} = \mathbb{H}^3$; thus $\tilde{S} = \partial_0 \tilde{E}$, where $\tilde{E}$ is a lift of E to $\mathbb{H}^3$ and $\partial_0 \tilde{E}$ is the lift of $\partial_0 E$. Suppose that the map $p_E : E \to M$ does not have finite multiplicity, then there exists an infinite sequence $\gamma_n \in \pi_1(M)$ such that:

- $\gamma_n \tilde{E} \cap \tilde{E} \neq \emptyset$;
- $\gamma_n \tilde{S} \cap \tilde{E} = \emptyset$ (according to Lemma 14.27);
- $d(\tilde{S}, \gamma_n \tilde{S}) \to \infty$ as $n \to \infty$.

Then $\tilde{S} \subset \gamma_n \tilde{E}$ for all n and hence $\gamma_n^{-1}(\tilde{S}) \subset E$. This implies that $p_E^{-1}(p(S))$ is not compact which contradicts Lemma 14.27. This contradiction concludes the proof of the Theorem. $\square$

One of the standard applications of Theorem 14.24 is strong convergence of sequences of representations. Consider Theorem 8.67, Assertion (1), under the assumption that $\Lambda(\Gamma) = \mathbb{S}^2$. Let G denote the geometric limit of the sequence $\rho_n(\Gamma)$, $\lim_n \rho_n = \rho, \rho(\Gamma) = \Gamma_\infty$. The group G is a discrete subgroup of $PSL(2, \mathbb{C})$ which contains Γ_∞. Thus we get a covering $p : M(\Gamma_\infty) \to M(G)$. Since $\Lambda(\Gamma_\infty) = \mathbb{S}^2$, both ends of the manifold $M(\Gamma_\infty)$ are geometrically infinite. Theorem 14.24 implies that either $M(G)$ has finite volume or p is a finite covering. If p is finite then Theorem 8.14 implies that $G = \Gamma_\infty$ and we are done. Otherwise $vol(M(G)) < \infty$ which contradicts the fact that infinite volume manifolds $M(F_n)$ converge to $M(\Gamma)$ in the quasi-isometric topology. $\square$

14.7. Geometric profile of algebraic convergence

In this section we analyze what happens to Scott compact cores of sequences of hyperbolic manifolds, whose fundamental groups converge algebraically.

This analysis will be used in §3.

Suppose that Γ is a nonelementary geometrically finite torsion-free subgroup in $\mathrm{Isom}(\mathbb{H}^3)$, $M = \mathbb{H}^3/\Gamma$ and $\overset{\bullet}{M} = \overset{\bullet}{M}(\Gamma)$ has incompressible boundary $S(\Gamma)$. Consider a sequence of discrete and faithful representations $\rho_n : \Gamma \to \Gamma_n \subset \mathrm{Isom}(\mathbb{H}^3)$ which is convergent algebraically to a homomorphism $\rho_\infty : \Gamma \to \Gamma_\infty$. Let $M_n := M(\Gamma_n) = \mathbb{H}^3/\Gamma_n$. We assume that each ρ_n is induced by a quasiconformal homeomorphism $\tilde{h}_n : \overline{\mathbb{H}^3} \to \overline{\mathbb{H}^3}$, in particular each group Γ_n is geometrically finite and ρ_n are type-preserving. Let $h_n : M \to M_n$ denote the projection of $\tilde{h}_n$. Pick a number $0 < \epsilon < \mu_3$. We have a sequence of truncated hyperbolic manifolds $M_n^0 := M(\Gamma_n)^0_\epsilon$, $M_\infty^0 := M(\Gamma_\infty)^0_\epsilon$ (obtained by removing Margulis cusps). The manifold M_n^0 has a relative compact core M_n^c (see Theorem 4.126). Our goal is to understand what happens to the compact cores $M_n^c \subset M_n^0$ as $n \to \infty$. Keep in mind, that we cannot choose compact cores M_n^c which converge to a compact core of $M(\Gamma_\infty)$ in (say) topology induced by Hausdorff distance between compact metric spaces. J. Anderson and D. Canary [AC96] constructed examples when compact core of $M(\Gamma_\infty)$ is not homeomorphic to M_n^c. Nevertheless, we will try to extract some geometric information about compact cores.

Let $\xi_n := \rho_n \circ \rho^{-1} : \Gamma_\infty \to \Gamma_n$, this sequence of homomorphisms converges algebraically to the identity representation. The manifold M_∞ could have more cusps than the manifolds $M_n = M(\Gamma_n)$. So, we consider the union C_∞ of those cusps in the ϵ-thin part of M_∞, which correspond to parabolic elements of Γ under the isomorphism $\rho_\infty : \Gamma \to \Gamma_\infty$ of the fundamental groups. Let M_∞^1 denote $M_\infty - C_\infty$ and $\partial_P M_\infty^1$ denote the intersection of ∂M_∞^1 and $cl(C_\infty)$, $\partial_P M_\infty^c := \partial M_\infty^c \cap cl(C_\infty)$; similarly the boundary $\partial_P M_n^0$ is the intersection of M_n^0 with the closure of the union of Margulis cusps in M_n.

Take a relative Scott compact core M_∞^c of the manifold M_∞^1. Let $\tilde{M}_\infty^c$ denote the lift of this compact core to the universal cover $\mathbb{H}^3$ of the manifold $M_\infty := M(\Gamma_\infty)$.

Lemma 14.28. *There exists a sequence of smooth ξ_n-equivariant maps*

$$\tilde{\lambda}_n : \tilde{M}_\infty^c \to \mathbb{H}^3$$

which C^1-converges to the identity uniformly on compacts in $\tilde{M}_\infty^c$. These maps project to **immersions** $\lambda_n : M_\infty^c \to M_n^0$, *which are injective at $\partial_P M_\infty^c$ and send $\partial_P M_\infty^c$ to $\partial_P M_n^0$.*

Proof: Repeat our arguments from the proof of Theorem 7.2. □

Lemma 14.29. *Suppose that the convergence $\xi_n \to id$ is strong. Then the maps λ_n are injective for all sufficiently large n.*

Proof: Choose a compact fundamental domain D for the action of Γ_∞ on $\tilde{M}_\infty^c$ (it is a compact domain) and let U be a the union of D and the images of D under face-pairing transformations (this domain is still compact). Since $\tilde{\lambda}_n$ converge to the identity on compacts in $\tilde{M}_\infty^c$, the restrictions of these maps to U are injective for large n. The only way the map λ_n can fail to be injective in this situation is that for each n there exists $\gamma_n \in \Gamma_n$ so that $\gamma_n \tilde{\lambda}_n(D) \cap \tilde{\lambda}_n(D) \neq \emptyset$ and $\xi_n^{-1}(\gamma_n)$ is not one of the face-pairing transformations in Γ_∞. If all $\xi_n^{-1}(\gamma_n)$ belong to a finite subset in Γ_∞, then after taking the limit as $n \to \infty$ we get a contradiction with the fact that D is a fundamental domain. If $\xi_n^{-1}(\gamma_n)$ do not belong to a finite subset of Γ_∞, they are still convergent to an element $\gamma_\infty \in \mathrm{Isom}(\mathbb{H}^3)$, however this limit is not in the group Γ_∞. This contradicts the assumption that the convergence is strong. $\square$

Let $M_n^d := \lambda_n(M_\infty^c)$. Note that for large n, the diameter of M_n^d is at most $2 + \mathrm{diam}(M_\infty^c)$. The manifolds M_n^0 admit compactifications to manifolds with boundary (since the groups Γ_n are geometrically finite). Hence, after taking sufficiently large Scott compact cores M_n^c of M_n^0, we can assume that $M_n^d \subset M_n^c$.

Now choose a component S of $\partial_0 M^c$. We know that $h_n : S \to h_n(S) \subset M_n$ is a homeomorphic embedding. By adjusting h_n we can assume that $h_n(S)$ is a component of $\partial_0 M_n^c$ and this surface is far away from the compact M_n^d. However we also have a continuous map $h_\infty : S \to M_\infty^c$ which induces the embedding $\rho_n : \pi_1(S) \to \pi_1(M_\infty^c)$. (Note that this map is in no sense a limit if h_n's). Since ρ_∞ at least sends parabolic elements to parabolic we conclude that h_∞ can be chosen so that $h_\infty : \partial S \to \partial_P M_\infty^c$ is injective.

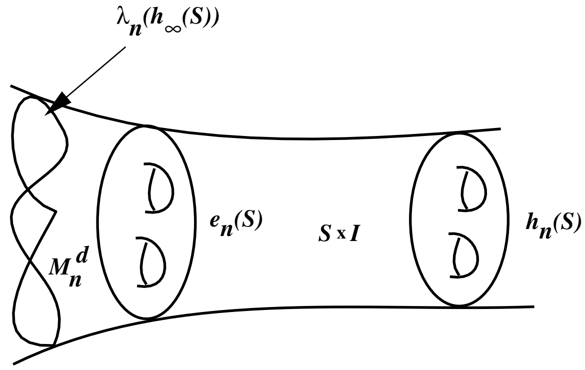

Figure 14.2: *Geometric profile of the algebraic convergence.*

Consider the composition $\lambda_n \circ h_\infty : S \to M_n^d$. This map is properly homotopic to $h_n : S \to h_n(S)$ as a map of pairs $(S, \partial S) \to (M_n^c, \partial_0 M_n^c)$. Hence by Theorem 1.27 there is a properly homotopic embedding $e_n : (S, \partial S) \to (M_n^c, \partial_P M_n^c)$ whose image is contained in the 1-neighborhood of M_n^d. Note that the embeddings e_n and $h_n|_S$ are homotopic as maps of pairs and their images are disjoint. Therefore (according to Theorem 1.29) they bound the product region $S \times I$ in M_n^c. See Figure 14.2.

We repeat this procedure for other components S' of $\partial_0 M^c$ so that images $e_n(S), e_n(S')$ are disjoint.

Now consider a special case: $\pi_1(S) = \Gamma$. The manifold M_∞^1 is naturally homeomorphic to the product $S \times \mathbb{R}$ via a homeomorphism $f_\infty : M^0 \cong$

$S \times \mathbb{R} \to M^1_\infty$ (see Theorem 14.17). We identify $S \times \{0\} \subset M^1_\infty$ with the surface $f_\infty(S)$. Take a sufficiently big compact core $M^c_\infty \subset M_\infty$ which contains the $4d$-neighborhood of $f_\infty(S)$, where $d = \operatorname{diam}(f_\infty(S))$. Then for large n we have: $\lambda_n(M^c_\infty) \supset e_n(S)$. Recall that λ_n is a local diffeomorphism in the interior of M^c_∞, thus $\partial(\lambda_n(M^c_\infty)) \subset \lambda_n(\partial M^c_\infty)$. This implies that λ_n-images of two components of $\partial_0(M^c_\infty)$ lie on different sides of the surface $e_n(S)$. We conclude that there is a natural 1-1 correspondence between the two topological ends of M^1_∞ and two topological ends of M^0_n for large n.

Chapter 15

Outline of the Proof of the Hyperbolization Theorem

Finally I can present an outline of the proof of Thurston's Hyperbolization Theorem. Our discussion mainly follows [Mor84]. By Theorem 8.36 it is enough to prove the Hyperbolization Theorem for a finite covering of M, thus we will consider only compact orientable pared manifolds (M, P), which contain orientable superincompressible surfaces and where $P = \partial M$ is the designated parabolic locus. Thurston's theorem is proven by induction on levels of the Haken hierarchy of M. It turns out that the main problem in proving Theorem 1.42 is the last step of induction:

Theorem 15.1. *(The last step of induction.) Suppose that (M, P) is an orientable pared atoroidal manifold, whose boundary P consists of incompressible tori. Suppose that $\Sigma_M \subset M$ is an orientable superincompressible surface which consists of either one or two components, so that the pared manifold (N, Q) obtained by splitting M along Σ_M, consists either of one or two components which admit geometrically finite hyperbolic structures with the parabolic locus Q. In the case when Σ_M consists of two components we assume that none of the components of $M - \Sigma_M$ is an interval bundle over a surface. Then the pared manifold (M, P) admits a complete hyperbolic structure of finite volume.*

There is a slight difference between this theorem and the one formulated in [Mor84], which deals with connected superincompressible surfaces only. The reason for necessity to consider disconnected superincompressible surfaces will become clear in §19.4.

Case A. The surface Σ_M is not a virtual fiber in a fibration of M over $\mathbb{S}^1$ (our assumptions imply that this is always the case provided that Σ_M is disconnected). It may happen that still one of the components of $(M - \Sigma_M, P - \Sigma_M)$ is an interval bundle over a surface. In this case we just take a double covering over M so that a component of the lift of Σ_M does not separate (see §1.9). Thus we can assume that in the Case A none of the components of $(M - \Sigma_M, P - \Sigma_M)$ is an interval bundle over a surface.

Case B. M (or its double cover) is fibered over a circle and Σ_M is a (virtual) fiber. This is exactly the case when N is an interval bundle over the a surface which is either homeomorphic to Σ_M or is 2-fold covered by Σ_M.

Steps of the proof:

Step 0. *The orbifold trick.* The main theorem of this step that will be proven in Chapter 19, Sections 19.2– 19.4, is the following:

Theorem 15.2. *Assume that Theorem 15.1 (Case A) is valid for all (M, P). Suppose that (M', P') is a compact pared manifold with the designated parabolic locus P', $\chi(M') \neq 0$; $L' \subset M'$ is an orientable connected superincompressible surface. Let (M, P) be the manifold obtained by splitting (M', P') along L', the surface $P \subset \partial M$ consists of a collection of simply-connected components as well as a union T of some tori and annuli. Suppose that the pared manifold (M, T) admits a geometrically finite hyperbolic structure with the parabolic locus T. Then the pared manifold (M', P') also admits a hyperbolic structure.*

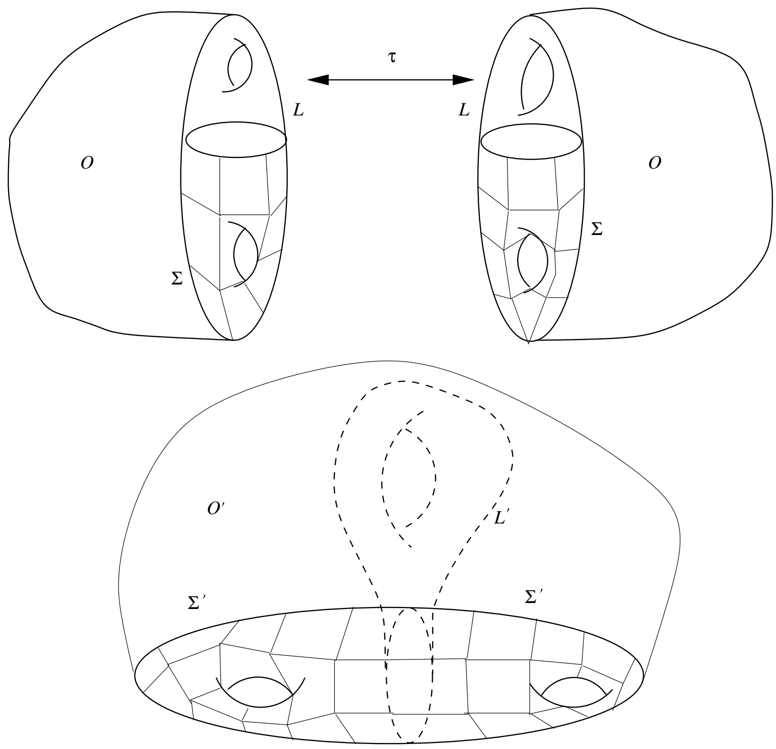

Figure 15.1: *The singular loci of the orbifolds O and O' are Σ and Σ' respectively. The faces of these loci are represented as quadrilaterals in this picture.*

We refer the reader to the Preface for the outline of the proof of this theorem. In §19.1 I explain how to construct finite manifold covers of the locally reflective bipolar orbifolds. The pared manifold (M', P') is obtained by gluing boundary subsurface $L \subset \partial M$ via a homeomorphism $\tau : L \to L$. In §19.2 we construct a reflective orbifold structure on $cl(\partial M - P - L)$ so that after we glue the boundary of the resulting orbifold O via τ we get an orbifold O' of zero Euler characteristic, i.e. $\partial O'$ consists only of TORI. See Figure 15.1. Using Brooks' theorem we choose the orbifold locus on $\partial M - P - L$ in such a way that O is a hyperbolic orbifold. Let Q denote the parabolic locus of O. The designated parabolic locus Q' in $\partial O'$ is obtained by gluing the pieces of the parabolic locus of (O, Q). In §19.3 we check that (O', Q') is atoroidal. We would like to apply Theorem 15.1 to the gluing $O \to O'$ to get a hyperbolic structure on O'. This theorem applies to manifolds, not to orbifolds. To be able to use Theorem 15.1 and construct a hyperbolic structure on (O', Q') we go through the following procedure described in §19.4. Using §19.1 we get finite manifold covers $(\tilde{O}', \tilde{Q}') \to (O', Q')$ and $(\tilde{O}, \tilde{Q}) \to (O, Q)$ and a lift $\tilde{\tau}$ of τ to a homeomorphism

$$\tilde{\tau} : \partial\tilde{O} - \tilde{Q} \to \partial\tilde{O} - \tilde{Q}$$

so that $(\tilde{O}', \tilde{Q}') = (\tilde{O}, \tilde{Q})/\tilde{\tau}$. Thus the pared manifold $(\tilde{O}', \tilde{Q}')$ is hyperbolic. The finite group of covering transformations $\Phi \curvearrowright \tilde{O}'$ does not act isometrically with respect to the hyperbolic structure on $(\tilde{O}', \tilde{Q}')$, however it is homotopic to an isometric action $\Psi \curvearrowright \tilde{O}'$. We then use Theorem 6.33 to prove that the hyperbolic orbifold $\tilde{O}'/\Psi$ is homeomorphic to $(\tilde{O}', \tilde{Q}')/\Phi = (O', Q')$. Hence O' is hyperbolic. It follows that the underlying pared manifold (M', P') is hyperbolic as well. This concludes the proof of Theorem 15.2.

Theorems 15.2 and 15.1 (Case A) imply the following:

Theorem 15.3. *Let (M, P) be an orientable pared atoroidal manifold, so that $\partial M = P$ and $\chi(P) = 0$. Suppose that $\Sigma_M \subset M$ is an orientable connected superincompressible surface, so that the pared manifold (N, Q) obtained by splitting M along Σ_M is not an interval bundle over a surface (equivalently, Σ_M is not a virtual fiber in a fibration over $\mathbb{S}^1$). Then the pared manifold (M, P) admits a complete hyperbolic structure of finite volume.*

Proof: Consider a Haken hierarchy for (M, P): $(M^1, P^1) := (N, Q)$, (M^2, P^2),..., M^h. Here M^h is a collection of balls; M^{j+1} is obtained from (M^j, P^j) by splitting along a connected superincompressible surface Σ^j and P^{j+1} is obtained from $P^j - \Sigma^j$ by removing contractible components. Clearly M^h admits a convex hyperbolic structure (take a collection of round balls in $\mathbb{H}^3$). Going backwards along the hierarchy and using Theorem 15.2 we construct geometrically finite hyperbolic structures on each (M^j, P^j), including (M, P). $\square$

Note that in the process of the proof we also derive Theorem 1.43 from Theorems 15.1 and 15.2. In Section 19.5 we will derive Theorem 1.43 directly from Thurston's Hyperbolization Theorem.

The proof in the Case B is very different and we first concentrate on the *generic* Case A, leaving discussion of the *exceptional* Case B to §15.2.

15.1. Case A: The generic case

Proof of Theorem 15.1 (Case A). It is very inconvenient to keep the distinction between the cases when Σ_M is connected and disconnected. Instead we formulate a theorem which covers both connected and disconnected cases.

Theorem 15.4. *Suppose that (N, Q) is a pared Haken 3-manifold (which consists either of one or of two components), we assume that $\partial_0 N$ is incompressible, (N, Q) admits a geometrically finite hyperbolic structure and no component of (N, Q) is an interval bundle over a surface. Suppose that Σ_1, Σ_2 are certain components of $\partial_0 N$ and $\tau : \Sigma_1 \cup \Sigma_2 \to \Sigma_2 \cup \Sigma_1$ is a gluing involution. Let $M = N/\tau$ and P be the image of Q in M. Suppose that the pared manifold (M, P) is atoroidal. Then (M, P) admits a geometrically finite hyperbolic structure.*

Note that once Theorem 15.4 is proven, Theorem 15.1 (Case A) follows by induction (which has either length 1 or 2). Thus, we are proving Theorem 15.4.

I recall that for a Kleinian group $G \subset PSL(2, \mathbb{C})$ we use the notation $\overset{\bullet}{M}(G)$ for the manifold with boundary $(\mathbb{H}^3 \cup \Omega(G))/G$. The compact manifold $\overset{\bullet}{M}(G)^0$ is the complement to the union of open Margulis cusps. Recall that $\partial_P \overset{\bullet}{M}(G)^0$ is the intersection of the boundary of $\overset{\bullet}{M}(G)^0$ with the union of (closed) Margulis cusps in $M(G)$. Recall also that $\mathcal{D}(G, PSL(2, \mathbb{C}))$ is the space of conjugacy classes of discrete and faithful representations of G to $PSL(2, \mathbb{C})$. The *relative* representation space

$$\mathcal{D}_{par}(G, PSL(2, \mathbb{C})) \subset \mathcal{D}(G, PSL(2, \mathbb{C}))$$

consists of (conjugacy classes) of representations whose restrictions to $\pi_1(Q_j)$ are parabolic for each components Q_j of Q.

According to the hypothesis, there exists a geometrically finite Kleinian group G (or a pair of such groups) such that $(N, Q) = (\overset{\bullet}{M}(G)^0, \partial_P \overset{\bullet}{M}(G)^0)$ (in the case when N is connected), otherwise

$$N = N_1 \sqcup N_2 = \overset{\bullet}{M}(G_1)^0 \sqcup \overset{\bullet}{M}(G_2)^0$$

and $G = (G_1, G_2)$. In any case the boundary of $\overset{\bullet}{M}(G)$ contains $\Sigma_1 \cup \Sigma_2 = \Sigma$. Let

$$\tau = \tau_1 \cup \tau_2, \quad \tau_1 : \Sigma_1 \to \Sigma_2, \quad \tau_2 : \Sigma_2 \to \Sigma_1$$

be the pair of gluing maps so that $\tau_2 = \tau_1^{-1}$. The Teichmüller space of G is the product of Teichmüller spaces of the boundary components $\partial_0 N$ and it is too large for us. So, we shall fix once and for all marked conformal structures on all components of $\partial_0 N$ except Σ, the resulting subvariety in $\mathcal{T}(G)$ will be denoted $\mathcal{T}_\Sigma(G)$ (if N is connected); in the case when N consists of two components we get $T_{\Sigma_j}(G_j) \subset \mathcal{T}(G_j)$ and $\mathcal{T}_\Sigma(G) := \mathcal{T}_{\Sigma_1}(G_1) \times \mathcal{T}_{\Sigma_2}(G_2)$. Bers' Theorem implies that $\mathcal{T}_\Sigma(G) \cong \mathcal{T}(\Sigma)$.

