WEAK HYPERBOLIZATION CONJECTURE FOR 3-DIMENSIONAL CAT(0) GROUPS

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Abstract. In this note we prove a weak hyperbolization conjecture for CAT(0) 3-dimensional Poincaré duality groups.

1. Introduction

Let G be a 3-dimensional Poincaré duality group over a commutative hereditary ring $\mathcal{R}$ with a unit; for instance, G could be the fundamental group of a closed aspherical 3-manifold. Suppose in addition that G is a $CAT(0)$–group, i.e., a group which admits a cocompact isometric properly discontinuous action $G \curvearrowright X$ on a locally compact $CAT(0)$ space X. The main result of this note is the following:

Theorem 1. *Under the above assumptions either G is Gromov–hyperbolic or G contains $\mathbb{Z}^2$.*

We note that special cases of this theorem were proven earlier by various people: S. Buyalo [8] and V. Schroeder [18] independently have proven that this theorem holds provided that X is the universal cover $\tilde{M}$ of a closed 3-manifold M, the $CAT(0)$–structure on $\tilde{M}$ is Riemannian and $G = \pi_1(M)$ acts on X by deck–transformations. L. Mosher [16] proved that Theorem 1 holds provided that $X = \tilde{M}$, $G = \pi_1(M)$, and the $CAT(0)$ metric on is obtained by lifting a piecewise-Euclidean (locally) $CAT(0)$–cubulation from M. M. Bridson and L. Mosher also have an unpublished proof of Theorem 1 under the assumption that $X = \tilde{M}$ has an arbitrary G-invariant $CAT(0)$–structure. Unlike all these proofs, our proof takes place on the ideal boundary of X; this allows us to treat 3-dimensional Poincaré duality groups.

Outline of the proof of Theorem 1.

In Section 3 we review the definition and properties of pretrees. We then show that, under certain conditions, one can associate an $\mathbb{R}$–tree to a pretree.

In Section 4.1 we prove that the ideal boundary of the $CAT(0)$ space X is homeomorphic to S^2.

We then assume that G is not Gromov hyperbolic, i.e., that X contains a 2–flat. Our proof exploits the geometry of flats and parallel sets in X, and the pattern of their boundaries in $\partial_\infty X$. The case breakdown goes as follows.

In Section 4.2 we analyze the case when the space X contains a 3-flat. In this case we show that G is commensurable to $\mathbb{Z}^3$.

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In Section 4.4 we analyze the case when the space X contains no 3-flats but some parallel subset $P \subset X$ has *full* ideal boundary, i.e. $\partial_\infty P = \partial_\infty X$. Then we prove that a finite index subgroup of G is isomorphic to the fundamental group of a 3-dimensional Seifert manifold.

In Section 4.5 we analyze the *generic case*, when X contains no 3-flat, and no parallel set has full boundary.

We show that every parallel set P in X is isometric to a product $\mathbb{R} \times Y$, where Y is Gromov hyperbolic. The ideal boundary of P is a suspension of the boundary $\partial_\infty Y$; when P contains a 2-flat, we identify certain topological circles in $\partial_\infty P$ which we call *peripheral*, and show that peripheral circles cannot *cross* one another. Next, we choose a flat $F \subset X$ whose boundary $\partial_\infty F \subset \partial_\infty X$ is a peripheral circle, and consider its orbit $\{g(F)\}_{g\in G}$. Because the circles $\{g(\partial_\infty F)\}_{g\in G}$ do not cross, we may use them to define a pretree $\mathcal{T}$ on which G has a natural action. Using a Plante-type construction, we associate to $\mathcal{T}$ an $\mathbb{R}$–tree T, which then inherits a nontrivial small stable G-action. By applying Rips' theory [3], we conclude that G admits a small nontrivial action on a simplicial tree. Using the fact that G is a $PD(3)$ group, we deduce that the edge groups must be virtually $\mathbb{Z}^2$.

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2. Geometric preliminaries

In this section we briefly review several notions of metric geometry. We refer the reader to [1], [6] for the detailed discussion.

A *geodesic metric space* is a metric space (X, d) such that any two points $x, y \in X$ in X are connected by geodesic, i.e. if $D := d(x, y)$ then there exists an isometric embedding

$$\gamma : [0, D] \to X,$$

so that $\gamma(0) = x, \gamma(D) = y$.

Let X be a metric space and $C \subset X$ be a subset. The r-neighborhood of C in X is defined as

$$N_r(C) := \{x \in X : d(x, C) < r\},$$

where $d(x, C) := \inf\{d(x, c) : c \in C\}$.

The *Hausdorff distance* between closed subsets of a metric space X is defined as

$$d_H(C_1, C_2) := \inf\{r : C_1 \subset N_r(C_2), \quad C_2 \subset N_r(C_1)\}.$$

Note that this distance is allowed to take infinite values. If X has finite diameter, the Hausdorff distance defines the *Hausdorff topology* on the set $\mathcal{C}(X)$ of closed subsets of X. More generally, even for unbounded metric spaces X one defines the *Gromov–Hausdorff topology* on $\mathcal{C}(X)$ as follows. We say that a sequence $C_n \in \mathcal{C}(X)$ converges (in the Gromov–Hausdorff topology) to a closed set $C \in \mathcal{C}(X)$ if for each closed metric ball $B \subset X$ the intersections

$$C_n \cap B \in \mathcal{C}(B)$$

converge to $C \cap B$ in the Hausdorff topology on $\mathcal{C}(B)$. Equivalently, C_n's converge to C if the corresponding distance functions $d(\cdot, C_n)$ converge to the distance function $d(\cdot, C)$ uniformly on bounded subsets in X.

