A survey of complex hyperbolic Kleinian groups

Michael Kapovich

Abstract This survey of discrete subgroups of isometries of complex hyperbolic spaces is aimed to discuss interactions between function theory on complex hyperbolic manifolds and the theory of discrete groups. We present a number of examples and basic results about complex-hyperbolic Kleinian groups. The appendix to the paper written by Mohan Ramachandran includes a proof of a result known as "Burns' Theorem" about ends of complex-hyperbolic manifolds.

Key words: Discrete groups; complex-hyperbolic geometry.

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1 Introduction

This survey is based on a series of lectures I gave at the workshop "Progress in Several Complex Variables," held in KIAS, Seoul, Korea, in October of 2019. It is useful to read it in conjunction with my (longer) survey of discrete isometry groups of real hyperbolic spaces, [55], since most issues in the real and complex hyperbolic settings are quite similar. The theory of complex hyperbolic manifolds and complex hyperbolic Kleinian groups (aka discrete holomorphic isometry groups of complex hyperbolic spaces $\mathbb{H}^n_{\mathbb{C}}$) is a rich mixture of Riemannian and complex geometry, topology, dynamics, symplectic geometry and complex analysis. The choice of topics covered in the survey is governed by my personal taste and is, by no means, canonical: It is geared towards a discussion of interactions between the function theory on complex hyperbolic manifolds and the geometry/dynamics of complex hyperbolic Kleinian groups (sections 9 and 10). I refer the reader to

Michael Kapovich @ Department of Mathematics, UC Davis, One Shields Avenue, Davis CA 95616, USA, e-mail: kapovich@math.ucdavis.edu

[18, 36, 37, 35, 70, 74, 75, 76, 85] for further discussion of the geometry of complex hyperbolic spaces and their discrete isometry groups. The bibliography of complex hyperbolic Kleinian groups appearing at the end of the survey is long but is not meant to be exhaustive, my apologies to everybody whose papers are omitted.

It should be pointed out that development of the theory of complex hyperbolic manifolds and complex hyperbolic Kleinian groups was influenced in part by the work of William Thurston on real hyperbolic manifolds and Kleinian groups (especially Thurston's approach to geometric finiteness and orbifolds), as well as by Thurston's work on general geometric structures on manifolds. On a personal note, Thurston's way of thinking about hyperbolic manifolds, discrete groups and geometric structures, was the single most importance influence on my mathematical work. Much of my past and present research is driven by attempting to understand and generalize Thurston's results.

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Contents

2 Complex hyperbolic space

Most of the basic material on geometry of complex hyperbolic spaces can be found in Goldman's book [37]; I also refer the reader to [35, 74, 76] for shorter introductions.

Consider the vector space $V = \mathbb{C}^{n+1}$ equipped with the pseudo-hermitian bilinear form

$$\langle z, w\rangle = -z_0\bar{w}_0 + \sum_{k=1}^{n} z_k\bar{w}_k.$$

Set $q(z) := \langle z, z\rangle$. This quadratic form has signature $(n, 1)$. Define the *negative light cone* $V_- := \{z : q(z) < 0\}$. Consider the complex projective space $\mathbb{P}^n := PV$, the projectivization of V, and the projection $p : z \mapsto [z] \in \mathbb{P}^n$. The projection $\mathbf{B}^n := p(V_-)$ is an open ball in $\mathbb{P}^n$. In order to see this, consider the affine hyperplane in $\mathbb{C}^{n+1}$ given by $A = \{z_0 = 1\}$ (and equipped with the standard Euclidean hermitian metric). Then $V_- \cap A$ is the open unit ball in A centered at the origin. This intersection projects diffeomorphically to $p(V_-)$.

The tangent space $T_{[z]}\mathbb{P}^n$ is naturally identified with $z^\perp$, the orthogonal complement of $\mathbb{C}z$ in V, taken with respect to $\langle\cdot,\cdot\rangle$. If $z \in V_-$, then the restriction of q to $z^\perp$ is positive-definite, hence, $\langle\cdot,\cdot\rangle$ project to a hermitian metric h (also denoted $\langle\cdot,\cdot\rangle_h$) on $\mathbf{B}^n$. From now on, I will always equip $\mathbf{B}^n$ with the hermitian metric h and let d denote the corresponding distance function on $\mathbf{B}^n$.

Definition 1 The *complex hyperbolic n-space* $\mathbb{H}^n_{\mathbb{C}}$ is $(\mathbf{B}^n, h)$.

I next describe the hermitian metric h on $\mathbf{B}^n$ using the coordinates $(z_1, ..., z_n)$ on A. First, regarding $\mathbf{B}^n$ as a subset of the affine hyperplane A, for a vector $y \in T_x\mathbf{B}^n$ we have

$$\langle y, y\rangle_h = \frac{\langle x, x\rangle\langle y, y\rangle - \langle x, y\rangle\langle y, x\rangle}{-\langle x, x\rangle^2}.$$

Setting $x = (1, z)$, $z \in \mathbb{C}^n$, and denoting $u \cdot v$ the standard Euclidean hermitian inner product on $\mathbb{C}^n$, we obtain:

$$\langle y, y\rangle_h = \frac{(-1 + |z|^2)|y|^2 - (z \cdot y)(y \cdot z)}{-(-1 + |z|^2)^2}, \quad y \in T_z\mathbf{B}^n.$$

In the differential form, the metric h is, thus, given by

$$ds^2_h = \frac{1}{1 - |z|^2}\sum_{k=1}^{n} dz_k d\bar{z}_k + \frac{1}{(1 - |z|^2)^2}\sum_{j,k=1}^{n} z_j\bar{z}_k dz_k d\bar{z}_j.$$

This hermitian metric is Kähler, with the Kähler potential (centered at the origin) equal to

$$f(z) = \log(1 - |z|^2),$$

and the Kähler form $\omega = \frac{i}{2}\partial\bar{\partial}f$ equal to

$$\omega = \frac{1}{1 - |z|^2}\sum_{k=1}^{n} dz_k \wedge d\bar{z}_k + \frac{1}{(1 - |z|^2)^2}\sum_{j,k=1}^{n} z_j\bar{z}_k dz_k \wedge d\bar{z}_j.$$

The complex hyperbolic metric on $\mathbf{B}^n$ (the unit ball in $\mathbb{C}^n$) is the Bergman metric with the Bergman kernel $K(z, \zeta)$ equal to

$$K(z,\zeta) = \frac{n!}{2\pi^n}(1-(z\cdot\zeta))^{-n-1},$$

where, as before, $z\cdot\zeta$ is the standard hermitian inner product on $\mathbb{C}^n$.

The distance function d on $\mathbb{H}^n_{\mathbb{C}}$ satisfies

$$\cosh^2(d([x],[y])) = \frac{\langle x,y\rangle\langle x,y\rangle}{\langle x,x\rangle\langle y,y\rangle}.$$

For example, specializing to the case when $[x]$ is the center of $\mathbf{B}^n$ and $[y]$ is represented by a point $z\in\mathbf{B}^n$, we obtain:

$$\cosh^2(d(0,z)) = (1-|z|^2)^{-1}.$$

See [37, pp. 72–79] and [63, §1.4]; note however that Goldman uses a different normalization of the metric on the complex hyperbolic space; with his normalization sectional curvature varies in the interval $[-2,-\frac{1}{2}]$.

A real linear subspace $W\subset V$ is said to be *totally real with respect to the form* $\langle\cdot,\cdot\rangle$ if for any two vectors $z,w\in W$, $\langle z,w\rangle\in\mathbb{R}$. Such a subspace is automatically totally real in the usual sense: $JW\cap W=\{0\}$, where J is the almost complex structure on V.

Real geodesics in $\mathbf{B}^n$ are projections (under p) of totally real indefinite (with respect to q) 2-planes in V (intersected with V_-). For instance, geodesics through the origin $0\in\mathbf{B}^n$ are Euclidean line segments in $\mathbf{B}^n$.

More generally, totally-geodesic real subspaces in $\mathbf{B}^n$ are projections of totally real indefinite subspaces in V (intersected with V_-). They are isometric to the real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$ of constant sectional curvature -1. Boundaries of real hyperbolic planes are called *real circles* in S^{2n-1}.

Complex geodesics in $\mathbf{B}^n$ are projections of indefinite complex 2-planes; boundaries of complex geodesics are called *complex circles* in S^{2n-1}. Complex geodesics are isometric to the unit disk with the hermitian metric

$$\frac{dzd\bar{z}}{(1-|z|^2)^2},$$

which has constant curvature -4. These are the extremal disks for the Kobayashi metric on $\mathbf{B}^n$, which coincides with the complex hyperbolic distance function d. It is also equal to the Caratheodory's metric on $\mathbf{B}^n$ (as is the case for all bounded convex domains in $\mathbb{C}^n$).

More generally, complex hyperbolic k-dimensional subspaces $\mathbb{H}^k_{\mathbb{C}}$ in $\mathbf{B}^n$ are projections of indefinite complex $k+1$-dimensional subspaces (intersected with V_-).

All complete totally-geodesic submanifolds in $\mathbb{H}^n_{\mathbb{C}}$ are either real or complex hyperbolic subspaces.

The holomorphic bisectional curvature of $\mathbb{H}^n_{\mathbb{C}}$ is constant, equal to -1. It turns out that $\mathbb{H}^n_{\mathbb{C}}$ has negative sectional curvature which varies in the interval $[-4,-1]$. Thus, $\mathbb{H}^n_{\mathbb{C}}$ is a *negatively pinched Hadamard manifold*:

Definition 2 1. A Hadamard manifold X is a simply-connected complete nonpositively curved Riemannian manifold.

2. A Hadamard manifold X is said to have *strictly negative curvature* if there exists $a < 0$ such that the sectional curvature of X is $\leq a$.

3. A Hadamard manifold X is said to be *negatively pinched* (has *pinched negative curvature*) if there exist two negative numbers $b \leq a < 0$ such that the sectional curvature of X lies in the interval $[b, a]$.

The group $U(n, 1) = U(q)$ of (complex) automorphisms of q projects to the group $G = PU(n, 1) = Aut(\mathbf{B}^n)$ of complex (biholomorphic) automorphisms of $\mathbf{B}^n$. This group acts transitively, with the stabilizer of the center of $\mathbf{B}^n$ equal to $K = U(n)$. Consequently, the metric d on $\mathbf{B}^n$ is complete. The group G is a Lie group, its Lie topology is equivalent to the topology of pointwise convergence, equivalently, the topology of uniform convergence on compacts in $\mathbf{B}^n$, equivalently, the quotient topology of the matrix group topology on $U(n, 1)$. The group G is linear, its matrix representation is given, for instance, by the adjoint representation, which is faithful since G has trivial center.

The Lie group G is connected and has real rank 1. Its Cartan decomposition is

$$G = KA_+K,$$

where A_+ is the semigroup of positive translations (transvections) along a chosen geodesic through 0.

Let $\overline{\mathbf{B}^n}$ denote the closure of $\mathbf{B}^n$ in $\mathbb{P}^n$. The boundary sphere $S^{2n-1} = \partial\mathbf{B}^n$ of $\mathbf{B}^n$ is the projection to $\mathbb{P}^n$ of the null-cone of the form q. The sphere S^{2n-1} is a *CR manifold*: It is equipped with a smooth totally nonintegrable hyperplane distribution $H_z, z \in S^{2n-1}$,

$$H_z = T_z S^{2n-1} \cap J(T_z S^{2n-1}),$$

where J is the almost complex structure on $\mathbb{P}^n$. The subspace H_z is a (complex) hyperplane in $T_z\mathbb{P}^n$. We let P_z denote the unique projective subspace in $\mathbb{P}^n$ passing through z and tangent to H_z. Thus, $P_z \cap \overline{\mathbf{B}^n} = \{z\}$.

One defines a *sub-Riemannian metric* d_C on S^{2n-1} as follows. Given points $\xi, \eta \in S^{2n-1}$, define $C_{p,q}$ as the collection of smooth paths $c : [0, 1] \to S^{2n-1}$ connecting p to q such that c is a *contact path*, i.e. $c'(t) \in H_{c(t)}$ for all $t \in [0, 1]$. Then the *Carnot metric* d_C on S^{2n-1} is

$$d_C(\xi, \eta) = \inf_{c \in C_{\xi,\eta}} \int_0^1 ||c'(t)||dt,$$

where $|| \cdot ||$ is a background Riemannian metric on S^{2n-1}, say, the unique metric of sectional curvature $+1$ invariant under $O(2n)$. It turns out that d_C is indeed a metric which topologizes S^{2n-1}. However, unlike a Riemannian metric on S^{2n-1}, which has Hausdorff dimension equal to the topological dimension, the metric space (S^{2n-1}, d_C) is *fractal*, its Hausdorff dimension $\dim_H$ is equal to

$$\dim_H(S^{2n-1}, d_C) = 2n.$$

Most of the following discussion is valid for general negatively pinched Hadamard spaces; I refer to the paper by Bowditch [13] for details, especially in the context of discrete isometry groups.

Since $\mathbb{H}^n_{\mathbb{C}}$ is a Hadamard manifold X, it has an *intrinsically defined* ideal (visual) boundary $\partial_\infty X$, defined as the set of equivalence classes of geodesic rays, where two rays are equivalent if and only if they are within finite Hausdorff distance. Every geodesic ray is equivalent to a geodesic ray emanating from a chosen base-point $o \in X$. The topology on $\partial_\infty X$ is the quotient topology, where the space of geodesic rays is equipped with the topology of uniform convergence on compacts. Equivalently, since the map from the unit tangent sphere UT_oX at o to $\partial_\infty X$ is bijective, $\partial_\infty X$ is homeomorphic to UT_oX. The union $\overline{X} := X \cup \partial_\infty X$ also has a natural topology with respect to which it is homeomorphic to the closed ball. Given a subset $Y \subset X$, we define $\partial_\infty Y$ as the intersection of the closure of Y in $\overline{X}$ with $\partial_\infty X$.

If X is strictly negatively curved, it satisfies the *visibility property*: Any two distinct points $\xi, \eta \in \partial_\infty X$ are connected by a unique geodesic, denoted $\xi\eta$.

In the case $X = \mathbb{H}^n_{\mathbb{C}}$, this abstract compactification is naturally homeomorphic to the closed ball compactification $\overline{\mathbf{B}^n}$: Two geodesic rays c_1, c_2 are equivalent if and only if they terminate at the same point of the boundary sphere S^{2n-1}.