Step 1. Since Σ_M is superincompressible, the surfaces Σ_j correspond to subgroups $F_j \subset G$ which are quasifuchsian because each F_j is geometrically finite, has no accidental parabolic elements and has simply-connected domain of discontinuity, see Corollary 4.112 and Theorem 8.17. Let Ω_j denote the component of $\Omega(F_j)$ which belongs to $\Omega(G)$, $j = 1, 2$ (or to $\Omega(G_j)$ if N is not connected). We use the notation ϕ for the isomorphism of the groups $F_1 \to F_2$ induced by τ_1. We let $\bar{\Omega}_j := \widehat{\mathbb{C}} - cl(\Omega_j)$.

Consider the case when N is connected.

Theorem 15.5. *In order to prove Theorem 15.4, it is enough to find a point $[\rho] \in \mathcal{T}_\Sigma(G)$ induced by a quasiconformal map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that: for $\Omega'_j = f(\Omega_j)$ and $\bar{\Omega}'_j = f(\bar{\Omega}_j)$, there is a Moebius transformation $A \in PSL(2,\mathbb{C})$*

$$A : \Omega'_1 \to \bar{\Omega}'_2, \quad A : \bar{\Omega}'_1 \to \Omega'_2$$

which induces the isomorphism $\rho\phi\rho^{-1}$ between $F'_1 = \rho(F_1)$ and $F'_2 = \rho(F_2)$.

Remark 15.6. The group uniformizing the manifold $M = N/\tau$ is the HNN extension of $\rho(G)$ via the Moebius transformation A.

In the case when N is not connected Theorem 15.5 should be reformulated as follows.

Theorem 15.7. *In order to prove Theorem 15.4, it is enough to find a pair of points $[\rho_j] \in \mathcal{T}_{\Sigma_j}(G_j)$ induced by quasiconformal maps $f_j : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that: for $\Omega'_j = f_j(\Omega_j)$ and $\bar{\Omega}'_j = f_j(\bar{\Omega}_j)$ there is a Moebius transformation*

$$A : \Omega'_1 \to \bar{\Omega}'_2, \quad A : \bar{\Omega}'_1 \to \Omega'_2$$

which induces the isomorphism $\rho_2\phi\rho_1^{-1}$ between $F'_1 = \rho_1(F_1)$ and $F'_2 = \rho_2(F_2)$.

Remark 15.8. In this case the amalgamated free product of $A^{-1}\rho_2(G_2)A$ and $\rho_1(G_1)$ is the subgroup of $PSL(2,\mathbb{C})$ uniformizing the manifold $M = N/\tau$.

Step 2. On this step, given the gluing map $\tau : \Sigma \to \Sigma$, we construct a map $\sigma = \sigma_\tau : \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma)$ such that existence of ρ in Theorems 15.5, 15.7 can be reformulated as follows.

Theorem 15.9. *(Fixed Point Theorem.) The map σ has a fixed point $X \in \mathcal{T}(\Sigma)$.*

Remark 15.10. If X is a fixed point, then the image of X under the Bers isomorphism $\beta : \mathcal{T}(\Sigma) \to \mathcal{T}_\Sigma(G)$ is the representation ρ required by Theorems 15.5 and 15.7.

In order to prove Theorem 15.9 we establish two facts about the map σ.

Step 3.

Theorem 15.11. *The map σ is a contraction of the Teichmüller metric.*

Step 4. Using Theorem 15.11 we prove here that the Fixed Point Theorem follows from the the following *Weak Bounded Image Theorem.*

Theorem 15.12. *(Weak Bounded Image Theorem.) Suppose that the gluing map τ produces an atoroidal 3-manifold $M = N/\tau$. Then the sequence $\sigma^n(Y)$ is precompact in $\mathcal{T}(\Sigma)$ for every point $Y \in \mathcal{T}(\Sigma)$.*

There is a stronger version of this result in the case when the pared manifold (N, Q) is acylindrical. We define $\overline{\Sigma}$ to be the quotient $\bar{\Omega}_1/F_1 \sqcup \bar{\Omega}_2/F_2$.

Theorem 15.13. *(Bounded Image Theorem for acylindrical manifolds.) Consider the embedding*

$$\beta : \mathcal{T}(\Sigma) \to \mathcal{T}_\Sigma(G) \hookrightarrow \mathcal{T}(F_1) \times \mathcal{T}(F_2) = \mathcal{T}(\Sigma) \times \mathcal{T}(\overline{\Sigma}).$$

Then the projection of $\beta(\mathcal{T}(\Sigma))$ to $\mathcal{T}(\overline{\Sigma})$ is precompact provided that (N, Q) is acylindrical.

I note here that Thurston claims to prove this theorem under the assumption that $M = N/\tau$ is atoroidal (without assuming that (N, Q) is acylindrical). However I do not know how to prove this stronger assertion.

Detailed discussion of Steps 1 through 4 occupies the Chapter 16. Proof of the Weak Bounded Image Theorem is "the heart of the Hyperbolization Theorem", this proof breaks into two major steps, which we discuss in details in the Chapter 17.

Step 5.

Theorem 15.14. *For every $Y \in \mathcal{T}(\Sigma)$ the sequence $\beta(\sigma^n(Y))$ is precompact in the space*

$$\mathcal{D}_{par}(G, PSL(2, \mathbb{C})) \subset \mathcal{D}(G, PSL(2, \mathbb{C})).$$

Remark 15.15. In the case when (N, Q) is acylindrical, this step is simplified by the fact that the space $\mathcal{D}_{par}(G, PSL(2, \mathbb{C}))$ is compact.

Step 6. Given Theorem 15.14, we have to prove that the sequence $\beta(\sigma^n(Y))$ is precompact not only in $\mathcal{D}(G, PSL(2, \mathbb{C}))$ but also in $T_\Sigma(G)$. Suppose that $[\rho_n] = \beta(\sigma^n Y)$ subconverges to $[\rho]$. We have to prove that:

(1) The restriction of ρ to each F_j has no accidental parabolic elements;

(2) Each group $\rho(F_j)$ is geometrically finite ($j = 1, 2$).

Once the above assertions are proven, it follows that the images $\rho(F_j)$ are quasifuchsian (see Theorem 8.17). Let $\psi_n := \rho_n|(F_1, F_2)$, $\psi := \rho|(F_1, F_2)$. Then Theorem 8.52 implies that (after taking a subsequence if necessary)

$$\lim_{n\to\infty} \beta^{-1}([\psi_n]) = \beta^{-1}([\psi])$$

where

$$\beta : \mathcal{T}(\Sigma) \times \mathcal{T}(\overline{\Sigma}) \to \mathcal{T}(F_1) \times \mathcal{T}(F_2).$$

Thus (after projecting to $\mathcal{T}(\Sigma)$) we conclude that the sequence $\sigma^n(Y)$ subconverges in $\mathcal{T}(\Sigma)$. This finishes the proof of the Hyperbolization Theorem in the generic case.

15.2. Case B: Manifolds fibered over the circle

It follows from Theorem 8.36 and the results of §1.9 that it is enough to consider the Case B when the surface Σ_M itself is a fiber in a fibration of M over $\mathbb{S}^1$.

Outline of the Thurston-Otal approach:

Suppose that M is a compact 3-manifold fibered over $\mathbb{S}^1$, Σ_M is a super-incompressible surface as above, $S := int(\Sigma_M)$. Let Γ denote the Fuchsian fundamental group of N, where $N = M - Nbd(\Sigma_M)$. We use the notation τ for the gluing map $\tau : \partial N \to \partial N$. This map corresponds to an *aperiodic* homeomorphism $\tau : S \to S$, which means that for any homotopically nontrivial nonperipheral loop $\gamma \subset S$ and any $n \neq 0$ the loops $\tau^n(\gamma)$ and γ are not freely homotopic (see §1.9). The surface S admits a complete hyperbolic metric of finite area which is uniformized by a Fuchsian group $\Gamma \subset PSL(2, \mathbb{R})$.

Step 7.

Theorem 15.16. *(Compactness theorem for aperiodic homeomorphisms.) For any aperiodic homeomorphism $\tau : S \to S$ and $\alpha \in \mathcal{T}(S)$, the sequence*

$$\beta^{-1}([\rho_n]) := (\tau^n(\alpha), \tau^{-n}(\alpha)) \in \mathcal{T}(S) \times \mathcal{T}(\overline{S}) = \mathcal{T}(\Gamma)$$

is subconvergent in $\mathcal{D}_{par}(\Gamma, PSL(2, \mathbb{C}))$.

Let ρ_∞ be the limit of a convergent subsequence in $\{\rho_n\}$, $\Gamma_\infty := \rho_\infty(\Gamma)$.

Theorem 15.17. *The limiting group Γ_∞ has empty discontinuity domain and there is a quasiconformal homeomorphism $f_\infty : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ so that:*

- $f_\infty \Gamma_\infty f_\infty^{-1} = \Gamma_\infty$.
- *The induced outer automorphism of Γ_∞ is the same as $\rho_\infty \tau_* \rho_\infty^{-1}$.*

Now, combination of Theorems 15.16, 15.17, 8.38 implies that f_∞ is Moebius and we extend Γ_∞ via f_∞. The group $H \subset \mathrm{Isom}(\mathbb{H}^3)$ generated by Γ_∞ and f_∞ is isomorphic to $\pi_1(M)$. Both manifolds M and $\mathbb{H}^3/H$ have zero Euler characteristic. Therefore by Theorem 1.31, the manifold M is homeomorphic to compact core of the hyperbolic manifold $\mathbb{H}^3/H$ (see Corollaries 4.63 and 4.66). This finishes the proof in the fibered case:

Theorem 15.18. *Suppose that (M, P) is a pared 3-manifold so that M fibers over the circle with the fiber S and $P = \partial M$. Then (M, P) admits a hyperbolic structure if and only if the monodromy homeomorphism $\tau : S \to S$ of this fibration is pseudo-Anosov.*

In §18.4 I describe an alternative approach to hyperbolization of 3-manifolds fibered over $\mathbb{S}^1$ via reduction to the *generic* Case A. I will show how this approach works in the case of manifolds with nonempty boundary.

Chapter 16

Reduction to the Bounded Image Theorem

Step 0. See Sections 19.2–19.4.

16.1. Step 1. The Maskit combination

Theorems 15.5, 15.7 directly follow from the Maskit Combination Theorems 4.101, 4.103. $\square$

16.2. Step 2. Formulation of the fixed–point theorem

Now we shall give yet another reformulation of the generic case of Theorem 15.4, using the language of the Teichmüller theory. I recall that there is a natural embedding $\alpha : \mathcal{T}_\Sigma(G) \hookrightarrow \mathcal{T}(F_1) \times \mathcal{T}(F_2)$, see §8.11.

Let c_j denote the projections from $\mathcal{T}_\Sigma(G)$ to $\mathcal{T}(F_j)$ ($j = 1, 2$). The gluing homeomorphism τ of $\Sigma \subset \partial_0 N$ reverses the induced orientation of the boundary. Consider the product manifold $\dot{M}(F_1) \sqcup \dot{M}(F_1) \cong [-1, 1] \times \Sigma$, where we identify $\{+1\} \times \Sigma$ with $\Omega_1/F_1 \cup \Omega_2/F_2$; Ω_j is contained in the domain of discontinuity of $\Omega(G_j)$ if N is not connected, and in $\Omega(G)$ if N is connected.

We extend τ to $[-1, 1] \times \Sigma$ as $\bar{\tau}(t, z) = (-t, \tau(z))$. Hence

$$\bar{\tau} : [-1, 1] \times \Sigma \to [-1, 1] \times \Sigma$$

preserves orientation and induces a map of the Teichmüller space $\bar{\tau}_* : \mathcal{T}(F_1) \times \mathcal{T}(F_2) \to \mathcal{T}(F_2) \times \mathcal{T}(F_1)$. (Namely, $\bar{\tau}$ is induced by a pair of quasiconformal homeomorphisms $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which conjugate F_1 to F_2 and F_2 to F_1. These homeomorphisms act on $\mathcal{T}(F_1) \times \mathcal{T}(F_2)$ by precomposition.)

Therefore, we can reformulate Theorem 15.4 as follows.

Theorem 16.1. *There exists a point* $[\rho] \in T_\Sigma(G)$ *such that*

$$\bar{\tau}_*(c_1([\rho])) = c_2([\rho]) \tag{16.1}$$

Proposition 16.2. *Theorem 16.1 implies Theorem 15.4.*

Proof: I will consider only the case of connected manifold N, the disconnected case is similar and is left to the reader. Suppose that there exists a representation ρ as above and $\rho(G) = \Gamma$, $\rho(F_j) = F_j'$.

Then the equality (16.1) implies that there exists an orientation–preserving isometry $a : \mathbb{H}^3/F_1' \to \mathbb{H}^3/F_2'$ which is properly homotopic to the homeomorphism

$$\bar{\tau} : \mathbb{H}^3/F_1' \to \mathbb{H}^3/F_2'.$$

(Here we use the fact that any 1-quasiconformal homeomorphism of $\widehat{\mathbb{C}}$ is a Moebius transformation.) The isometry a lifts to a Moebius transformation A in the hyperbolic 3-space so that we have:

(a) A conjugates F_1' to F_2' and the conjugation induces the isomorphism $\phi : F_1 \to F_2$.

(b) Since τ flips the boundary components of $\Sigma \times [-1, 1]$, the transformation A must send Ω_1' to $\overline{\Omega}_2'$.

Therefore we are in the situation of the 2-nd Maskit Combination Theorem and the discrete subgroup of $\mathrm{Isom}(\mathbb{H}^3)$ generated by Γ and A uniformizes the manifold $M = N/\tau$. $\square$

We have the embedding

$$\mathcal{T}(\Sigma) \xrightarrow{\beta} \mathcal{T}_\Sigma(G) \xrightarrow{\alpha} \mathcal{T}(\Sigma) \times \mathcal{T}(\overline{\Sigma}).$$

Define a map $\bar{s}_G : \mathcal{T}(\Sigma) \to \mathcal{T}(\bar{\Sigma})$ as the composition $\pi^- \circ \beta$, where π^- is the projection to the second factor in $\mathcal{T}(\Sigma) \times \mathcal{T}(\overline{\Sigma})$. The map $\bar{s}_G = (\bar{s}_1, \bar{s}_2)$ is called the *skinning map* associated with the group G. Here $\bar{s}_j : \mathcal{T}(\Sigma) \to \mathcal{T}(\overline{\Sigma}_j)$. On the other hand, the (orientation-reversing) gluing involution $\tau : \Sigma \to \Sigma$ induces a map $\bar{\tau}_* : \mathcal{T}(\overline{\Sigma}) \to \mathcal{T}(\Sigma)$, which is an isometry of the Teichmüller spaces. Recall that $\tau = (\tau_1, \tau_1^{-1}) : \Sigma_1 \cup \Sigma_2 \to \Sigma_2 \cup \Sigma_1$.

Theorem 16.3. *Let* $[\rho] = \beta(\theta)$, $\theta \in \mathcal{T}(\Sigma)$. *The following two conditions are equivalent:*

(C1) $\bar{\tau}_*(c_1([\rho])) = c_2([\rho])$.

(C2) $\bar{\tau}_* \circ \bar{s}_G(\theta) = \theta$.

Proof: In this proof we will identify $\mathcal{T}_\Sigma(G)$ with $\mathcal{T}(\Sigma)$ using the Bers' map β. Thus $[\rho]$ is identified with θ. Let $[\rho] = \theta = (\theta_1, \theta_2) \in \mathcal{T}(\Sigma_1) \times \mathcal{T}(\Sigma_2) = \mathcal{T}(\Sigma)$. Consider (C1). We have

$$c_1(\theta) = (\theta_1, \bar{s}_1(\theta_1, \theta_2)) \tag{16.2}$$

therefore

$$\bar{\tau}_{1*}(c_1(\theta)) = \bar{\tau}_{1*}(\theta_1, \bar{s}_1(\theta_1, \theta_2)) = (\tau_{1*}\bar{s}_1(\theta_1, \theta_2), \tau_{1*}(\theta_1)) \tag{16.3}$$

Then

$$c_2(\theta_1, \theta_2) = (\theta_2, \bar{s}_2(\theta_1, \theta_2)) \tag{16.4}$$

Thus (C1) is equivalent to

$$\tau_{1*}\bar{s}_1(\theta_1, \theta_2) = \theta_2 \text{ and } \tau_{1*}(\theta_1) = \bar{s}_2(\theta_1, \theta_2) \tag{16.5}$$

Now consider (C2):

$$\begin{aligned} \bar{\tau}_* \circ \bar{s}(\theta_1, \theta_2) &= \bar{\tau}_*(\bar{s}_1(\theta_1, \theta_2), \bar{s}_2(\theta_1, \theta_2)) = \\ &= (\tau_{1*}^{-1}\bar{s}_2(\theta_1, \theta_2), \tau_{1*}\bar{s}_1(\theta_1, \theta_2) = (\theta_1, \theta_2) \end{aligned}$$

by (C2). Therefore we get the same system of equations as in (16.5). □

16.3. Step 3. Proof of the contraction theorem

The goal of this step is to prove that the map $\sigma = \bar{\tau}_* \circ \bar{s}_G : \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma)$ is a contraction of the Teichmüller metric $d_{\mathcal{T}}$. The transformation $\bar{\tau}_*$ is just an isometry between the Teichmüller spaces $\mathcal{T}(F_1)$ and $\mathcal{T}(F_2)$. Therefore, we have to prove that the skinning map $\bar{s}$ is a contraction. We shall consider only the case when N is connected, the proof in the second case is similar. Let $X = (X_1, X_2), Y = (Y_1, Y_2) \in \mathcal{T}(G) = \mathcal{T}(\Sigma_1) \times \mathcal{T}(\Sigma_2)$ be two distinct points corresponding to Kleinian groups G, Γ. We use the notation $V = V_1 \cup V_2, U = U_1 \cup U_2$ for the lifts of Σ to the discontinuity domains $\Omega(G), \Omega(\Gamma)$. Each U, V consists of a disjoint union of two open topological disks stabilized by quasifuchsian subgroups $F_j \subset G, F_j' \subset \Gamma$.

Then the distance between the points X and Y is calculated as follows. Let $f_j : X_j \to Y_j$ be the extremal Teichmüller maps which realize distances between X_1, Y_1 and X_2, Y_2. Then $d_{\mathcal{T}}(X, Y) = \frac{1}{2}\log K$, where $K = \max(K(f_1), K(f_2)) \neq 1$.

Now there are two ways to argue, first we will give a proof assuming that the Riemann surfaces under consideration are compact.

The quasiconformal map $f = f_1 \cup f_2$ is not conformal, therefore it has at least one *singular point* (i.e. a point where f is not smooth). Let $\mu = t_1\bar{\phi}_1/|\phi_1| + t_2\bar{\phi}_2/|\phi_2|$ be the Beltrami differential of f, zeroes of the quadratic differentials $\phi_j \in Q(X_j), j = 1, 2$, correspond to the singular points of f. Now we associate to μ a quasiconformal homeomorphism $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ (which induces $\rho : G \to \Gamma$) as in §8.7. Namely, extend μ by zero to other components of $S(G)$, then lift μ to a Beltrami differential $\tilde{\mu}$ in $\Omega(G)$, extend it by zero on the limit set and let h be a solution of the Beltrami equation $\overline{\partial}h = \mu\partial h$. It follows from Bers' Theorem that after composing h with a Moebius transformation we get a quasiconformal homeomorphism (again denoted by h) inducing ρ. Note that $K(h) = K$. Each $[\Omega(G) - V_j]/F_j$ has infinitely many components, which correspond to F_j-orbits of the lifts of X_j which are distinct from V_j. Each of these components contains at least one point at which $\tilde{\mu}$ is not differentiable. Thus h is *singular* at every of these points. Therefore the projection $\widehat{h}$ of h to a quasiconformal homeomorphism between $\bar{\Sigma}_j = [\widehat{\mathbb{C}} - cl(V_j)]/F_j$ and $\bar{\Sigma}'_j = [\widehat{\mathbb{C}} - cl(U_j)]/F'_j$ is not smooth in infinitely many points and, hence, is not a Teichmüller mapping. This implies that $\widehat{h}$ is not extremal in its homotopy class and the extremal map has smaller coefficient of quasiconformality than K.

This proof fails in the case of surfaces with punctures and we need a different argument to prove that $\widehat{h}$ is not a diffeomorphism in infinitely many points. Note that the domain $\widehat{\mathbb{C}} - cl(V_j)$ contains continuum of conical limit points of the group G; Theorem 8.34 implies that h is not differentiable at every of these points. It follows that $\widehat{h}$ is not a Teichmüller mapping.

In any case we conclude that

$$d_j = d_{\mathcal{T}}([\bar{\Sigma}_j, id], [\bar{\Sigma}'_j, \widehat{h}]) < \log K(\widehat{h})/2 = \log(K)/2.$$

However the maximum of the distances d_1, d_2 is exactly $d_{\mathcal{T}}(\bar{s}_G(X), \bar{s}_G(Y))$. So, the skinning map $\bar{s}_G$ is a contraction of the Teichmüller distance. $\square$

Remark 16.4. Note that we get strict inequality because $\Omega(G)$ is not the disjoint union of two disks. If G is quasifuchsian then the above computations yield:

$$d_j \leq \log(K)/2$$

and the skinning map is actually an isometry.

The above calculations actually imply more. Pick a base-point $X \in \mathcal{T}(\Sigma)$. Consider a sequence of admissible isomorphisms $\rho_m : G \to \Gamma_m \subset PSL(2, \mathbb{C})$ which corresponds to the sequence of iterations $X_m := (\sigma)^m(X) \in \mathcal{T}(\Sigma)$. Let

$$\psi_m := \rho_m|_{\pi_1(\Sigma)}.$$

Each ψ_m is a pair of isomorphisms which define markings on the Riemann surfaces $\Omega(\Gamma_m)/\Gamma_m$. Thus, $X_m = [\Omega(\Gamma_m)/\Gamma_m, \psi_m]$. For each m we have an isomorphism $(\phi_m, \phi_m^{-1}) : \pi_1(X_m) \to \pi_1(X_{m+1})$ so that

$$(\phi_m, \phi_m^{-1}) \circ \psi_m = \psi_m \circ \tau_*$$

and (ϕ_m, ϕ_m^{-1}) maps X_m to X_{m+1} as an element of the Teichmüller modular group acting on $\mathcal{T}(\Sigma)$. Our calculations above show that $d_{\mathcal{T}(\Sigma)}(X_m, X_{m+1}) < \log(K)/2$. Hence we can realize each (ϕ_m, ϕ_m^{-1}) by a K-quasiconformal homeomorphism. Lifting this homeomorphism to $\Omega(\Gamma_m)$ and extending it to the limit set we get a sequence of quasiconformal homeomorphisms

$$f_m : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}, \quad \text{so that } \; K(f_m) \leq K, \quad f_{m*}(\Gamma_m) := f_m \Gamma_m f_m^{-1} = \Gamma_m$$

and the restriction of f_{m*} to the quasifuchsian groups stabilizing components of $\Omega(\Gamma_m)$ is the same as τ_{m*}. Hence by applying Theorem 8.57 we get the following

Proposition 16.5. *For each loxodromic element $g \in \pi_1(\Sigma)$ we have:*

$$K^{-1} \leq \ell(\rho_n(g))/\ell(\rho_n(\tau_*(g))) \leq K$$

where ℓ denotes the translation length in $\mathbb{H}^3$.