Given a number $\kappa \in \mathbb{R}$ let M_κ denote the (unique up to isometry) complete simply-connected surface of the constant curvature κ. A geodesic metric space X is said to be a $CAT(\kappa)$ space if X is complete as a metric space and geodesic triangles in X are "thinner" than triangles in M_κ. More precisely, consider a geodesic triangle $T = [x, y, z] \subset X$ (with the vertices x, y, z), in case when $\kappa > 0$ (and M_κ is a sphere) we assume that the perimeter of this triangle is less than the circumference of the great circle in M_κ. Consider a triangle $T' = [x', y', z'] \subset M_\kappa$ whose side-lengths are equal to the corresponding side-lengths of the triangle T. Let p be a point in the geodesic side $\overline{xy}$ of T and let $p' \in \overline{x'y'}$ be such that

$$d(x', p') = d(x, p).$$

Then we require

$$d(x, p) \leq d(x', p').$$

In this paper we will also need a generalization of the concept of a $CAT(1)$ space to metric spaces X which are not geodesic. We assume that X is a disjoint union of geodesic metric spaces X_α, $\alpha \in J$, where each X_α is a geodesic $CAT(1)$ metric space and if $\alpha \neq \beta$ the distance between any $x \in X_\alpha, y \in X_\beta$ equals π. Then X will be also referred to as a $CAT(1)$ space. An example of such a space is a space with discrete metric where distance between any pair of distinct points equals π.

If $X$is a $CAT(1)$ space, we call points $x, y \in X$ *antipodal* if $d(x, y) = \pi$.

Suppose that X is a $CAT(0)$ space. Then the distance function on X is *convex*, i.e., its restriction to each geodesic in X is convex.

A space X is called $CAT(-\infty)$ if it is $CAT(\kappa)$ for each $\kappa \in \mathbb{R}$. A *metric tree* is a $CAT(-\infty)$; in other words, it is a complete geodesic metric space where each geodesic triangle is isometric to a tripod.

A group G is called a $CAT(0)$-*group* if it admits an isometric properly discontinuous cocompact action on a locally compact $CAT(0)$-space.

Suppose that X is a $CAT(0)$ space and $F \subset X$ is a k-*flat*, i.e., an isometrically embedded copy of a Euclidean space $\mathbb{R}^k$. Then the *parallel set* P_F of F in X is the union of all k-flats $F' \subset X$ which are within finite distance from F. The parallel set P_F is closed, convex and is isometric to a product

$$F \times Y$$

where Y is a $CAT(0)$ space, see for instance [6, Theorem II.2.14].

Remark 2. Theorem II.2.14 in [6] is stated in the case $k = 1$. The general case follows, for instance, by induction on the dimension of the flat.

We will say that a parallel set is *trivial* if $k = 1$ and Y is bounded.

Given a $CAT(0)$ space one defines the *ideal boundary* of X as the collection of equivalence classes of geodesic rays in X, where rays are equivalent if they are within finite Hausdorff distance from each other. This boundary has two (typically distinct) topologies:

1. The *visual topology*, in which case the ideal boundary is denoted $\partial_\infty X$ and is called the *geometric boundary* of X.

2. The *Tits topology*, which is defined via the *Tits angular metric*, in which case the ideal boundary is denoted $\partial_{Tits} X$.

The second boundary is called *Tits boundary* of X; this boundary is always a $CAT(1)$ space.

For instance, in the case when $X = \mathbb{H}^2$, $\partial_\infty X$ is homeomorphic to S^1, while $\partial_{Tits} X$ has discrete metric: the distance between distinct points equals π. A $CAT(0)$ space is called a *visibility space* if any pair of distinct points in $\partial_{Tits} X$ are antipodal.

A subset $C \subset Z := \partial_{Tits} X$ is called *convex* if for any two non-antipodal points $x, y \in Z$, the geodesic segment $\overline{xy}$ connecting x to y, is entirely contained in C. Intersection of two convex subsets of Z is also convex. If $Y \subset X$ is a convex subset then $\partial_{Tits} Y \subset Z$ is convex as well.

Let $\delta \in [0, \infty)$ and consider a geodesic metric space X. A triangle $T \subset X$ is called *δ-thin* if there exists a point $p \in X$ which is within distance $\leq \delta$ from all three sides of T. A complete geodesic metric space X is called *δ-hyperbolic* if each geodesic triangle T in X is δ-thin. A space X is called *Gromov–hyperbolic* if it is δ-hyperbolic for some δ. A finitely generated group G is called *Gromov–hyperbolic* if its Cayley graph is Gromov–hyperbolic. One again defines the ideal boundary $\partial_\infty X$ by looking at the equivalence classes of geodesic rays in X.

Suppose that G is a group acting isometrically, properly discontinuously and cocompactly on a $CAT(0)$ space X. Then the group G is Gromov–hyperbolic iff X is a visibility space.

Let X be a Gromov-hyperbolic geodesic metric space which admits a cocompact isometric group action. We assume that the ideal boundary of X consists of more than 2 points; it then follows that $\partial_\infty X$ has the cardinality of the continuum. The *displacement function* of an isometry $g : X \to X$ is

$$dis(g) : x \to d(x, g(x)), x \in X.$$

Lemma 3. *Under the above assumptions there exists a constant $D = D(X)$ such that for each $g \in \mathrm{Isom}(X)$ which fixes $\partial_\infty X$ pointwise, the displacement of g is bounded from above by D.*

Proof. Let $G \curvearrowright X$ be a cocompact isometric group action; pick a metric ball $B = B(o, R) \subset X$ so that the G–orbit of B equals X. It then suffices to prove that there exists $D < \infty$ such that for each isometry g of X fixing $\partial_\infty X$ pointwise,

$$d(o, g(o)) \leq D.$$

Since the ideal boundary of X contains at least 4 points, there exists a pair of geodesics $\gamma_1, \gamma_2 \subset X$ which have disjoint ideal boundaries. Without loss of generality we may assume that both γ_1, γ_2 pass through the ball B.