Suppose that X is a Hadamard manifold. Given a closed subset $\Lambda \subset \partial_\infty X$, one defines the *closed convex hull*, denoted $\mathrm{hull}(\Lambda)$, of Λ in X as the intersection of all closed subsets $C \subset X$ such that $\partial_\infty C \supset \Lambda$. For $\eta > 0$ we will use the notation $\mathrm{hull}_\eta(\Lambda)$ to denote the closed η-neighborhood of $\mathrm{hull}(\Lambda)$ in X.

Theorem 1 (M. Anderson, [3]) *If X has pinched negative curvature then for every closed subset $\Lambda \subset \partial_\infty X$ which is not a singleton,* $\mathrm{hull}(\Lambda)$ *is a (closed, convex) subset of X such that* $\partial_\infty \mathrm{hull}(\Lambda) = \Lambda$.

Remark 1 (a) Assuming that X is negatively curved:

1. $\mathrm{hull}(\Lambda) = \emptyset$ if and only if Λ consists of at most one point.
2. For any two distinct points $\xi, \eta \in \partial_\infty X$, $\mathrm{hull}(\{\xi, \eta\}) = \xi\eta$.

(b) Anderson's theorem fails for the Euclidean plane $X = E^2$.

Anderson's theorem requires negative pinching: It fails if X merely has strictly negative curvature, see [2].

The geometry of convex hulls remains a bit of a mystery, for instance we still do not entirely understand volumes of convex hulls of finite subsets. The best known result seems to be:

Theorem 2 (A. Borbély, [10]) *If X is m-dimensional, has curvature in the interval $[-k^2, -1]$ and Λ has cardinality $\leq n$, then $Vol(\mathrm{hull}(\Lambda)) \leq Cn^{2-\eta}$, where $C = C(m, k)$, while*

$$\eta = \frac{1}{1 + 4k(m-1)}.$$

For a closed subset $\Lambda \subset \partial \mathbf{B}^n$, define its *tangent hull* $\hat{\Lambda}$ as the union of hyperplanes P_λ, $\lambda \in \Lambda$. I will refer to the hyperplanes P_λ, $\lambda \in \Lambda$ as the *complex support hyperplanes* of Λ. Similarly, for an open subset $\Omega = \partial \mathbf{B}^n - \Lambda$, define

$$\check{\Omega} = \mathbb{P}^n - \hat{\Lambda}. \tag{1}$$

Remark 2 $\hat{\Lambda}$ is also closed and $\hat{\Lambda} \cap \overline{\mathbf{B}^n} = \Lambda$.

See Appendix A for a discussion of *horospheres* and *horoballs* in Hadamard manifolds X and the *horofunction compactification* of X, which leads to an alternative description of the topology on $\overline{X}$.

Isometries of X extend to homeomorphisms of $\overline{X}$; in the setting of $\mathbf{B}^n$, this is just the fact that all automorphisms of $\mathbf{B}^n$ are restrictions of projective transformations:

$$PU(n, 1) < PGL(n + 1, \mathbb{C}).$$

The group $G = PU(n, 1)$ acts doubly transitively on the boundary sphere S^{2n-1}: Given two pairs of distinct points $\xi_i, \eta_i, i = 1, 2$, we connect these points by unique biinfinite (unit speed) geodesics $c_i = \xi_i\eta_i$. Set $z_i := c_i(0), v_i := c'(0) \in T_{z_i}\mathbf{B}^n$. Then, since G acts transitively on the unit tangent bundle $UT\mathbf{B}^n$, there exists $g \in G$ sending $v_1 \mapsto v_2$. Thus, $g(c_1) = c_2$ and, consequently, $g(\xi_1) = \xi_2, g(\eta_1) = \eta_2$.

Classification of isometries. Every isometry $g \in G = Aut(\mathbf{B}^n)$ is continuous on the closed ball $\overline{\mathbf{B}^n}$ and, hence, has at least one fixed point there. Accordingly, automorphisms $g \in G$ are classified as:

1. **Elliptic**: g has a fixed point z in $\mathbf{B}^n$. After conjugating g via $h \in Aut(\mathbf{B}^n)$ which sends z to 0,
$$hgh^{-1} \in K = U(n).$$
2. **Parabolic**: g has a unique fixed point in $\overline{\mathbf{B}^n}$ and this is a boundary point $z \in S^{2n-1}$. Equivalently,
$$\inf\{d(z, gz) : z \in \mathbf{B}^n\} = 0$$
and the infimum is not realized.
3. **Hyperbolic**: g has exactly two fixed points ξ, η in $\overline{\mathbf{B}^n}$, both are in S^{2n-1}. (In particular, g preserves the unique geodesic $\xi\eta$ in $\mathbf{B}^n$ and acts as a translation along this geodesic. This geodesic is called the *axis* of g.) Equivalently,
$$\inf\{d(z, gz) : z \in \mathbf{B}^n\} \neq 0.$$
This infimum is realized by any point on the axis of g.

The fixed point λ of a hyperbolic isometry γ is called *attractive* (resp. *repulsive*) if for some (every) $x \in X$, $\gamma^i(x) \to \lambda$ as $i \to \infty$ (resp. $i \to -\infty$).

An elliptic automorphism of $\mathbf{B}^n$ is called a *complex reflection* if its fixed-point set is a complex hyperbolic hyperplane in $\mathbb{H}^n_{\mathbb{C}}$.

As any strictly negatively curved Hadamard manifold, $\mathbb{H}^n_{\mathbb{C}}$ satisfies the *convergence property:*

Theorem 3 *For every sequence $g_i \in G = PU(n,1)$, after extraction, the following dichotomy holds:*

(a) Either g_i converges to an isometry $g \in G$.

(b) Or there is a pair of points $\xi, \eta \in S^{2n-1}$ such that $g_i|_{\overline{\mathbf{B}^n}-\{\eta\}}$ converges uniformly on compacts to the constant ξ.

Proof First, one proves the convergence property for sequences of hyperbolic isometries with a common axis. The Cartan decomposition of G then concludes the proof.

In the case (b), I will say that (g_i) converges to the *quasiconstant map* ξ_η. (The point η is the *indeterminacy point* of ξ_η.)

It turns out that most elementary properties of discrete isometry groups of strictly negatively curved Hadamard manifolds can be derived just from the Convergence Property! See [14, 90, 91] for a development of the theory of *convergence group actions* on compact metrizable spaces, i.e. topological group actions satisfying the Convergence Property.

Remark 3 1. If $g_i \to \xi_\eta$ then $g_i^{-1} \to \eta_\xi$.

2. If $g_i \to \xi_\eta$, then (g_i) converges (again, uniformly on compacts) to the constant map ξ on $\mathbb{P}^n - P_\eta$.

3 Basics of discrete subgroups of $PU(n,1)$

Almost all the properties of discrete subgroups $\Gamma < G = PU(n,1)$ stated in this section hold for discrete isometry groups of negatively pinched Hadamard manifolds.

Definition 3 A subgroup $\Gamma < \mathrm{Isom}(X)$ of isometries of a Riemannian manifold X is called *discrete* if it is discrete as a subset of $\mathrm{Isom}(X)$. Discrete subgroups $\Gamma < PU(n,1)$ are *complex hyperbolic Kleinian groups*.

Here, all *reasonable* topologies on $\mathrm{Isom}(X)$ agree. For instance, one can use the topology of uniform convergence on compact subsets, or the topology of pointwise convergence.

Recall that a group Γ of homeomorphisms of a topological space X is said to act *properly discontinuously* on X if for every compact $C \subset X$,

$$\mathrm{card}\{\gamma \in \Gamma : \gamma C \cap C \neq \emptyset\} < \infty.$$

Remark 4 Suppose that X is a Riemannian manifold and $G = \mathrm{Isom}(X)$ is the isometry group of X.

(a) The following are equivalent for subgroups $\Gamma < G$:

1. Γ is a discrete subgroup of G.

2. Γ acts properly discontinuously on X.

3. For one (equivalently, every) $x \in X$ the function $\Gamma \to \mathbb{R}_+$, $\gamma \mapsto d(x, \gamma x)$ is proper (with Γ equipped with discrete topology), i.e. if γ_i is a sequence consisting of distinct elements of Γ, then

$$\lim_{i\to\infty} d(x, \gamma_i x) = \infty.$$

(b) Every discrete subgroup of G is at most countable.

A group Γ is said to act *freely* on X is for every $x \in X$, the Γ-stabilizer

$$\Gamma_x = \{\gamma \in \Gamma : \gamma x = x\}$$

is the trivial subgroup of Γ.

If X is a manifold and Γ is a group acting freely and properly discontinuously, then the quotient space X/Γ is a manifold and the projection map $X \to X/\Gamma$ is a covering map. If one does not assume freeness of the action then X/Γ is an *orbifold* and the projection map $X \to X/\Gamma$ is an *orbi-covering map*. If X is simply-connected, the group Γ is the (orbifold) fundamental group of X/Γ. See Appendix D for a discussion of orbifolds and related concepts.

In the case when X is a Hadamard manifold, a subgroup $\Gamma < \mathrm{Isom}(X)$ acts freely on X if and only if Γ is *torsion-free*, i.e. every nontrivial element of Γ has infinite order. If Γ acts on X isometrically/holomorphically, the Riemannian metric/complex structure on X descends to the quotient manifold (orbifold) X/Γ.

Definition 4 A complex hyperbolic n-dimensional orbifold (manifold) is the quotient of $\mathbb{H}^n_{\mathbb{C}}$ by a discrete (torsion-free) subgroup of $PU(n,1)$, $M_\Gamma = \mathbb{H}^n_{\mathbb{C}}/\Gamma$.

Remark 5 If X is a Hadamard manifold and $\Gamma < \mathrm{Isom}(X)$ is discrete, then Γ is torsion-free if and only if it contains no elliptic elements, besides the identity.

For *finitely generated subgroups* $\Gamma < PU(n,1)$, one can eliminate torsion by passing to a finite index subgroup:

Theorem 4 (Selberg's Lemma, see e.g. [46] or [78]) *If* $\mathbf{k}$ *is a field and* $\Gamma < GL(n, \mathbf{k})$ *is a finitely generated subgroup, then* Γ *is* virtually torsion-free*, i.e. contains a torsion-free subgroup of finite index.*

In particular, every complex hyperbolic orbifold O with finitely generated (orbifold) fundamental group, admits a finite-sheeted manifold orbi-covering $M \to O$.

Remark 6 Selberg's theorem fails for discrete finitely generated groups of isometries of negatively pinched Hadamard manifolds, see [56].

Definition 5 Given a Hadamard manifold X, a discrete subgroup $\Gamma < \mathrm{Isom}(X)$ and a point $x \in X$, the *limit set* $\Lambda = \Lambda_\Gamma$ is the accumulation set of the orbit Γx in $\partial_\infty X$, i.e.

$$\Lambda = \partial_\infty(\Gamma x).$$

The complement $\Omega := \partial_\infty X - \Lambda$ is called the *discontinuity domain* of Γ.

Remark 7 Suppose that Γ is a discrete subgroup of $\mathrm{Isom}(X)$ and X is strictly negatively curved. Then:

1. Λ is independent of $x \in X$.[1]
2. Λ is closed and Γ-invariant. Accordingly, Ω is open in $\partial_\infty X$ and is Γ-invariant as well.
3. Ω is either empty or is dense in $\partial_\infty X$.
4. Either Λ consists of at most two points or it is perfect, i.e. contains no isolated points.
5. If Γ' is a subgroup of Γ, then $\Lambda_{\Gamma'} \subset \Lambda_\Gamma$.
6. If $\Gamma' \triangleleft \Gamma$ is an infinite normal subgroup then $\Lambda_{\Gamma'} = \Lambda_\Gamma$.
7. If $\Gamma' < \Gamma$ is a subgroup of finite index then $\Lambda_{\Gamma'} = \Lambda_\Gamma$.

Example 1 Let $\gamma \in \mathrm{Isom}(X)$ be a non-elliptic element. Then the limit set of the group $\Gamma = \langle \gamma \rangle$ generated by γ is equal to the fixed-point set of γ in $\partial_\infty X$.

Lemma 1 *If* $\Gamma < \mathrm{Isom}(X)$ *is a discrete subgroup and* X *is a strictly negatively curved Hadamard manifold, then* Γ *acts properly discontinuously on* $Y = X \cup \Omega$.

Proof Let C be a compact subset of Y. Suppose there exists a sequence consisting of distinct elements $\gamma_i \in \Gamma$ such that for each i, $\gamma_i C \cap C \neq \emptyset$. In view of the Convergence Property, after extraction, the sequence γ_i either converges to an isometry $\gamma \in \mathrm{Isom}(X)$ (which would contradict the discreteness of Γ) or to a quasiconstant map ξ_η, with $\xi, \eta \in \Lambda$. Since (γ_i) converges to ξ uniformly on compacts in $\overline{X} - \{\eta\}$ and $C \subset Y \subset \overline{X} - \{\eta\}$ is compact, there exists a neighborhood U of ξ disjoint from C; thus, for all but finitely many values of i, $\gamma_i(C) \subset U$. A contradiction.

A more difficult result is

Theorem 5 (A. Cano, J. Seade, see [19, 18]) *Every discrete subgroup* $\Gamma < PU(n, 1)$ *acts properly discontinuously on* $\check{\Omega} := \mathbb{P}^n - \hat{\Lambda}$ *(see* (1)*).*

Remark 8 An alternative proof of this result is an application of a proper discontinuity theorem in [58]. More precisely, let $F_{1,n}$ be the flag-manifold consisting of flags (V_1, V_n) in $V = \mathbb{C}^{n+1}$, where V_1 is a line and V_n is a hyperplane (containing V_1). We have a G-equivariant holomorphic fibration $\pi : F_{1,n} \to \mathbb{P}^n$ sending each pair (V_1, V_n) to V_1. The tangent hull $\hat{\Lambda}$ of Λ defines a natural continuous map $\theta : \Lambda \to F_{1,n}$ sending each $\lambda \in \Lambda$ to the pair (V_1, V_n) consisting of the preimages of λ and P_λ in V. Let $\tilde{\Lambda}$ be the image of θ and let $Th(\tilde{\Lambda})$ be the *thickening* of $\tilde{\Lambda}$ in $F_{1,n}$, consisting of flags (V_1', V_n') such that either V_1' belongs to Λ or V_n' is a complex support hyperplane of Λ. Then Γ acts properly discontinuously on $\Omega_{Th} = F_{1,n} - Th(\tilde{\Lambda})$; see [58]. Since $\pi^{-1}(\check{\Omega}) \subset \Omega_{Th}$, the action of Γ on $\check{\Omega}$ is properly discontinuous as well.