16.4. Step 4. End of the proof of the fixed–point theorem

Suppose for a moment that we already have proved that the sequence $\sigma^n(X)$ is bounded in the locally compact metric space $\mathcal{T}_\Sigma(G) \cong \mathcal{T}(\Sigma)$. Note that σ is continuous (since it is 1-Lipschitz). Let L denote the closure of the sequence $\sigma^n(X)$ in $\mathcal{T}(\Sigma)$. Clearly L is compact and $\sigma : L \to L$. Then σ has a fixed point according to the following

Lemma 16.6. *Suppose that $\sigma : L \to L$ is a mapping of a compact metric space (L, d) such that*

$$d(\sigma(x), \sigma(y)) < d(x, y), \quad \textit{if} \;\; x \neq y.$$

Then σ has a fixed point in L.

Proof: Suppose that the mapping $\sigma : L \to L$ does not have a fixed point. Then the continuous function $d(\sigma(x), x)$ has a positive minimum in L realized by a point $x \in L$. However the contraction property of σ implies that $d(\sigma^2(x), \sigma(x)) < d(\sigma(x), x)$. This contradiction shows that σ has a fixed point. □

Chapter 17

The Bounded Image Theorem

17.1. Limit exists

I will assume that the manifold N is connected, the other case is completely similar (with exception of few notational differences).

Proof of Theorem 15.14.

Recall that all the representations $[\rho_n] \in \mathcal{D}(G, PSL(2,\mathbb{C}))$, that we constructed in the Chapter 16, are induced by quasiconformal conjugations of a single Kleinian group G (since we assume that N is connected). Each nontrivial element in $\pi_1(Q_j)$ is parabolic (for every component $Q_j \subset Q$), thus

$$[\rho_n] \in \mathcal{D}_{par}(G, PSL(2,\mathbb{C})), \quad \rho_n : G \to \Gamma_n$$

where $G = \pi_1(N) \subset PSL(2,\mathbb{C})$. Every component of the boundary $\partial_0 N := \partial N - Q$ is incompressible. If the pared manifold (N, Q) would be acylindrical as well, then the Compactness Theorem 12.91 would imply that the sequence $[\rho_n]$ is subconvergent in $\mathcal{D}(G, PSL(2,\mathbb{C}))$ since the space $\mathcal{D}_{par}(G, PSL(2,\mathbb{C}))$ is compact.

Assume that the sequence $\beta(\sigma^n(X)) = [\rho_n] \in T_\Sigma(G)$ is not relatively compact in $\mathcal{D}(G, PSL(2,\mathbb{C}))$. Therefore (according to Theorem 10.24) we can extract from it a subsequence ρ_{n_k} which (after appropriate rescaling) converges to a small relatively elliptic (with respect to Q) action of G on a metric tree T (see Theorem 10.24). We shall establish contradiction with Theorem 10.24 by proving that the action of G on T has a global fixed point. We retain the notation ρ_n for ρ_{n_k}.

I recall that (N, Q) contains a submanifold W which is called *the window* (see §1.8). The submanifold W has codimension zero and is a disjoint union of interval bundles over surfaces. Therefore the boundary of W consists of two parts: $\partial_0 W \subset \partial N$ and $\partial_1 W = cl(\partial W - \partial_0 W)$, the latter is the union of *vertical* cylinders.

For each component of $\partial_0 N$ the action of its fundamental group on T is small and relatively elliptic (with respect to the components of Q);

the action need not be minimal, so we will have to take minimal invariant subtrees in T. Then using Theorem 11.34 we define a *maximal elliptic* subsurface $B \subset \partial_0 N$ (with respect to the action of the fundamental groups of components of $\partial_0 N$ on T). The surface B contains (up to conjugation) all elements $g \in \pi_1(\partial_0 N)$ which have fixed points in the tree T. In particular, the subsurface B entirely contains each of those components of $\partial_0 N$ whose fundamental group has a global fixed points on T. According to Theorem 12.1, for each $g \in \pi_1(\partial_1(W))$ the translation length $\ell_T(g)$ is zero and g has a fixed point in T.

Proposition 16.5 implies that for every representation ρ_n as above and for any $g \in \pi_1(\Sigma) = \pi_1(\Sigma_1) \cup \pi_1(\Sigma_2) \subset G$ we have:

$$\text{either} \quad K^{-1} \le \ell(\rho_n(g))/\ell(\rho_n(\tau_*(g)) \le K \quad \text{or} \quad \ell(\rho_n(g)) = \ell(\rho_n(\tau_*(g)) = 0.$$

Therefore $\ell_T(g) = 0$ if and only if $\ell_T(\tau_* g) = 0$.

If Σ''' is a component of $\partial_0 N - \Sigma$, then the marked conformal structure on this surface does not change under quasiconformal deformations in $T_\Sigma(G)$. Thus (according to Theorem 4.84) its fundamental group has a global fixed point in T, hence $\Sigma''' \subset B$.

This implies that the surface B must be invariant (up to isotopy) under the involution $\tau : \Sigma \to \Sigma$; after composing τ with an isotopy of Σ we can assume that $\tau(B) = B$.

Proposition 17.1. $B = \partial_0 N$.

Proof: We define the operation π of *pushing through the window* on $\partial_0 W = \partial W \cap \partial_0 N$ as follows. Suppose that $\gamma \subset \partial_0 W$ is simple loop. Let $\pi_i : W_i \to S_i$ be the projection to the base of this interval bundle, we let π denote the union of these maps. Then $C_\gamma := \pi^{-1}(\pi(\gamma_i))$ is either a cylinder or a Moebius band bounded by γ. The cylinder (or the Moebius band) C_γ is the *trace of pushing*. Clearly the loops $\gamma, \pi(\gamma)$ are freely homotopic in N, thus the surface B is invariant under the *pushing*.

Suppose that $B \ne \partial_0 N$; then ∂B is contained in $\partial_2 W$ (according to Theorem 12.1), this finite collection of loops is invariant under both pushing and τ. Note that if we start pushing a loop in ∂B, then we never get a loop on $\partial_0 N - \Sigma$ since $\partial_0 N - \Sigma \subset B$ and B is pushing-invariant.

Assume first that none of the components of ∂B bounds an essential Moebius band in W. Then take a loop $\gamma_0 \subset \partial B$ and apply to it the alternating sequence of τ and *pushing* π:

$$\gamma_k = \underbrace{\pi \circ \tau \circ \ldots \circ \pi \circ \tau}_{k \text{ times}}(\gamma_0).$$

Since each γ_j is contained in ∂B, there is $k > 0$ so that $\gamma_k = \gamma_i$, where $i \ne k$. The union of traces of pushing of $\gamma_i, \gamma_{i+1}, \ldots, \gamma_{k-1}$ is a collection of annuli. This collection projects to an incompressible torus in N/τ which is not isotopic to any boundary component of N/τ. This contradicts the assumption that N/τ is atoroidal. Now consider the case when a loop $\gamma_0 \subset \partial B$ bounds an essential Moebius band $C_{\gamma_0} \subset W$. We apply an alternating sequence of τ and pushing to this loop as above. Then either some γ_k $(k > 0)$

again bounds an essential Moebius band $C_{\gamma_k} \subset W$, or (before meeting such loop) we get a cycle $\gamma_i = \gamma_k$ as before. In the latter case we repeat our argument above. In the former case we take the union of cylinders and Moebius bands

$$C_{\gamma_0} \cup \ldots \cup C_{\gamma_k}.$$

The image of this union in the manifold N/τ bounds a Klein bottle which is not isotopic to a boundary component. Contradiction. □

Let H_j denote the images of the fundamental groups of components of $\partial_0 N$ in $G = \pi_1(N)$. Each of these groups has a global fixed point in T. Thus, according to Theorem 12.93, the group G fixes a point in T as well. This contradicts Theorem 10.24. □

17.2. Geometry of the limit

1. Preliminaries and notation

Before starting the proof we will reorganize notation for the objects that will appear along the way. Start with the geometrically finite Kleinian group $\Gamma := G$. We have the quotient manifold $M = M(\Gamma) = \mathbb{H}^3/\Gamma$, the surface $\Omega(\Gamma)/\Gamma$ which contains a subsurface Σ (this surface consists of two components). From now on I will use the notation Σ', Σ'' for the components of Σ and $\tau : \Sigma' \to \Sigma''$ for the *gluing map*. Inside $M(\Gamma)$ we have a compact core M^c, which is the convex core of M with the open Margulis cusps removed. The partial boundary $\partial_0 M^c$ contains two compact incompressible subsurfaces S', S'' (they are compact cores of the images of Σ', Σ'' under the retraction $r : \Omega(\Gamma)/\Gamma \to \partial C\Lambda(\Gamma)/\Gamma$). Thus the surface Σ may have punctures and the surface $S = S' \sqcup S''$ may have boundary. Each surface Σ', Σ'' is the quotient $\Omega'/F', \Omega''/F''$, where Ω', Ω'' are simply-connected components of $\Omega(\Gamma)$, and the subgroups $F', F'' \subset \Gamma$ stabilizing these components are quasifuchsian. The surface $\Omega(\Gamma)/\Gamma$ may contain two more components, the notation that we reserve for any of them is: $\Sigma''' = \Omega'''/F'''$, the group F''' is quasifuchsian as well. The corresponding component of $\partial_0 M^c$ will be denoted S'''.

The rest of the boundary $Q = \partial_P M^c$ is a collection of annuli and tori, they correspond to the Margulis cusps. Let $\widehat{M}' := \mathbb{H}^3/F', \widehat{M}'' := \mathbb{H}^3/F''$, these manifolds cover M_0. Let $p = (p', p'') : \widehat{M} = \widehat{M}' \sqcup \widehat{M}'' \to M$ denote the covering map. Each of the manifolds $\widehat{M}', \widehat{M}''$ topologically is the product $\Sigma' \times \mathbb{R}, \Sigma'' \times \mathbb{R}$. This identification is chosen so that the Margulis cusps in $\widehat{M}'$, $\widehat{M}''$ correspond to the punctured disks in Σ multiplied by the $\mathbb{R}$–factor. Thus, both manifolds $\widehat{M}'$, $\widehat{M}''$ have exactly two geometrically finite ends: $\widehat{E}'_+, \widehat{E}'_-, \widehat{E}''_+, \widehat{E}''_-$. The difference between the *positive* and *negative* ends is that the *positive* ends 1-1 cover actual ends of the manifold M (which are the product regions bounded by the surfaces $\Sigma', r(\Sigma')$ and $\Sigma'', r(\Sigma'')$).

We have an isomorphism $\phi : F' \to F''$ (coming from the gluing map τ), it is induced by a quasiconformal homeomorphism $\tilde{f} : \bar{\mathbb{H}}^3 \to \bar{\mathbb{H}}^3$ such that $K_O(\tilde{f}) \leq K^3$ (this is a G-equivariant extension of the K-quasiconformal homeomorphism $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, see Theorem 8.21). The homeomorphism $\tilde{f}$ projects to a quasiconformal homeomorphism

$$\widehat{f} : \widehat{M} \to \widehat{M}.$$

The latter has the property that it "switches" negative and positive ends (image of negative is positive and vice-versa).

Recall that we have a sequence of representations

$$\rho_n : \Gamma \to \Gamma_n = \rho_n(\Gamma) \subset PSL(2, \mathbb{C})$$

whose restrictions to F', F'' we denote $\psi_n := (\psi_n', \psi_n'')$. The isomorphisms ρ_n are induced by quasiconformal homeomorphisms $\tilde{h}_n : \bar{\mathbb{H}}^3 \to \bar{\mathbb{H}}^3$. Thus $\tilde{h}_n$ project to quasiconformal homeomorphisms $h_n : M := M(\Gamma) \to M(\Gamma_n) = M_n$. Using these homeomorphisms we "transfer" the notation that we have for the manifold $M(\Gamma)$ (and groups, manifolds, maps etc. related to it) to the manifolds $M(\Gamma_n)$ by adding the subscript n. For instance, for each n we have a quasiconformal homeomorphism $\tilde{f}_n : \bar{\mathbb{H}}^3 \to \bar{\mathbb{H}}^3$, so that $K_O(\tilde{f}_n) \le K^3$. The mapping $\tilde{f}_n$ induces the isomorphism $\phi_n : F_n' \to F_n''$, so that $\phi_n \circ \psi_n' = \psi_n'' \circ \phi$. The manifolds $\widehat{M}_n', \widehat{M}_n''$ have *positive* and *negative* ends $\widehat{E}_{\pm n}', \widehat{E}_{\pm n}''$ which are switched by $\widehat{f}_n$. The compact manifolds M_n^c are defined as complements to the Margulis cusps (we use the same Margulis constant for each n) in the convex cores of $M(\Gamma_n)$.

The situation becomes more delicate when we pass to the algebraic limit:

$$\lim_{n\to\infty} \rho_n = \rho_\infty, \quad \rho_\infty : \Gamma \to \Gamma_\infty = \rho_\infty(\Gamma) \subset PSL(2, \mathbb{C}).$$

The main problem is that this limit (a priori) does not have to be *strong* (see §8.17), thus it is not obvious how to transfer the geometric/topological information from $M(\Gamma_n)$ to $M(\Gamma_\infty)$. The coefficients of quasiconformality of $\tilde{f}_n$ are uniformly bounded from above by K^3, thus (after taking a subsequence if necessary) we get a limit (see §5.2):

$$\lim_{n\to\infty} \tilde{f}_n = \tilde{f}_\infty.$$

(Note that we do not worry about normalization at three points in $\widehat{\mathbb{C}}$ since the mappings $\tilde{f}_n$ are ρ_n-equivariant and the homomorphisms ρ_n converge to ρ_∞.) By continuity, $\tilde{f}_\infty$ is quasiconformal, $K_O(\tilde{f}_\infty) \le K^3$ and $\tilde{f}_\infty \Gamma_\infty \tilde{f}_\infty^{-1} = \Gamma_\infty$. Now we introduce the "limiting" notation whenever it is possible and add the subscript ∞ to the old notation. Let

$$\psi_\infty = (\psi_\infty', \psi_\infty'') := \lim_{n\to\infty} (\psi_n', \psi_n''), (F_\infty', F_\infty'') := (\psi_\infty'(F'), \psi_\infty''(F'')).$$

and

$$\widehat{M}_\infty = \widehat{M}_\infty' \cup \widehat{M}_\infty'' = \mathbb{H}^3/F_\infty' \cup \mathbb{H}^3/F_\infty''$$

$\phi_\infty := \psi_\infty'' \circ \phi \circ (\psi_\infty')^{-1}$ is an isomorphism between F_∞' and F_∞'' which is induced by $\tilde{f}_\infty$.

The isomorphism ρ_∞ maps parabolic elements to parabolic, however the problem is that ρ_∞ (a priori) can have accidental parabolic elements.

We will see that it suffices to consider only accidental parabolic elements for ψ_∞. So, as in §14.7, we take $\widehat{C}_\infty$ to be the union of those (open) Margulis cusps in $\widehat{M}_\infty$ which correspond to the cusps in $\widehat{M}$. Let $\widehat{A}_\infty$ denote the rest of Margulis cusps in $\widehat{M}_\infty$, we call them *accidental* since they correspond to accidental parabolic elements in the groups F'_∞, F''_∞. These cusps project to the Margulis cusps $A_\infty \subset M_\infty$.

How many accidental cusps we may have? According to our calculations in §4.23, there could be not more than $4(3g-4+m)$ of them, where g is the genus of S' and m is the number of its boundary components. Clearly accidental cusps in $\widehat{M}'$ are "pared" with the accidental cusps in $\widehat{M}''$, since there is an quasiconformal homeomorphism from the one manifold to the other. Thus we get a collection of $2k \le 4(3g-4+m)$ primitive elements

$$\gamma'_{(\pm j)} \in F', \quad \gamma''_{(\pm i)} \in F''$$

which represent simple nonperipheral loops on S', S'' and are mapped to accidental parabolic elements via ρ_∞. Let $\Pi = \Pi^+ \sqcup \Pi^- \subset F' \sqcup F''$ denote the union of F' and F''-conjugacy classes of these elements. (We will see in a moment how to assign $\pm$ signs to the ends on $\widehat{M}$. According to this labeling we will partition Π into Π^+ and Π^- as in §4.23.) Clearly the involution

$$\tau_* = (\phi, \phi^{-1}) : (F', F) \to (F'', F')$$

just permutes the elements of Π.

Recall that the manifolds M_n may have other ends besides $\widehat{E}_{\pm n}$. Namely, the quasiconformal homeomorphisms $\tilde{h}_n$ conjugating Γ to Γ_n are conformal on the components Ω''' of $\Omega(\Gamma)$ corresponding to surfaces $\Sigma''' = \Omega'''/F''' \subset \Omega(\Gamma)/\Gamma - \Sigma$. Let $\tilde{h}'''_n$ denote the restriction of $\tilde{h}_n$ to Ω'''. Then the conformal mappings $\tilde{h}'''_n$ are subconvergent uniformly on compacts to a conformal mapping $\tilde{h}'''_\infty : \Omega''' \to \Omega'''_\infty$. Thus the group $F'''_\infty = \rho_\infty(F''')$ acts properly discontinuously on Ω'''_∞ and the quotient $\Sigma'''_\infty := \Omega'''_\infty/F'''_\infty$ is naturally biholomorphic to Σ'''. Let G_∞ denote the geometric limit of the groups Γ_n (we may have to pass to a subsequence first). Recall that $G_\infty \subset PSL(2,\mathbb{C})$ is a discrete subgroup containing Γ_∞, see §8.2. The fact that we have (uniform on compacts) convergence of the mappings $\tilde{h}'''_n$ implies that each Ω'''_∞ is precisely invariant under the subgroup F'''_∞ in G_∞. Finally, the surface Σ'''_∞ projects to a surface $r_\infty(\Sigma'''_\infty) \subset C(\Lambda(G_\infty)/G_\infty)$ under the nearest-point retraction r_∞; the end E'''_∞ of the manifold $M(G_\infty)$ bounded by $r_\infty(\Sigma'''_\infty)$ and by the surface at infinity Σ'''_∞ is geometrically finite. All cusps in this end correspond to punctures on the surface $\Sigma'''_\infty \cong \Sigma'''$. In particular, if the groups F'_∞, F''_∞ have some accidental parabolic elements, then the corresponding Margulis cusps are disjoint from E'''_∞ (provided that the Margulis constant is chosen sufficiently small).

Thus, the geometry of the ends of M_n which correspond to Σ''' does not change much as we take the limit. This means that we will concentrate mainly on the ends corresponding to $\widehat{E}_{\pm n}$.

According to Theorem 14.17, the manifold $\widehat{M}^1_\infty := \widehat{M}_\infty - C_\infty$ is homeomorphic to the product $S \times \mathbb{R}$, where $S = S' \cup S'' \subset \partial_0 M^c$ (recall that this is a compact surface which may have boundary) and we assume that $S \times [0, +\infty)$ corresponds to $\widehat{E}_{+\infty}$ and $S \times (-\infty, 0]$ corresponds to $\widehat{E}_{-\infty}$ under the orientation-preserving homeomorphism $\widehat{M}^0 \to \widehat{M}^1_\infty$ which induces the map ψ_∞ of the fundamental groups. In particular, this determines the *positive* and the *negative* labels of the ends of $\widehat{M}^1_\infty$. Now we can use Lemma 14.28 to construct immersions

$$\lambda_n : M^c_\infty \to M_n \quad \widehat{\lambda}_n : \widehat{M}^c_\infty \to \widehat{M}_n$$

where $\widehat{M}^c_\infty$, M^c_∞ are compact cores in $\widehat{M}^1_\infty, M_\infty$, so that:

- $\widehat{\lambda}_n$ is a lift of λ_n.
- The lifts of $\widehat{\lambda}_n$ to the universal covers are C^1-convergent to the identity uniformly on compacts.

Warning 17.2. Obviously $\widehat{M}^c_\infty$ does not cover M^c_∞, unless M^c is an interval bundle over a surface.

Identify a compact core $\widehat{M}^c_\infty$ with $S \times [-1, 1]$ and the surface S with $S \times \{0\}$. Assuming that $\widehat{M}^c_\infty$ contains $S = S \times \{0\}$ we get a sequence of maps $\widehat{\lambda}_n : S \to \widehat{M}_n$ which are homeomorphic embeddings near the boundary of S. According to our analysis in §14.7, for sufficiently large n we can find embeddings $\widehat{e}_n : S \times \{0\} \to \widehat{M}_n$ in the 1-neighborhood of each $\widehat{\lambda}_n(S)$ so that:

(a) $\widehat{e}_n$ is homotopic to $\widehat{\lambda}_n$ (rel. boundary).

(b) The compositions $e_n := p_n \circ \widehat{e}_n$ are embeddings as well.

Thus the surfaces $\widehat{e}_n(S)$ split $\widehat{M}_n$ into *positive* and *negative* ends. In the end of §14.7 we have established a natural 1-1 correspondence between the ends of the manifold $\widehat{M}^0_n$ and the topological ends of the manifold $\widehat{M}^1_\infty$ so that $\widehat{\lambda}_n$ sends positive ends to positive and negative ends to negative in the following sense:

$$\widehat{\lambda}_n(S \times \{+1\}) \subset \widehat{E}_{+n}, \quad \widehat{\lambda}_n(S \times \{-1\}) \subset \widehat{E}_{-n}.$$

Note that this labeling (a priori) has nothing to do with topology of the manifold M_∞, since its compact core is only homotopy-equivalent to the compact core M^c_n and does not have to be homeomorphic, see [AC96]. (For instance we do not yet know that the *positive* ends of $\widehat{M}^1_\infty$ cover the ends of M^c_∞ with finite multiplicity.)

The $\pm$ labeling of the ends is consistent with the orientation: the surfaces S', S'' are canonically oriented as well as the manifold $\widehat{M}^c_\infty$, if the end $\widehat{E}'_+$ lies to the "right" from $S' \subset \widehat{M}^c_\infty$, then the end $\widehat{E}'_{+\infty}$ lies to the "right" from the corresponding surface $S' \times \{0\} \subset \widehat{M}^c_\infty$.

The limiting map $\widehat{f}_\infty$ switches positive to negative ends and vice-versa because the distance between the mappings

$$\widehat{\lambda}_n \circ \widehat{f}_\infty, \quad \widehat{f}_n \circ \widehat{\lambda}_n$$

tends to zero as $n \to \infty$ and the mapping $\widehat{f}_n$ switches the ends of $\widehat{M}_n$. Accordingly, the *accidental cusps* in $\widehat{A}_\infty$ group into the *positive cusps* $\widehat{A}_{+\infty} = \widehat{A}'_{+\infty} \cup \widehat{A}''_{+\infty}$ which belong to $\widehat{E}_{+\infty}$ and the *negative cusps* $\widehat{A}_{-\infty} = \widehat{A}'_{-\infty} \cup \widehat{A}''_{-\infty}$ which belong to $\widehat{E}_{-\infty}$. The map $\widehat{f}_\infty$ sends parabolic elements to parabolic and does not mix accidental and regular parabolic elements. (Since it induces τ_* which maps parabolic elements of F' to parabolic elements of F''.) Thus (up to isotopy) the homeomorphism $\widehat{f}_\infty$ will map the *positive* accidental cusps to the *negative* accidental cusps. Note that the fundamental group of each *negative* accidental cusp in $\widehat{E}_{-\infty}$ is not conjugate in $\pi_1(\widehat{M}_\infty)$ to the fundamental group of any *positive* cusp since they are disjoint (see Lemma 4.62).