Since X is δ–hyperbolic, there exists a number $r = r(\delta) < \infty$ such that if geodesics $\alpha, \beta \subset X$ are within finite Hausdorff distance, then

$$d_H(\alpha, \beta) \leq r,$$

see for instance [6]. For every isometry g as above, the geodesics

$$\gamma_i, g(\gamma_i)$$

are within finite Hausdorff distance from each other; therefore

$$d_H(\gamma_i, g(\gamma_i)) \le r, i = 1, 2.$$

Then

$$d(g(o), g(\gamma_i)) \le R \Rightarrow d(g(o), \gamma_i) \le R + r, \quad i = 1, 2.$$

However, since the geodesics γ_1, γ_2 have disjoint ideal boundaries, the diameter of

$$S := N_{R+r}(\gamma_1) \cap N_{R+r}(\gamma_2)$$

is finite. Therefore, if we take $D := \text{diam}(S)/2$, the distance between o and $g(o)$ is at most D. $\square$

Remark 4. An analogue of Lemma 3 holds for quasi-isometries of X with uniformly bounded quasi-isometry constants.

3. Pretrees

In what follows we will need definitions and basic facts about pretrees; the definitions which we give follow [5].

A *pretree* is a set T together with a ternary relation (the *betweenness relation*)

"y is between x and z",

to be denoted $\beta(xyz)$, satisfying the following axioms:

Axiom 1. $\beta(xyz)$ implies that $x \ne y \ne z$.

Axiom 2. $\beta(xyz) \iff \beta(zyx)$.

Axiom 3. $\beta(xyz)$ and $\beta(yxz)$ cannot hold simultaneously.

Axiom 4. If $w \ne y$ then $\beta(xyz)$ implies that either $\beta(xyw)$ or $\beta(wyz)$.

Given a pretree T one can define *closed*, *open* and *half-open* intervals in T by

$$(x, z) := \{y \in T : \beta(xyz)\}, [x, z] := (x, z) \cup \{x, z\}, \text{etc.}$$

Given an increasing union of intervals

$$[x_1, y_1] \subset [x_2, y_2] \subset ... \subset [x_i, y_i] \subset ...$$

we will also refer to the union of these intervals as a (possibly infinite) interval in T.

We note that β defines an order on each interval in T.

Define a "triangle" in T with vertices a, b, c to be the union of the segments (called "sides" of the triangle) $[a, b], [b, c], [c, a]$.

Lemma 5. *Each triangle Δ in T is 0-thin, i.e., each side of Δ is contained in the union of the two other sides.*

Proof. Follows immediately from Axiom 4. $\square$

Suppose that T is a pretree which is given a measure μ (without atoms) defined on closed intervals in T and the σ-algebra which these intervals generate. Define a function $d(x, y)$ on T by $d(x, y) := \mu([x, y])$.

Lemma 6. *d is a pseudo-metric on T.*

Proof. It is clear that d is symmetric and $d(x,x)=0$ (since μ has no atoms). The triangle inequality follows because for each triangle with the vertices a,b,c we have (see Lemma 5)

$$[a,b]\subset[a,c]\cup[b,c]. \quad \square$$

We note that if for each interval $[a,b]\subset T$, with $a\neq b$, $\mu(a,b)>0$ then d is a metric. Moreover, it follows that $(a,b)\neq\emptyset$ for each $a\neq b$. If the restriction of the metric d to each interval $[x,y]$ is complete then $[x,y]$ is order isomorphic to an interval in $\mathbb{R}$ and moreover, $([x,y],d)$ is isometric to an interval in $\mathbb{R}$. We thus get:

Lemma 7. *Suppose that for each interval $[x,y]\subset T$, with $x\neq y$, $\mu[x,y]>0$, and that the restriction of the metric d to each interval in T is complete. Then (T,d) is a metric tree.*

Proof. It is clear from the above discussion that T is a geodesic metric space. Since each triangle in T is 0-thin, it follows that each triangle in T is isometric to a tripod. Finally, let's check completeness of T: Suppose that $x_i, i\geq 0$, is a Cauchy sequence in T. Then there exists an increasing sequence of intervals $I_i\subset T$ such that

$$\lim_i \mu([x_0,x_i]\cap I_i)=\lim_i d(x_0,x_i).$$

Then completeness of d restricted to the union I of I_i's implies that (x_i) converges to a point in the interval I. $\square$

4. Proof of the main theorem

1. **Ideal boundaries of $CAT(0)$ Poincaré duality groups.** Let $G\curvearrowright X$ be a discrete cocompact action of a $PD(3)$ group G on a $CAT(0)$-space X. In this section we show that the ideal boundary of the $CAT(0)$ space X is homeomorphic to S^2.

We refer the reader to [4], [7] for the background on the cohomology of groups. Recall [4], that an n-dimensional Poincaré duality group over a ring $\mathcal{R}$ (for short, $PD(n)$ group over $\mathcal{R}$), is an FP-group over $\mathcal{R}$ such that $H^i(G,\mathcal{R}G)$ is isomorphic to $\mathcal{R}$ as an $\mathcal{R}$-module when $i=n$ and is trivial otherwise.

Let $Z:=\partial_\infty X$ be the ideal boundary of a locally compact $CAT(0)$ space. M. Bestvina in [2] proved that the compactification

$$\bar{X}:=X\cup Z$$

satisfies the axioms of the $\mathcal{Z}$-set compactification. Instead of listing all the axioms of the $\mathcal{Z}$-set compactification we mention only several properties:

1. If $G\curvearrowright X$ is an isometric group action then this action extends to a topological action of G on $\bar{X}$.

2. There exists a natural isomorphism

$$H_c^*(X)\to\tilde{H}_c^{*-1}(Z),$$

which is compatible with inclusions of closed convex subsets $X'\subset X$.

3. We state the third property as a lemma:

Lemma 8. *If G is a $PD(3)$ group acting isometrically, properly discontinuously and cocompactly on a $CAT(0)$ space X, then the ideal boundary Z of X is homeomorphic to S^2.*

Proof. Bestvina proves, [2, Theorem 2.8], that if G is a $PD(3)$ group over $\mathcal{R}$, then Z is homeomorphic to S^2. We note that Bestvina proves the latter theorem under more restrictive assumptions than we are working with (although, his class of groups G includes 3-manifold groups):

1. Bestvina assumes that the commutative ring $\mathcal{R}$ is a PID. However this assumption is used only to apply the Universal Coefficients Theorem, which works for hereditary rings as well, see [9].