In particular, the quotient $\overline{M}_\Gamma := (\mathbf{B}^n \cup \Omega)/\Gamma$ embeds as an orbifold with boundary in the complex orbifold without boundary $\check{\Omega}/\Gamma$. The boundary of $\overline{M}_\Gamma$ (equal to Ω/Γ) is strictly Levi-convex in $\check{\Omega}/\Gamma$.

Notation The *boundary* ∂M_Γ of a complex hyperbolic orbifold M_Γ is Ω_Γ/Γ; in other words, this is the boundary of $\overline{M}_\Gamma$. □

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[1] This also holds for general Hadamard manifolds even though the convergence property fails.

We now return to the discussion of discrete subgroups of general negatively pinched Hadamard manifolds X.

Theorem 7 *If α, β are hyperbolic elements of a discrete subgroup of* $\mathrm{Isom}(X)$*, then their fixed-point sets are either equal or disjoint.*

Corollary 1 *If $\Gamma < \mathrm{Isom}(X)$ is discrete and fixes a point $\lambda \in \partial_\infty X$ then Λ_Γ either equals to $\{\lambda\}$ and Γ contains no hyperbolic element, or Λ_Γ consists of two points, $\Lambda_\Gamma = \{\lambda, \lambda'\}$ and Γ contains no parabolic element.*

Definition 6 A discrete subgroup $\Gamma < \mathrm{Isom}(X)$ is called *elementary* if $\mathrm{card}(\Lambda_\Gamma) \leq 2$. It is said to be *nonelementary* otherwise.

Elementary subgroups are, in many ways, exceptional, among discrete subgroups.

In view of Remark 7(4), the limit set of every nonelementary subgroup is perfect. In particular, it has the cardinality of the continuum. Hence:

Proposition 1 *The limit set of a discrete subgroup of* $\mathrm{Isom}(X)$ *consists of 0, 1, 2 or a continuum of points.*

Proposition 2 *The limit set of a nonelementary discrete group Γ is the smallest nonempty closed Γ-invariant subset of $\partial_\infty X$. In particular, every orbit in Λ_Γ is dense.*

Proof Suppose that $L \subsetneq \Lambda_\Gamma$ is a closed nonempty and Γ-invariant subset. Take a point $\xi \in \Lambda_\Gamma - L$ and let (γ_i) be a sequence in Γ converging to a quasiconstant map ξ_η. Then for every $\lambda \in L - \{\eta\}$, $\lim_{i\to\infty} \gamma_i(\lambda) = \xi$. Since L is closed and $\xi \notin L$, for all sufficiently large i, $\gamma_i(\lambda) \notin L$, contradicting the invariance of L. This leaves us with the possibility that L is the singleton $\{\xi\}$ and ξ is fixed by the entire Γ. It then follows that Γ is elementary.

Theorem 8 *Suppose that Γ is an elementary subgroup of* $\mathrm{Isom}(X)$.

1. If Λ_Γ is a singleton then every element of Γ is elliptic or parabolic.

2. If Λ_Γ consists of two points then every element of Γ is elliptic or hyperbolic. Hyperbolic elements fix Λ_Γ pointwise. Elliptic elements can swap the two limit points.

3. Γ is a virtually nilpotent[2] group.

See [7] for a more detailed discussion of elementary groups and their quotient spaces M_Γ. Here we only note that discrete elementary subgroups of $PU(n, 1)$ are virtually *2-step* nilpotent.

Proposition 3 *Suppose that X is a strictly negatively curved Hadamard manifold. If ξ, η are distinct limit point of a discrete subgroup $\Gamma < \mathrm{Isom}(X)$ then there exists a sequence $\gamma_k \in \Gamma$ of hyperbolic elements whose attractive (resp. repulsive) fixed points converge to ξ (resp. η).*

[2] I.e. contains a nilpotent subgroup of finite index

Proof Since ξ, η are limit points of Γ, there exist sequences $(g_i), (h_j)$ in Γ which converge, respectively, to the quasiconstant maps ξ_α and β_η. By precomposing these sequences with suitable elements of Γ, we can assume that the points ξ, η, α, β are pairwise distinct. Let $U_\alpha, U_\beta, U_\xi, U_\eta$ be pairwise disjoint open ball neighborhoods in $\partial_\infty X$ of α, β, ξ, η respectively. In view of the convergence $g_i \to \xi_\alpha, h_j \to \beta_\eta$, for all sufficiently large i we have

$$h_i(\partial_\infty X - U_\eta) \subset U_\beta, \quad g_i(\partial_\infty X - U_\alpha) \subset U_\xi,$$

and, hence,

$$g_i \circ h_i(\partial_\infty X - U_\eta) \subset U_\xi.$$

In particular, the composition $f_i = g_i \circ h_i$ has an attractive fixed point in U_ξ. Similarly, f_i^{-1} has an attractive fixed point in U_η.

Corollary 2 *If* Γ *is nonelementary then the set of hyperbolic fixed points of elements of* Γ *is dense in* Λ_Γ.

Corollary 3 *If a discrete group* Γ *contains a parabolic element then parabolic fixed points are dense in* Λ_Γ.

The following theorem provides a converse to Theorem 8(3):

Theorem 9 *Each nonelementary discrete subgroup* $\Gamma < \mathrm{Isom}(X)$ *contains a non-abelian free subgroup whose limit set is homeomorphic to the Cantor set.*

Definition 7 The *convex core*, $Core(M)$, of $M = M_\Gamma = X/\Gamma$ is the projection to M_Γ of the closed convex hull $\mathrm{hull}(\Lambda_\Gamma)$ of the limit set of Γ.

Given $\eta > 0$, define $Core_\eta(M)$ as the projection to M_Γ of $\mathrm{hull}_\eta(\Lambda_\Gamma)$. Intrinsically, the convex core can be defined as:

Remark 9 $Core(M)$ is the intersection of all closed convex suborbifolds $M' \subset M$ such that $\pi_1(M') \to \pi_1(M)$ is surjective.

Conical limit points. I conclude this section with a discussion of a classification of limit points of discrete subgroups of $\mathrm{Isom}(X)$.

Definition 8 A sequence (x_i) in X is said to converge to a point $\xi \in \partial_\infty X$ *conically* if there exists a geodesic ray $x\xi$ in X and a constant $R < \infty$ such that:
$d(x_i, x\xi) \le R$ for all i and $\lim_{i\to\infty} x_i = \xi$.

Remark 10 Let $\lambda \in \Lambda_\Gamma$ be a limit point. The following are equivalent:
1. There exists a sequence $\gamma_i \in \Gamma$ such that the sequence $(\gamma_i(x))$ converges to ξ conically.
2. The projection of the ray $x\lambda$ to M_Γ defines a non-proper map $\mathbb{R}_+ \to M_\Gamma$.

Definition 9 A limit point $\lambda \in \Lambda_\Gamma$ is called *conical* or *radial* if it satisfies one of the two equivalent properties in the above remark. The set of conical limit points of Γ is denoted $\Lambda^c = \Lambda^c_\Gamma$.

Example 2 1. If Γ is an elementary hyperbolic subgroup of $\mathrm{Isom}(X)$ then $\Lambda_\Gamma = \Lambda^c_\Gamma$.
2. If Γ is an elementary parabolic subgroup of $\mathrm{Isom}(X)$ then $\Lambda^c_\Gamma = \emptyset$.

4 Margulis Lemma and thick-thin decomposition

In this section, X is a negatively pinched Hadamard manifold. For each discrete subgroup $\Gamma < \mathrm{Isom}(X)$, a point $x \in X$ and a number $\epsilon > 0$, define $\Gamma_{x,\epsilon}$ to be the subgroup of Γ generated by the (necessarily finite) set

$$\{\gamma \in \Gamma : d(x, \gamma x) < \epsilon\}.$$

This subgroup is the "almost-stabilizer" of x in Γ.

Let $U_{\Gamma,\epsilon}$ denote the subset of X consisting of points x for which the almost-stabilizer $\Gamma_{x,\epsilon}$ is infinite.

The components of $U_{\Gamma,\epsilon}$ need not be convex (already for $X = \mathbb{H}^2_{\mathbb{C}}$), but each component is contractible:

Proposition 4 *Each component of $U_{\Gamma,\epsilon}$ is contractible.*

In view of the contractibility of X and of hull Λ_Γ, it follows that $X - U_{\Gamma,\epsilon}$ and hull $\Lambda_\Gamma - U_{\Gamma,\epsilon}$ are both contractible. Furthermore, if X has curvature ≤ -1, each component U of $U_{\Gamma,\epsilon}$ is *uniformly quasiconvex*:

Theorem 10 *There exist universal constants δ_0, η_0 such that each component U of $U_{\Gamma,\epsilon}$ satisfies:*

1. For any two points $x, y \in U$, the geodesic xy is contained in the δ_0-neighborhood of U.

2. The η_0-neighborhood of U is convex.

Theorem 11 (Kazhdan–Margulis; Margulis; see e.g. [5]) *Let X be an n-dimensional Hadamard manifold of sectional curvature bounded below by $b \le 0$. Then there exists $\epsilon = \epsilon(n, b)$ such that for every discrete subgroup $\Gamma < \mathrm{Isom}(X)$ and every $x \in X$, the subgroup $\Gamma_{x,\epsilon}$ is virtually nilpotent. In particular, if X is negatively curved, then $\Gamma_{x,\epsilon}$ is elementary.*

Corollary 4 *For each discrete torsion-free subgroup $\Gamma < \mathrm{Isom}(X)$, the set $U_{\Gamma,\epsilon}$ breaks into connected components $X_{\Gamma,\epsilon,i}$ each of which is stabilized by some elementary subgroup Γ_i of Γ and for each $x \in X_{\Gamma,\epsilon,i}$ the stabilizer Γ_i contains the "almost stabilizer" $\Gamma_{x,\epsilon}$. (The index can be infinite.)*

As a corollary, one obtains the *thick-thin decomposition* of the orbifold $M = M_\Gamma$:

$M_{(0,\epsilon)}$ is the projection of $U_{\Gamma,\epsilon}$ to M. It consists of all points $y \in M$ for which there exists a homotopically nontrivial loop based at y of length $< \epsilon$. Define also $M_{(0,\epsilon]}$ as the closure of $M_{(0,\epsilon)}$ in M. Both $M_{(0,\epsilon)}$ and $M_{(0,\epsilon]}$ are called the ϵ-*thin* parts of M. The complement $M_{[\epsilon,\infty)} = M - M_{(0,\epsilon)}$ and its interior $M_{(\epsilon,\infty)}$ are called the ϵ-*thick* parts of M.

One defines the ϵ-thick, resp. thin, part of the convex core $Core(M)$ as the intersection $Core(M) \cap M_{[\epsilon,\infty)}$, resp. $Core(M) \cap M_{(0,\epsilon)}$.

Components of the thin parts M and $Core(M)$ come in two shapes:

(a) **Tubes**. Suppose that U is a component of $U_{\Gamma,\epsilon}$ whose stabilizer Γ_U in Γ is virtually hyperbolic, i.e. contains a cyclic hyperbolic subgroup of finite index. In other words, the limit set of Γ_U consists of two points ξ, η. The geodesic $\xi\eta$ is then invariant under Γ_U; it is also contained in U and projects to a closed geodesic $c \subset U/\Gamma_U$. The quotient U/Γ_U is a *tube*: If Γ_U is torsion-free then this quotient is homeomorphic to an $\mathbb{R}^k$-bundle over S^1, with the base of the fibration corresponding to the closed geodesic c.

(b) **Cusps**. Suppose that U is a component of $U_{\Gamma,\epsilon}$ whose stabilizer Γ_U in Γ is virtually parabolic, i.e. contains a parabolic subgroup of finite index. In other words, the limit set of Γ_U consists of a single point η. The group Γ_U preserves horoballs B_η based at η. The subsets $U_{\Gamma,\epsilon}$ are typically strictly smaller (not even Hausdorff-close) than any of the horoballs B_η.

5 Geometrically finite groups

The notion of geometrically finite Kleinian group was introduced by Lars Ahlfors in the mid 1960s for the real hyperbolic space and later generalized (by William Thurston and Brian Bowditch) to manifolds of negative curvature: The discrete groups in this class are the *nicest-behaving* among discrete isometry groups of negatively pinched Hadamard manifolds.

Definition 10 Let X be a negatively pinched Hadamard manifold. A discrete subgroup $\Gamma < G = \mathrm{Isom}(X)$ is called *geometrically finite* if:

(a) The orders of elliptic elements of Γ are uniformly bounded (from above), and

(b) the volume of $Core_\eta(M_\Gamma)$ is finite for some (equivalently, every, $\eta > 0$).

A discrete subgroup $\Gamma < G$ is called *convex-cocompact* if $\mathrm{card}(\Lambda_\Gamma) \neq 1$ and $Core(M_\Gamma)$ is compact.

For instance, if $\Lambda_\Gamma = \partial_\infty X$ then $\mathrm{hull}(\Lambda_\Gamma) = X$ and, thus, Γ is geometrically finite if and only if $\Gamma < G$ is a *lattice*, i.e. $vol(M_\Gamma) < \infty$. Under the same assumption, Γ is convex-cocompact if and only if $\Gamma < G$ is a *uniform lattice*, i.e. M_Γ is compact.

Theorem 12 *1. (B. Bowditch, [13]) A discrete subgroup $\Gamma < G$ is geometrically finite if and only if the ϵ-thick part of $Core(M_\Gamma)$ is compact.*

2. (B. Bowditch, [13]) A discrete subgroup $\Gamma < G$ is convex-cocompact if and only if $\overline{M}_\Gamma$ is compact.

3. (B. Bowditch, [13]) A discrete subgroup $\Gamma < G$ is convex-cocompact if and only if every limit point of Γ is conical.

4. (M. Kapovich, B. Liu, [59]) A discrete subgroup $\Gamma < G$ is geometrically finite if and only if every limit point of Γ is either conical or a parabolic fixed point.

In particular, (1) implies that geometrically finite groups are finitely presentable (since $\mathrm{hull}\,\Lambda_\Gamma - U_{\Gamma,\epsilon}$ is contractible).