What happens in the "pre-limit" hyperbolic manifolds $\widehat{M}_n$ with the *accidental* cusps? They correspond to the Margulis tubes around very short geodesics (whose lengths tend to zero as $n \to \infty$); for these tubes we shall use the notation

$$\widehat{A}_n = \widehat{A}'_{+n} \cup \widehat{A}''_{+n} \cup \widehat{A}'_{-n} \cup \widehat{A}''_{-n}.$$

Inside the manifold M^1_∞ we have a compact core M^c_∞ of the diameter d. Let ν be the injectivity radius at M^c_∞. Recall that M^1_∞ was constructed given the Margulis constant μ_3. A priori this compact core can nontrivially intersect the union of accidental Margulis cusps A_∞. However we can choose a smaller constant $\epsilon < \mu_3$ so that:

- The $(d+2)$-neighborhood of M^c_∞ is disjoint from $A_\infty \cap M(\Gamma_\infty)_{(0,\epsilon]}$.
- $\mu_3 > \nu > \theta(K^{6k}, \epsilon)$, where $2k$ is the cardinality of Π and θ is the function introduced in §3.11 (this function describes distortion of the hyperbolic metric under quasiconformal mappings).

We shrink the Margulis cusps A_∞ to the corresponding components $A_\infty \cap M(\Gamma_\infty)_{(0,\epsilon]}$. To simplify the notation we will retain the notation A_∞ for the smaller accidental cusps. Similarly we reduce μ_3 to ϵ for the Margulis tubes $A_n \subset M_n$, which converge to A_∞. Again we retain the old notation for the new (smaller) Margulis tubes. However we retain the old Margulis constant for the regular cusps of $\widehat{M}_n, \widehat{M}_\infty$. Thus the 2-neighborhood of $\lambda_n(M^c_\infty)$ is disjoint from A_n for all sufficiently large n. Moreover, for each $i \leq 2k$ we have:

$$p_n(\widehat{f}^i_n(\widehat{A}_n)) \cap Nbd_2(\lambda_n(M^c_\infty)) = \emptyset$$

since $K(\widehat{f}^i_n) \leq K_O(\widehat{f}^i_n) \leq K^{3i}$ and the distortion of hyperbolic length by quasiconformal mappings is bounded from above according to the estimates of §3.11. Thus $p_n(\widehat{f}^i_n(\widehat{A}_n))$ is also disjoint from the embedded surfaces $e_n(S'), e_n(S'')$; therefore each component of $p_n(f^i(\widehat{A}_n))$ is forced to be contained in one of the ends E'_{+n}, E''_{+n} bounded by $e_n(S'), e_n(S'')$.

Now, at last, we are ready to discuss the idea of the proof.

2. Idea of the proof

We have to prove that the pair of isomorphisms

$$(\psi'_\infty, \psi''_\infty) : (F', F'') \to (F'_\infty, F''_\infty)$$

has no accidental parabolic elements and the groups in the image are geometrically finite.

I will outline the proof assuming that $Q = \emptyset$ (i.e. Γ has no cusps), $\partial M^c = S \cong \Sigma$ and the manifold M^c is acylindrical. Let M^c_∞ be Scott compact core of the manifold $\mathbb{H}^3/\Gamma_\infty$. Theorem 1.31 implies that the isomorphism $\rho_\infty : \pi_1(M^c) \to \Gamma_\infty = \pi_1(M^c_\infty)$ is realized by a homeomorphism

$$h_\infty : M^c \to M^c_\infty.$$

In particular, $h_\infty(S) = S_\infty$ consists of the pair of boundary surfaces of M^c_∞, whose union is the entire boundary of this manifold.

The manifold $\widehat{M}_\infty$ has four ends: $\widehat{E}'_{\pm\infty}, \widehat{E}''_{\pm\infty}$ (see §1). Since the sequences of maps $\tilde{\lambda}_n$ converge to the identity uniformly on compacts, we conclude that the *positive* ends of $\widehat{M}_\infty$ 1-1 cover ends of the manifold $M_\infty = M(\Gamma_\infty)$. The quasiconformal homeomorphism $\widehat{f}_\infty$ switches positive and negative ends of $\widehat{M}_\infty$.

Suppose that $\widehat{M}'_\infty$ has cusps on the positive side, thus $\widehat{M}''_\infty$ must have cusps on the negative side as well. Suppose that such cusp $\widehat{C}^- \subset \widehat{E}_{-\infty} \subset \widehat{M}''_\infty$ projects to a cusp in the end of M_∞ bounded by S'_∞. Then $\pi_1(\widehat{C}^-) \subset \pi_1(S''_\infty) = F''_\infty$ is conjugate (in Γ_∞) to an infinite cyclic subgroup of $\pi_1(S'_\infty) = F'_\infty$, which is impossible since M^c is acylindrical. Suppose that $\widehat{C}^-$ projects to a cusp C^- in the *positive* end $E''_{+\infty}$ of M_∞ bounded by S''_∞. Recall that the covering $\widehat{E}''_{+\infty} \to E''_{+\infty}$ is 1-1 and, hence, we can lift C^- to a cusp $\widehat{C}^+$ in the positive end $\widehat{E}''_{+\infty}$. The cyclic groups $\pi_1(\widehat{C}^+), \pi_1(\widehat{C}^-) \subset F''_\infty$ cannot be conjugate in F''_∞ since these cusps are disjoint (see Lemma 4.62). However, they both project to C^+, hence these cyclic groups are conjugate in Γ_∞ which contradicts our assumption that M^c is acylindrical.

But where the projection C^- can be in M_∞? It cannot lie in any of the two ends and it cannot lie in the compact M^c_∞. Therefore we are left with only one possibility: there are no cusps on the positive side of $\widehat{M}_\infty$. The same argument shows that there are no cusps on the negative side of $\widehat{M}_\infty$ as well. So we conclude that the groups F'_∞, F''_∞ have no parabolic elements at all.

Remark 17.3. Clearly the above arguments cannot be repeated verbatim in the case when M^c is not acylindrical, thus instead of working with the Margulis cusps in the limit M_∞ we will give an argument which works with the Margulis tubes in M_n. See §3 below.

The next stage of the proof is to establish that the groups F'_∞, F''_∞ are geometrically finite. Theorem 14.24 implies that the negative ends $\widehat{E}'_{-\infty}, \widehat{E}''_{-\infty}$ cannot be geometrically infinite since the fundamental group of M^c is not a finite extension of the surface group. The limiting quasiconformal homeomorphism $\widehat{f}_\infty$ switches positive and negative ends $\widehat{E}_{\pm\infty}$. This implies that all ends of $\widehat{M}_\infty$ are geometrically finite. Thus F'_∞, F''_∞ are both geometrically finite and have no parabolic elements. Therefore they are quasifuchsian groups and hence belong to $\mathcal{T}(F'), \mathcal{T}(F'')$.

We conclude that the sequence $\sigma^n(X) = \beta^{-1}[\rho_n]$ converges in $\mathcal{T}(\Sigma)$, hence we are done.

3. The limit has no accidental parabolics

Theorem 17.4. *The subgroups F'_∞, F''_∞ of Γ_∞ have no accidental parabolic elements.*

Proof: Suppose that the set of accidental parabolic loops Π is not empty. To simplify the discussion let me first assume that Π consists of exactly one positive loop $[\gamma_+] \in \Pi'_+$ and one negative loop $[\gamma_-] \in \Pi''_-$, where $\gamma_\pm$ denote the elements of F', F'' which represent the corresponding loop. Since τ_* switches Π_+ and Π_-, we conclude that $\tau_*(\gamma_+) = \gamma_-$ and $\tau_*(\gamma_-) = \gamma_+$. Start with the Margulis tube $\widehat{A}_{+n} \subset \widehat{M'_n}$ which corresponds to γ_+, the image $p_n(\widehat{f}_n \widehat{A}_{+n})$ is contained either in E'_{+n} or in E''_{+n} (see our discussion in Section 1). In any case there exists an isometric lift of $p_n(\widehat{f}_n \widehat{A}_{+n})$ to $\widehat{E}_{+n}$, which must belong to the $\theta(K^3, \epsilon)$-thin part of $\widehat{E}_{+n}$, i.e. is contained in one of the Margulis tubes $\widehat{B}_{+n}$. Algebraically this means the following. The element $\tau_*(\gamma_+) = \gamma_-$ is conjugate to an element α_+ which belongs either to F' or to F'' (depending on in which of the two ends E'_{+n} or E''_{+n} the tube $p_n(\widehat{f}_n \widehat{A}_{+n})$ is). This element is nonperipheral in F', F''. Let me consider first the case $\alpha_+ \in F'$ (see Figure 17.1). The F'-conjugacy class of α_+ depends only on topology on M^c and does not depend on the number n at all. So, let us choose α_+ within its conjugacy class once and for all (independently on n). The translation length $\ell(\rho_n(\alpha_+)) \neq 0$ is at most $K\ell(\rho_n(\gamma_+))$, hence it tends to zero as $n \to \infty$. Therefore α_+ is an accidental parabolic element for ρ_∞ (i.e. $\rho_\infty(\alpha_+)$ is parabolic). However we have a complete list of (conjugacy classes) of accidental parabolic elements: we conclude that $\alpha_+ = (\gamma_+)^s$ (up to conjugation in F'). The elements γ_+, γ_- are primitive in the groups F', F''. Thus Lemma 1.52 implies that $s = \pm 1$.

Recall that to get a presentation for the HNN extension $\pi_1(M^c/\tau)$ of $\Gamma = \pi_1(M^c)$, we add to the list of generators in Γ a letter t and add the relations:

$$tgt^{-1} = \phi(g), g \in F'.$$

Therefore in the group $\pi_1(M^c/\tau)$ we get a trivial Baumslag-Solitar relation

$$t\gamma_+ t^{-1} = (\gamma_+)^s.$$

This means that we have got a nonperipheral subgroup $\mathbb{Z} \times \mathbb{Z} \subset \pi_1(M^c/\tau)$ (which has finite index in the subgroup generated by t, γ_+), this contradicts the assumption that M^c/τ is atoroidal.

Our argument in the case $\alpha_+ \in F''$ is similar. After conjugation in F'' the element α^+ equals $(\gamma_-)^s$. However the Margulis tube corresponding to γ_- is contained in the *negative* part of $\widehat{M''_n}$, meanwhile the Margulis tube $\widehat{B}_{+n}$ corresponding to α^+ must be contained in the "positive part". This is impossible.

Now we consider the general case (when the cardinality of Π does not have to be 2). Let $\Pi_+ = \{\gamma_{+1}, ..., \gamma_{+k}\}, \Pi_- = \{\gamma_{-1}, ..., \gamma_{-k}\}$. The argument remains pretty much the same. We start with the Margulis tube $\widehat{A}^1_{+n} \subset \widehat{M'_n}$

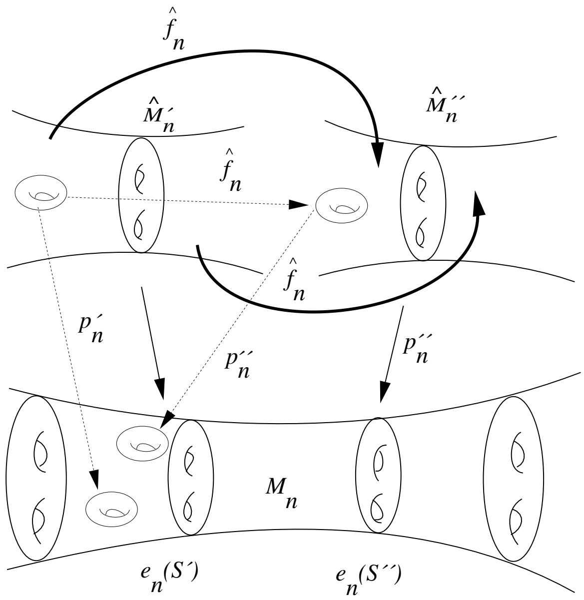

Figure 17.1: *Projections of Margulis tubes.*

which corresponds to γ_{+1}. Then $p_n(\widehat{f}_n \widehat{A}^1_{+n})$ is contained either in E'_{+n} or in E''_{+n}. Suppose that it is contained in E'_{+n}. We lift it isometrically into one of the Margulis tubes $\widehat{B}^1_{+n}$ which is contained in the $\theta(K^3, \epsilon)$-thin part of $\widehat{E}'_{+n}$. Algebraically this corresponds to the fact that $\gamma_{-1} := \tau_*(\gamma_{+1})$ is conjugate in Γ to an accidental parabolic element $\alpha_{+1} = (\gamma_{+i_1})^{s_1}$, where $\gamma_{+i_1} \in F'$ (for some i_1).

Both loops $[\alpha_{+1}], [\gamma_{-1}]$ are simple, they are homotopic in M^c but not in $\partial_0 M^c$, so we connect them by an essential annulus $L_1 \subset M^c$. If $i = 1$ then we are done as above. Otherwise we let $i = 2$ and apply the same procedure as before to the element γ_{+2}, etc. On each step $j \leq k$ of induction, $p_n(\widehat{f}_n \widehat{B}^j_{+n})$ is contained in the $\theta(K^{3j}, \epsilon)$-thin part of M_n, therefore it is in one of the product ends E'_{+n}, E''_{+n} and can be isometrically lifted to $\widehat{E}_{+n}$. After at most k iterations we get stuck: $\alpha_{+i} = \gamma_{+j}$ (for certain choices of i and j). This means that inside of the manifold M^c/τ we have constructed an incompressible nonperipheral torus or Klein bottle (see Figure 17.2); this contradicts the assumption that M^c/τ is atoroidal. Therefore $\psi'_\infty, \psi''_\infty$ do not have accidental parabolic elements. $\square$

4. The endgame: the limit is geometrically finite

We proved that the groups F'_∞, F''_∞ do not have accidental parabolic elements. We know that the limiting quasiconformal homeomorphism $\tilde{f}_\infty$ conjugates F'_∞ to F''_∞.

There are several way to argue now. **The 1-st argument.** According to Theorem 8.67, the convergence $\rho_n \to \rho_\infty$ is strong.

Then our discussion in Section 1 implies that the isomorphism ρ_∞ :

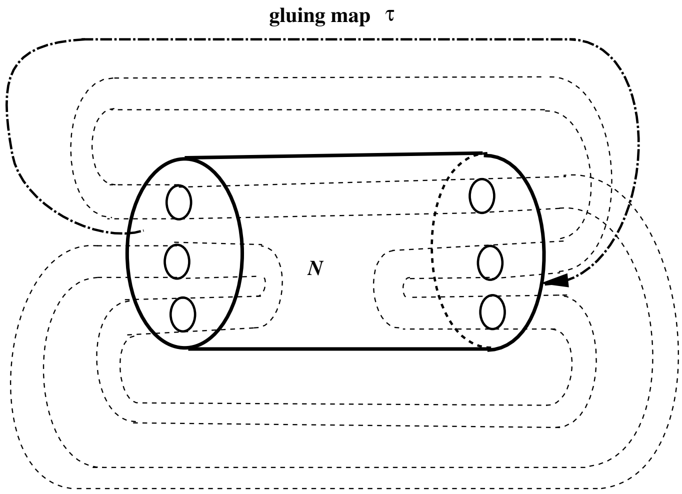

Figure 17.2: *Construction of a torus or a Klein bottle.*

$\Gamma \to \Gamma_\infty$ preserves the peripheral structure. Hence the ends $\widehat{E}'_{+\infty}, \widehat{E}''_{+\infty}$ project 1-1 to the ends of the manifold M_∞. Since the peripheral subgroups F', F'' do not have finite index in Γ it follows that the restriction of the covering $\widehat{p}_\infty : \widehat{E}_- \to M_\infty$ has infinite multiplicity in both $\widehat{E}'_{-\infty}, \widehat{E}''_{-\infty}$. Thus, according to Theorem 14.24, the ends $E'_{-\infty}, E''_{-\infty}$ are geometrically finite. Suppose that one of the two other ends (say $\widehat{E}'_{+\infty}$) is geometrically infinite. The limiting quasiconformal map $\widehat{f}_\infty$ sends $\widehat{E}_{+\infty}$ to $\widehat{E}_{-\infty}$, hence $\widehat{E}_{-\infty}$ is geometrically infinite. Contradiction, see Figure 17.3. Similarly, the end $\widehat{E}''_{+\infty}$ is geometrically finite as well, which implies that both groups F'_∞, F''_∞ are geometrically finite and we are done.

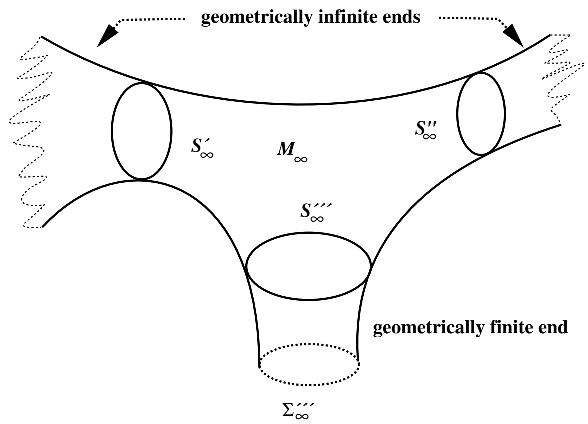

Figure 17.3: *Ends of the limiting manifold.*

The 2-nd argument. We apply Theorem 8.67 to the restrictions of the homomorphisms ρ_n to the peripheral (quasifuchsian) subgroups. Then the end $\widehat{E}'_{+\infty}$ (resp. $\widehat{E}''_{+\infty}$, $\widehat{E}'_{-\infty}, \widehat{E}''_{-\infty}$) is geometrically finite if and only if the corresponding sequence of marked Riemann surfaces Ω'_n/F'_n converges in $\mathcal{T}(\Omega'/F')$. Thus, if $\widehat{E}'_{+\infty}$ is geometrically finite, then it 1-1 covers an end $E'_{+\infty}$ (the argument is essentially the same as in the case of the surfaces S'''). Whence the end $\widehat{E}'_{-\infty}$ cannot be geometrically infinite (see Theorem 14.24). Thus the group F'_∞ is geometrically finite, it is quasiconformally conjugate to the group F'_∞, which implies that F'_∞ is geometrically infinite as well.

The 3-rd argument. This argument does not rely upon theorems about strong convergence, however our arguments will be similar to the ones that are used in the proof of Theorem 8.67. Let G_∞ denote the geometric limit of Γ_n (after passing to a subsequence if necessary this limit always exists). The group G_∞ is discrete according to Proposition 8.9. Our goal is to show that the ends $\widehat{E}'_{-\infty}, \widehat{E}''_{-\infty}$ are geometrically finite. Consider the covering $q : \widehat{M}_\infty \to M(G_\infty)$ (induced by the inclusion $\Gamma_\infty \hookrightarrow G_\infty$). Suppose that the end $\widehat{E}'_{-\infty}$ is geometrically infinite, then the restriction of q to this end is m-to-1.

Pick a base-point $x_\infty \in M(G_\infty)$, which is the projection of a point in $S \times \{0\} \subset \widehat{M}'_\infty$. According to Theorem 8.11 there is a choice of base-points $x_n \in M_n$ so that the manifolds (M_n, x_n) converge to $(M(G_\infty), x_\infty)$ in the refined Gromov-Hausdorff topology. I.e. for any sufficiently large R and $\kappa > 1$ there exists i_0 and an open neighborhood U of the R-ball in $M(G_\infty)$ centered at x_∞ so that for each $i \geq i_0$ there is a map $\alpha_i : (U_\infty, x_\infty) \to (M(G_i), x_i)$ which is κ-bilipschitz diffeomorphism onto its image. Moreover, for sufficiently large R the map α_i induces the homomorphism ρ_i^{-1} on the subgroup $\Gamma_\infty \subset G_\infty$. The restriction of the covering p'_i to $\widehat{E}'_{-i}$ has infinite multiplicity. Hence the multiplicity of the map

$$\alpha_i \circ q | Nbd_R(S \times \{0\}) \cap \widehat{E}'_{-\infty} \tag{17.1}$$

is unbounded as $i \to \infty, R \to \infty$. On the other hand, for each given R and $i > i_0$ the restriction $\alpha_i | B_R(x_\infty)$ is injective, thus geometric infiniteness of $\widehat{E}'_{-\infty}$ implies that the multiplicity of the map (17.1) is at most m. Contradiction. Thus the end $\widehat{E}'_{-\infty}$ is geometrically finite. Similarly, the end $\widehat{E}''_{-\infty}$ is geometrically finite as well. However the mapping $\widehat{f}_\infty$ sends $\widehat{E}_{+\infty}$ to $\widehat{E}_{-\infty}$ which implies that all the ends of $\widehat{M}_\infty$ are geometrically finite. This finishes the proof of the Bounded Image Theorem. $\square$

Chapter 18

Hyperbolization of Fibrations

18.1. Compactness theorem for aperiodic homeomorphisms

Let S be a Riemann surface of finite hyperbolic type, $\Gamma = \pi_1(S)$. We assume that Γ is embedded in $PSL(2,\mathbb{R})$ is a Fuchsian group so that $S = \mathbb{H}^2/\Gamma$. Recall that $\bar{S} = (\widehat{\mathbb{C}} - cl(\mathbb{H}^2))/\Gamma$ has the same marked hyperbolic structure as S and the opposite orientation. These surfaces are identified via projection of the complex conjugation $z \mapsto \bar{z}$ to a map $S \to \bar{S}$. There is a continuous length function

$$\ell : \mathcal{T}(S) \times \mathcal{ML}(S) \to \mathbb{R}$$

which extends the length function for the closed geodesics in S, see §11.16. Suppose that

$$(\psi_n^+, \psi_n^-) \in \mathcal{T}(S) \times \mathcal{T}(\overline{S})$$

is a sequence and (λ^+, λ^-) is a pair of elements in $\mathcal{ML}(S) \times \mathcal{ML}(\overline{S})$ so that there is a number $L < \infty$ such that for all $n \geq 0$

$$\ell(\psi_n^\pm, \lambda^\pm) \leq L.$$

Consider the sequence $[\rho_n] \in \mathcal{T}(\Gamma)$ which corresponds to (ψ_n^+, ψ_n^-) under the Bers isomorphism.

Theorem 18.1. *(Double Limit Theorem, W. Thurston, J.-P. Otal.) Suppose that the pair of measured geodesic laminations λ^+, λ^-* **binds** *the surface S. Then the sequence $[\rho_n]$ is subconvergent in the character variety $\mathcal{R}(\Gamma, PSL(2,\mathbb{C}))$.*

In the next section I will give an outline of the proof, for the details the reader is referred to [Ota96] and [Thu87a].