2. Bestvina's definition of an n-dimensional Poincaré duality group is more restrictive than the usual one: Instead of the FP property he assumes that a group G acts freely, properly discontinuously, cocompactly on a contractible cell complex Y. Note however that Bestvina in his proof uses only the fact that $G \curvearrowright Y^{(i)}$ is cocompact on each i-skeleton of Y. Then existence of such an action for the $CAT(0)$-groups follows from a general construction described in [14]. Namely, if a group G admits a properly discontinuous cocompact action on a contractible space X (e.g. the $CAT(0)$-space in our case) then it also admits a free, properly discontinuous action on a contractible cell complex Y (possibly of infinite dimension) such that $Y^{(i)}/G$ is compact for each i.

3. Bestvina assumes that the image of the orientation character χ of the Poincaré duality group G is finite (he then passes to a finite index subgroup in G which is the kernel of χ). However this assumption can be omitted from his theorem using *twisting* of the action $G \curvearrowright C_*(Y)$ by the character χ as it is done in [14, Section 5.1].

With the above modifications, Bestvina's arguments apply in our case and it follows that $\partial_\infty X$ is homeomorphic to the 2-sphere. ☐

2. **Case 1: X contains a 3-flat.** The main goal of this section is to show that, in case X contains a 3-flat, the group G contains a finite index subgroup isomorphic to $\mathbb{Z}^3$.

Lemma 9. *Suppose that S is a convex subset in X such that $\partial_\infty S = \partial_\infty X$. Then S is within finite Hausdorff distance from X.*

Proof. Pick a base-point $o \in X$. If S is not within finite Hausdorff distance from X then there exists a sequence of isometries $g_i \in G$ such that $d(o, g_i S)$ diverges to infinity. Consider the functions $f_i := d(x, g_i S) - d(o, g_i S)$. Then, according to Lemma 2.3 in [15], the sequence of functions f_i subconverges to a Busemann function b on X. Clearly, the sublevel sets $\{f_i \le 0\}$ subconverge into the horoball $U := \{b \le 0\}$ in X. Since $\partial_\infty \{f_i \le 0\} = \partial_\infty g_i S = \partial_\infty X$, it follows that $\partial_\infty X = \partial_\infty U$.

Let F be a 2-flat in X. Then $\partial_\infty F \subset \partial_\infty U$ and convexity of horoballs in X imply that for each $x \in F$

$$t = f(x) \Rightarrow F \subset \{z : b(z) \le t\}.$$

It follows that the restriction $b|F$ is constant and thus F is contained in the horosphere $\{x : b(x) = t\}$ for some $t \in \mathbb{R}$. Then Lemma 2.2 in [15] implies that X contains a half-space $H := \mathbb{R}_+ \times F$. Then, by taking an appropriate limit of the half-spaces $h_j(H), h_j \in G$, we see that X contains the 3-flat $F' := F \times \mathbb{R}$. By Lemma 8, $\partial_\infty F' = \partial_\infty X$. Suppose that F' is not within finite Hausdorff distance from X. Then, by repeating the same argument as above with S replaced with F'

and then F replaced with F', we see that X contains a 4-flat, which contradicts Lemma 8.

Therefore X is within finite Hausdorff distance from the 3-flat F'; in particular, there are no horoballs in X which have the same ideal boundary as X. Contradiction. □

Corollary 10. *If X contains a 3-flat then the group G is virtually abelian, in particular, it contains $\mathbb{Z} \times \mathbb{Z}$.*

Proof. If F is a 3-flat in X then, by Lemma 8, $\partial_\infty F = \partial_\infty X$ and, by Lemma 9, F is within finite Hausdorff distance from X. It follows that the group G is isomorphic to a lattice in $Isom(\mathbb{R}^3)$ and hence it is virtually abelian and contains $\mathbb{Z}^3$ as a subgroup of finite index. □

Assumption 11. From now on we will assume that X contains no 3-flats.

3. **Metric balls and parallel sets in X.** In this section we establish certain geometric properties of X which follow from Assumption 11.

Lemma 12. *There exists $r_0 \in \mathbb{R}$ such that the following holds. For each ball $B(x,r) \subset X$, isometric to a disk of the radius r in $\mathbb{R}^3$, we have: $r \le r_0$.*

Proof. If the assertion is false then there exists a sequence of balls $B(x_i, r_i)$ with $\lim_i r_i = \infty$. Let $g_i \in G$ be such that $g_i(x_i)$ is a bounded sequence in X. Then the balls $g_i(B(x_i, r_i))$ subconverge to a 3-flat in X. Contradiction. □

Corollary 13. *The set of 2-flats $F' \subset X$ which are parallel to a flat F is compact in Gromov–Hausdorff topology.*

Proof. If not then X contains convex subsets isometric to $[0,r] \times \mathbb{R}^2$ for arbitrarily large r. This contradicts the previous lemma. □

Lemma 14. *Suppose that $Y \times \mathbb{R}$ is a parallel set in X. Then Y is Gromov–hyperbolic.*

Proof. We repeat the arguments in [6, Theorem 9.33]. If Y is not Gromov–hyperbolic then there exists a pair of points $\xi, \eta \in \partial_\infty Y$ so that the Tits angle between ξ, η is positive but less than π. Pick a point $o \in Y$ and consider a sequence of points $y_i \in \overline{o\xi}$ which converge to ξ and the geodesic rays $\overline{y_i\eta}$. We identify the rays $\overline{y_i\xi}, \overline{y_i\eta}$ with geodesic rays in $Y \times \mathbb{R} \subset X$ (that share common point y_i). Then, by applying an appropriate sequence of elements $g_i \in G$ (for which $\{g_i(y_i)\}$ is bounded in X) to $Y \times \mathbb{R}$ and to the rays $\overline{y_i\xi}, \overline{y_i\eta}$ and passing to the limit of a subsequence, we get:

1. The sets $g_i(Y \times \mathbb{R})$ subconverge to a parallel set $Y' \times \mathbb{R}$.
2. Y' contains two geodesic rays $\overline{y\xi'}$, $\overline{y\eta'}$ (limits of the sequences of rays $g_i(\overline{y_i\xi})$, $g_i(\overline{y_i\eta})$) which bound a flat sector in Y'.