In particular, a convex-cocompact subgroup $\Gamma < PU(n,1)$ acts properly discontinuously and cocompactly on $\mathbb{H}^n_{\mathbb{C}} \cup \Omega$. The action of Γ on $\check{\Omega}$ is properly discontinuous but not cocompact. If becomes cocompact if we lift it to the flag-manifold $F_{1,n}$ (see [58]):

Theorem 13 *The Γ-action on the domain $\Omega_{Th} \subset F_{1,n}$ is properly discontinuous and cocompact.*

6 Ends of negatively curved manifolds

Let X be a negatively pinched Hadamard manifold and let Λ be a closed subset of $\partial_\infty X$ consisting of at least two points. Set $\Omega = \partial_\infty X - \Lambda$. The nearest-point projection $\Pi : X \to \mathrm{hull}(\Lambda)$ extends continuously to a map $\Pi : X \cup \Omega \to \mathrm{hull}(\Lambda)$: While for $x \in X$, $\Pi(x)$ is defined by minimizing the distance function $d_x = d(x, \cdot)$ on hull(Λ), for $\xi \in \Omega$, the projection $\Pi(\xi)$ is defined by minimizing the Busemann function b_ξ based at ξ. For a component $\Omega_0 \subset \Omega$ we define a subset $X_0 \subset X$ as the union of open geodesic rays $x\xi - \{x\}$, where $\xi \in \Omega_0$, $x = \Pi(\xi)$. The union of these geodesic rays is an open subset of $X - \mathrm{hull}(\Lambda)$ whose closure in $X \cup \Omega$ equals $X_0 \cup \Omega_0 \cup \Pi(\Omega_0)$.

We now specialize to the setting when $\Lambda = \Lambda_\Gamma$ is the limit set of a discrete subgroup $\Gamma < \mathrm{Isom}(X)$. If Ω_0 has cocompact stabilizer Γ_0 in Γ, then Γ_0 also acts cocompactly on $X_0 \cup \Omega_0 \cup \Pi(\Omega_0)$. Thus, M_Γ has an *isolated end* E_0 corresponding to Ω_0/Γ_0, with isolating neighborhood X_0/Γ_0.

Definition 11 Ends E_0 of $M = M_\Gamma$ which have this form are called *convex ends* of M.

From the analytical viewpoint, the advantage of working with convex ends E_0 is that they admit *convex exhaustion functions*: For every convex end E_0 there exists a convex function $\phi : M \to \mathbb{R}_+$ which is proper on the closure of E_0 and vanishes on $M - E_0$.

Suppose that C is an unbounded component of the thin part $M_{(0,\epsilon)}$ of $M = M_\Gamma$, and C has compact boundary. Then C also defines an isolated end E_C with an isolating neighborhood given by $C \cap M_{(0,\epsilon)}$.

Definition 12 Ends E_C of M_Γ which have this form are called *cuspidal ends* of M_Γ.

Remark 11 1. Γ is convex-cocompact if and only if M_Γ has only convex ends.
2. If M_Γ has only convex and cuspidal ends then Γ is geometrically finite.

One can refine (cf. [53]) the above definitions in two ways:
(a) Considering unbounded components of the thin part of $Core(M_\Gamma)$ and, thus, defining cuspidal ends of the convex core.
(b) Removing from M_Γ its cuspidal ends and their preimages under the nearest-point projection $M_\Gamma \to Core(M_\Gamma)$, one then defines *relative* convex ends of M_Γ.

One can also classify ends of M_Γ using potential theory as *hyperbolic* and *parabolic* ends, see [72]. Note that if $M = M_\Gamma$ is a complex hyperbolic manifold, then every convex end E of M is hyperbolic.

7 Critical exponent

Notation Let $B(x, r)$ denote the open ball of radius r and center at x in a metric space. □

I will discuss the critical exponent mostly in the case of complex hyperbolic Kleinian groups; for a discussion in the broader context of negatively curved Hadamard manifolds and Gromov-hyperbolic spaces see e.g. [21, 27, 28, 64, 79].

The *critical exponent* of a discrete isometry group Γ of a Hadamard manifold X (typically, satisfying some further curvature restrictions) is, probably, the single most important numerical invariant of Γ: It reflects the geometry of Γ-orbits in X, the geometry of the limit set of Γ, the ergodic theory of the action of Γ on the limit set and analytic properties of the quotient space X/Γ. Its origin goes back to the 19th century and the work of Poincaré (among others), who was interested in constructing *automorphic functions* (and forms) on the hyperbolic plane by "averaging" a certain holomorphic function (or a form) over a discrete isometry group Γ. The resulting infinite series (the *Poincaré series*) may or may not converge, depending on the *weight* of the form, leading to the notion of *critical exponent* or *exponent of convergence* of Γ.

Let $\Gamma < \mathrm{Isom}(X)$, a discrete isometry group of a Hadamard manifold. Pick points $x, y \in X$. The *entropy* of Γ is defined as

$$\delta = \delta_\Gamma = \limsup_{r\to\infty} \frac{1}{r}\, \mathrm{card}(B(x, r) \cap \Gamma y).$$

Thus, the entropy measures the rate of exponential growth of Γ-orbits in X. It turns out that δ is equal to the *critical exponent* of Γ, defined as

$$\delta = \inf\{s : \sum_{\gamma\in\Gamma} \exp(-sd(x, \gamma y)) < \infty\},$$

i.e. δ is the *exponent of convergence* of the Poincaré series $\sum_{\gamma\in\Gamma} \exp(-sd(x, \gamma y))$. Furthermore, δ is independent of the choice of $x, y \in X$. If

$$\sum_{\gamma\in\Gamma} \exp(-\delta d(x, \gamma y)) < \infty$$

(which depends only on Γ and not on the choice of x, y), then Γ is said to be a subgroup of *convergence type*; otherwise, Γ is said to be of *divergence type*.

Below are equivalent characterizations of δ in the case $X = \mathbb{H}^n_{\mathbb{C}}$:

Theorem 15 *Suppose that* $\Gamma < PU(n, 1)$ *is a discrete subgroup.Then:*

1. (Corlette [23]; Corlette–Iozzi [25], Theorem 6.1) $\delta = \delta_\Gamma$ *is equal to the Hausdorff dimension* $\dim_H \Lambda^c_\Gamma$, *where the conical limit set* Λ^c_Γ *is equipped with the restriction of the Carnot metric on* S^{2n-1}. *In particular, if* Γ *is geometrically finite then* $\delta = \dim_H \Lambda$.

2. (Elstrodt–Patterson–Sullivan–Corlette–Leuzinger, see [65, Corollary 1]) Let $\lambda = \lambda(M_\Gamma)$ denote the bottom of the L^2-spectrum of the Laplacian on M_Γ. Then

$$\begin{cases} \lambda = n^2 & \text{if } 0 \le \delta \le n \\ \lambda = \delta(2n-\delta) & \text{if } n \le \delta \le 2n \end{cases}$$

8 Examples

I will say that a discrete torsion-free subgroup $\Gamma < G = PU(n,1)$ is *Stein* if the complex manifold M_Γ is Stein.

I will start with two elementary examples.

Example 3 **Cyclic hyperbolic groups.** Let $\gamma \in PU(n,1)$ be a hyperbolic isometry fixing points $\lambda_\pm \in S^{2n-1} = \partial_\infty \mathbb{H}^n_{\mathbb{C}}$ and let $\Gamma = \langle\gamma\rangle$ be the cyclic subgroup of $PU(n,1)$ it generates. Then Γ is an elementary subgroup with the limit set $\Lambda = \{\lambda_-, \lambda_+\}$. The quotient manifold $M_\Gamma = \mathbb{H}^n_{\mathbb{C}}/\Gamma$ is diffeomorphic to the product $\mathbb{R}^{2n-1} \times S^1$ while $\overline{M}_\Gamma$ is diffeomorphic to the product $\bar{D}^{2n-1} \times S^1$, where $\bar{D}^{2n-1}$ is the closed disk of real dimension $2n-1$.

Example 4 **Integer Heisenberg groups.** Given a natural number n, define the $2n+1$-dimensional real Lie group H_{2n+1} as the group of $(n+2)\times(n+2)$-matrices

$$\begin{bmatrix} 1 & \mathbf{a} & c \\ 0 & I_n & \mathbf{b} \\ 0 & 0 & 1 \end{bmatrix},$$

where I_n is the identity $n\times n$ matrix, $\mathbf{a} \in \mathbb{R}^n$ is a row-vector, $\mathbf{b} \in \mathbb{R}^n$ is a column-vector and $c \in \mathbb{R}$. This group is 2-step nilpotent with 1-dimensional center consisting of the matrices with $\mathbf{a} = \mathbf{b} = 0$ and $c \in \mathbb{R}$. The quotient of H_{2n+1} by its center is the $2n$-dimensional commutative Lie group isomorphic to $\mathbb{R}^{2n}$. The real Heisenberg group H_{2n+1} contains the integer Heisenberg group $H_{2n+1}(\mathbb{Z})$, defined as the intersection

$$H_{2n+1} \cap SL(n+2, \mathbb{Z}).$$

The quotient $N = H_{2n+1}/H_{2n+1}(\mathbb{Z})$ is a compact *nil-manifold*, which is a nontrivial circle over the torus T^{2n}. Algebraically, in terms of its presentation, $H_{2n+1}(\mathbb{Z})$ is given by

$$\langle x_1, y_1, \ldots, x_n, y_n, t | [x_i, t] = [y_j, t] = 1, [x_i, y_i] = t, i = 1, \ldots, n, j = 1, \ldots, n\rangle.$$

The Heisenberg group H_{2n+1} embeds in $PU(n+1,1)$, fixing a point ξ in $\partial_\infty \mathbb{H}^{n+1}_{\mathbb{C}}$ and acting simply-transitively on every horosphere in $\mathbb{H}^{n+1}_{\mathbb{C}}$ centered at ξ. Thus, $H_{2n+1}(\mathbb{Z})$ embeds as a discrete elementary subgroup $\Gamma < PU(n+1,1)$ such that M_Γ

is diffeomorphic to $N \times (0, \infty)$. The partial compactification $\overline{M}_\Gamma$ is diffeomorphic to $N \times [0, \infty)$.

The rest of our examples are nonelementary.

Example 5 **Schottky groups.** These are convex-cocompact subgroups $\Gamma < G$ isomorphic to free nonabelian groups F_k of finite rank k. The limit set Λ_Γ is homeomorphic to the Cantor set. Its Hausdorff dimension is positive but can be arbitrarily close to 0. Schottky groups are always Stein. Every nonelementary discrete subgroup contains a Schottky subgroup. Schottky subgroups can be found via the following procedure. Let $\gamma_1, \ldots, \gamma_k$ be hyperbolic isometries with pairwise disjoint fixed-point sets. Then there exists t_0 such that for each integer $t \geq t_0$, the subgroup generated by $s_1 = \gamma_1^t, \ldots, s_k = \gamma_k^t$ is a Schottky group with free generating set $s_1, \ldots, s_k$.

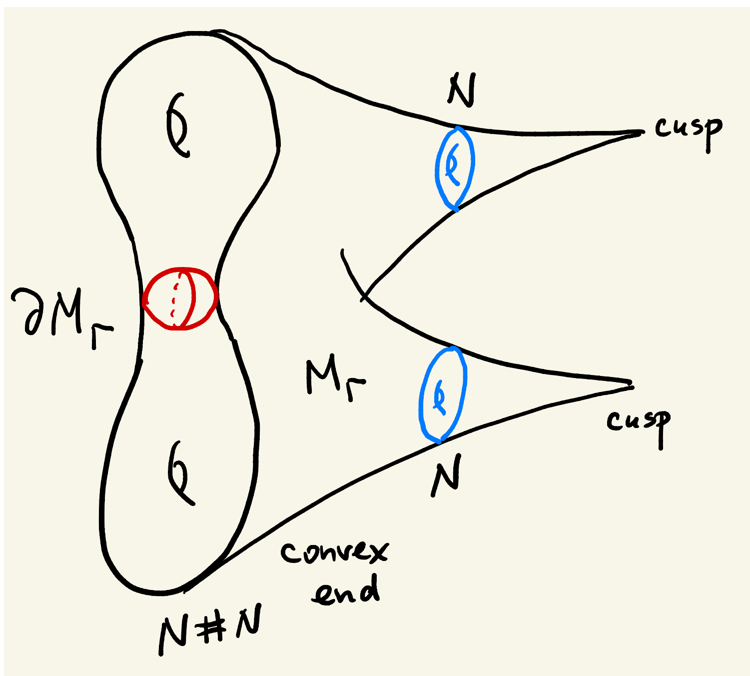

Fig. 1 Quotient manifold of a Schottky-type group with $k = 2$.

Example 6 **Schottky-type groups.** These are geometrically finite subgroups $\Gamma < G$ isomorphic to free products of elementary subgroups of G, such that the limit set Λ_Γ is homeomorphic to the Cantor set. Schottky-type subgroups can be found via the following procedure. Let $\Gamma_1, \ldots, \Gamma_k$ be elementary subgroups with pairwise disjoint limit sets. Then there exist torsion-free finite-index subgroups $\Gamma_i^\ell < \Gamma_i, i = 1, \ldots, k$, such that the subgroup generated by

$$\Gamma_1^\ell, \ldots, \Gamma_k^\ell$$

is Schottky-type and the homomorphism

$$\Gamma_1^\ell \star \ldots \star \Gamma_k^\ell \to \Gamma = \langle \Gamma_1^\ell, \ldots, \Gamma_k^\ell \rangle$$

sending $\Gamma_i^\ell \to \Gamma_i^\ell, i = 1, \ldots, k$, is an isomorphism. For instance, suppose that $\Gamma_1, \ldots, \Gamma_k$ are integer Heisenberg subgroups of G. Then M_Γ has k cuspidal ends (diffeomorphic to $N \times (0, \infty)$) and one convex end, with ∂M_Γ diffeomorphic to the k-fold connected sum of N with itself, where $N = H_{2n-1}/H_{2n-1}(\mathbb{Z})$. See Figure 1.

Real and complex Fuchsian groups defined below were introduced by Burns and Shnider in [17].

Example 7 **Real-Fuchsian subgroups.** Let $\mathbb{H}^2_{\mathbb{R}} \subset \mathbb{H}^n_{\mathbb{C}}$ be a real hyperbolic plane in $\mathbb{H}^n_{\mathbb{C}}$. Let $\Gamma < PU(n,1)$ be a geometrically finite subgroup whose limit set is $\partial_\infty \mathbb{H}^2_{\mathbb{R}}$. Then Γ preserves $\mathbb{H}^2_{\mathbb{R}}$ and acts on it with quotient of finite area. The quotient surface-orbifold Σ is the convex core of M_Γ. The limit set of Γ has Hausdorff dimension 1. Assume now that $n = 2$, Γ is torsion-free and Σ is compact. Then M_Γ is diffeomorphic to the tangent bundle of Σ and is Stein.