Now we shall see how Theorem 15.16 can be deduced from the Double Limit Theorem. Suppose that $\tau : S \to S$ is an aperiodic homeomorphism. It has stable and unstable measured geodesic laminations λ^+, λ^- which are binding the surface S. Recall that there is a constant $k > 1$ so that $\tau(\lambda^{\pm}) = k^{\pm 1}\lambda^{\pm}$. Length of the lamination $\tau(\lambda^{\pm})$ in the marked hyperbolic surface $\psi_n^{\pm} := \tau^{\pm n}(S)$ is the same as the length $L^{\pm} = \ell_S(\lambda^{\pm})$ of $\lambda^{\pm}$ on S. Thus the length of $\lambda^{\pm}$ in $\psi_n^{\pm}$ is $L^{\pm}/k^n$, which tends to zero as $n \to \infty$. So we are in the situation when we can apply Theorem 18.1. $\square$

18.2. The double limit theorem: an outline

We start the outline with the example where $\lambda^+ = \alpha$, $\lambda^- = \beta$ is a binding pair of simple closed geodesics in S. We will regard α, β as elements of the fundamental group Γ of S. Suppose that the sequence of conjugacy classes of representations $[\rho_n] \in \mathcal{D}_{par}(\Gamma, SL(2,\mathbb{C}))$ is not precompact in the character variety. Then (after rescaling by suitable factors D_n) the sequence $\{\rho_n\}$ subconverges to a minimal, small, relatively elliptic and nontrivial action of the group Γ on a tree T (see Theorem 10.24). Theorem 4.84 claims that the translation lengths $\ell(\rho_n(\alpha))$ and $\ell(\rho_n(\beta))$ are uniformly bounded from above by $2L$. However, according to the construction of the tree T, this implies that both α and β have fixed points on T:

$$\ell_T(\alpha) = \lim_{n\to\infty} \ell(\rho_n(\alpha))/D_n = 0, \quad \ell_T(\beta) = \lim_{n\to\infty} \ell(\rho_n(\beta))/D_n = 0$$

since $D_n \to \infty$. It follows from Theorem 11.34 that there is a (maximal elliptic) subsurface $B \subset S$ which contains homotopy classes of all loops whose representatives in $\pi_1(S) = \Gamma$ have fixed points in T. Our assumption that the pair α, β is binding implies that $S = B$ and therefore Γ has a global fixed point in T. Contradiction. $\square$

Now we consider the general case. Suppose that the sequence $[\rho_n]$ is not precompact in the character variety. Recall that for each representation $\rho_n : \Gamma \to PSL(2,\mathbb{C})$ we have a number $D_n = D_{\rho_n}$ such that after rescaling $\mathbb{H}^3$ by D_n^{-1}, the sequence of representations ρ_n is subconvergent to the action of the group Γ on a tree T. We identified the space of projective classes of laminations $\mathcal{PML}(S)$ with a "unit sphere" Θ_S in $\mathcal{ML}(S)$ (Θ_S consists of normalized laminations $\mu/p(\mu)$, see §11.15). The subset $\mathcal{C}(S)$ of simple closed geodesics in S is embedded into Θ_S as: $\gamma \mapsto \gamma/p(\gamma)$. Let Θ_S^0 denote the image. According to Theorem 11.25 the subset Θ_S^0 is dense in Θ_S. The translation length $\ell_\rho : \Gamma \to \mathbb{R}$ defines a function

$$\ell : \Theta_S^0 \times \mathcal{T}(\Gamma) \to \mathbb{R}$$

by the formula

$$\ell(\gamma/p(\gamma), \rho) = \ell(\rho(\gamma))/[p(\gamma) \cdot D_\rho].$$

Theorem 18.2. *There exists a continuous extension of the function ℓ to*

$$\ell : \Theta_S \times \mathcal{T}(\Gamma) \to \mathbb{R}.$$

Proof: The function D_ρ depends continuously on the conjugacy class of the representation ρ, thus it suffices to show that the function $\ell(\rho(\gamma))/p(\gamma)$ admits a continuous extension. (F. Bonahon in [Bon97] actually proves that the extension is in a certain sense Hölder.) The proof of this fact requires detailed discussion of pleated surfaces, we refer the reader to [EM87], [Bon97], [Ohs98] and [Bro98] for the proof of this result and its generalization to the case of the function

$$\ell : \Theta_S^0 \times \mathcal{D}_{par}(\Gamma, PSL(2,\mathbb{C})) \to \mathbb{R}. \quad \square$$

Recall that $\mathcal{P}Tree(\Gamma)$ is the projectivization of the space $\mathcal{T}ree(\Gamma)$ which consists of small, minimal and relatively elliptic actions of the group Γ on metric trees. Every such action is uniquely determined by the corresponding translation length function $\ell_T : \Gamma \to \mathbb{R}$. The space $\mathcal{T}ree(\Gamma)$ has the topology of convergence of the length functions on a finite set of elements of Γ, and it is naturally homeomorphic to $\mathcal{ML}(S)$ (see §11.12). Embed the space $\mathcal{T}(\Gamma)$ into the space

$$\mathcal{D}_{par}(\Gamma, PSL(2,\mathbb{C})) \cup \mathcal{T}ree(\Gamma)$$

(see §10.4) and let $\widehat{\mathcal{T}}(\Gamma)$ denote the union of $\mathcal{T}(\Gamma)$ with its accumulation set in $\mathcal{T}ree(\Gamma)$. Note that the projectivization of $\widehat{\mathcal{T}}(\Gamma) - \mathcal{T}(\Gamma)$ is $\mathcal{P}\mathcal{T}ree(\Gamma)$ (see Corollary 11.51). The main technical result of Otal [Ota96] [1] is the following:

Theorem 18.3. *For each $\theta \in \Theta_S$ the function $\ell(\theta, \bullet)$ admits an extension $\bar{\ell} : \widehat{\mathcal{T}}(\Gamma) \to \mathbb{R}$ which is lower semicontinuous at each point $T_\lambda \in \mathcal{T}ree(\Gamma)$. Moreover, $\bar{\ell}(\theta, T_\lambda) = i(\theta, \lambda)$ is the geometric intersection number.*

Now we can deduce the *Double Limit Theorem* from the above information. Recall that for each $(\psi_n^+, \psi_n^-) \in \mathcal{T}(S) \times \mathcal{T}(\overline{S})$ (which corresponds to $[\rho] \in \mathcal{T}(\Gamma)$), and for each simple closed geodesic γ on the marked hyperbolic surface (S, ψ_n^+) (and $(\bar{S}, \psi_n^-)$), the translation length of the $\rho(\gamma)$ on $\mathbb{H}^3$ is at most $2Length_{\psi_n^\pm}(\gamma)$, according to Theorem 4.84. (Here we regard γ as an element of $\pi_1(\bar{S}) = \Gamma$ and $Length_{\psi_n^\pm}$ is the length with respect to the hyperbolic metric corresponding to $\psi_n^\pm$.) On the other hand, multiples of simple closed geodesics are dense in Θ_S and the length function $\ell : \Theta_S \times \mathcal{T}(\Gamma) \to \mathbb{R}$ is continuous. Thus

$$\ell(\lambda^\pm, [\rho_n]) \le 2L.$$

Hence, after taking a subsequence, we get:

$$\lim_{n\to\infty} D_n^{-1}[\rho_n(\Gamma) \curvearrowright \mathbb{H}^3] = [G \curvearrowright T_\xi]$$

where $T = T_\xi$ is a Γ-tree dual to a measured geodesic lamination ξ and $D_n := D_{\rho_n}$. Therefore

$$i(\xi, \lambda^\pm) \le \lim_{n\to\infty} D_n^{-1} \ell(\lambda^\pm, [\rho_n])/p(\lambda^\pm) \le \lim_{n\to\infty} D_n^{-1} 2L/p(\lambda^\pm) = 0.$$

This means that $i(\xi, \lambda^\pm) = 0$. However, the pair of laminations λ^+, λ^- is binding the surface S, and we get a contradiction. $\square$

[1] See also the paper of Ohshika [Ohs90a] who uses Thurston's preprint [Thu87a].

18.3. Proof of Theorem 15.17

We already proved that the sequence of representations $\rho_n : \Gamma \to \Gamma_n \subset PSL(2,\mathbb{C})$ subconverges to a representation $\rho_\infty : \Gamma \to \Gamma_\infty$. Note that the K-quasiconformal homeomorphism $\tau : S \to S$ is realized by K-quasiconformal homeomorphisms of the pair of marked Riemann surfaces $(\tau^n(\alpha), \tau^{-n}(\alpha))$ (see §16.3). Recall that we are dealing with a sequence of representation $\rho_n : \Gamma \to \Gamma_n$, which correspond to the points $(\tau^n(\alpha), \tau^{-n}(\alpha)) \in \mathcal{T}(S) \times \mathcal{T}(\overline{S})$ under the Bers isomorphism. Thus for each n we get a K-quasiconformal homeomorphism $f_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ so that

$$f_{n*} \circ \rho_n = \rho_n \circ \tau_*$$

where $f_{n*} : \Gamma_n \to \Gamma_n$ is the automorphism induced by f_n and $\tau_* : \Gamma \to \Gamma$ is the automorphism induced by τ. Hence, by the convergence property of sequences of uniformly quasiconformal homeomorphisms (§5.2), the sequence f_n is subconverges to a quasiconformal homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which conjugates Γ_∞ to itself and

$$f_{\infty*} \circ \rho_\infty = \rho_\infty \circ \tau_*.$$

The quasiconformal homeomorphism f_∞ conjugates parabolic elements to parabolic. On the other hand, according to Sullivan Finiteness Theorem there are only finitely many conjugacy classes of primitive accidental parabolic elements[2] in Γ_∞ and they all correspond to simple loops $\gamma_j^\pm$ in S (see §4.23 for the discussion of these loops). The mapping τ has to permute the isotopy classes of the loops $\gamma_j^\pm$; hence, if there is at least one such loop, these isotopy classes are preserved by some iteration of τ. This contradicts the assumption that τ is aperiodic. Thus the group Γ_∞ contains no accidental parabolic elements whatsoever and we can apply Theorem 8.67. Each component Ω^0_∞ of $\Omega(\Gamma_\infty)$ has to be simply-connected (see the proof of Theorem 8.65). Hence, according to Theorem 8.67, marked conformal structures on the surfaces $\Omega(\Gamma_n)^0/\Gamma_n$ converge to the marked conformal structure on the surface $\Omega^0_\infty/\Gamma_\infty$. This contradicts the fact that the marked conformal structures $\tau^{\pm n}(S)$ do not accumulate in the Teichmüller space $\mathcal{T}(S)$ (and $\mathcal{T}(\overline{S})$), because the action of Mod_S on $\mathcal{T}(S)$ is properly discontinuous. Therefore $\Omega(\Gamma_\infty)$ is empty. □

18.4. An alternative approach

In this section we describe a possible alternative approach to hyperbolization of 3-manifolds fibered over the circle.

Theorem 18.4. *Suppose that M is a compact atoroidal 3-manifold fibered over $\mathbb{S}^1$ so that the fiber has negative Euler characteristic. Assume that M admits a finite covering M' such that:*

$$\text{rank } H_2(M, \partial M; \mathbb{Z}) \geq 2.$$

Then M is hyperbolic.

[2] Recall that these are parabolic elements α of Γ_∞ so that $\rho_\infty^{-1}(\alpha)$ is loxodromic.

Proof: We apply Theorem 2.1 to the manifold M'. Note that, according to the discussion in §1.9, the only case when an embedded (connected) superincompressible surface S is not a fiber in a fibration of M over $\mathbb{S}^1$, but is a *virtual fiber*, is: S separates M in two components. Such surface is homologically trivial, thus, the superincompressible surface given by Theorem 2.1 is not a virtual fiber. Therefore we apply the *generic case* of Thurston's Hyperbolization Theorem to the manifold M' and conclude that M' is hyperbolic. Thus Theorem 8.36 implies that M is hyperbolic as well. □

Corollary 18.5. *Suppose that M fibers over $\mathbb{S}^1$, is atoroidal and has nonempty boundary. Then M is hyperbolic.*

Proof: Combine Theorem 18.4 and Corollary 2.15. □

Remark 18.6. Conjecturally, any atoroidal Haken 3-manifold admits finite covers M' with arbitrarily large $\mathrm{rank}(H_2(M', \partial M'; \mathbb{Z}))$. See [CLR94, CLR97] for the recent progress in this direction.

Chapter 19

The Orbifold Trick

19.1. Manifold coverings of bipolar orbifolds

Suppose that O is a locally-reflective 3-dimensional orbifold, I will assume that the interior of O admits a complete hyperbolic structure and $|O|$ is orientable. The universal cover $\mathbb{H}^3$ of $int(O)$ has a partition into convex domains which are the components of the preimages of $X_O - \Sigma_O - \partial O$ under the universal cover $\mathbb{H}^3 \to int(O)$. If C is the closure of one of those domains and $G := G_C$ is its stabilizer in $\pi_1(O)$, then G_C acts freely on C and $|int(O)| = C/G_C$ (see §6.5). Let $R := R_C$ be the subgroup in $\pi_1(O)$ generated by reflections in the *faces* of C. Clearly $gRg^{-1} = R$ for every $g \in G$. According to the Poincare's theorem on fundamental polyhedra, C is a fundamental domain for the action of R on $\mathbb{H}^3$. We conclude that the fundamental group $\pi_1(O)$ is generated by R and G and hence $\pi_1(O) = R \ltimes G$. In particular, $g \in \pi_1(O)$ is orientation-reversing iff it is the product of an odd number of reflections in R and of an element of G.

Suppose that $\gamma \in \pi_1(O)$ is a nontrivial element of finite order. Then there is an element $g \in G$ such that $Fix(g\gamma g^{-1}) \cap C \neq \emptyset$. This implies that γ is either a reflection or a rotation of order 2, which is conjugate to the product of two reflections in R.

Our next goal is to give an explicit construction of a finite index torsion-free subgroup in $\pi_1(O)$ provided that O is an *all right bipolar* orbifold. Recall that this implies that the reflections in R are labeled *white* and *red* (and the distinct reflection of the same color generate an infinite subgroup of $\mathrm{Isom}(\mathbb{H}^3)$, reflections of distinct colors either generate an infinite subgroup or commute). The action of G by conjugations preserves this coloring. We define an epimorphism $\psi : \pi_1(O) \to \mathbb{Z}/2 \times \mathbb{Z}/2$ as follows. The group $\mathbb{Z}/2 \times \mathbb{Z}/2$ has two generators of order two, let's call them *white* and *red*. We send the reflections in R to the generators of $\mathbb{Z}/2 \times \mathbb{Z}/2$ which have the same color. Note that the group R has the presentation

$$\langle \tau_j : \tau_j^2 = 1, (\tau_i\tau_j)^2 = 1, i, j = 1, 2, ...\rangle$$

where the index j runs over all faces of C, τ_j is the reflection in that face and we have the relations $(\tau_i\tau_j)^2 = 1$ iff the corresponding faces are orthogonal (which implies that they have distinct colors). Thus we defined a homomorphism $\psi : R \to \mathbb{Z}/2 \times \mathbb{Z}/2$. Extend this homomorphism to the group G as $\psi(G) = \{1\}$. Since conjugation by the elements of G preserves white/red labels, we have a well-defined epimorphism $\psi : \pi_1(O) \to \mathbb{Z}/2 \times \mathbb{Z}/2$. Because of our description of elements of finite order in $\pi_1(O)$, the kernel K of ψ is torsion-free. This gives us a regular 4-fold manifold covering $p : N \to O$. Clearly N is orientable.

Here is a geometric description of the manifold N. Take four copies $X_i, i = 1, .., 4$, of X_O each labeled by an element of $\mathbb{Z}/2 \times \mathbb{Z}/2$, where X_1 is labeled by the unit, X_2 is labeled by the white generator $x_2 \in \mathbb{Z}/2 \times \mathbb{Z}/2$ and X_3 is labeled by the red generator x_3 of $\mathbb{Z}/2 \times \mathbb{Z}/2$. The group $\mathbb{Z}/2 \times \mathbb{Z}/2$ acts naturally on $X_1 \sqcup ... \sqcup X_4$. Then glue the faces on ∂X_i's in a $\mathbb{Z}/2 \times \mathbb{Z}/2$-equivariant manner so that the gluing preserves the coloring of faces and if (say) a face F of X_1 is labeled white then we glue F to the face $x_2 F \subset X_2$, etc. The result of this gluing is a hyperbolic 3-manifold N which covers O.

We need to understand the behavior of this cover over the boundary components of the orbifold

$$\overset{\bullet}{O} = (\mathbb{H}^3 \cup \Omega(\pi_1(O)))/\pi_1(O).$$

Recall that each component S_j of $\Omega(\pi_1(O)))/\pi_1(O)$ is a locally reflective 2-orbifold without corner reflectors so that all components of the singular locus Σ_{S_j} inherit the same labels (say white) from the orbifold O (see Definition 6.28). Therefore $\psi(\pi_1(S_j)) \subset \mathbb{Z}/2, p^{-1}(S_j)$ consists of two components S'_j, S''_j; each component is the double of $|S_j|$ along Σ_{S_j}. Hence each covering $S'_j \to S_j, S''_j \to S_j$ is *characteristic*, i.e. if $\tau : S_j \to S_i$ is a diffeomorphism then τ lifts to a diffeomorphism

$$\tilde{\tau} : S'_j \cup S''_j \to S'_i \cup S''_i.$$

19.2. Building hyperbolic orbifolds of finite type

In this section we start the proof of Theorem 15.2, which will be finished in §19.4.

We start with a compact orientable atoroidal Haken pared manifold (M', P') and a connected properly embedded superincompressible surface $S' \subset M'$. I recall that P' is a collection of incompressible boundary tori and annuli. In each of the annuli we draw the middle loop. The compact manifold M is obtained from M' by splitting along S'. As the result we get the following combinatorial data on the boundary of M:

- A surface $L \subset \partial M$ and an involution $\tau : L \to L$ so that after gluing along τ we get $M/\tau = M'$, $L/\tau = S'$.
- A collection P of subsurfaces in $\partial M - L$ which we get after splitting P' along S'; P consists of tori, annuli and squares. The latter have distinguished segments A_j connecting the opposite sides (these segments correspond to the middle loops on the cylindrical components of P').

Given this data we will construct a bipolar orbifold O of finite type with the underlying space M. Gluing of O via τ gives rise to an orbifold O' which is *all right* and has zero Euler characteristic; the underlying space of O' is M'.

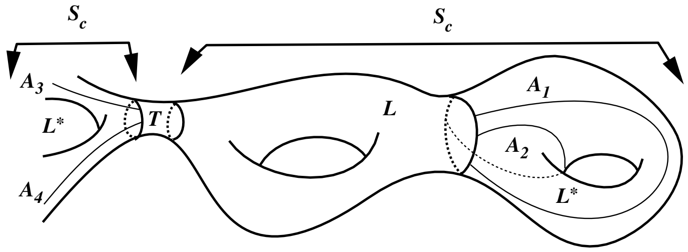

Figure 19.1:

Let Q be the collection of toroidal and cylindrical components of P. By the hypothesis of Theorem 15.2, the pared manifold (M, Q) admits a complete geometrically finite hyperbolic structure, where Q is the designated parabolic locus. As usual there are two cases depending on the number of components in M, I will consider the case when M is connected, the other case is similar and is left to the reader. Thus there exists a geometrically finite torsion-free Kleinian group $G \subset PSL(2, \mathbb{C})$ with the quotient $\overset{\bullet}{M}(G) = [\mathbb{H}^3 \cup \Omega(G)]/G$. Then M is a (relative) Scott compact core of $\overset{\bullet}{M}(G)$. The components of Q correspond to the cusps of the manifold $\overset{\bullet}{M}(G)$. We assume that $M \cap S(G)$ is a subsurface S_c obtained by removing from $S(G)$ punctured disks which are cuspidal neighborhoods . Then the compact core M (rel. S_c) is given by Theorem 4.126.

Note that $L \subset S_c$ is an incompressible surface so that no component of

$$L^* := cl(S_c - L)$$

is an annulus (which implies that if a boundary component of L is isotopic to a boundary curve of Q, then these loops are actually equal). See Figure 19.1.

Our next goal is to construct a certain 3-valent graph in L^* which cuts this surface into simply-connected components. If a component $\partial_j M'$ of $\partial M'$ is disjoint from S' then we pick a trivalent graph in $\partial_j M'$ which cuts this subsurface into simply-connected components, this graph projects homeomorphically to a graph in the corresponding component of ∂M. We now consider the more complicated case of a component $\partial_j M'$ of $\partial M'$ which intersect S' nontrivially. Recall that $\partial_j M'$ may contain an annulus from P' which is cut by S' into squares. In this annulus we already have chosen a central loop (which we call a *parabolic loop*). Let C_j denote a disjoint union of polygonal loops in $\partial_j M'$ which contains the union of parabolic loops so that C_j is transversal to $\partial S'$ and the complement $\partial_j M' - C_j - \partial S'$ consists only of simply-connected components. We will assume that $C_j \cap P'$ consists only of the *parabolic loops*, we call the components of $C_j - P'$ the *hyperbolic loops*). When we cut $\partial_j M'$ open along S' we get a component of the surface L' and the graph $C_j \cup \partial S'$ descends to a 3-valent graph in L^*. This graph is the union of $\mathcal{A} := A_1 \cup ... \cup A_k$ which are obtained by splitting *parabolic* and *hyperbolic* loops. The parabolic loops split into the *parabolic arcs* A_j, $1 \leq j \leq p$, the *hyperbolic loops* produce the *hyperbolic arcs* A_j, $p+1 \leq j \leq k$. We restrict the discussion to the more complicated case when each component of L^* has nonempty boundary and leave the remaining case to the reader.

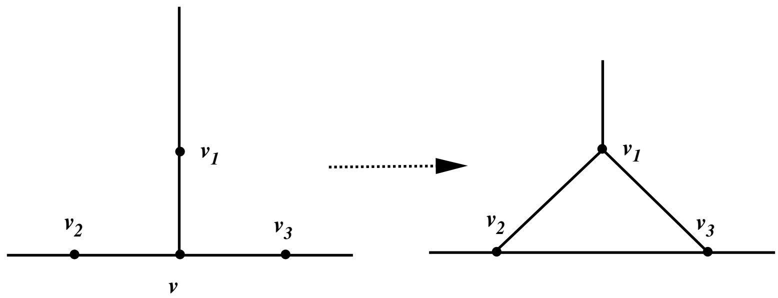

Figure 19.2: *Blowing up the vertex v.*

The union $\mathcal{B} := \mathcal{A} \cup \partial L^*$ is a graph in the surface S_c. By construction the restriction of τ to $\partial L^* \cap \partial L$ preserves the cellular structure of this graph. Then we add (in a τ-invariant way) two vertices to the interior of each edge of $\mathcal{B}$ contained in ∂L^*. For *parabolic* arcs A_j we take their disjoint regular neighborhoods $\mathcal{N}(A_j)$ in L^* to be the square components of $P - Q$.

We form another graph $\hat{\mathcal{B}}$ as follows. For every end-point v of each *hyperbolic arc* we take the vertices $v_2, v_3 \in \partial L^*$ and v_1 of the edges emanating from v (they have valence 2) and connect them to form a triangle as in Figure 19.2. Then we erase the vertex v and the edge $[v_1 v]$ from the graph. As the result we "blow up" the vertex v into a *hyperbolic triangle* $[v_1, v_2, v_3]$.