This contradicts Lemma 12. □

4. **Case 2: X contains a parallel set with the full boundary.** In this section we prove the main theorem under the assumption that X contains a parallel set P whose ideal boundary is the entire $\partial_\infty X$.

Proposition 15. *Suppose that there is a convex product subset $P = \mathbb{R} \times Y$ such that $\partial_\infty S = \partial_\infty X$. Then G is commensurable to the fundamental group of a 3-dimensional Seifert manifold. In particular, G contains $\mathbb{Z}^2$.*

Proof. We will assume that P is a maximal convex product subset in X. Since Y is Gromov–hyperbolic, it follows that the Tits boundary of S is the suspension of a discrete metric space which is the ideal boundary of Y. Therefore, since $\partial_\infty P = \partial_\infty X$, the group G preserves the ideal boundary of the geodesic $l = \mathbb{R} \times \{y\}$. Hence for each $g \in G$ the geodesic $g(l)$ is parallel to l, which (by the maximality assumption) implies that $g(P) = P$.

We have an induced isometric action $\rho : G \curvearrowright Y$. Since the suspension of $\partial_\infty Y$ is homeomorphic to the 2-sphere $\partial_\infty X$, the ideal boundary of Y is homeomorphic to S^1. Thus the cocompact isometric action $\rho : G \curvearrowright Y$ extends to a uniform (topological) convergence action $G \curvearrowright \partial_\infty Y = S^1$. Therefore, according to [10, 12, 13, 19], the action $G \curvearrowright S^1$ is topologically conjugate to a Moebius action ρ'.

Let K denote the kernel of ρ'.

Lemma 16. *K contains an infinite cyclic subgroup of finite index.*

Proof. Let $D = D(Y)$ denote the constant given by Lemma 3. Pick a point $y \in Y$. Then for each $g \in K$,

$$d(y, g(y)) \leq D.$$

Therefore the K–orbit of y is contained in the metric ball $B(y, D)$. Thus for every $x \in X$, the K–orbit of x is contained in a D–neighborhood of the geodesic $l = \{y\} \times \mathbb{R}$ (passing through x). Therefore K is quasi–isometric to $\mathbb{Z}$ and hence is virtually $\mathbb{Z}$. □

Lemma 17. *The action $G \curvearrowright S^1$ is topologically conjugate to an action of a uniform lattice in $Isom(\mathbb{H}^2)$.*

Proof. The action $\rho'(G) \curvearrowright \mathbb{H}^2$ is cocompact, therefore we have the following possibilities:

(a) $\rho'(G)$ is a cocompact discrete subgroup in $Isom(\mathbb{H}^2)$.

(b) $\rho'(G)$ is a solvable subgroup in $Isom(\mathbb{H}^2)$, which fixes a point in S^1. Then $\rho'(G)$ is not virtually abelian which contradicts the fact that G is a CAT(0) group.

(c) $\rho'(G)$ is dense in $PSL(2, \mathbb{R})$. Then, the group $\rho'(G)$ contains a nontrivial elliptic element $\hat{g}$ and it also contains a sequence of elements $\hat{h}_i$ which converge to $1 \in PSL(2, \mathbb{R})$. Let $g, h_i \in G$ be elements which map (via ρ') to $\hat{g}$ and $\hat{h}_i$ respectively. Clearly, $\rho(g) \in Isom(Y)$ is elliptic as well, let $y \in Y$ be its fixed point. By taking conjugates $g_i := h_i g h_i^{-1}$, we get an infinite collection of distinct elements $\{g_i : i \in \mathbb{N}\}$ of G such that for each $n \in \mathbb{Z}$, $g_i(y \times \mathbb{R})$ is contained in $N_R(y \times \mathbb{R})$ where $R \in \mathbb{R}_+$ is independent of i. We note that since all g_i are pairwise conjugate, there exists $C < \infty$ such that $d(x, g_i(x)) < C$ for each $x \in y \times \mathbb{R}$ and $i \in \mathbb{N}$. This contradicts discreteness of the action of G on X. □

The above two lemmas imply that the kernel of ρ is commensurable to $\mathbb{Z}$ and the quotient $\rho(G)$ is commensurable to the fundamental group of a 2-dimensional hyperbolic surface. Thus, after passing to a finite index subgroup in G we obtain a short exact sequence

$$1 \to K \to G \to Q \to 1 \tag{18}$$

where Q is the fundamental group of a closed oriented surface.

Lemma 19. *Suppose that for a group H we have a short exact sequence*

$$1 \to \mathbb{Z}/n\mathbb{Z} \to H \to Q \to 1.$$

Then H contains a finite index surface subgroup.

Proof. Let t denote the generator of $\mathbb{Z}/n\mathbb{Z}$. Let $a_i, b_i, i = 1, ..., n$ denote the lifts to H of the standard generators of Q. It suffices to consider the case when

$$[a_1, b_1] \cdots [a_n, b_n] = t$$

and t belongs to the center of H. Consider the finite Heisenberg group

$$H_n := \langle a, b, t : [a, b] = t, a^n = b^n = t^n = 1, [a, t] = 1, [b, t] = 1\rangle.$$

Define the homomorphism $\phi : H \to H_n$ by

$$\phi(a_1) = a, \phi(b_1) = b, \phi(a_i) = \phi(b_i) = 1, \forall i \geq 2.$$

Then the kernel H' of ϕ is a torsion-free subgroup of finite index in H. It follows that the map $H \to Q$ sends H' injectively to a finite index subgroup in Q. Therefore H' is a surface group. ☐

We now return to the exact sequence (18). As in the above lemma we let $a_i, b_i, i = 1, ..., n$ denote the lifts to G of the standard generators of Q. Let $H \subset G$ denote the subgroup generated by these elements. If

$$t := [a_1, b_1] \cdots [a_n, b_n]$$

is an infinite order element of K then H is isomorphic to the fundamental group of a Seifert manifold (whose base is a surface with the fundamental group Q). It is clear that H has finite index in G.