Example 8 **Real quasi-Fuchsian subgroups.** Let $\Gamma_t, t \in [0,1]$, be a continuous family of discrete convex-cocompact subgroups of $PU(n,1)$ such that Γ_0 is real-Fuchsian but other subgroups $\Gamma_t, t > 0$ are not.[3] The subgroups Γ_t are real-quasi-Fuchsian subgroups. Their limit sets are topological circles of Hausdorff dimension > 1.

Assume that $n = 2$, Γ is torsion-free and Σ is compact. Then M_Γ is diffeomorphic to the tangent bundle of Σ and is Stein.

Example 9 **Complex-Fuchsian subgroups.** Let $\mathbb{H}^1_{\mathbb{C}} \subset \mathbb{H}^n_{\mathbb{C}}$ be a complex hyperbolic line in $\mathbb{H}^n_{\mathbb{C}}$. Let $\Gamma < PU(n,1)$ be a geometrically finite subgroup whose limit set is $\partial_\infty \mathbb{H}^1_{\mathbb{C}}$. Then Γ preserves $\mathbb{H}^1_{\mathbb{C}}$ and acts on it with quotient of finite area. The quotient surface-orbifold Σ is the convex core of M_Γ. The limit set of Γ has Hausdorff dimension 2. Let $W \subset V = \mathbb{C}^{n+1}$ be the 2-dimensional complex linear subspace such that the projection of $W \cap V_-$ to $\mathbf{B}^n$ equals $\mathbb{H}^1_{\mathbb{C}}$. The $W^\perp \subset V$ (the complex orthogonal complement with respect to the form q on V) has the property that q restricted to $W^\perp$ is positive-definite. The projection $[W^\perp]$ of $W^\perp$ to $\mathbb{P}^n$ is Γ-invariant. The pair $([W], [W^\perp])$ defines a linear holomorphic fibration of $\mathbb{P}^n - [W^\perp]$ over $[W]$: The fiber through $x \in \mathbb{P}^n - [W^\perp]$ is the unique projective hyperplane passing through x and intersecting transversally both $[W]$ and $[W^\perp]$. Restricting to $\mathbf{B}^n$ we obtain a Γ-invariant holomorphic fibration $\mathbf{B}^n \to \mathbb{H}^1_{\mathbb{C}}$. Projecting to M_Γ we obtain a holomorphic orbi-fibration $M_\Gamma \to \Sigma$, whose fibers are biholomorphic to quotients of $\mathbf{B}^{n-1}$ by finite subgroups of $Aut(\mathbf{B}^{n-1})$. Assume now that $n = 2$, Γ is torsion-free and Σ is compact. Then M_Γ is diffeomorphic to the square root of the tangent bundle of Σ (the spin-bundle) and is not Stein (it contains the compact complex curve Σ).

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[3] Such deformations exist as long as Γ_t is, say, torsion-free. More generally, such deformations exist if Γ has trivial center and is not isomorphic to a von Dyck group. See e.g. [94].

Convex-cocompact complex Fuchsian groups are locally rigid in the sense that any small deformation of such a group is again complex Fuchsian, [89]. The complex Fuchsian examples generalize to the case of geometrically finite subgroups of $PU(n,1)$ whose limit sets are ideal boundaries of k-dimensional complex hyperbolic subspaces $\mathbb{H}^k_{\mathbb{C}} \subset \mathbb{H}^n_{\mathbb{C}}$. The rigidity theorem holds in this case as well, see [39, 22, 16].

Example 10 **Hybrid groups.** One can combine, say, torsion-free, real and complex Fuchsian groups in a variety of ways. For instance, one can form free products of such groups. The nature of the quotient manifolds will depend on the precise way in which the free factors are interacting with each other. For instance, in the case $n = 2$ the boundary of M_Γ can be either a connected sum, or the *toral sum* of certain circle bundles over surfaces. One can also break real and complex Fuchsian groups into smaller pieces and consider amalgams over $\mathbb{Z}$ of these pieces. As a result, one can get for instance, circle bundles over surfaces other than the unit tangent bundle and its square root, see [38] and [1] for more detail.

Example 11 **AGG subgroups: Anan'in–Grossi–Gusevskii**, [1]. These interesting examples of convex-cocompact subgroups of $PU(2,1)$ are isomorphic images of *von Dyck groups* $D(2,n,n)$, for $n \in \{10\} \cup [13, 1001]$. None of these subgroups is complex Fuchsian or real quasi-Fuchsian. According to Proposition 6, these subgroups are locally rigid in $PU(2,1)$: Every small deformation is conjugate in $PU(2,1)$ to the original subgroup. The limit set is a topological circle but is neither a complex nor a real circle. Fix a (unique up to conjugation) discrete, faithful and isometric action of $D(2,n,n)$ on $\mathbb{H}^1_{\mathbb{C}}$. For each embedding $\rho : D(2,n,n) \to \Gamma < PU(2,1)$ constructed in Section 3.3 of [1], the complex hyperbolic orbifold M_Γ is diffeomorphic to the total space of an orbifold bundle over the complex 1-dimensional orbifold $\mathcal{B} = \mathbb{H}^1_{\mathbb{C}}/D(2,n,n)$ with fibers given by projections to M_Γ of some complex geodesics in $\mathbb{H}^2_{\mathbb{C}}$. It follows from the local rigidity of each ρ, combined with [86, Lemma 4.5], that there exists an equivariant holomorphic map

$$\tilde{f} : \mathbb{H}^1_{\mathbb{C}} \to \mathbb{H}^2_{\mathbb{C}}.$$

(I owe this observation to Ludmil Katzarkov.)[4] Since the orbi-bundle $M = M_\Gamma \to \mathcal{B}$ has holomorphic fibers, it follows that $\tilde{f}$ descends to a holomorphic map $f : \mathcal{B} \to M$ which has only positive, zero-dimensional intersections with the fibers. Composing with the projection $M \to \mathcal{B}$, we obtain a self-map $h : \mathcal{B} \to \mathcal{B}$ which is a branched covering. Since $\mathcal{B}$ is a hyperbolic orbifold, it follows that $h = \mathrm{id}$. In other words, $M \to \mathcal{B}$ admits a holomorphic section. In particular, M (and any of its finite manifold-covering spaces, given by Selberg's Lemma) is non-Stein.

Example 12 **Polygon-groups, J. Granier,** [41].

The polygon-group $\Gamma_{6,3}$ (see Example 17) embeds as a convex-cocompact subgroup in $PU(2,1)$ via the reflection representation $\rho_{6,3}$. Thus, the limit set of $\Gamma_{6,3} < PU(2,1)$ is homeomorphic to the Menger curve.

[4] I refer the reader to the book [20] for a gentle introduction to Simpson's results, discussion of variations of Hodge structures and period domains.

Conjecturally, the same holds for all polygon-groups $\Gamma_{n,3}, n \geq 6$, cf. [11, 54, 30] for a discussion of isometric actions on real hyperbolic spaces.

Example 13 **Complex-hyperbolic manifolds which are singular fibrations with compact fibers.**

Definition 13 A *singular Kodaira fibration* is a surjective holomorphic map with connected fibers $f : M \to B$ between connected complex manifolds/orbifolds, where $0 < \dim B < \dim M$. (Usually, it is required that no two generic fibers are biholomorphic to each other, but, in order to simplify the discussion, I will omit this condition.)

Singular Kodaira fibrations need not be locally trivial in the holomorphic or even topological sense; a *Kodaira fibration* is a holomorphic map $f : M \to B$ which is a smooth fiber bundle.

In the context of complex hyperbolic manifolds, the first example of a singular Kodaira fibration appeared in Ron Livne's PhD thesis, [67]. Many more examples are now known. Below we discuss one example which (to my knowledge) first appeared in the work of Mostow [71] with an explicit description of the quotient-orbifold given by Hirzebruch in [50].

Consider the *standard quadrangle* in $\mathbb{P}^2_{\mathbb{C}}$, which is a configuration A of six lines $A_1, A_2, A_3, B_1, B_2, B_3$ with four triple intersection points a_1, a_2, a_3, a_4 and three double intersection points b_1, b_2, b_3, see Figure 2. Let Y denote the complex surface obtained via blow-up of the four triple intersection points of A; let $\beta : Y \to \mathbb{P}^2_{\mathbb{C}}$ denote the blow-down map. Then Y contains a configuration $\tilde{A}$ of eight distinguished smooth rational curves $C_1, ..., C_{10}$: The four exceptional divisors $E_1, ..., E_4$ of the blow-up and six lifts $\tilde{A}_i, \tilde{B}_i, i = 1, 2, 3$, of the original projective lines in the arrangement A. (See Figure 3.) The configuration $\tilde{A}$ is a divisor D with simple normal crossings: Any two curves intersect in at most one point and at every intersection point only two curves intersect. Our next goal is to define a *complex orbifold* O with underlying space Y and the singular/orbifold locus Σ_O equal to the union of curves in $\tilde{A}$ (the preimage under β of the union of lines in A). The local complex orbifold-charts of O are defined as follows.

1. At every point $z \in O - \Sigma_O$ the local chart is given by the restriction of β to a suitable neighborhood of z.

2. At every point $z \in \Sigma_O$ which is not a (double) intersection point of the divisor D but $z \in C_i, i = 1, ..., 10$, the local chart is the holomorphic 5-fold branched covering over a suitable neighborhood of z, ramified over C_i.

3. Suppose that z is an intersection point of D, $z \in C_i \cap C_j$, $i \neq j$. Choose local holomorphic coordinates at z where C_i, C_j correspond to the coordinate lines in $\mathbb{C}^2$ and z corresponds to the origin; $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$. Each factor $\mathbb{C}$ in this product decomposition is biholomorphic to the quotient $\mathbb{C}/\mathbb{Z}_5$, with $\mathbb{Z}_5$ acting linearly on $\mathbb{C}$. Thus, a small neighborhood U of z in Y is biholomorphic to the quotient of the bi-disk, $\Delta^2/\mathbb{Z}_5^2$. This yields the local orbifold-chart at z, $\Delta^2 \to \Delta^2/\mathbb{Z}_5^2 \cong U$.

The result is a complex orbifold O with the underlying space Y. Hirzebruch then proves that the orbifold O is biholomorphic to the orbifold-quotient $M_\Gamma = \mathbf{B}^2/\Gamma$

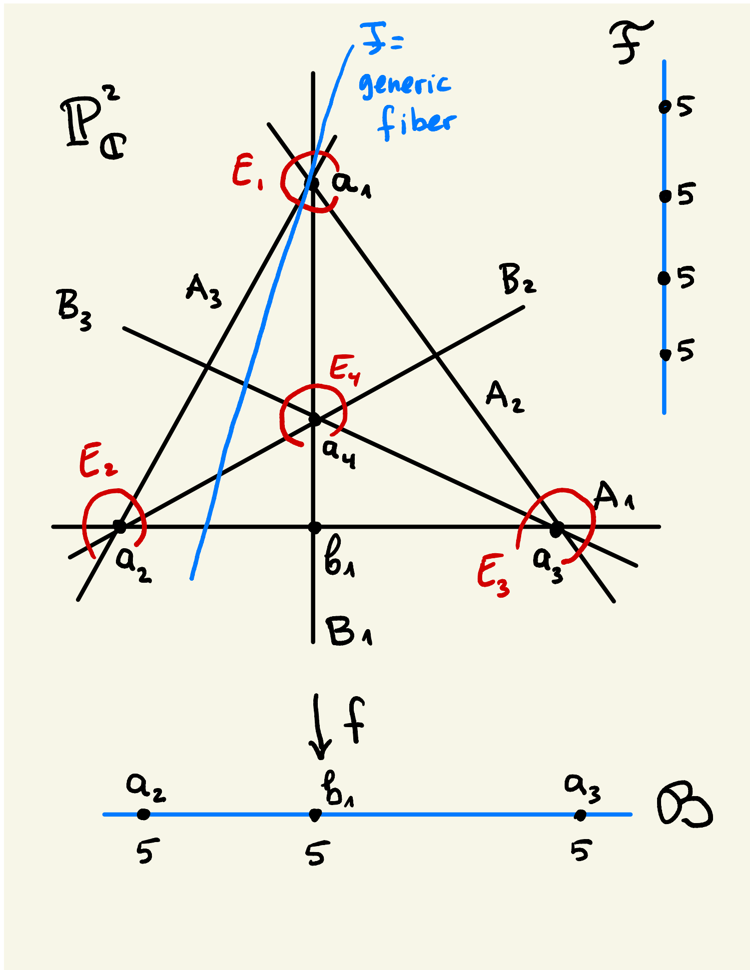

Fig. 2 Orbi-Kodaira fibration

of the complex 2-ball, by appealing to Yau's Uniformization Theorem, [95]: He verifies that the orbifold O admits a finite holomorphic orbifold-covering $M \to O$, where M is a complex surface of general type satisfying the equality of characteristic classes $3c_2 = c_1^2$; equivalently, $3\sigma(M) = \chi(M)$, where σ is the signature and χ is the Euler characteristic. Yau's theorem implies that M admits a Kähler metric of constant bisectional curvature -1, i.e. is a ball-quotient. Mostow Rigidity Theorem then implies that O is a complex hyperbolic orbifold as well. A bit more streamlined version of this argument was later developed by Barthel–Hirzebruch–Hofer, [6], and Holzapfel, [51], who defined orbifold-characteristic classes directly computable from lines arrangement A in $\mathbb{P}^2_{\mathbb{C}}$ (as well as $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$) and the orbifold-ramification numbers assigned to rational curves in the corresponding post-blow-up divisor.