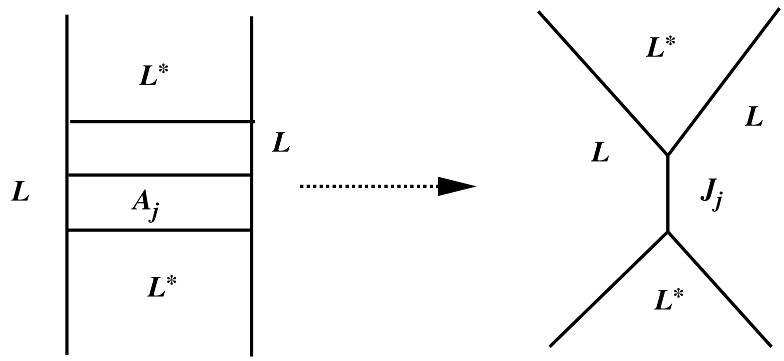

Figure 19.3: *Collapse of a regular neighborhood of a parabolic arc A_j to a parabolic segment J_j.*

At the next step we collapse the regular neighborhood $\mathcal{N}(A_j)$ of each *parabolic arc* A_j along the family of segments parallel to A_j. See Figure

19.3. As the result all what is left from the rectangle $\mathcal{N}(A_j)$ is a *parabolic* segment J_j, we erase all the interior vertices of these segments. The result is the graph $\hat{\mathcal{B}} \subset S_c$.

Suppose that $\mathcal{P} = \{B_j, j = 1, ..., r\}$ is a *partial packing* of the surface $S(G)$ (i.e. it is a collection of *round disks* in $S(G)$ whose interiors are pairwise disjoint). The combinatorial structure of $\mathcal{P}$ is described by its *dual graph* $\mathcal{P}^{\#} \subset S(G)$ which is defined as follows. The vertices of $\mathcal{P}^{\#}$ are the points in the interior of B_j (one vertex for each disk). The edges of $\mathcal{P}^{\#}$ are segments connecting distinct vertices such that corresponding closed disks have nonempty intersection. Note that embedding of this graph into $S(G)$ is well-defined up to isotopy.

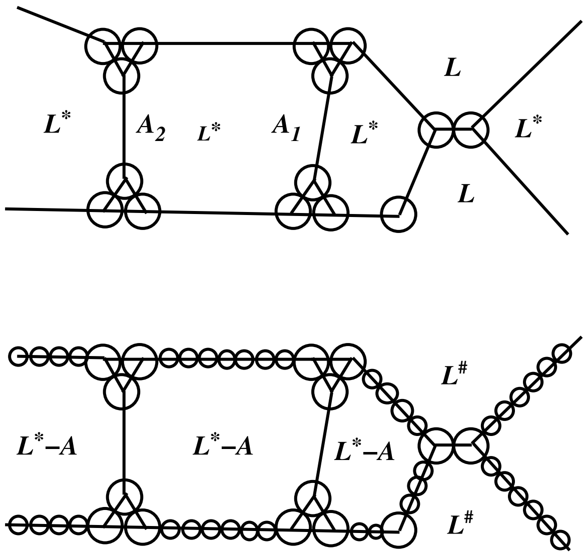

Figure 19.4:

Theorem 19.1. *For the given Kleinian group G, surfaces L, S_c, graph $\mathcal{B}$, etc., as above, there exists a subdivision $\mathcal{B}'$ of $\mathcal{B}$ and a partial packing $\mathcal{P}$ of S_c such that:*

- *The graph $\hat{\mathcal{B}}'$ is isotopic to a graph $\mathcal{P}^{\#} \subset S_c$ which is dual to $\mathcal{P} = \{B_j, j = 1, ..., r\}$.*
- *After isotopy, the involution $\tau : L \to L$ preserves the cellular structure of $\hat{\mathcal{B}}' \cap \partial L$.*

Proof: After subdividing $\mathcal{B}$ an applying an isotopy of S_c, we can assume that:

(1) The graph $\hat{\mathcal{B}}$ is a cell complex where cells are points and circular segments in S.

(2) Each hyperbolic triangle and parabolic segment is contained in a small round disk in S_c, which has homeomorphic lift to $\Omega(G)$.

Then we put round disks with centers at the vertices of *parabolic segments* and *hyperbolic triangles* of $\hat{\mathcal{B}}$ so that every pair of distinct disks is either disjoint or tangent (the latter happens if vertices belong to the same *parabolic segment* or *hyperbolic triangle*). Next we put small disjoint round disks in each valence 2 vertex of $\mathcal{B}$. We assume that these disks are so small that they are disjoint from any previously constructed disks. In the isotopy class of τ we choose an involution which preserves the cellular structure of $\hat{\mathcal{B}}$, carries $\hat{\mathcal{B}} \cap L$ onto itself and sends vertices of *hyperbolic triangles* to vertices of *hyperbolic triangles* and *parabolic edges* to *parabolic edges*.

Then, we subdivide the portion of the graph $\hat{\mathcal{B}}$ which is not already covered by disks and cover these edges by round disks, each of which are either disjoint or tangent. By choosing a sufficiently fine subdivision we assure that τ preserves the cellular structure of $\hat{\mathcal{B}}$. See Figure 19.4. $\square$

Note that the surface

$$S_c - \bigcup_{B_j \in \mathcal{P}} B_j$$

consists of a subsurface $L^{\#}$ isotopic to L and a collection of contractible components corresponding to $L^* - \mathcal{A}$. Henceforth we let $L := L^{\#}$, thus the cellular structure on $\partial L \cap \partial L^*$ is given by circular arcs and ideal vertices (points of tangency between round disks in $\mathcal{P}$). After appropriate isotopy we assume that $\tau : L \to L$ preserves *this* cellular structure and we retain the notation: $L^* := S_c - L$.

Apply Theorem 13.2 to the group G and the partial packing $\mathcal{P}$. As the result we find a small deformation $G^\epsilon \subset \mathrm{Isom}(\mathbb{H}^3)$ of the group G so that the corresponding partial packing $\mathcal{P}^\epsilon$ of $S(G^\epsilon)$ can be extended to a finite pattern $\mathcal{D}$ of round disks in $S(G^\epsilon)$ which has the properties:

- The pattern $\mathcal{D}$ is obtained from the pattern $\mathcal{R}$ (given by Theorem 13.2) by removing those disks of $\mathcal{R} - \mathcal{P}^\epsilon$ which intersect $L^{\#}$.
- The union of disks in $\mathcal{D}$ is the surface L^*.
- Any two disks in this pattern are either disjoint, equal, tangent or their boundaries are orthogonal.

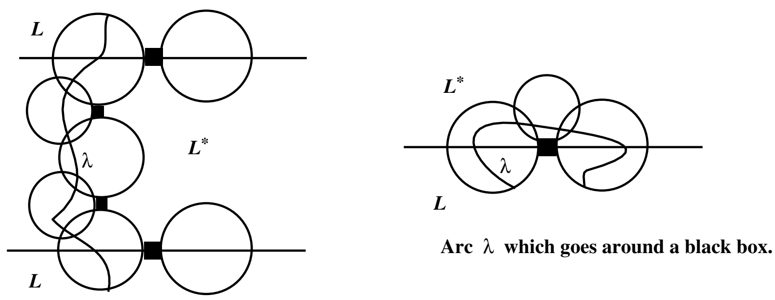

Figure 19.5:

Theorem 19.2. *The compact locally reflective orbifold O associated with $\mathcal{D}$ as in §13.1 is a bipolar orbifold of finite type.*

Proof: It is easy to see that O satisfies the Axioms 1, 2 and 3. The orbifold O is very good since it is obtained by *chopping off* cusps of a geometrically finite hyperbolic orbifold.

What requires verification are the Axioms 4, 5. We will verify the Axiom 5, the proof of the Axiom 4 is similar. Consider an arc $\lambda \subset L^*$ with the end-points in the boundary of L^*. Then, unless λ goes around a "black box" (a point of intersection of three circles in the pattern which will be blown up to a 3-edged square orbifold), it will intersect at least four bigons formed by the boundary circles of the disks in $\mathcal{D}'$. These bigons will become edges of the graph Γ associated with the pattern of disks, so we get at least four vertices on λ. See Figure 19.5. $\square$

19.3. Gluing orbifolds of zero Euler characteristic

Our next step is to glue the *all right* orbifolds of zero Euler characteristic from the orbifold O constructed in the previous section. Recall that $L = \partial O - Q$ has natural structure of a 2-dimensional orbifold, where Q is the parabolic locus of O; $\tau : L \to L$ is a homeomorphism preserving this structure. Let O' be the orbifold obtained from O by gluing via τ. We let $L' = S'$ denote the image of L in M'.

We describe the combinatorial structure of the orbifold O' as follows. The underlying space of O' is the manifold M'. The image of the parabolic locus Q in M' is the parabolic locus Q' of the orbifold O'. Let $\hat{\Gamma}$ be the graph of the orbifold structure of O as in §6.3.

Glue the graph $cl(\hat{\Gamma} - L) \subset \partial M$ to itself via τ. The result is a graph $\hat{\Gamma}'$ in the boundary of M'. Each vertex of $\hat{\Gamma}'$ has valence 3 and belongs to the boundary of Q'. Let $\Gamma' := \hat{\Gamma}' - Q$.

We use the function

$$\theta' : \text{ edges of } \Gamma' \to \{\pi/2\}$$

to give the space M' the structure of a locally reflective orbifold O'. The boundary of O' equals Q' and consists of Euclidean 2-orbifolds.

Proposition 19.3. *The orbifold O' above is Waldhausen, acylindrical and atoroidal.*

Proof: We first note that the orbifold O' is not Seifert since its fundamental group contains rank 2 abelian subgroups which have trivial intersection (take any two subgroups corresponding to two distinct rank 2 cusps in O). The surface L' has a natural orbifold structure $O_{L'}$. The fact that the suborbifold $O_{L'} \subset O'$ is incompressible directly follows from the assumption that L' is incompressible in the manifold M'. Since $O - \mathcal{F}(P^{(1)})$ is acylindrical we conclude that $O_{L'}$ is also superincompressible. Recall that the

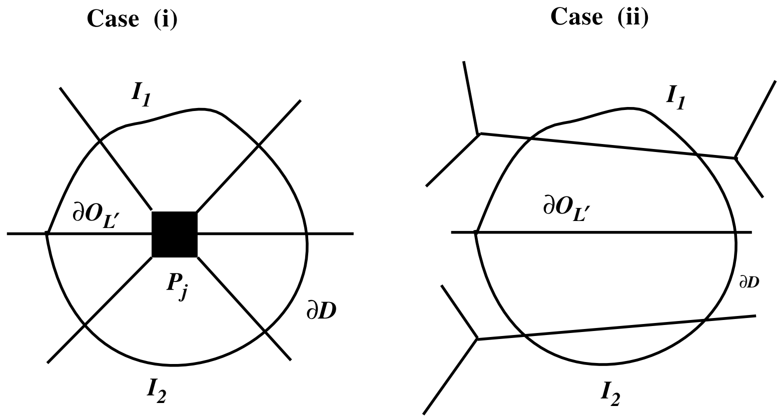

Figure 19.6: *Possible positions for the loop* ∂D*.*

manifold $M' = |O'|$ was irreducible and atoroidal by the assumption. Thus, to verify that O' is irreducible and acylindrical we need to consider only bad, spherical and Euclidean suborbifolds whose underlying sets are either disks or annuli, or Moebius bands properly embedded in M'. We will discuss the case of the underlying sets which are disks and leave two remaining (easy) cases to the reader.

Suppose that $(D, \partial D) \subset (M', \partial M' - Q')$ is a properly embedded 2-disk such that D inherits from $\partial M'$ the orbifold structure of an elliptic (or Euclidean) 2-orbifold O_D which does not bound a compact elliptic (or Euclidean) 3-suborbifold in O'. By altering the orbifold O_D within its isotopy class we first minimize the number of points of intersection of ∂D with $\partial L'$.

The boundary of D is transversal to the graph Γ', the intersection of D with Γ' determines the set of vertices in ∂D (which are corner reflectors). The number of these vertices is at most four (otherwise the orbifold O_D is hyperbolic). The boundary ∂D corresponds to a collection of arcs I_j in the surface $\Sigma = \Sigma_O$. The end-points of these arcs correspond to the intersection of ∂D with $\partial L'$. Consider one of these arcs $I := I_j$. Since the number of points of intersection $\partial D \cap \partial L'$ is minimal within the isotopy class of O_D, we conclude that the arc I has at least one interior vertex (i.e. a point in $I \cap \Gamma$). The number of the interior vertices is at least two (by the Axioms 4 and 5 of orbifolds of finite type). Since O_D is not hyperbolic, we conclude that the number of the interior vertices in each arc I_j is exactly two and there are precisely two of these arcs I_1, I_2. The Axiom 5 implies that both arcs are isotopic into L relatively to ∂I_j. By the above analysis, the arc I_1 satisfies the property (i) (on the page 145) if and only if the second arc I_2 satisfies the same property (i). Hence there are exactly two combinatorial possibilities for the loop ∂D (see Figure 19.6).

In the 1-st case the loop ∂D is a circuit which goes around one of the mirror squares $P_j \subset Q' \cap \partial L'$ and it intersects exactly four edges of Γ': those which emanate from the square P_j. Thus D is isotopic to the suborbifold P_j in the parabolic locus of O'. In the second case the orbifold O_D bounds a 3-dimensional Euclidean suborbifold in O', which is the *SOLID TORUS.*

Hence O_D is compressible. $\square$

Corollary 19.4. *The orbifold O' is an all right orbifold of zero Euler characteristic.*

Proof: It follows from Theorem 6.14 that O' is very good. In the following section we will give another proof of the fact that O' is very good by constructing a finite manifold covering over O'. $\square$

19.4. Hyperbolization of locally reflective orbifolds

As in the previous section we assume that O is a locally reflective orbifold of finite type, Q is the parabolic locus of this orbifold, $L := \partial O - Q$; $\tau : L \to L$ is a gluing involution involution. In the previous section we have constructed a geometrically finite hyperbolic structure on (O, P). We also proved that $O' = O/\tau$ is an all right orbifold of zero Euler characteristic, the image Q' of Q in O' is the *designated parabolic locus* of O'. Recall that $\partial O' = Q'$.

Theorem 19.5. *The orbifold $O' - Q'$ admits a hyperbolic structure of finite volume, where Q' corresponds is the parabolic locus.*

Proof: Let $N = \tilde{O} \to O$ denote the finite regular manifold cover of O constructed in §19.1, the manifold $N = \tilde{O}$ is orientable. Let Φ denote the finite group of covering transformations for $\tilde{O} \to O$, let $\tilde{Q}$ be the lift of Q to N. We get a pared manifold $(\tilde{O}, \tilde{Q})$. Note that the relative boundary $\partial_0 \tilde{O} = \partial N - \tilde{Q}$ has exactly four components. The homeomorphism τ lifts to a homeomorphism $\tilde{\tau} : \partial_0 \tilde{O} \to \partial_0 \tilde{O}$, since the covering $\partial_0 \tilde{O} \to \partial_0 O$ is characteristic (see § 19.1).

Let $N' = N/\tilde{\tau}$. The gluing homeomorphism $\tilde{\tau}$ is Φ-equivariant. Thus we have a well-defined action of Φ on N' so that $N'/\Phi = O'$. The manifold N' is Haken, atoroidal and $\tilde{Q}' := \partial N'$ consists of incompressible tori (lifts of the Euclidean orbifolds in the boundary $Q' = \partial O'$).

Therefore the gluing involution $\tilde{\tau} : \tilde{O} \to \tilde{O}$ satisfies the assumptions of Theorem 15.1 (Case A). The pared manifold $(\tilde{O}, \tilde{Q})$ clearly is not an interval bundle over surface. Note that the image of $\partial_0 N$ in N' consists of two components, that is why we have to treat the case of a disconnected superincompressible surface in Theorem 15.1.

Hence, according to Theorem 15.1 (Case A), the pared manifold $(\tilde{O}', \tilde{Q}')$ admits a complete hyperbolic metric of finite volume. By Mostow Rigidity Theorem, the action of the finite group Φ on N' is homotopic to the action of a finite group of isometries Ψ of the hyperbolic metric. Hence the orbifolds O' and N'/Φ have isomorphic fundamental groups and the orbifold $(N - \tilde{Q}')/\Phi$ is complete hyperbolic (see Theorem 8.36). Namely, if $N' - \tilde{Q}' = \mathbb{H}^3/G$, where G is a torsion-free lattice in $PSL(2, \mathbb{C})$, then we lift Ψ to $\mathbb{H}^3$ and get a finite extension $G^+ \subset \mathrm{Isom}(\mathbb{H}^3)$ of the group G so that $\mathbb{H}^3/G^+ = (N' - \tilde{Q}')/\Psi$.

Thus Theorem 6.33 implies that the orbifolds N'/Ψ and $O' = N'/\Phi$ are homeomorphic, hence the pared orbifold (O', Q') also admits a complete hyperbolic structure. $\square$

Now we can finish the proof of Theorem 15.2. Theorem 19.5 implies that the pared orbifold (O', Q') admits a complete hyperbolic structure of finite volume. Therefore the underlying space X of the orbifold $O' - Q'$ admits a metrically complete convex hyperbolic structure of finite volume. However X is homeomorphic to the manifold $M' - P' - E$ (where P' is the parabolic locus of M' and E is a finite collection of points which correspond to the mirror square components of Q'). Since X has finite volume, its convex core $C(X)$ also has finite volume and finitely generated fundamental group. Annular and toroidal components of P' correspond to cusps of the complete hyperbolic manifold $C(X)$. Thus we have constructed a geometrically finite hyperbolic structure on the pared manifold (M', P'). $\square$

19.5. Hyperbolic design

Theorem 19.6. *Suppose that $G \subset PSL(2, \mathbb{C})$ is a finitely generated discrete subgroup. Then there is a geometrically finite group $\Gamma \subset PSL(2, \mathbb{C})$ and a type-preserving isomorphism $\rho : \Gamma \to G$.*

Remark 19.7. The same theorem is valid for discrete finitely generated subgroups of $\mathrm{Isom}(\mathbb{H}^3)$.

Proof: Let G^0 be a normal torsion-free subgroup of finite index in G; $F := G/G^0$. According to Theorem 4.129, there exists a relative compact core $M^c \subset M(G^0)$ which is invariant under the action of F. Thus we get an orbifold $O = O_G = M^c/F$ whose fundamental group is isomorphic to G. Cusps of $\mathbb{H}^3/G$ define the parabolic locus $P \subset \partial O_G$. Note that (O, P) does not have to be a pared orbifold, for instance it could happen that O contains a properly embedded essential ANNULUS one of whose boundary components is in P and the other is not. Nevertheless, we can assume that there are no essential ANNULI in O whose boundary components are in P. Note that if $\partial O = P$, then ∂M^c consist only of tori, which means that G^0 is a lattice and there is nothing to prove. So we shall assume that $\partial O \neq P$.

Step 1. First we reduce the problem to the class of discrete groups G such that $\partial O - P$ is incompressible. Suppose that we have proved Theorem for all orbifolds with incompressible boundary. Then the proof is finished by induction on the number of nonparallel compressing DISKS for $\partial O - P$. Suppose that $(O_1, P_1), (O_2, P_2)$ are orbifolds with the designated parabolic loci P_1, P_2, and Γ_1, Γ_2 are geometrically finite Kleinian groups such that the isomorphisms $\Gamma_j \to \pi_1(O_j)$ preserve type of elements. Let $D_1 \subset (\partial O_1 - P_1), D_2 \subset (\partial O_2 - P_2)$ be DISKS and the orbifold (O, P) is obtained by gluing these orbifolds via a homeomorphism $h : D_1 \to D_2$. Let me first explain what to do in the case when $D_1 \cong D_2$ are nonsingular disks. Then $\pi_1(O) = \pi_1(O_1) * \pi_1(O_2) = \Gamma_1 * \Gamma_2$. Thus we take $\Gamma = \Gamma_1 * \Gamma_2$ be a Klein combination of the groups Γ_1, Γ_2. The isomorphism $\Gamma \to \pi_1(O)$ is type-preserving.

Now consider the case when $D_1 \cong D_2$ is the quotient of the round disk D by the cyclic group J or order m. Then the groups Γ_j contain cyclic elliptic subgroups J_j which correspond to $\pi_1(D_j)$ under the isomorphism $\pi_1(O_j) \to \Gamma_j$, $j = 1, 2$. Choose an isomorphism $\phi : J_1 \to J_2$ of the cyclic groups so that a generator $\gamma_1 \in J_1$ is carried by ϕ to a generator $\gamma_2 \in J_2$ which has the same angle of rotation as γ_1.

The elliptic elements γ_j have the property: their axes A_j contain at least one end-point a_j in $\Omega(\Gamma_j)$. Take fundamental domains $\Phi_j \subset \widehat{\mathbb{C}}$ of the groups Γ_j so that $a_j \in \Phi_j$. By conjugating Γ_1, Γ_2 in $PSL(2, \mathbb{C})$ we arrange that $\gamma_1 = \gamma_2$ and $a_1 \neq a_2$ are distinct end-points of the common axis A of γ_1, γ_2. Next, we take hyperbolic translations h_1, h_2 along A so that h_j has the repulsive fixed-point a_j and the sets $\widehat{\mathbb{C}} - h_1(\Phi_1), \widehat{\mathbb{C}} - h_2(\Phi_2)$ are disjoint. Let $\Gamma_j := h_j \Gamma_j h_j^{-1}$, they have fundamental domains $\Phi_j := h_j(\Phi_j)$ and share a common cyclic elliptic subgroup $J = J_1 = J_2$. This is the situation when we can apply Maskit Combination Theorem 4.104 and conclude that the group $\Gamma := \langle \Gamma_1, \Gamma_2 \rangle$ is geometrically finite, isomorphic to $\Gamma_1 *_J \Gamma_2$ and any parabolic element of Γ is conjugate either to Γ_1 or to Γ_2.

This finishes the reduction to the case of orbifolds with incompressible boundary.

Step 2. Now we assume that (O, P) has incompressible boundary. Let $M := M^c$ and let Q denote the lift of P to ∂M^c. Then the manifold $M - Q$ also has incompressible boundary.

Theorem 19.8. *There is a geometrically finite Kleinian group $\Gamma^0 \subset PSL(2, \mathbb{C})$ and an isomorphism $\psi : \Gamma^0 \to \pi_1(M)$ which preserves type of elements.*

Proof: Note that the manifold M is atoroidal (since any $\mathbb{Z} \oplus \mathbb{Z}$ subgroup of G is parabolic). Suppose for a moment that (M, Q) is a pared acylindrical manifold. Take the double DM of this manifold along the partial boundary $\partial_0 M := \partial M - int(Q)$. The manifold DM is irreducible (since M is irreducible and $\partial_0 M$ is incompressible), it is Haken (the incompressible surface is $\partial_0 M$) and atoroidal (since (M, Q) is acylindrical). Thus we can apply Thurston's Hyperbolization Theorem to DM and conclude that DM admits a complete hyperbolic metric of finite volume. The pared manifold (M, Q) is not an interval bundle over a surface. Therefore Theorem 14.24 implies that $\Gamma^0 = \pi_1(M) \subset \pi_1(DM)$ is a geometrically finite subgroup. An element of Γ^0 is parabolic if and only if it is conjugate to $\pi_1(Q)$. This finishes the proof in the case of acylindrical pared manifolds.

This argument obviously does not work if (M, Q) is either not pared or not acylindrical. There are several ways to deal with this problem. In any case all what we need is a compact (not necessarily connected) pared manifold (or locally reflective orbifold) (B, T) which is irreducible, acylindrical and has incompressible boundary, so that $\partial_0 B = \partial B - T$ is diffeomorphic to $\partial_0 M = \partial M - Q$.

Then we glue the manifold M to B along the boundary $\partial_0 M \cong \partial_0 B$. The resulting manifold (or locally reflective orbifold) H has the following properties:

- H contains the incompressible surface $\partial_0 M$;
- H is irreducible, atoroidal and each boundary component of H is a TORUS.