It t has finite order then, according to Lemma 19, after passing to a finite index subgroup in Q) we can assume that $t = 1$. Pick an infinite order element $k \in K$ which belongs to the center of G. Then the subgroups H and $\langle k\rangle$ generate the product

$$\mathbb{Z} \times Q \subset G.$$

Again, clearly, this subgroup has finite index in G. Thus, in the both cases, G is commensurable to the fundamental group of a 3-dimensional Seifert manifold. ☐

Thus, the conclusion of Theorem 1 holds provided that X contains a parallel set with the full boundary.

Assumption 20. From now on we will assume that the ideal boundary of each parallel set of X is a proper subset of $\partial_\infty X$.

5. **Case 3: The ideal boundary of every parallel set in X is a proper subset of $\partial_\infty X$.** In this section we show that the *peripheral circles* of the ideal boundaries of nontrivial parallel sets in X can be used to construct a *small stable nontrivial isometric action* of G on an $\mathbb{R}$-tree. Then, by Rips theory, G admits a nontrivial splitting as an amalgam with virtually abelian edge groups. This, in turn, implies that the edge groups are virtually $\mathbb{Z}^2$.

According to Eberlein's theorem (see [11] in the smooth case and [6, Theorem 9.33] in general), the $CAT(0)$ space X is either a visibility space or it contains a 2-flat F. Since in the former case, G is Gromov–hyperbolic, we assume that X contains a 2-flat F. In particular, X contains *nontrivial parallel sets*.

Lemma 21. *Suppose that $P = Y \times \mathbb{R}$ is a nontrivial parallel set in X. Then $\partial_\infty P$ contains a topological circle S which is geodesic in the Tits metric so that S bounds a disk in $\partial_\infty X \setminus \partial_\infty P$.*

Proof. Let $\xi, \eta \in \partial_\infty P$ be the ideal points of a geodesic $y \times \mathbb{R} \subset Y \times \mathbb{R} = P$. Then the Tits boundary $\partial_{Tits} P$ is the metric join $S^0 \star \partial_{Tits} Y$, which is the union of geodesic segments L_μ of length π connecting η and ξ and passing through $\mu \in \partial_{Tits} Y \subset \partial_{Tits} X$. Clearly, if $\mu \neq \mu'$ then $S := L_\mu \cup L_{\mu'}$ is a topological circle which is geodesic in the Tits metric.

Let D be a component of $\partial_\infty X \setminus \partial_\infty P$. Then there is a point $\zeta \in \partial D$ which belongs to $L_\mu \setminus \{\xi, \eta\}$ for some $\mu \in \partial_{Tits} Y$. Clearly, ∂D is not contained in L_μ, therefore there exists a point $\zeta' \in \partial D$ which belongs to $L_{\mu'} \setminus \{\xi, \eta\}$ for some $\mu' \in \partial_{Tits} Y \setminus \{\mu\}$. The reader will verify that the circle $S = L_\mu \cup L_{\mu'}$ bounds D. $\square$

We will refer to these circles S as in Lemma 21, as *peripheral circles* of $\partial_\infty P$. A flat in X whose boundary is a peripheral circle will be called a *peripheral flat.*

It follows from the properties of the Tits metric (discussed in section 2) that if $F, F' \subset X$ are 2-flats then the intersection $\partial_{Tits} F \cap \partial_{Tits} F' \subset \partial_{Tits} X$ is convex and either consists of two antipodal points or is a circular arc in $\partial_{Tits} F$ of the length $\leq \pi$.

Definition 22. We say that totally-geodesic circles $S, S' \subset Z$ *cross* if S contains points from each component of $Z \setminus S'$ (in the visual topology). Note that *crossing* is a symmetric relation. We will say that the ideal boundaries of two parallel sets P, P' *cross* if at least one circle in $\partial_{Tits} P$ crosses a circle in $\partial_{Tits} P'$.

Observe that if S and S' cross, the intersection $S \cap S'$ consists of a pair of antipodal points.

Lemma 23. *Suppose that $P = l \times Y \subset X$ is a parallel set for which $\partial_\infty Y$ consists of at least 3 points (i.e., P is not within finite Hausdorff distance from a flat) and $F \subset X$ is a 2-flat which is not contained in P. Then $\partial_\infty P$ and $S = \partial_\infty F$ do not cross.*

Proof. Suppose to the contrary that $\partial_\infty P$ and $S = \partial_\infty F$ do cross. Recall that $\partial_\infty P$ is the metric join of $\{\eta, -\eta\} = \partial_\infty l$ and $\partial_\infty Y$. If S were to pass through η then, by convexity, S passes through $-\eta$ as well and hence F would be contained in the parallel set P. Therefore, S does not pass through $\partial_\infty l$ and the configuration $\{\partial_\infty P, S\}$ has to look like the one in Figure 1, where x, y, z denote the distances

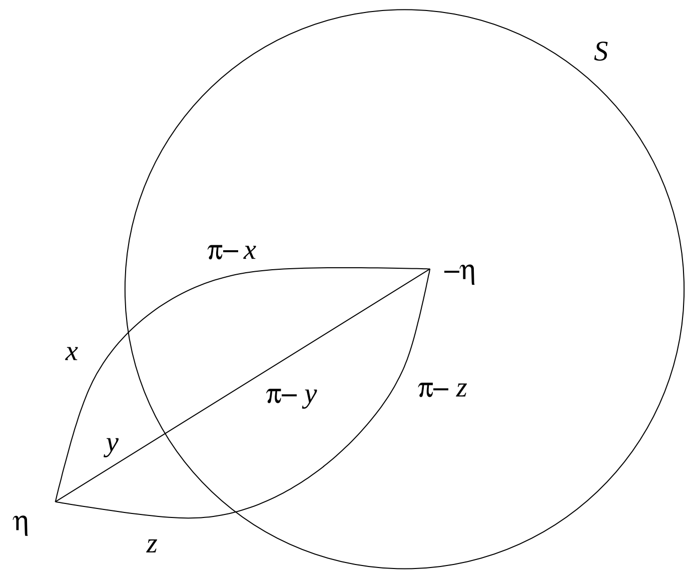

FIGURE 1.