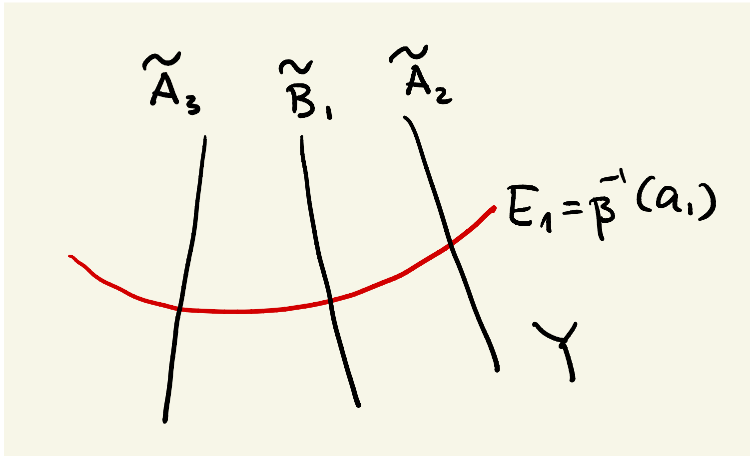

Fig. 3 Blow-up

I next describe a singular orbi-Kodaira fibration on O. Pick one of the triple intersection points, say, a_1, of the arrangement A and let A_1 be a line in A not passing through a_1. Consider the pencil of projective lines passing through a_1. This pencil defines a (nonsingular) holomorphic fibration of $\mathbb{P}^2_{\mathbb{C}} - \{a_1\}$ with the base A_1; the fibration map sends $z \in \mathbb{P}^2_{\mathbb{C}} - \{a_1\}$ to the point of intersection of the line za_1 with the line A_1. This fibration becomes a holomorphic map $f : Y \to \tilde{A}_1$ when we lift it to Y. Some fibers of f are, however, singular: These are the three singular fibers corresponding to the lifts of the three lines A_2, A_3, B_1 passing through a_1 and other points of triple intersection of A: a_2, a_3, a_4. The corresponding fibers are reducible rational curves (with the extra components corresponding to the exceptional divisors E_2, E_3, E_4). The line A_1 has an orbifold structure induced from O: The corresponding orbifold $\mathcal{B}$ has three singular points a_2, a_3, b_1, with local isotropy groups $\mathbb{Z}_5$ for each of them. The map f defined above respects the orbifold structure of O and $\mathcal{B}$ and, hence, we obtain a singular Kodaira orbi-fibration $f : O \to \mathcal{B}$. This fibration is nonsingular away from the preimages of the points a_2, b_1, a_3, with the generic fiber(s) $\mathcal{F}$ diffeomorphic to the orbifold with the underlying space $\mathbb{P}^1_{\mathbb{C}}$ and four singular points of order 5.

The restriction of f to $O' = f^{-1}(\{a_2, b_1, a_3\})$ is a nonsingular Kodaira fibration, i.e. a smooth (orbifold) fiber bundle; accordingly, $\pi_1(\mathcal{F})$ embeds as a normal subgroup in $\pi_1(O')$. Since the inclusion $O' \to O$ induces an epimorphism of fundamental groups $\pi_1(O') \to \pi_1(O) = \Gamma$, the image N of $\pi_1(\mathcal{F})$ in $\pi_1(O) = \Gamma$ is a normal finitely-generated subgroup $N \triangleleft \Gamma$. By passing to the universal covering of $\mathcal{B}$, we obtain a holomorphic map $h : \mathbb{H}^2_{\mathbb{C}}/N \to \mathbb{H}^1_{\mathbb{C}}$. The fibers of this map are compact and, generically, diffeomorphic to $\mathcal{F}$. The map h has infinitely many critical values in $\mathbb{H}^1_{\mathbb{C}}$ which break into finitely many $\pi_1(\mathcal{B})$-orbits and accumulate to the entire circle $\partial_\infty \mathbb{H}^1_{\mathbb{C}}$. Lifting h further to an N-invariant holomorphic function

$\mathbb{H}^2_{\mathbb{C}} \to \mathbb{H}^1_{\mathbb{C}}$ and extending this function to a measurable N-invariant nonconstant function $S^3 = \partial_\infty \mathbb{H}^2_{\mathbb{C}} \to S^1$, we conclude that the action of N on S^3 is non-ergodic.

The group Γ in the above example is a special case of:

Example 14 **Arithmetic lattices of simplest type.** Let K be a totally real number field, i.e. a finite extension of $\mathbb{Q}$ such that the image of every embedding $K \to \mathbb{C}$ lies in $\mathbb{R}$. Take an imaginary quadratic extension L/K, i.e. an extension which does not embed in $\mathbb{R}$. Since K is totally-real and L is its imaginary extension, all embeddings $L \to \mathbb{C}$ come in complex conjugate pairs:

$$\sigma_1, \bar{\sigma}_1, ..., \sigma_k, \bar{\sigma}_k.$$

Next, take a hermitian quadratic form in $n+1$ variables

$$\varphi(z, \bar{z}) = \sum_{p,q=1}^{n+1} a_{pq} z_p \bar{z}_q$$

with coefficients in L. I require the forms $\varphi^{\sigma_1}, \varphi^{\sigma_2}$ to have signature $(n, 1)$ and the forms $\varphi^{\sigma_j}, \varphi^{\bar{\sigma}_j}$ to be definite for the rest of the embeddings. I will identify L with $\sigma_1(L)$, so $\sigma_1 = \mathrm{id}$. Let $SU(\varphi)$ denote the group of special unitary automorphisms of the form φ on L^{n+1}. The embedding σ_1 defines a homomorphism $SU(\varphi) \to SU(n, 1)$ with relatively compact kernel.

A subgroup Γ of $SU(n, 1)$ is said to be an arithmetic lattice of the *simplest type* (or *of type I*) if it is commensurable[5] to $SU(\varphi, O_L) = SU(\varphi) \cap SL(n+1, O_L)$, where O_L is the ring of integers of L. For every such Γ the quotient $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ has finite volume. I refer to [70] for more detail on arithmetic subgroups of $SU(n, 1)$.

It is known that every arithmetic lattice Γ of the simplest type contains a finite index congruence-subgroup Γ' with infinite abelianization, [60] (see also [93]). Equivalently, the quotient-space $\mathbf{B}^n/\Gamma'$ has positive 1st Betti number. In contrast, Rogawski, [80], proved that for *type II* arithmetic lattices in $SU(2, 1)$, every congruence-subgroup has finite abelianization. It is unknown if such a lattice contain finite index subgroups with infinite abelianization. Furthermore, certain classes of non-arithmetic lattices in $SU(2, 1)$ (the ones violating the integrality condition for arithmetic groups) are proven to have positive virtual first Betti number by the work of S.-K. Yeung, [96].

I now discuss the existence of (singular) Kodaira fibrations of *compact* complex hyperbolic manifolds $M = \mathbb{H}^n_{\mathbb{C}}/\Gamma$.

1. Suppose that $b_1(M) > 0$. Since M is Kähler, $b_1(M)$ is even, hence, there exists an epimorphism $\phi : \Gamma \to \mathbb{Z}^2$. If the kernel of ϕ is not finitely-generated, then, according to a theorem of Delzant, [29], the manifold M admits a singular Kodaira fibration over a 1-dimensional complex hyperbolic orbifold.

2. If M and B are both complex hyperbolic, then there are no (nonsingular) Kodaira fibrations $M \to B$: This was first proven in the case when M is a surface by

[5] I.e. the intersection of the two groups has finite index in both.

Liu, [66], and then generalized to arbitrary dimensions by Koziarz and Mok, [62]. They also prove nonexistence of Kodaira fibrations $M \to B$ when $\dim(B) \geq 2$ and M merely has finite volume. Furthermore, if M is 2-dimensional, for every singular Kodaira fibration $M \to B$, the kernel of the homomorphism $\pi_1(M) = \Gamma \to \pi_1(B)$ is finitely generated but is not finitely-presentable, [52, 49].

Problem 1 Is there a discrete subgroup $\Gamma < PU(2,1)$ isomorphic to the fundamental group of a compact real hyperbolic surface, such that $M = M_\Gamma$ admits a Kodaira fibration (with compact fibers) $M \to \mathbb{H}^1_{\mathbb{C}}$? Is there a singular Kodaira fibration (with compact fibers) $\mathbb{H}^2_{\mathbb{C}}/\Gamma \to \mathbb{H}^1_{\mathbb{C}}$ which has only finitely many singular fibers?

9 Complex hyperbolic Kleinian groups and function theory on complex hyperbolic manifolds

In this section we discuss some interesting interactions between the general theory of holomorphic functions on complex manifolds (which I review in Section 16) and geometry/topology of complex Kleinian groups.

Proposition 5 *If $\Gamma < PU(n,1)$ is a discrete, torsion-free subgroup such that $M = M_\Gamma$ admits a surjective holomorphic map with compact fibers $f : M \to B$, where B is a complex manifold satisfying* $\dim(B) < n$*, then $\Omega_\Gamma = \emptyset$. In particular, M cannot have convex ends.*

Proof Suppose, to the contrary, that $\Omega_\Gamma \neq \emptyset$. Then $Core_\eta(M)$ is a proper submanifold (with boundary) in M. Since $\mathbb{H}^n_{\mathbb{C}}$ is strictly negatively curved, the nearest-point projection $\Pi : \mathbb{H}^n_{\mathbb{C}} \to \mathrm{hull}_\eta(\Lambda_\Gamma)$ is strictly contracting away from $\mathrm{hull}_\eta \Lambda_\Gamma$. By the Γ-equivariance, Π descends to a strictly contracting projection $\pi : M \to Core_\eta(M)$. Therefore, if Y is a compact complex k-dimensional subvariety in M of positive dimension not contained in $Core_\eta(M)$ then $\pi(Y)$ has k-volume strictly smaller than that of Y. This is a contradiction since $\pi : Y \to \pi(Y)$ is homotopic to the identity inclusion map $\mathrm{id}_Y : Y \to M$ and compact complex subvarieties in Kähler manifolds are volume-minimizers in their homology classes. Taking a generic fiber Y of $f : M \to B$ through a point $x \in M - \mathrm{hull}_\eta(\Lambda_\Gamma)$ concludes the proof.

I next discuss geometry and topology of quotient-orbifolds M_Γ, primarily for convex-cocompact subgroups $\Gamma < PU(n,1)$.

A classical example of a complex submanifold with strictly Levi-convex boundary is a closed unit ball $\overline{\mathbf{B}^n}$ in $\mathbb{C}^n$. Suppose that $\Gamma < Aut(\mathbf{B}^n)$ is a discrete torsion-free subgroup of the group of holomorphic automorphisms of $\mathbf{B}^n$ with (nonempty) domain of discontinuity $\Omega \subset \partial \mathbf{B}^n$. The quotient

$$\overline{M}_\Gamma = (\mathbf{B}^n \cup \Omega)/\Gamma$$

is a smooth submanifold with strictly Levi-convex boundary in the complex manifold $\check{\Omega}_\Gamma/\Gamma$ (see (1)). Thus, we conclude:

Lemma 2 *If* $\overline{M}_\Gamma = (\mathbf{B}^n \cup \Omega)/\Gamma$ *has compact boundary, then* M *is strongly pseudoconvex.*

Consequently:

Theorem 16 *Let* $\Gamma < PU(n, 1)$*,* $n \geq 2$*, be a convex-cocompact discrete subgroup. Then* ∂M_Γ *is connected.*

Proof Since Γ is convex-cocompact, it is also finitely generated. Hence, by Selberg's Lemma, the orbifold M_Γ is very good. Therefore, it suffices to consider the case when Γ is torsion-free, i.e. M_Γ is a complex n-manifold. Since $\overline{M}_\Gamma$ is strongly pseudoconvex, connectedness of its boundary is an immediate consequence of Theorem 26.

Theorem 17 *Let* $\Gamma < PU(n, 1)$*,* $n \geq 2$*, be a convex-cocompact discrete subgroup which is not a lattice, i.e.* $\Omega_\Gamma \neq \emptyset$*. Then* $\dim(\Lambda_\Gamma) \leq 2n - 3$*, equivalently,* $\mathrm{cd}_{\mathbb{Q}}(\Gamma) \leq 2n - 2$*.*

Proof As before, it suffices to consider the case of torsion-free groups Γ. According to Corollary 6, M_Γ is homotopy-equivalent to a CW complex of dimension $\leq 2n-2$. It follows that $\mathrm{cd}_{\mathbb{Q}}(\Gamma) \leq 2n-2$ and, by the Bestvina-Mess theorem, $\dim(\partial_\infty\Gamma) \leq 2n-3$. Since $\partial_\infty\Gamma$ is homeomorphic to Λ_Γ, $\dim(\Lambda_\Gamma) \leq 2n - 3$ as well.

In particular, Λ_Γ does not separate S^{2n-1} (even locally) and, hence, Ω_Γ is connected, which gives another proof of the fact that ∂M_Γ is connected.

Specializing to the case $n = 2$, we obtain: If $\Gamma < PU(2, 1)$ (for simplicity, torsion-free) is convex-cocompact and is not a lattice, then Λ_Γ is at most 1-dimensional. In particular, according to [57], Γ admits an iterated amalgam decomposition over trivial and cyclic subgroups, so that the terminal groups are either cyclic, or isomorphic to Fuchsian groups (and the limit set is a topological circle) or groups whose limit sets are Sierpinski carpets or Menger curves.

Theorem 18 *Suppose that* Γ *is torsion-free convex cocompact,* $n > 1$ *and* M_Γ *contains no compact complex subvarieties of positive dimension. Then* M_Γ *is Stein.*

Proof This is an immediate consequence of Theorem 27.

One way to prove that M_Γ contains no compact complex subvarieties of positive dimension is to argue that $\Gamma = \pi_1(M)$ is free: This implies that $H_i(M_\Gamma) = 0, i \geq 2$, but, since M_Γ is Kähler, every compact complex k-dimensional subvariety of M_Γ would define a nonzero $2k$-dimensional homology class. For instance, if Γ is convex-cocompact, $\delta_\Gamma < 1$ then $\dim \Lambda_\Gamma \leq \dim_H(\Lambda_\Gamma) < 1$, which implies that $\dim \Lambda_\Gamma = 0$ and, hence, Γ is a virtually free group. However, even when $H_2(M) \neq 0$, one can still, sometimes, prove that M_Γ contains no compact complex curves. For instance, let $L \to M_\Gamma$ be the canonical line bundle. If $C \subset M_\Gamma$ is an (even singular) complex curve, the pull-back of L to C has nonzero 1st Chern class. Assuming that $H_2(M_\Gamma) \cong \mathbb{Z}$ (e.g. if Γ is isomorphic to the fundamental group of a compact Riemann surface), if the 1st Chern class of L evaluated on the generator of $H_2(M_\Gamma)$ is zero, then M_Γ contains no complex curves. This argument applies in the case of real-Fuchsian groups and their quasi-Fuchsian deformations.

Observe that if $\Gamma < PU(2, 1)$ is a complex Fuchsian group, then $\dim_H(\Lambda_\Gamma) = 2$.