If H is a manifold, then according to the Hyperbolization Theorem, its interior admits a complete hyperbolic structure of finite volume. The fundamental group $\pi_1(M)$ injects into $\pi_1(H) = \Gamma^{\#}$. The image under this homomorphism of the fundamental group of each component of Q is a parabolic subgroup of $\Gamma^{\#} \subset PSL(2, \mathbb{C})$. Finally, Theorem 14.24 implies that $\Gamma^0 \subset \Gamma^{\#}$ is a geometrically finite subgroup.

In the case when H is a locally reflective orbifold the situation is slightly more delicate. According to Theorem 6.14, the orbifold H is very good. Thus it admits a finite regular manifold covering H'. Then we apply the Hyperbolization Theorem to H' and using Mostow rigidity conclude that at least $\pi_1(H)$ is isomorphic to a lattice in Isom$(\mathbb{H}^3)$ (in fact H itself admits a complete hyperbolic structure). Then we repeat our arguments in the manifold case.

So, what is left is to construct the manifold (or orbifold) B.

The 1-st way: construction of an orbifold B.
We will use Theorem 13.2. Namely, let $S_1, ..., S_q$ be the collection of (closed) boundary surfaces of the manifold M (at this moment we ignore the orbifold locus Q) which have negative Euler characteristic. For each of these surfaces we find a Fuchsian group F_j so that $\mathbb{H}^2/F_j = S_j$. Take the inversion σ_j in the limit set of F_j and consider the group $\hat{F}_j = F_j \times \langle \sigma_j \rangle \subset$ Isom$(\mathbb{H}^3)$. Clearly $\Omega(\hat{F}_j)/\hat{F}_j = S_j$. According to Theorem 13.2, there exists a small deformation $F_j^\epsilon \times \langle \sigma_j \rangle = \hat{F}_j^\epsilon \subset$ Isom$(\mathbb{H}^3)$ of the group $\hat{F}_j$, so that $\hat{F}_j^\epsilon \subset \Gamma_j$, where Γ_j is a lattice in Isom$(\mathbb{H}^3)$. The convex hull $C\Lambda(\hat{F}_j^\epsilon)$ is a hyperbolic hyperplane $\mathbb{H}^2 \cong L_j \subset \mathbb{H}^3$, according to Theorem 13.2 this convex hull is precisely-invariant under $\hat{F}_j^\epsilon$ in Γ_j. Hence, $L_j/\hat{F}_j^\epsilon$ is a boundary component of the underlying set $|\mathbb{H}^3/\Gamma_j|$ of the orbifold $\mathbb{H}^3/\Gamma_j$. Thus we let B_j be the orbifold $\mathbb{H}^3/\Gamma_j$ with the "forgotten" orbifold structure on $L_j/\hat{F}_j^\epsilon$. The orbifold B_j is locally reflective. The boundary of B_j consists of the incompressible surface S_j and a collection of TORI T_j. There is a subgroup $G_j \subset \Gamma_j$ so that $B_j - T_j = C\Lambda(G_j)/G_j$; the boundary of the convex hull $C\Lambda(G_j)$ is the G_j-orbit of the hyperbolic plane L_j. Clearly B_j is atoroidal and irreducible (since it is hyperbolic). Let us verify that B_j is acylindrical. It is enough to check the following:

Suppose that $g \in G_j$ is such that $gL_j \neq L_j$, then

$$F_j \cap gF_jg^{-1} = \{1\} \tag{19.1}$$

Suppose that the intersection (19.1) contains an element $\gamma \neq 1$. Then γ is a hyperbolic element of F_j. Therefore the intersection

$$g(\partial_\infty L_j) \cap \partial_\infty L_j$$

contains at least two points which are the fixed points of γ. However $\partial_\infty L_j$ is a round circle, thus $gL_j \cap L_j \neq \emptyset$, which contradicts the fact that L_j is precisely invariant under F_j. Hence B_j is acylindrical. Fix a diffeomorphism $f_j : S_j \to \partial_0 B_j$. The boundary surface S_j of M could contain a portion Q_j of the designated parabolic locus Q. Let $T_j' := f_j(Q_j)$. The pared orbifold $(B_j, T_j \cup T_j')$ is still acylindrical and has incompressible relative boundary $\partial_0 B_j$. This means that we can take

$$(B,T) := \bigcup_{j=1}^{q} (B_j, T_j \cup T_j').$$

This finishes the first construction.

The 2-nd way: construction of a manifold B**.** We will use the following

Theorem 19.9. *(R. Myers [Mye83].) Suppose that M is a compact 3-manifold (possibly with boundary), no boundary component of M has positive Euler characteristic. Then there exists a link $L \subset int(M)$ so that the manifold $M^{\#} := M - Nbd(L)$ is Haken, atoroidal, acylindrical and has incompressible boundary.*

We apply this theorem to the manifold M at hand and let $B := M^{\#}$.
□

Step 3. Now we have a geometrically finite Kleinian group $\Gamma^0 \subset PSL(2,\mathbb{C})$ which is isomorphic to G^0. Since $\partial_0 M$ is incompressible, all components of $\Omega(\Gamma^0)$ are simply-connected. Consider the Teichmüller space

$$\mathcal{T}(\Gamma^0) \cong \mathcal{T}(\Sigma_1) \times ... \times \mathcal{T}(\Sigma_m)$$

where $\Sigma_j = \Omega_j/\Gamma_j^0$, Ω_j are components of $\Omega(\Gamma^0)$ stabilized by subgroups Γ_j^0 of Γ^0. Note that these subgroups do not have to be quasifuchsian because of accidental parabolic elements. Not all surfaces Σ_j are diffeomorphic, let $P_m \subset S_m$ be the subgroup of the group of permutations on m symbols so that each element σ of P_m can be realized as a diffeomorphism of M permuting the components of $\partial_0 M$ according to σ. The finite group F acts on $\mathcal{T}(\Gamma^0)$, a fixed point for such action corresponds to a representation $\rho : \Gamma^0 \to \Gamma^1 \subset PSL(2,\mathbb{C})$ so that:

- $[\rho] \in \mathcal{T}(\Gamma^0)$.
- Γ^1 admits a finite extension Γ in $PSL(2,\mathbb{C})$, which is isomorphic to G.
- The isomorphism $\Gamma \to G$ is type-preserving.

(Cf. our proof of Theorem 8.36). To prove that F has a fixed point we give $\mathcal{T}(\Gamma^0)$ the Riemannian metric d_{WP} which is the direct product of Weil-Petersson metrics on the Teichmüller spaces $\mathcal{T}(\Sigma_j)$. The group F acts on $\mathcal{T}(\Gamma^0)$ as a subgroup of $P_m \ltimes (Mod_{\Sigma_1} \times ... \times Mod_{\Sigma_m})$. It follows that F is a finite group of isometries of $(\mathcal{T}(\Gamma^0), d_{WP})$, hence F has a fixed point on $\mathcal{T}(\Gamma^0)$ according to Corollary 5.11. □

Chapter 20

Beyond the Hyperbolization Theorem

20.1. Thurston's geometrization conjecture

In this section we discuss various conjectures related to Thurston's Hyperbolization Theorem. Our discussion is far from being comprehensive, we refer the interested reader to Kirby's list of problems [Kir97] for more details.

Some hypothesis of the hyperbolization theorem are necessary for hyperbolicity of the 3-manifold M (e.g. being atoroidal), other are not. Most importantly, the assumption that M is Haken (conjecturally) should be replaced by the assumption that M is irreducible and has infinite fundamental group:

Conjecture 20.1. *(Thurston's Hyperbolization Conjecture.) Suppose that M is a closed irreducible 3-manifold such that $\pi_1(M)$ is infinite and M is atoroidal. Then M admits a hyperbolic metric.*

Recall that each Haken 3-manifold of zero Euler characteristic admits a canonical (JSJ) decomposition along incompressible tori and Klein bottles into pieces each of which is either hyperbolic or Seifert. Thurston suggested a further generalization of this. Let X be a simply-connected 3-dimensional Riemannian manifold which is *homogeneous*, i.e. the group of isometries $Isom(X)$ acts transitively on X. Such manifolds X are classified by Scott [Sco83a] and Thurston [Thu97a] under the extra assumption that X admits a cocompact discrete group of isometries; it turns out that there are eight such spaces (up to isomorphism of their isometry groups):

(a) Spaces of constant curvature: $\mathbb{H}^3, \mathbb{E}^3, \mathbb{S}^3$.

(b) Product spaces: $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}$.

(c) Twisted product spaces, each of which is a Lie group with a left-invariant metric: $\widetilde{SL}_2(\mathbb{R})$, i.e. the universal cover of $SL_2(\mathbb{R})$; Nil, i.e. the group of upper triangular 3-by-3 matrices with 1's on the diagonal; Sol, i.e. the semi-direct product of $\mathbb{R}^2$ and $\mathbb{R}$ where $t \in \mathbb{R}$ acts on $\mathbb{R}^2$ as the diagonal matrix:

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

Each of the above spaces is called a 3-dimensional *geometry*. A 3-manifold M is said to be *geometric*, *modelled on* X if $M = X/\Gamma$, where Γ is a discrete subgroup of $Isom(X)$ acting freely on X. Similarly, a 3-dimensional orbifold O is said to be *geometric* if $O \cong X/\Gamma$ where $\Gamma \subset Isom(X)$ is a discrete subgroup. For each geometry X, except $\mathbb{H}^3$ and *Sol*, any closed manifold modelled on X is Seifert.

Conjecture 20.2. *(Thurston's Geometrization Conjecture.) Let M be a closed 3-manifold which does not contain 2-sided projective planes. Then M admits a connected sum decomposition and a decomposition along disjoint incompressible tori and Klein bottles into pieces M_j such that each M_j is* **geometric**.

In the case when M is nonorientable and contains 2-sided projective planes the formulation of this conjecture is more complicated, it requires considering not only manifolds but also orbifolds, see Conjecture 20.7 below. We refer the reader to [Sco83a] for details. In particular, the above conjecture would imply that any homotopy equivalence of closed aspherical 3-manifolds is induced by a homeomorphism. This conjecture also includes the Poincare Conjecture and the Spherical Space Forms Conjecture as special cases:

Conjecture 20.3. *(3-dimensional Poincaré Conjecture.) Any 3-manifold which is homotopy equivalent to the 3-sphere is homeomorphic to $\mathbb{S}^3$.*

Conjecture 20.4. *(Spherical Space Forms Conjecture.) Let Γ be a finite group acting freely on the 3-sphere. Then this action preserves a metric of constant positive sectional curvature on $\mathbb{S}^3$.*

It is clear that Conjectures 20.1, 20.3 and 20.4 imply Conjecture 20.2. We note that Conjecture 20.2 would imply that if G is the fundamental group of a closed 3-manifold, G' is a torsion free group which contains G as a subgroup of finite index then G' is also a 3-manifold group. This conjecture is false if we allow G to be the fundamental group of a compact 3-manifold with nonempty incompressible boundary, see Exercise 1.48 and [KK99].

Conjecture 20.5. *(Thurston's Realization Conjecture.) Let M be a geometric 3-manifold and F be any finite group of diffeomorphisms of M. Then there is a geometric structure on M so that F acts by isometries of this structure.*

This conjecture was proven by Meeks and Scott [MS86] in the case when M is not modelled on $\mathbb{H}^3$ and $\mathbb{S}^3$. A special case of Conjecture 20.5 is the following

Conjecture 20.6. *(Generalized Smith's Conjecture.) Let Γ be a finite group acting smoothly on the 3-sphere. Then this action preserves a metric of constant positive sectional curvature on $\mathbb{S}^3$.*

Smith's conjecture itself which deals with smooth cyclic group actions where the generator fixes a smooth circle in $\mathbb{S}^3$, was proven in [MB84] as the result of collective efforts of a large number of people. The key part of the proof was Thurston's hyperbolization theorem.

Gabai in [Gab94] proved conjecture 20.5 for hyperbolic manifolds under the assumption that the action is free. Thurston claims to have a proof this conjecture in the case when the dimension of the fixed-point set of some nontrivial element of F is at least 1; he also claims the following. Suppose that M is a closed irreducible 3-manifold which admits a finite group of diffeomorphisms such that the dimension of the fixed-point set of some nontrivial element is at least 1, then the Conjecture 20.2 is valid for M. More general conjecture is

Conjecture 20.7. *(Thurston's geometrization conjecture for orbifolds.) Let O be a closed 3-dimensional orbifold which is not a manifold and which does not contain bad two-dimensional suborbifolds. Then O admits a decomposition along incompressible spherical and Euclidean suborbifolds into suborbifolds O_j so that the following holds. If we attach a 3-dimensional BALL[1] to each boundary component of O_j which is a spherical 2-orbifold, then each resulting orbifold $\widehat{O}_j$ is geometric.*

If in the above conjecture O is a manifold M then we get the geometrization conjecture for manifolds, however some $\widehat{O}_j$'s still could have nonempty singular locus. The reason for that is the fact that when we cut $M = O$ along 2-spheres and projective planes we get 3-manifolds whose boundaries might contain projective planes. To such boundary components we have to attach the cone over the projective plane which is a 3-BALL in our terminology.

Recently there was a considerable progress in proving Conjecture 20.7 under extra assumptions including the assumption that each component of the singular locus is one-dimensional, see [BP98], [CHK98] and [Zho99]. However at the present moment Conjecture 20.7 remains unproven even under the above dimension assumption.

Non-Haken hyperbolic manifolds. Thurston's hyperbolization conjecture is supported by the fact that large number of non-Haken aspherical 3-manifolds admit hyperbolic structure. One way to produce such examples is as follows. Let M^0 be a noncompact complete orientable hyperbolic 3-manifold of finite volume. Let M denote a compact core of M^0 (e.g. the thick part of M^0); let $T_1, ..., T_m$ be the boundary tori of M. For each $i = 1, ..., m$ pick a basis a_i, b_i in $H_1(T_i)$. We let $P \subset \mathbb{S}^2 = \overline{\mathbb{R}^2}$ denote the subset which consists of: pairs of coprime integers (p_i, q_i), pairs of the form $(0, \pm 1), (\pm 1, 0)$, and the point $\infty := (\infty, \infty)$. We give P the subset topology induced from $\mathbb{S}^2$. For a pair $(p_i, q_i) \in P - \{(\infty, \infty)\}$ let c_i denote a simple loop in T_i which represents the homology class of $p_i a_i + q_i b_i$. In the case $(p_i, q_i) = (\infty, \infty)$ we do not choose any c_i at all. Let

$$M_{(p_1/q_1, \ldots, p_m/q_m)}$$

[1] Recall that a BALL is a 3-dimensional orbifold which is the quotient of a 3-ball by a finite group of Euclidean isometries.

denote the 3-manifold obtained by attaching to M solid tori $\widehat{T}_i$ along those T_i's for which $(p_i, q_i) \neq (\infty, \infty)$, so that the loop c_i bounds a disk in $\widehat{T}_i$. In the case of $(p_i, q_i) = (\infty, \infty)$ we do not do anything with the torus T_i. The manifold $M_{(p_1/q_1,\ldots,p_m/q_m)}$ is said to be obtained from M by a *Dehn filling*. The following is known as *Thurston's hyperbolic Dehn surgery theorem*:

Theorem 20.8. *(W. Thurston [Thu81], see also [BP98].) There exists a neighborhood U of the point*

$$((\infty,\infty), \ldots, (\infty,\infty)) \in P^m \subset (\mathbb{S}^2)^m$$

such that for each $(p_1, q_1, \ldots, p_m, q_m) \in U$ the manifold $M_{(p_1/q_1,\ldots,p_m/q_m)}$ is hyperbolic.

This theorem allows to construct hyperbolic metrics on many non-Haken manifolds. For instance, if $K \subset \mathbb{S}^3$ is the "figure 8 knot", then $M^0 := \mathbb{S}^3 - K$ is hyperbolic [Thu81]. However for all but finitely many pairs $(p, q) \in P$, the resulting manifold $M_{p/q}$ is irreducible, aspherical but not Haken (see [HT85]).

Other related conjectures and possible approaches to the Geometrization Conjecture.

Conjecture 20.9. *(W.Jaco, W.Thurston, D.Gabai.) Let M be a closed 3-manifold whose fundamental group is not almost solvable. Then M admits finite coverings with the nonzero 1-st Betti numbers/ arbitrarily large 1-st Betti numbers.*

Note that if M is a closed irreducible 3-manifold satisfying the above conjecture, then M admits a finite cover M' which is Haken. Thus we can apply Thurston's Geometrization Theorem to the manifold M'. If M' contains an incompressible torus, then the manifold M is either Seifert or Haken, hence Conjecture 20.2 holds for M. If M' is atoroidal, then M is homotopy-equivalent to a hyperbolic 3-manifold N in which case Theorem 20.14 implies that M is hyperbolic itself.

Thus Conjecture 20.9 gives a possible approach to the Geometrization Conjecture. The problem however is that it is unclear why the manifold M should have any finite nontrivial coverings at all. Even if M itself is hyperbolic (and thus admits infinitely many finite coverings) Conjecture 20.9 is open even in the case when the fundamental group of M is arithmetic, see [LM93], [Lub96] for the recent progress in this direction.

Conjecture 20.10. *(D.Gabai.) Let M be a complete hyperbolic 3-manifold of finite volume. Then M admits a sequence of finite coverings $M_i \to M$ so that rank$(H_2(M_i, \partial M_i; \mathbb{Z}))$ tends to infinity.*

Recently the above conjecture (and the assertion about existence of closed incompressible surfaces in finite coverings over M) was proven for noncompact manifolds M in a joint work of Cooper, Long and Reid [CLR97].

The following related conjecture of Thurston seems the most doubtful of all his conjectures:

Conjecture 20.11. *(W.Thurston.) Any hyperbolic 3-manifold of finite volume is finitely covered by a surface bundle over* $\mathbb{S}^1$.

The large scale geometry approach.

The following is known as the *Weak Hyperbolization Conjecture:*

Conjecture 20.12. *(G.Mess, L.Mosher, U.Oertel.) Suppose that M is a closed aspherical 3-manifold. Then either $\pi_1(M)$ contains $\mathbb{Z}^2$ (i.e. M is* **toroidal***) or $\pi_1(M)$ is Gromov-hyperbolic.*

Note that one can drop the assumption that M is aspherical in this conjecture, it would still follow from Thurston's Geometrization Conjecture since the free product of Gromov-hyperbolic groups is Gromov-hyperbolic.

It is known that failure of Gromov-hyperbolicity of $\pi_1(M)$ implies existence of a "least area" map from a certain compact measured surface lamination of zero Euler characteristic into M (see [MO98], [Gab98]). Less technically, assume that M is given a Riemannian metric; if $\pi_1(M)$ is not Gromov-hyperbolic then there exists a conformal mapping f from $\mathbb{R}^2$ to the universal cover X of M so that the image of f is an immersed area-minimizing surface, i.e. for each disk $D^2 \subset \mathbb{R}^2$ with smooth boundary, the area of the smallest immersed disk in X with the boundary $f(\partial D^2)$ equals the area of $f(D^2)$, see [Gro87].

One way to prove Conjecture 20.12 would be to show that existence of such map implies that $\pi_1(M)$ contains $\mathbb{Z} \times \mathbb{Z}$. Along these lines it was proven in [Ka95a] that if M is a closed aspherical Riemannian manifold whose universal cover contains a totally geodesic Euclidean plane, then $\pi_1(M)$ contains $\mathbb{Z} \times \mathbb{Z}$. Gabai and Kazez [Gab98, GK98] recently proved Conjecture 20.12 for 3-manifolds which contain *genuine essential laminations.* This conjecture is also known to hold provided that M admits a metric of nonpositive curvature, see [Ebe72, Buy88, Sch90a].

Suppose that M is a closed aspherical 3-manifold whose fundamental group G is Gromov-hyperbolic. In this case one can compactify the Cayley graph of G by the *sphere at infinity* $\partial_\infty G$ on which G acts by homeomorphisms. It is known that $\partial_\infty G$ is homeomorphic to the 2-dimensional sphere (see [BM91]).

Conjecture 20.13. *(Cannon's Conjecture, see [CS98].) Under the above conditions the action of G on $\mathbb{S}^2$ is topologically conjugate to the action of a group G' that consists of holomorphic and antiholomorphic transformations of the standard conformal structure on* $\mathbb{S}^2$.

Suppose for a moment that Conjectures 20.12 and 20.13 are valid. Then each closed irreducible aspherical 3-manifold M is either toroidal or its fundamental group acts on $\mathbb{S}^2$ as a discrete group of conformal and anticonformal transformations. In the former case M is either Haken or Seifert (it follows from the results [Mes90], [Gab92], [CJ94], [Sco80] and Thurston's Geometrization Theorem for Haken manifolds). In the latter case, G is isomorphic to a discrete subgroup G' of $\mathrm{Isom}(\mathbb{H}^3)$, i.e. M is homotopy-equivalent to a closed hyperbolic 3-manifold $N = \mathbb{H}^3/G'$. Thus, according to Theorem 20.14 below, the manifold M is hyperbolic.

Theorem 20.14. *(D.Gabai, R.Meyerhoff, N.Thurston, see [Gab97], [GMT97].) Any closed irreducible 3-manifold M which is homotopy-equivalent to a hyperbolic manifold, is hyperbolic itself.*

Thus (assuming Conjectures 20.12 and 20.13) in both *toroidal* and *Gromov-hyperbolic* cases of the Conjecture 20.12, the manifold M satisfies Thurston's Hyperbolization Conjecture.

Thurston's approach to hyperbolization of manifolds admitting taut foliations.

Suppose that M is a closed (irreducible aspherical atoroidal) 3-manifold and $\mathcal{F}$ is a *taut foliation* on M, i.e. a smooth foliation such that M contains a smooth loop which intersects transversally each leaf of $\mathcal{F}$. For instance, if M fibers over the circle then the fibers of this fibration form a taut foliation of M. Currently it is unknown if every closed irreducible atoroidal aspherical 3-manifold admits a taut foliation; Thurston's approach deals only with the manifolds which do admit taut foliations. Thurston suggests to construct a hyperbolic structure on M by imitating construction of a hyperbolic structure on 3-manifolds fibered over the circle. A bit more precisely, put a Riemannian metric on M, let $\tilde{\mathcal{F}}$ denote the lift of $\mathcal{F}$ to the universal cover $\tilde{M}$ of M. Assume that the induced metric on each leaf of $\tilde{\mathcal{F}}$ is conformal to the hyperbolic plane. Thurston proposes to identify (in $\pi_1(M)$-invariant way) the ideal boundaries of the leaves of $\tilde{\mathcal{F}}$ and construct a geodesic lamination L_F on each leaf F of $\tilde{\mathcal{F}}$ for which the following holds. Take two sequences F_n, F'_n of leaves of the lift of $\mathcal{F}$ to $\tilde{M}$ assuming that their minimal distance tends to infinity as $n \to \infty$. Realize each F_n, F'_n as a convex pleated surface Φ_n, Φ'_n in $\mathbb{H}^3$ with the pleated locus given by $L_{F_n}, L_{F'_n}$ so that Φ_n, Φ'_n bound a convex region Θ_n in $\mathbb{H}^3$. Thurston suggests that this could be done in such a way that the region R_n is (L_n, A_n)-quasi-isometric to the corresponding region $T_n \subset \tilde{M}$ with (L_n, A_n) uniformly bounded as n tends to infinity. As the result, the universal cover $\tilde{M}$ is quasi-isometric to $\mathbb{H}^3$, so the fundamental group of M is isomorphic to a cocompact discrete subgroup of $Isom(\mathbb{H}^3)$ (see [CC92]). By Theorem 20.14, M is a hyperbolic manifold. Part of this program was carried out by Thurston [Thu97b] in the case when foliation comes from a *slithering* over the circle and by Calegari [Cal99] in a more general case when the foliation $\tilde{\mathcal{F}}$ is dual to either a line or 1-rooted tree.