from η to the points of intersection between $\partial_\infty P$ and S. It follows that $x + y = \pi$, $y + z = \pi$, $x + z = \pi$ and thus

$$x = y = z = \pi/2.$$

This implies that the circle S is contained in $\partial_\infty Y$, thus Y cannot be Gromov–hyperbolic. This contradicts Lemma 14. □

We observe that, since $G \curvearrowright X$ is properly discontinuous, the stabilizer of each flat $F \subset X$ in the group G is virtually abelian. We assume that this stabilizer is virtually cyclic (possibly finite)–otherwise G contains $\mathbb{Z}^2$.

Suppose that we have three flats $F, F', F'' \subset X$ with pairwise distinct ideal boundaries. We will say that F' *separates* F from F'' if the following holds:

$$\partial_\infty F \subset \overline{D}, \partial_\infty F'' \subset \overline{D}'',$$

where $D \sqcup D'' = Z \setminus \partial_\infty F'$. We set the ternary relation β by: $\beta(FF'F'')$ if F' separates F from F''.

We leave it to the reader to verify that with this ternary relation the set $\mathcal{P}$ of all peripheral flats in X satisfies the axioms of a pretree.

Lemma 24. *If U_0 is a horoball in X then $W := \partial_\infty U_0$ does not separate $\partial_\infty X$.*

Proof. Let $\xi \in \partial_\infty X$ and consider the horoballs $U_t = \{b_\xi(x) \le t\}$, $t \in \mathbb{R}$, where b_ξ is the appropriately normalized Busemann function at ξ. Clearly $\partial_\infty U_t = W$ for each t. Property (2) of the $\mathcal{Z}$-set compactification applied to the pairs (U_t, W) means that we have natural isomorphisms

$$H^i_c(U_t) \to \tilde{H}^{i-1}(W). \tag{25}$$

Suppose that $[\zeta] \in H^i_c(U_t)$. Then there exists $s < t$ such that U_s is disjoint from the support set of the cocycle ζ. Therefore $[\zeta]$ maps trivially to $H^i_c(U_s)$ and hence, by naturality of (25), it maps trivially to $\tilde{H}^{i-1}(W)$. We conclude that $\tilde{H}^*(W) = 0$. Therefore, by the Alexander duality on $\partial_\infty X$, the subset $W = \partial_\infty U_0$ cannot separate $\partial_\infty X$. □

Proposition 26. *Let F, F'' be flats in X. Then the set $S(F, F'')$ of flats F' separating F from F'' is compact with respect to the Gromov–Hausdorff topology.*

Proof. If $\partial_\infty F = \partial_\infty F''$ then for each flat F' separating F and F'' we have: $\partial_\infty F' = \partial_\infty F$. Therefore, $S(F, F'')$ is compact by Corollary 13.

Whence we can assume that $\partial_\infty F' \neq \partial_\infty F''$. Suppose that F_i is a sequence of 2-flats in X which diverge to infinity, i.e.

$$\lim_i d(o, F_i) = \infty$$

where $o \in X$ is a base-point. Then, as in the proof of Lemma 9, the limit of the distance functions to F_i (normalized at o) subconverge to a Busemann function b_ξ in X. Let U be the horoball $\{x : b_\xi(x) \leq 0\}$.

If, say, $\partial_\infty F \subset \partial_\infty U$ then the flat F is contained in the sublevel set of the Busemann function b_ξ and therefore X would contain a flat half-space $\mathbb{R}^3_+$, which contradicts Lemma 12. Thus both complements

$$\partial_\infty F \setminus \partial_\infty U, \quad \partial_\infty F'' \setminus \partial_\infty U$$

are nonempty.

Lemma 27. *1. In the Hausdorff topology on the set of closed subsets of $X \cup \partial_\infty X$, the sets $F_i \cup \partial_\infty F_i$ subconverge into $\partial_\infty U$.*

2. $\partial_\infty F \cap \partial_\infty F'' \subset \partial_\infty U$.

Proof. 1. Suppose that the assertion is false. Then there exists a sequence of points $x_i \in \partial_\infty F_i$ such that

$$\eta = \lim_i x_i \notin \partial_\infty U.$$

Clearly, $\eta \in \partial_\infty X$. Consider a parametrization $\rho(t), t \in \mathbb{R}_+$ of the geodesic ray $\overline{o\eta}$. Then, since $\eta \notin \partial_\infty U$, there exists $T \geq 0$ such that

$$b_\xi(\rho(t)) \geq 1, \quad \forall t \geq T. \tag{28}$$

The Busemann function b_ξ is the limit of the normalized distance functions

$$d_i(x) = d(x, F_i) - d(o, F_i).$$

Then $d_i(o) = 0, d_i(x_i) \leq 0$ for all i and hence, by convexity,

$$d_i(y_i) \leq 0, \quad \forall y_i \in \overline{ox_i}.$$

This, together with the inequality (28), contradicts the assumption that the geodesics $\overline{ox_i}$ converge to the geodesic ray $\overline{o\eta}$.