Theorem 19 (S. Dey, M. Kapovich, [33])

If $\Gamma < PU(n,1)$ is discrete, torsion-free and M_Γ contains a compact complex subvariety of positive dimension, then $\delta_\Gamma \geq 2$.

Corollary 5 *Suppose that $\Gamma < PU(n,1)$ is torsion-free, convex-cocompact and $\delta_\Gamma < 2$, then M_Γ is Stein.*

Burns' Theorem. I now drop the convex-cocompactness assumption and consider general discrete, torsion-free subgroups $\Gamma < PU(n,1)$. Theorem 16 has the following striking generalization. It was first stated by Dan Burns, who, as it appears, never published a proof; a published proof is due to Napier and Ramachandran, [73, Theorem 4.2]:

Theorem 20 *Suppose that $n \geq 3$, $\Gamma < PU(n,1)$ is discrete, torsion-free and ∂M_Γ has at least one compact component S. Then:*
1. $\partial M_\Gamma = S$.
2. Γ is geometrically finite.

A good example illustrating this theorem is that of a Schottky-type group (Example 6), where the limit set is totally disconnected, the quotient manifold Ω_Γ/Γ is compact and M_Γ has k cusps. In particular, $\overline{M}_\Gamma$ is noncompact in this example.

It is unknown if Burns' theorem holds for $n = 2$, but Mohan Ramachandran proved the following:

Theorem 21 *Suppose that $\Gamma < PU(2,1)$ is discrete, torsion-free, the injectivity radius of M_Γ is bounded away from zero, and ∂M_Γ has at least one compact component. Then Γ is convex-cocompact.*

The proof of this theorem is given in Appendix G.

10 Conjectures and questions

In this section I collect some conjectures and questions in addition to those scattered throughout the survey.

The first conjecture is a generalization of Burns' theorem, Theorem 20:

Conjecture 1 Suppose that $\Gamma < PU(n,1), n \geq 2$, is such that for $M = M_\Gamma$ the thick part $M_{[\epsilon,\infty)}$ has a convex end. Then Γ is geometrically finite and Ω_Γ is connected.

The next two conjectures are motivated by Theorem 19:

Conjecture 2 If $\Gamma < PU(n,1)$ is discrete, torsion-free, $\delta_\Gamma = 2$ and M_Γ contains a compact complex subvariety of positive dimension, then Γ is a complex Fuchsian group.

Conjecture 3 If $\Gamma < PU(n,1)$ is discrete, torsion-free and $\delta_\Gamma < 2k$, then M_Γ cannot contain a compact complex subvariety of dimension k.

Conjecture 4 (Chengbo Yue's Gap Conjecture, [97]) Suppose that $\Gamma < G = Aut(\mathbf{B}^n)$ is a convex-cocompact torsion-free subgroup. Then either Γ is a uniform lattice in G (and, thus, $\delta_\Gamma = 2n$) or $\delta_\Gamma \le 2n-1$.

Note that the two other conjectures about nonelementary convex-cocompact subgroups $\Gamma < PU(n,1)$ made in the introduction to [97] fail already in dimension $n = 2$:

(a) The inequality $\dim_H \Lambda_\Gamma > n-1$ does not imply that M_Γ is non-Stein. For instance, a real-hyperbolic quasifuchsian subgroup of $PU(2,1)$ serves as an example.

(b) Even if M_Γ is non-Stein, a compact complex curve in M_Γ need not be a finite union of totally geodesic complex curves, as it is shown by the AGG-examples.

Problem 2 1. Investigate which polygon-groups embed discretely in $PU(2,1)$.

2. Is there a convex-cocompact subgroup of $PU(2,1)$ with the limit set homeomorphic to Sierpinski carpet?

While "most" compact 3-dimensional manifolds are hyperbolic, very few examples of hyperbolic 3-manifolds which are of the form Ω_Γ/Γ, $\Gamma < PU(2,1)$ are known, see the book by Richard Schwartz [85] for further discussion.

Conjecture 5 The Menger curve limit set in Example 12 is "unknotted" in S^3, i.e. is ambient-isotopic to the standard Menger curve $\mathcal{M} \subset \mathbb{R}^3 \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$. Furthermore, in this example, the quotient 3-dimensional manifold Ω_Γ/Γ is hyperbolic.[6]

Problem 3 Prove the existence of discrete geometrically infinite subgroups of $PU(2,1)$ which are isomorphic to fundamental groups of compact surfaces.[7]

Note that such subgroups do not exist in $PU(1,1)$ but abound in $PO(3,1)$. Furthermore, the only known examples of finitely generated geometrically infinite subgroups of $PU(2,1)$ come from singular Kodaira fibrations and are not finitely-presentable, see Example 13.

The conjectures and questions appearing above, deal with discrete subgroups Γ of $PU(n,1)$ which are not lattices, i.e. the $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ has infinite volume. Below, I discuss two problems regarding lattices.

Arithmeticity. The most famous open problem regarding lattices in $PU(n,1)$ deals with the existence problem of nonarithmetic subgroups and was first raised in Margulis' ICM address [68]. It is known (due to the work of Margulis [69], Corlette [24], Gromov–Schoen [45], and Gromov–Piatetski-Shapiro [44]) that:

(a) For each n, the Lie group $SO(n,1)$ contains non-arithmetic lattices.

[6] It suffices to show that Ω_Γ/Γ contains no incompressible tori, which is closely related to the unknottedness problem of the Menger-curve limit set.

[7] Cf. section 11.4 in [55].

(b) For every simple noncompact connected linear Lie group G which is not locally isomorphic to $SO(n,1)$ and $SU(n,1)$, every lattice $\Gamma < G$ is arithmetic.

This leaves out the series of Lie groups $PU(n,1)$, $n \geq 2$. Currently, primarily due to the work of Deligne and Mostow, see [31], there are known examples of nonarithmetic lattices in $PU(2,1)$ and $PU(3,1)$. Loosely speaking there are three approaches to constructing nonarithmetic lattices:

(a) As monodromy groups of some linear holomorphic ODEs, see [31, 26], as well as [88] for a geometric interpretation.

(b) By constructing the corresponding complex hyperbolic orbifolds M_Γ whose underlying space is a blown-up $\mathbb{P}^n$, see [6, 84, 26, 32].

(c) By constructing a Dirichlet fundamental domain of Γ in $\mathbb{H}^2_{\mathbb{C}}$, see [71, 34].

But using these techniques becomes increasingly difficult (or even impossible) as the dimension n increases, which means that different approaches are needed.

Conjecture 6 For each n, $PU(n,1)$ contains a nonarithmetic lattice.

By analogy with the construction of non-arithmetic lattices in [44], one can hope for a similar "hybrid" construction of nonarithmtic lattices in $PU(n,1)$, leading to a conjecture due to Bruce Hunt:

Conjecture 7 For every $n \geq 2$, there exists a quadruple of arithmetic lattices $\Gamma_1, \Gamma_2 < SU(n-1,1)$ and $\Gamma_3 < SU(n-2,1)$ such that:

(1) Γ_3 is isomorphic to subgroups in Γ_1, Γ_2; hence, we obtain an amalgam $\Gamma_0 = \Gamma_1 \star_{\Gamma_3} \Gamma_2$.

(2) There exists an epimorphism $\rho : \Gamma_0 \to \Gamma < SU(n,1)$ injective on Γ_1, Γ_2, whose image is a nonarithmetic lattice $\Gamma < SU(n,1)$.

Unlike [44], where nonarithmetic lattices in $SO(n,1)$ were constructed via a similar process, with an *isomorphism* $\Gamma_0 \to \Gamma < SO(n,1)$, in the complex hyperbolic setting there is no hope for an injective homomorphism ρ (a lattice in $SU(n,1)$ cannot be isomorphic to an amalgam Γ_0 as above).

Nonexistence of reflection lattices. The known examples of nonarithmetic lattices Γ in $PU(n,1)$, $n = 2, 3$, are all commensurable to *complex reflection subgroups*, i.e. discrete subgroups of $PU(n,1)$ generated by complex reflections. Furthermore, up to commensuration, the underlying spaces of their quotient orbifolds $M_\Gamma = \mathbb{H}^n_{\mathbb{C}}/\Gamma$ are *rational projective varieties*.

Conjecture 8 There exists N such that for all $n \geq N$ the following holds:

1. If $\Gamma < PU(n,1)$ is a lattice then Γ cannot be a reflection subgroup.
2. If $\Gamma < PU(n,1)$ is a lattice then the underlying space of the orbifold M_Γ cannot be a rational algebraic variety. More ambitiously, it has to be a variety of general type.

The motivation for this conjecture comes from theorems due to Vinberg, [92], and Prokhorov, [77], establishing nonexistence of reflection lattices in $PO(n,1)$, when n is sufficiently large.

11 Appendix A. Horofunction compactification

A metric space (Y, d) is called *geodesic* if any two points x, y in X are connected by a geodesic segment, denoted xy. (This notation is a bit ambiguous since in many cases such a segment is non-unique.) A *geodesic triangle*, denoted xyz, in a metric space (X, d) is a set of three geodesic segments xy, yz, zx connecting cyclically the points x, y, z, the *vertices* of the triangle; the segments xy, yz, zx are the *edges* of the triangle. Thus, geodesic triangles are 1-dimensional objects.

Let (Y, d) be a locally compact geodesic metric space. For each $y \in Y$ define the 1-Lipschitz function $d_y = d(y, \cdot)$ on Y. This leads to the *Kuratowski embedding* $\kappa : Y \to C(Y) = C(Y, \mathbb{R})$, $y \mapsto d_y$. Let $\mathbb{R} \subset C(Y)$ denote the linear subspace of constant functions. Composing the embedding κ with the projection $C(Y) \to C(Y)/\mathbb{R}$ (where $\mathbb{R}$ acts additively on $C(Y)$) we obtain the *Kuratowski embedding* of Y,

$$Y \hookrightarrow C(Y)/\mathbb{R}.$$

Then $\overline{Y}$, the closure of Y in $C(Y)/\mathbb{R}$, is the *horofunction compactification* of Y. The *horoboundary* of Y is $\overline{Y} \setminus Y$.

Functions representing points in $\partial_\infty Y = \overline{Y} - Y$ are the *horofunctions* on Y. In other words, horofunctions on Y are limits (uniform on compacts in Y) of sequences of normalized distance functions $d_{y_i} - d_{y_i}(o)$, where $y_i \in Y$ are divergent sequences in Y. Each geodesic ray $r(t)$ in Y determines a horofunction in Y called a *Busemann function* b_r, which is the subsequential limit

$$\lim_{i\to\infty} d_{r(i)} - d_{r(i)}(o).$$

If Y is a Hadamard manifold, then each limit as above exists (without passing to a subsequence). Furthermore, each horofunction is a Busemann function. This yields a topological identification of the visual compactification of Y and its horofunction compactification. Level sets of Busemann functions are called *horospheres* in X. The point $r(\infty) \in \partial_\infty Y$ is the *center* of the horosphere $\{b_r = c\}$. Sublevel sets $\{b_r < c\}$ are called *horoballs*. The point $r(\infty)$ represented by the ray r is the *center* of the corresponding horospheres/horoballs.

12 Appendix B: Two classical Peano continua

A *Peano continuum* is a compact connected and locally path-connected metrizable topological space. I will need two examples of 1-dimensional Peano continua. Both are obtained via a procedure similar to the construction of the "ternary" Cantor set.

Sierpinski carpet. Let $I = [0, 1]$ denote the unit interval. Start with the unit square $Q_0 = I^2 \subset \mathbb{R}^2$. Divide I in three congruent subintervals and, accordingly, divide I^2 in 9 congruent subsquares. Remove the interior of the "middle subsquare", the one disjoint from the boundary of Q. Call the result Q_1. Now, repeat this procedure for each of the remaining 8 subsquares in Q_1, to obtain a planar region Q_2, etc. The *standard Sierpinski carpet* in $\mathbb{R}^2$ is the intersection

$$\mathcal{S} := \bigcap_{i=0}^{\infty} Q_i.$$

Menger curve. Consider the unit cube $C = I^3 \subset \mathbb{R}^3$. Let $\pi_i, i = 1, 2, 3$ denote the orthogonal projections of $\mathbb{R}^3$ to the coordinate hyperplanes $P_i, i = 1, 2, 3$, in $\mathbb{R}^3$. In all three planes we take the Sierpinski carpets $\mathcal{S}_i \subset P_i$, constructed from the unit squares $Q_i = C \cap P_i$, $i = 1, 2, 3$. Then the *standard Menger curve* in $\mathbb{R}^3$ is defined as

$$\mathcal{M} := \bigcap_{i=1}^{3} \pi_i^{-1}(\mathcal{S}_i).$$

Alternatively, $\mathcal{M}$ can be described as follows. First, divide $C = C_0$ in 27 congruent subcubes with the edge-length $1/3$ and remove from C the "middle " open cube" Q_1 as well as the 8 open subcubes which share with Q_1 2-dimensional faces; remove those open faces as well. Continue inductively constructing a sequence of nested compacts $C_0 \supset C_1 \supset C_2 \supset ..$ and. Lastly,

$$\mathcal{M} = \bigcap_{i=0}^{\infty} C_i.$$

13 Appendix C: Gromov-hyperbolic spaces and groups

A geodesic metric space (X, d) is called *δ-hyperbolic* if every geodesic triangle xyz in X is δ-slim, i.e. every edge of xyz is contained in the closed δ-neighborhood of the union of the other two edges. A geodesic metric space is called *Gromov-hyperbolic* if it is δ-hyperbolic for some $\delta < \infty$.

Examples of Gromov-hyperbolic spaces are *strictly negatively curved Hadamard manifolds*: If X is a Hadamard manifold of sectional curvature ≤ -1 then X is δ_0-hyperbolic with $\delta_0 = \operatorname{arccosh}(\sqrt{2})$.

Let Γ be a group with finite generating set S. Given S, one defines the *Cayley graph* $C_{\Gamma,S}$. This graph is connected and Γ acts on it with finite quotient (the quotient graph has a single vertex and $\operatorname{card}(S)$ edges). The graph $C_{\Gamma,S}$ has a graph-metric, where every edge has unit length.

Definition 14 A finitely generated group Γ is called *Gromov-hyperbolic* or simply *hyperbolic* if one (equivalently, every) Cayley graph of Γ is a Gromov-hyperbolic metric space.

The *Gromov boundary* $\partial_\infty\Gamma$ of Γ is the horoboundary of one (any) Cayley graph of Γ: Gromov boundaries corresponding to different Cayley graphs are equivariantly homeomorphic.