Differential geometry approaches.

The general idea behind the differential geometric approaches to the geometrization conjecture is to start with a more or less randomly chosen Riemannian metric g on a closed 3-manifold M and then deform g to a metric of constant sectional curvature along a sequence (or a continuous family) of Riemannian metrics g_t. Of course, M does not have to admit a metric of constant sectional curvature. Conjecturally, this corresponds to a particular degeneration of the family of metrics g_t:

(i) Either g_t degenerates along a globally defined foliation of M with compact leaves (this would correspond to existence of a *geometric structure* on M which is either *Sol* or one of the structures on Seifert manifolds).

(ii) Or g_t degenerates along a collection C of incompressible tori/Klein bottles and homotopically nontrivial spheres/projective planes. On the complement to C the metrics g_t are supposed to either converge to complete metrics of constant sectional curvature or degenerate as in (i).

The question is how to choose the sequence/family g_t and how to verify that the degeneration really corresponds to the decomposition of M in geometric components. I refer the reader to the survey papers [Yau88], [Ham95], [Ham99], [And93], [And97], [And99] for more detailed discussion of these issues, I present here only a very brief overview.

(a) **The Ricci flow**. This approach was introduced by R. Hamilton in [Ham82]. Let $g(t)$ be a smooth 1-parameter family of Riemannian metrics on M satisfying the *Ricci flow equation*:

$$\frac{\partial}{\partial t} g(t) = 2(r(t)g(t)/3 - Ric(t))$$

where $Ric(t)$ is the Ricci curvature and $r(t)$ is the average scalar curvature of $g(t)$ with randomly chosen initial Riemannian metric $g_{ij}(0)$. The family of metrics $g(t)$ is called the *Ricci flow*. Using this flow Hamilton proved in [Ham82] that a Riemannian metric of positive Ricci curvature on a closed 3-manifold M can be deformed to a metric of constant positive sectional curvature, in particular, M is a spherical space-form. One of the difficulties of this approach is the fact that in general the solution $g(t)$ can *blow up* even in a finite amount of time. See [Ham95].

(b) **Minimax sequences.** I will describe here only one example of the minimax problem, the reader is referred to the Anderson's survey papers [And93], [And97] and the paper of Schoen [Sch89] for more detailed discussion. Consider the space $\mathcal{M}_1$ of all Riemannian metrics on M with unit volume modulo the diffeomorphism group of M. The *total scalar curvature functional* $\mathcal{S} : \mathcal{M}_1 \to \mathbb{R}$ is given by the formula:

$$\mathcal{S}(g) = \int_M s_g dV_g$$

where s_g is the scalar curvature of the metric g and dV_g is the volume form of g. The critical points of this functional are the Einstein metrics, i.e. metrics of constant sectional curvature in the dimension 3. Here is the *the minimax procedure for finding the critical points of* $\mathcal{S}$. Consider the *conformal class* $[g]$ of g, i.e. collection of metrics on M of the form $\lambda \cdot g$, where λ is a positive function on M. Define the *Yamabe constant* of $[g]$ as

$$\mu[g] = \inf_{g \in [g]} \mathcal{S}(g)$$

and the σ-constant of M as

$$\sigma(M) = \sup_{[g] \in \mathcal{C}(M)} \mu[g]$$

where $\mathcal{C}(M)$ is the collection of conformal structures on M, i.e. the equivalence classes $[g]$. According to the solution of the Yamabe problem, there is a Riemannian metric $g_\mu \in [g] \cap \mathcal{M}_1$ such that $\mathcal{S}(g_\mu) = \mu[g]$. The solutions g_μ of the Yamabe minimization problem are called *Yamabe metrics.* The *minimax sequence* is a sequence of Yamabe metrics $\{g_j\}$ in $\mathcal{M}_1$ such that $\lim_j \mu[g_j] = \sigma(M)$. The problem is to show that the sequence g_j:

(1) Either converges to a Riemannian metric g (in which case g should be an Einstein metric, and hence a metric of constant sectional curvature).

(2) Or g_j degenerates either along a foliation with compact leaves on M or according to a decomposition of M along a collection of incompressible tori/Klein bottles and homotopically nontrivial spheres/projective planes so that the complementary regions admit geometric structures.

(c) **Alexandrov spaces approach.** Let $\mathcal{N}_{-1}(M)$ denote the collection of Riemannian metrics of the sectional curvature ≥ -1 on the given closed n-manifold M. Consider the problem of minimizing the volume of metrics in $\mathcal{N}_{-1}$. It is known that this minimum is achieved in the compactification $\overline{\mathcal{N}_{-1}}$ of $\mathcal{N}_{-1}$ in the topology of *Gromov-Hausdorff convergence.* The elements of $\overline{\mathcal{N}_{-1}}$ are *Alexandrov spaces* of the curvature bounded from below by -1 (see [BGP92]). Now restrict to the case when M is 3-dimensional. The Gromov-Hausdorff limit of a volume-minimizing sequence $(M, g_j), g_j \in \mathcal{N}_{-1}(M)$, depends on the choice of base-points $x_j \in M_j$. The limit breaks into components depending on the location of x_j's. This decomposition into components conjecturally corresponds to the decomposition of M along spheres, projective planes, incompressible tori and Klein bottles into *geometric components.* Suppose that the limit of the volume-minimizing sequence (M, g_j) is a compact connected 3-dimensional Alexandrov space X (which does not depend on the choice of the base-points). Conjecturally X is a Riemannian manifold of the constant sectional curvature -1. This conjecture is open even if the limit is a Riemannian manifold. It is proven by Besson, Courtois and Gallot in [BCG95] (see also [BCG96] for an easier proof) that if (M, g) is a closed Riemannian n-manifold of the sectional curvature -1 then the metric g minimizes volume in the class of Riemannian metrics $\mathcal{N}_{-1}(M)$. One can compare this with the paper of Bonahon [Bon98], who proves similar minimization result in a special class of Alexandrov metrics on a compact hyperbolic 3-manifold.

20.2. Other structures

Thurston's hyperbolization theorem and Jaco-Shalen-Johannson decomposition theorem imply that each compact Haken 3-manifold M whose boundary consists of tori and Klein bottles is either is geometric itself or admits a canonical decomposition (along incompressible tori and Klein bottles) into geometric components. The issue which we address in this section is what kind of "geometric structure" could be introduced on the whole manifold M. There are several kinds of structures one can play with.

(a) Metrics of nonpositive curvature on M. Although not as useful as metrics of negative curvature, such metrics exist on a wider class of manifolds. We will assume that with respect to the metric in question the boundary of M is totally geodesic. We first note that the standard metrics on $\mathbb{H}^3$, $\mathbb{E}^3$ and $\mathbb{H}^2 \times \mathbb{R}$ are nonpositively curved. Thus they model manifolds which admit metrics of nonpositive curvature. On the other hand, it was known since the early 1980's that closed manifolds modelled on Nil, Sol and $\widetilde{SL}_2(\mathbb{R})$ geometry do not admit metrics of nonpositive curvature [CE80], [Ebe82].

Theorem 20.15. *(B. Leeb, [Lee95]) Suppose that M is a Haken manifold which either has nonempty incompressible boundary or its JSJ decomposition contains a hyperbolic component. Then M admits a metric of nonpositive curvature.*

A closed 3-manifold is called a *graph-manifold* if it is neither Seifert, nor is modelled on Sol, but its JSJ decomposition contains only Seifert components. Existence of a metric of nonpositive curvature on graph-manifolds is a nontrivial combinatorial issue, it was resolved by Buyalo and Kobelsky [BK95] in terms of the gluing data. Below are two examples. Suppose that $f : S \to S$ is the Dehn twist along a homotopically nontrivial loop on a closed hyperbolic surface S. Then the mapping torus of f is a graph-manifold which does not admit a metric of nonpositive curvature. On the other hand, if M is a graph-manifold whose JSJ decomposition contains a nonorientable Seifert component then M admits a metric of nonpositive curvature [KL96].

(b) Birational geometry. Recall that a homeomorphism between connected open subsets of $\mathbb{S}^3$ is called *birational* if its components are rational functions. A *birational structure* on a closed 3-manifold M is a maximal atlas where all transition maps are birational. A special case of birational structures consists of *uniformizable* birational structures: $M = \Omega/G$ where Ω is an open subset of $\mathbb{S}^3$ and G is a group of birational homeomorphisms acting freely properly discontinuously on Ω. For instance, if $M = \mathbb{H}^3/G$ is a hyperbolic manifold where G is a discrete torsion-free group of isometries of $\mathbb{H}^3$ then we get a natural uniformizable birational structure on M. It is conceivable that all closed 3-manifolds admit uniformizable birational structures, this was conjectured by J. Hempel in [Hem76]. One can show that if M satisfies Thurston's geometrization conjecture and no component of the decomposition of M along spheres and projective planes is a non-Haken Seifert manifold, then Hempel's conjecture holds for M [Ka92a].

(c) Non-Riemannian locally-homogeneous spaces. Suppose that X is a smooth manfold which admits a smooth transitive action of a Lie group G. Call the pair (X, G) a *non-Riemannian geometry.* Such geometries can be described as quotients $(G/H, G)$, where G is a Lie group and H is a closed subgroup in G. A closed manifold is said to be modelled on a non-Riemannian geometry (X, G) if it admits an atlas (with values in X) whose transition maps are restrictions of the elements of G to open subsets of X. Most pairs (X, G) are too *rigid* to admit closed manifolds modelled on them. Among important exceptions to this rule are:

(1) *The flat conformal geometry*, $X = \mathbb{S}^n$, G is the group of Moebius transformations of X.

(2) *The spherical CR geometry*, $X = \mathbb{S}^{2n-1}$, G is the group $PU(n, 1)$ of holomorphic automorphisms of the closed unit ball in $\mathbb{C}^n$, restricted to

$\mathbb{S}^{2n-1}$.

(3) *The projective geometry*, $X = \mathbb{S}^n$, $G = GL(n+1, \mathbb{R})$. (Strictly speaking this is a 2-fold covering of the projective geometry.)

Similarly to the case of complex-projective structures, each (X, G)-structure on a manifold M gives rise to a developing mapping $d : \tilde{M} \to X$ from the universal covering of M. This implies that for a simply-connected X (like in all the above examples), the only compact simply-connected manifold M modelled on (X, G) is X itself. For instance, if a counterexample Σ to the 3-dimensional Poincaré conjecture exists, then Σ does not admit a geometric structure.

Conjecture 20.16. *Suppose that M is a closed connected orientable 3-manifold. Then for each 3-dimensional non-Riemannian geometry (X, G) (from the above list) there is a closed 3-manifold N (depending on M and (X, G)) so that the connected sum $M \# N$ is modelled on (X, G).*

Not much is known about manifolds which admit flat projective structures, i.e. are modelled on the projective geometry $(\mathbb{S}^3, GL(4, \mathbb{R}))$ apart from a number of examples. Each manifold modelled on any 3-dimensional (Riemannian) geometry, with the possible exception of $\widetilde{SL}(2, \mathbb{R})$, admits a flat projective structure. For instance, to get hyperbolic manifolds we can use the projective model of the hyperbolic 3-space. I will discuss the examples (1) and (2) in the 3-dimensional case.

1. It is known [Gol83], [Mat92] that no closed manifold modelled on Sol or Nil geometry admits a flat conformal structure, i.e. is modelled on the flat conformal geometry. Some graph-manifolds are also known not to admit a flat conformal structure [Ka90]. On the positive side, one conjectures the following:

Conjecture 20.17. *Suppose that M is a closed 3-manifold satisfying Thurston's geometrization conjecture so that no component of the decomposition of M along spheres and projective planes is modelled on Sol or Nil. Then there is a finite covering $M' \to M$ such that M' admits a flat conformal structure.*

This conjecture was proven [Ka89, Ka93, Ka90] under the extra assumption that M is a Haken manifold such that in the geometric decomposition of M no two hyperbolic components are adjacent (in particular, no hyperbolic component is glued to itself).

2. It is known that no closed 3-manifold modelled on $\mathbb{E}^3$ or Sol admits a *locally spherical CR-structure*, i.e. is modelled on the spherical CR geometry. Much less is known on the positive side. For instance it is unknown if a product manifold $F \times \mathbb{S}^1$ (where F is a closed hyperbolic surface) can be modelled on $(\mathbb{S}^3, PU(2,1))$. Some Seifert manifolds modelled on $\widetilde{SL}_2(\mathbb{R})$ are known to admit a locally spherical CR-structure, for instance, the total space of the unit tangent bundle to a closed hyperbolic surface [BS76]. Only recently R. Schwartz constructed the first example of a closed hyperbolic 3-manifold admitting a locally spherical CR-structure [Sch98]. Currently there are no counterexamples to the following analogue of the conjecture 20.17:

Conjecture 20.18. *Suppose that M is a closed 3-manifold satisfying Thurston's geometrization conjecture so that no component of the decomposition of M along spheres and projective planes is modelled on Sol or $\mathbb{E}^3$. Then there is a finite covering $M' \to M$ such that M' admits a locally spherical CR-structure.*

However this conjecture is unknown even for graph-manifolds. It is known [Min90] that if a closed 3-manifold M has almost solvable fundamental group and M admits a locally spherical CR-structure then M is either modelled on $\mathbb{S}^3$ or $\mathbb{S}^2 \times \mathbb{R}$ or on the Nil-geometry.

20.3. Higher dimensions

One of the main points of Thurston's work on the geometry of 3-manifolds is that "generic" closed 3-manifolds are hyperbolic. The situation is much different in higher dimensions. Note that a special case of Thurston's geometrization conjecture is that a closed 3-manifold which admits a Riemannian metric of negative curvature also admits a metric of constant negative curvature. Starting with the dimension 4, there are symmetric spaces which are negatively curved but their metrics do not have constant curvature. A typical example of such space is the complex-hyperbolic n-space $\mathbb{CH}^n$. According to a theorem of Borel's (see for instance [Rag72]) the isometry group of each symmetric space X contains a discrete cocompact subgroup. Mostow rigidity theorem for locally symmetric spaces [Mos73] (which is a far-reaching generalization of Mostow rigidity theorem for the hyperbolic space $\mathbb{H}^n$) implies that for two *irreducible* noncompact symmetric spaces X, X' the isometry groups $\mathrm{Isom}(X), \mathrm{Isom}(X')$ are isomorphic if and only if they contain isomorphic discrete subgroups Γ, Γ' with quotients $X/\Gamma, X'/\Gamma'$ of finite volume (i.e. Γ, Γ' are lattices). In particular, unless $n = 1$, no quotient of $X = \mathbb{CH}^n$ by a discrete torsion-free cocompact subgroup of $Isom(X)$ admits a metric of constant (negative) sectional curvature. Moreover, Mostow and Siu [MS80] constructed the first example of a negatively curved compact Kähler manifold of complex dimension 2 whose fundamental group is not isomorphic to a lattice in *any* symmetric space. Examples of closed negatively curved manifolds of arbitrary dimension $n \geq 4$ which are not homeomorphic to locally symmetric manifolds, were constructed later by Gromov and Thurston [GT87]. On the other hand, something very strange happens in higher dimensions:

There are embarrassingly few known constructions of negatively curved (even in $CAT(-1)$ sense) higher-dimensional manifolds. The known constructions either use discrete subgroups of Lie groups in some way, or they are based on Davis' orbifold trick [Dav83]. The only known examples of compact higher dimensional aspherical n-manifolds M ($n \geq 6$) are either obtained from compact locally symmetric spaces by a cut-and-paste operation or by taking certain ramified coverings (as in [GT87], [CD95]), or $\pi_1(M)$ contains Baumslag-Solitar subgroups (in the last case M does not admit a metric of a negative curvature)[2]. It remains to be seen if one can

[2]Although Gromov in his recent preprint [Gro99] promises constructions of higher-

produce higher-dimensional closed negatively curved manifolds (or higher-dimensional Gromov-hyperbolic groups) via a construction which does not rely on existence of arithmetic subgroups in Lie group one way or another. This brings us to the next subject.

20.4. Hyperbolic groups

This subject (and, more generally, the geometric group theory) was developing rapidly since the mid 1980's. Its development, especially on the early stages, was greatly motivated by the Kleinian groups theory and by Thurston's hyperbolization theorem, and the technique used in its proof. Our discussion of hyperbolic groups is of course biased and far from comprehensive, I will mostly consider the aspects of the theory of hyperbolic groups which are related to Thurston's hyperbolization theorem.

Purely group-theoretic analogues of the gluing theorems 15.1 and 15.18 (under the assumption $P = \emptyset$) were proven by Bestvina and Feighn in [BF92], [BF96]. Namely, they proved that under appropriate assumptions the fundamental group of a graph of hyperbolic groups is Gromov-hyperbolic. The whole procedure of assembly of the fundamental group a closed atoroidal Haken 3-manifold from pieces was done by Swarup in [Swa98] who used [BF92], [BF96] and Mitra's paper [Mit97] to prove (by purely group-theoretic methods) that the fundamental group of each closed atoroidal Haken 3-manifold is Gromov-hyperbolic.

As we noted above, philosophically speaking, Thurston's hyperbolization theorem states that "generic" closed 3-manifolds are hyperbolic. Something analogous is true for hyperbolic groups. The following was stated by Gromov in [Gro87] and proven by Olshansky in [Ol'92] and (in a special case) by Champetier in [Cha95]:

Theorem 20.19. *Let $k \in \mathbb{Z}$ ($k \geq 2$) and let $X = \{x_1^{\pm 1}, ..., x_k^{\pm 1}\}$ be an alphabet. Let $i \in \mathbb{Z}_+$ and let $(n_1, ..., n_i)$ be a sequence of positive integers. Let $N = N(k, i, n_1, ..., n_i)$ denote the number of group presentations*

$$G = \langle x_1, \ldots, x_k | R_1, \ldots, R_i \rangle$$

such that $R_1, \ldots, R_i$ are reduced words in the alphabet X such that the length of R_j is n_j for $j = 1, 2, \ldots, i$. Let N_h be the number of hyperbolic groups in this collection and $n := \min\{n_1, \ldots, n_i\}$. Then

$$\lim_{n \to \infty} N_h / N = 1.$$

Thus, with the "probability 1" a finitely-presented group is Gromov-hyperbolic. Note however, that Thurston's hyperbolization theorem is not a "probabilistic" assertion: it describes precise obstructions to hyperbolicity of the fundamental group G of a Haken manifold (in terms of subgroups of G). There are only few such results in the purely group-theoretic context.

dimensional aspherical manifolds which we have not seen before.

Below we list the basic properties of Gromov-hyperbolic groups G which could be viewed as "obstructions" to hyperbolicity. We refer to [Gro87], [GdlH91], [ABC], [CDP90] for details; the reader should compare these properties with the properties of discrete convex-cocompact subgroups of $\mathrm{Isom}(\mathbb{H}^n)$.

1. G admits a properly discontinuous simplicial cocompact action on a contractible simplicial complex X. In particular, G is finitely presentable and if G is torsion-free then there exists a finite $K(G,1)$ space.

2. G contains only finitely many conjugacy classes of finite subgroups.

3. The group $\mathbb{Z}^2$ cannot embed into G. Moreover, if $a, b \in G$ satisfy the Baumslag-Solitar relation for certain p, q,

$$ba^p b^{-1} = a^q$$

and a has infinite order then the group generated by a and b is almost cyclic.

Only recently N. Brady [Bra99] constructed the first example of a finitely presentable group G which satisfies the properties 2 and 3 but is not Gromov-hyperbolic; in Brady's example G is torsion-free but does not admit finite $K(G,1)$, so the 1-st property fails. It is almost surely true that there are groups satisfying the properties (1)–(3) which are not Gromov-hyperbolic. However for certain special classes of groups the properties (1)–(3) might imply hyperbolicity. This is known for instance for extensions of finitely generated free groups via automorphisms [BF92, BF96, BH92]. Certain generalizations of this result were obtained by P. Brinkmann in [Bri99] and I. Kapovich in [Kap].

It is conjectured by S. Gersten that for a finitely generated 1-relator torsion-free group G, Gromov-hyperbolicity is equivalent to the assumption that G contains no Baumslag-Solitar subgroups. One-relator groups with torsion all have the form:

$$\langle x_1, ..., x_n | R(x_1, ..., x_n)^p = 1\rangle$$

where $p \geq 2$ and R is a word in $x_1, ..., x_n$. Every such group is Gromov-hyperbolic according to a result of B.B.Newman [New68]. In general there are some interesting similarities between the theory of 1-relator groups and the theory of Haken 3-manifolds. Namely, there is a hierarchy of finitely generated 1-relator groups $G_0 = G, G_1, ..., G_m$, such that G_m is a free group (or a free product of a finite cyclic group with a free group); each G_i is obtained from G_{i+1} by one of the following two procedures:

(a) either by HNN extension along a certain finitely generated free subgroup (so called *Magnus subgroup*), or

(b) $G_i \subset \langle z\rangle *_{g=z^k} G_{i+1}$ where $g \in G_{i+1}$ and G_{i+1} a 1-relator group which is obtained from G_{i+2} by the procedure described in (a). (Note that if G_{i+1} happens to be Gromov-hyperbolic then G_i is also hyperbolic.)

We refer the reader to [LS77] for a detailed description of this hierarchy.

One of the implications of Thurston's hyperbolization theorem is the fact that the fundamental groups of Haken manifolds are residually finite. Recall that for finitely generated groups residual finiteness implies the Hopf property. Although it is unknown if all Gromov-hyperbolic groups are residually finite (which is very unlikely), Z.Sela proved in [Sel99] that each hyperbolic group satisfies the Hopf property.

Algorithmic aspects. One of the important features of Gromov-hyperbolic groups is solvability of the word and conjugacy problems for their elements (these problems are known to be unsolvable for general finitely presented groups). A much deeper result is a theorem of Z.Sela [Sel95] who proved solvability of the isomorphism problem for the class of torsion-free word-hyperbolic groups that do not split (as an amalgamated product or an HNN extension) over the trivial or the infinite cyclic group. Some ideas of Sela's work come from the Rips' theory of group actions on trees. Sela's work also implies solvability of the homeomorphism problem for 3-manifolds satisfying Thurston's geometrization conjecture.

The class of Gromov-hyperbolic groups has several generalizations, most important is probably the class of *automatic groups*, see [ECH92]. For such groups the word problem is known to be solvable, but the conjugacy problem is currently open. The conjugacy problem is known to be solvable for the (conjecturally smaller) class of bi-automatic groups. Although automatic groups have much less structure than the hyperbolic ones, they are much easier to find. For instance, if M is any 3-manifold satisfying Thurston's geometrization conjecture which has no *Nil* and *Sol* components, then $\pi_1(M)$ is automatic, see [ECH92]. If $G \subset \mathrm{Isom}(\mathbb{H}^n)$ is a geometrically finite group then G is bi-automatic [ECH92], in particular the conjugacy problem for G is solvable.

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