2. Observe that $\partial_\infty F \cap \partial_\infty F'' \subset \partial_\infty F_i$ for each i. Thus (2) follows from (1). □

We continue the proof of Proposition 26. Pick points

$$\eta \in \partial_\infty F \setminus \partial_\infty U, \quad \eta'' \in \partial_\infty F'' \setminus \partial_\infty U.$$

Previous lemma implies that

$$\eta, \eta'' \notin \partial_\infty F \cap \partial_\infty F''$$

and that (since $\partial_\infty U$ does not separate $\partial_\infty X$) for large i the points η, η'' belong to the same connected component of $\partial_\infty X \setminus \partial_\infty F_i$. This contradicts the assumption that F_i is between F, F'' for all i. □

Now, let's pick a peripheral 2-flat $F_0 \in \mathcal{P}$, consider the set $\{gF_0, g \in G\}$ and its closure $\mathcal{F}$ in the Gromov–Hausdorff topology. The elements of $\mathcal{F}$ are peripheral 2-flats in X and the group G acts naturally on $\mathcal{F}$. We note that since no flat in $\mathcal{F}$ has cocompact stabilizer, $\mathcal{F}$ contains no isolated points. After passing to a smaller G-invariant subset in $\mathcal{F}$ we may assume that the action $G \curvearrowright \mathcal{F}$ is minimal. The union

$$\tilde{\mathcal{L}} := \cup_{F \in \mathcal{F}} F$$

equipped with the Gromov–Hausdorff topology becomes a locally compact 2-dimensional lamination, the topological action $G \curvearrowright \tilde{\mathcal{L}}$ is properly discontinuous and cocompact. The lamination $\tilde{\mathcal{L}}$ has a continuous G-invariant leafwise flat metric. Therefore, since each leaf of $\tilde{\mathcal{L}}$ is amenable, Plante's construction (see [17]) implies existence of a transversal G-invariant measure μ on $\tilde{\mathcal{L}}$; minimality of $G \curvearrowright \mathcal{F}$ implies that this measure has full support.

Lemma 29. *Suppose that $F \in \mathcal{F}$, $g_n \in G$ is a sequence such that $\lim_{n\to\infty} g_n F = F_\infty \in \mathcal{F}$. Then there exist $x_-, x_+ \in \mathcal{F}$ such that for all sufficiently large n, $g_n F \in [x_-, x_+]$ and $F_\infty \in [x_-, x_+]$.*

Proof. Since $\lim_{n\to\infty} g_n F = F_\infty$, the circles $\partial_{Tits}(g_n F)$ converge to the circle $\partial_{Tits} F_\infty$ in the Chabauty topology (we again are using here the visual topology on Z). The circles in the collection

$$\{\partial_{Tits}(g_n F), \partial_{Tits} F_\infty, n \in \mathbb{N}\}$$

are all peripheral and hence do not cross each other (by Lemma 23). This implies that for all large n, m either $\partial_{Tits}(g_n F)$ separates $\partial_{Tits}(g_m F)$ from $\partial_{Tits} F_\infty$ or $\partial_{Tits} F_\infty$ separates $\partial_{Tits}(g_n F)$ from $\partial_{Tits}(g_m F)$. □

The above lemma implies that the natural projection $p : \tilde{\mathcal{L}} \to \mathcal{F}$ is continuous, where we give $\mathcal{F}$ the order topology, whose basis consists of the open intervals (a, b). It is also clear that p is a proper map in the sense that for each interval $[a, b]$ the inverse image $p^{-1}([a, b])$ consists of leaves of $\tilde{\mathcal{L}}$ which intersect a certain compact subset in X: If a sequence of flats F_j leaves every compact subset in X then this sequence subconverges to a point in $\partial_\infty X$, but a point cannot separate one circle in $\partial_{Tits} X$ from another.

The measure μ on the pretree $\mathcal{F}$ has no atoms and (since the measure μ transversal to $\tilde{\mathcal{L}}$ has full support) for each pair of distinct points $x, x' \in \mathcal{F}$, $\mu([x, x']) = 0$ iff the corresponding flats F, F' in X are not separated by any flat in $\mathcal{F}$. We let T be the quotient of $\mathcal{F}$ by the equivalence relation: Points $x, x' \in \mathcal{F}$ are equivalent iff $\mu([x, x']) = 0$. The G-action, the pretree structure, and the measure μ project to T (we retain the notation μ for the projection of the measure). As it was explained in section 3, the measure μ yields a metric d on T. Local compactness of $\tilde{\mathcal{L}}$ implies that the restriction of d to each interval in T is a complete metric. It is clear that the group G acts isometrically on T.

Remark 30. The map $\mathcal{F} \to T$ has at most countable multiplicity. Moreover, all but countably many points in T have a unique preimage in $\mathcal{F}$.

Lemma 31. *1. T is an uncountable metric tree.*

2. Stabilizers of nondegenerate arcs in T are virtually cyclic and the action $G \curvearrowright T$ is stable.

3. G does not have a global fixed point in T.

Proof. 1. Follows from Lemma 7.

2. By our hypothesis, for each point $F \in \mathcal{F}$ its G-stabilizer is virtually cyclic. Since $\mathcal{F}$ is prefect, it is uncountable; hence, by Remark 30, uncountably many points in each nondegenerate arc $[x, y] \subset T$ have a virtually cyclic stabilizer. Thus the action $G \curvearrowright T$ is *small.* Since G is a $CAT(0)$ group, each virtually cyclic subgroup of G is contained in a maximal virtually cyclic subgroup. Therefore, if $I_1 \supset I_2 \supset ...$ is a descending chain of arcs in T, then the sequence of their stabilizers in the group G

$$G_{I_1} \subset G_{I_2} \subset ...$$

is eventually constant. Thus the action $G \curvearrowright T$ is *stable.*

3. The action $G \curvearrowright \mathcal{F}$ is minimal, hence the action $G \curvearrowright T$ is minimal as well. Since T is not a point it follows that G cannot fix a point in T. □

Since G acts properly discontinuously and cocompactly on the contractible space X, this group is finitely-presented. Therefore, by Lemma 31, we can apply [3] to conclude that the group G splits as an amalgam with a virtually solvable edge subgroup A. Since G is a $CAT(0)$ group, the subgroup A is virtually abelian. Since G is a 3-dimensional Poincaré duality group over $\mathcal{R}$, it follows from the Meyer-Vietoris sequence that A is a 2-dimensional Poincaré duality group over $\mathcal{R}$. This implies that A is virtually a surface group, see for instance [14]. Since A is virtually abelian, it follows that A contains $\mathbb{Z}^2$ as a subgroup of finite index. This proves the main theorem. □

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