Examples of hyperbolic groups are given by:

Example 15 Let X be a strictly negatively curved Hadamard manifold, $Y \subset X$ is a closed convex subset and $\Gamma < \mathrm{Isom}(X)$ acts properly discontinuously and cocompactly on Y. Then Γ is hyperbolic and the ideal boundary $\partial_\infty Y$ of Y is equivariantly homeomorphic to the Gromov boundary of Γ.

In particular, every convex-cocompact discrete subgroup $\Gamma < \mathrm{Isom}(X)$ is Gromov-hyperbolic and $\partial_\infty\Gamma$ is equivariantly homeomorphic to the limit set of Γ.

Cohomological dimension (with respect to the Chech cohomology) of the Gromov boundary of a hyperbolic group is closely related to the rational cohomological dimension of Γ itself:

Theorem 22 (Bestvina–Mess, [8]) $\dim(\partial_\infty\Gamma) = cd_{\mathbb{Q}}(\Gamma) - 1$.

In particular, by the Stallings–Swan–Dunwoody Theorem, Γ is *virtually free* (i.e. contains a free subgroup of finite index) if and only if $\partial_\infty\Gamma$ is zero-dimensional, if and only if $\partial_\infty\Gamma$ is totally disconnected, equivalently, it is either empty, or consists of two points or is homeomorphic to the Cantor set.

One classifies 1-dimensional boundaries of hyperbolic groups as follows:

Theorem 23 (Kapovich–Kleiner, [57]) *Suppose that* Γ *is a hyperbolic group with connected 1-dimensional Gromov boundary. Then either* $\partial_\infty\Gamma$ *is homeomorphic to* S^1, *or* Γ *splits as a finite graph of groups with virtually cyclic edge groups,* [8] *or* $\partial_\infty\Gamma$ *is homeomorphic to the Sierpinski carpet or the Menger curve.*

Example 16 Hyperbolic von Dyck groups $D(p,q,r)$,

$$D(p,q,r) = \langle a,b,c | a^p = b^q = c^r = 1, abc = 1\rangle, p^{-1} + q^{-1} + r^{-1} < 1.$$

These are hyperbolic groups with Gromov boundary homeomorphic to S^1. Moreover, each $D(p,q,r)$ admits a unique (up to conjugation in $\mathrm{Isom}(\mathbb{H}^2)$) isometric conformal action on the hyperbolic plane.

Representations of von Dyck groups to $PU(2,1)$**.** Given an element $g \in G = PU(2,1)$ we let $\zeta(g)$ denote the codimension in G of the centralizer of g in G. In other words, $\zeta(g)$ is the local dimension near g of the subvariety of elements of G conjugate to g. Thus, $\zeta(g) \geq 2$ for every $g \in G$. Furthermore, if g is an involution then $\zeta(g) = 4$. The paper [94] by Andre Weil describes the local geometry of the *character variety*

$$Hom(D(p,q,r), G)//G$$

[8] Hence, its Gromov boundary can be inductively described using boundaries of vertex groups.

as follows:

Suppose that $\rho : D(p,q,r) \to G$ is a *generic* representation, i.e. one whose image has trivial centralizer in G. For instance, any representation whose image is discrete, nonelementary, not stabilizing a complex geodesic, will satisfy this condition. Then, near $[\rho]$, the real-algebraic variety $Hom(D(p,q,r),G)//G$ is smooth of dimension

$$\zeta(\rho(a)) + \zeta(\rho(b)) + \zeta(\rho(c)) - 2\dim(G) = \zeta(\rho(a)) + \zeta(\rho(b)) + \zeta(\rho(c)) - 16.$$

Assuming that $p = 2$, $\zeta(\rho(a)) = 4$, which implies that

$$\zeta(\rho(a)) + \zeta(\rho(b)) + \zeta(\rho(c)) - 16 \le 4 + 12 - 16 = 0.$$

Combined with an easy analysis of non-generic representations, one obtains:

Proposition 6 *If $p = 2$ then $Hom(D(p,q,r),G)//G$ is zero-dimensional.*

Example 17 (Polygon-groups)

Fix two natural numbers $p \ge 5$ and $q \ge 3$. Define the *polygon-group* $\Gamma_{p,q}$ via the presentation

$$\langle a_1, ..., a_p | a_i^q = 1, [a_i, a_{i+1}] = 1, i = 1, ..., p\rangle,$$

where i is taken mod p. Each $\Gamma_{p,q}$ is hyperbolic with $\partial_\infty \Gamma_{p,q}$ homeomorphic to the Menger curve.

Every $\Gamma_{p,q}$ admits a canonical *reflection representation* $\rho_{p,q}$ to $PU(2,1)$ constructed as follows:

Pick a real hyperbolic plane $\mathbb{H}^2_{\mathbb{R}} \subset \mathbb{H}^2_{\mathbb{C}}$ and a regular right-angled p-gon $P = z_1...z_p$ in $\mathbb{H}^2_{\mathbb{R}}$. Let C_i denote the complex geodesic through the edge $z_i z_{i+1}$ of P (i is taken mod p). For each i let g_i be the order q complex reflection with the fixing C_i, with the rotation in the hyperplanes normal to C_i through the angle $2\pi/q$. Then $[g_i, g_{i+1}] = 1$ and, hence, we obtain a representation

$$\rho_{p,q} : \Gamma_{p,q} \to PU(2,1).$$

14 Appendix D: Orbifolds

The notion of *orbifold* is a generalization of the notion of a *manifold* which appears naturally in the context of properly discontinuous non-free actions of groups on manifolds. Orbifolds were first invented by Satake [83] in the 1950s under the name of V-manifolds, they were reinvented under the name of orbifolds by Thurston in the 1970s (see [87]) as a technical tool for proving his Hyperbolization Theorem. I refer the reader to [9] for a detailed treatment of orbifolds.

Before giving a formal definition we start with basic examples of orbifolds. Suppose that M is a smooth connected manifold and G a discrete group acting smoothly, faithfully[9] and properly discontinuously on M. Then the quotient $O = M/G$ is an *orbifold*, such orbifolds are called *good*. The quotient M/G, considered as a topological space X_O, is the *underlying space* of this orbifold. If S is a set of points in M where the action of G is not free, then its projection $\Sigma = S/G$ is the *singular locus* of the orbifold O.

To be more concrete, consider 2-dimensional orbifolds. Suppose that $M = \mathbb{H}^2_{\mathbb{R}}$ and G is a discrete subgroup of $PSL(2, \mathbb{R})$. Then the quotient $O = \mathbb{H}^2/G$ is a Riemann surface X_O with a discrete collection of *cone points* z_j which form the singular locus Σ of the orbifold O. The projection $p : \mathbb{H}^2 \to O$ is the *universal cover* of the orbifold O. The Riemann surface X_O has a natural hyperbolic metric which is singular in the discrete set Σ. Metrically, the points z_j are characterized by the property that the total angles around these points are $2\pi/n_j$. The numbers n_j are the orders of cyclic subgroups G_{z_j} of G which stabilize the points in $p^{-1}(z_j)$, they are called *the local isotropy groups*. The projection p is a *ramified covering* from the point of view of Riemann surfaces. From the point of view of orbifolds this is an (orbi) *covering*. Thus, the singular locus of the orbifold O consists of the points z_j in Σ equipped with the extra data: The $PSL(2, \mathbb{R})$-conjugacy classes of the local isotropy groups G_{z_j} (of course, each local isotropy group G_{z_j} is determined by the number n_j).

I now discuss the general definition. A (smooth) n-dimensional orbifold O is a pair: A Hausdorff paracompact topological space X (which is called the *underlying space* of O and is denoted X_O) and an *orbifold-atlas* $\mathcal{A}$ on X. The atlas $\mathcal{A}$ consists of:

- A collection of open sets $U_i \subset X$,which is closed under taking finite intersections, such that $X = \bigcup_i U_i$.
- A collection of open sets $\tilde{U}_i \subset \mathbb{R}^n$.
- A collection of finite groups of diffeomorphisms Γ_j acting on $\tilde{U}_i$ so that each nontrivial element of Γ_j acts nontrivially on each component of $\tilde{U}_j$.
- A collection of homeomorphisms

$$\phi_i : U_i \to \tilde{U}_i/\Gamma_i.$$

The atlas $\mathcal{A}$ is required to behave well under inclusions. Namely, if $U_i \subset U_j$, then there is a smooth embedding

$$\tilde{\phi}_{ij} : \tilde{U}_i \to \tilde{U}_j$$

and a monomorphism $f_{ij} : \Gamma_i \to \Gamma_j$ such that $\tilde{\phi}_{ij}$ is f_{ij}-equivariant.

The open sets U_j are the *coordinate* neighborhoods of the points $x \in U_j$ and $\tilde{U}_j$ are their *covering coordinate* neighborhoods.

Similarly to orbifolds, one defines the class of *orbifolds with boundary*; just instead of *open* sets $\tilde{U}_j \subset \mathbb{R}^n$ we use open subsets in

$$\mathbb{R}^n_+ \cup \mathbb{R}^{n-1} = \{(x_1, ..., x_n) : x_n \geq 0\}.$$

[9] i.e. each nontrivial element of G acts nontrivially

The *boundary* of such an orbifold consists of points $x \in X_O$ which correspond to $\mathbb{R}^{n-1}$ under the identification $U_i \cong \tilde{U}_i/\Gamma_i$. As in the case of manifolds, the boundary of each orbifold is an orbifold without boundary. By abusing notation we will call an *orbifold with boundary* simply an *orbifold*. A compact orbifold without boundary is called *closed*.

To each point $x \in X$ we associate a germ of action of a finite group of diffeomorphisms Γ_x at a fixed point $\tilde{x}$. If $\phi_j(x)$ is covered by a point $\tilde{x}_j \in \tilde{U}_j$, then we have the isotropy group $\Gamma_{j,x}$ of $\tilde{x}_j$ in Γ_j. Note that if $U_i \subset U_j$, then the map $\tilde{\phi}_{ij} : \tilde{U}_i \to \tilde{U}_j$ induces an isomorphism from the germ of the action of $\Gamma_{j,x}$ at $\tilde{x}_j$ to the germ of the action of $\Gamma_{i,x}$ at $\tilde{x}_i$. Thus we let the germ $(\Gamma_x, \tilde{x})$ be the equivariant diffeomorphism class of the germ $(\Gamma_{j,x}, \tilde{x}_j)$. The group Γ_x is called the *local isotropy group* of O at x. The set of points x with nontrivial local isotropy group is called *the singular locus* of O and is denoted by Σ_O. Note that the singular locus is nowhere dense in X_O. An orbifold with empty singular locus is called *nonsingular* or a *manifold*.

The main source of examples of orbifolds is:

Example 18 Let M a smooth connected n-manifold and Γ is a discrete group acting smoothly and faithfully on M. Then $X = M/\Gamma$ has a natural orbifold structure. The atlas A on X is given as follows: Each $y \in M$ admits a coordinate neighborhood $\tilde{U}$ (identified with an open subset of $\mathbb{R}^n$) such that for every $g \in \Gamma$ either $g\tilde{U} \cap \tilde{U} = \emptyset$ or $g \in \Gamma_y$ (the stabilizer of y in Γ) and $g(\tilde{U}) = \tilde{U}$. Then let $\phi : \tilde{U} \to U = \phi(\tilde{U})$ be the quotient map. One verifies that A indeed satisfies axioms of an orbifold-atlas. The groups G_j in the definition of an atlas are just the stabilizers Γ_y as above.

Since Γ_x acts smoothly near the fixed point $\tilde{x}$, the germ $(\Gamma_x, \tilde{x})$ is linearizable: Equip a neighborhood of $\tilde{x}$ with a Γ_x-invariant Riemannian metric. Then the exponential map (with the origin at $\tilde{x}$) conjugates the orthogonal action of Γ_x on the tangent space $T_{\tilde{x}}\mathbb{R}^n$ to the germ of the action of Γ_x at $\tilde{x}$.

Definition 15 A *Riemannian metric* ρ on an orbifold O is a usual Riemannian metric on $X_O - \Sigma_O$, such that after we lift ρ to the local covering coordinate neighborhoods $\tilde{U}_i$, it extends to a Γ_i-invariant Riemannian metric on the whole $\tilde{U}_i$.

The same definition applies to *complex structures*.

Remark 12 Each orbifold O admits a Riemannian metric: The proof is the same as for smooth manifolds, using a partition of unity.

A *homeomorphism* (resp. *diffeomorphism*) between orbifolds O, O' is a homeomorphism $h : X_O \to X_{O'}$ such that for all points $x \in O, y = h(x) \in O'$, there are coordinate neighborhoods $U_x \cong \tilde{U}_x/\Gamma_x, V_y \cong \tilde{V}_y/\Gamma_y$ such that h lifts to an equivariant homeomorphism (resp. diffeomorphism)

$$\tilde{h}_{xy} : \tilde{U}_x \to \tilde{V}_y.$$

Note that to describe a smooth orbifold O up to homeomorphism it suffices to describe the topology of the pair (X_O, Σ_O) and the homeomorphic equivalence classes of the germs $(\Gamma_x, \tilde{x})$ for the points $x \in \Sigma_O$.

Remark 13 Let O be a connected compact 1-dimensional orbifold without boundary which is not a manifold. Then O is homeomorphic to the closed interval $[a, b]$ where $(\Gamma_a, \tilde{a})$, $(\Gamma_b, \tilde{b})$ are the germs $(\mathbb{Z}_2, 0)$ of the reflection group $\mathbb{Z}_2$ acting isometrically on $\mathbb{R}$ near its fixed point $0 \in \mathbb{R}$.

A *smooth* map between orbifolds O and O' is a continuous map

$$g : O \to O'$$

which can be (locally) lifted to smooth equivariant maps between pairs of coordinate covering neighborhoods

$$\tilde{g}_{ij} : \tilde{U}_j \to \tilde{V}_i$$

Similarly we define *immersions* and *submersions* between orbifolds as smooth maps between orbifolds which locally lift to immersions and submersions respectively.

Suppose that O', O are orbifolds and $p : X_{O'} \to X_O$ is a continuous map. The map p is called a *covering map* or *orbi-covering* between the orbifolds O', O if the following property is satisfied:

For each point $x \in X_O$ there exists a chart $U = \tilde{U}/G_x$ such that for every component V_i of $p^{-1}(U)$, the restriction map $p : V_i \to U$ is a quotient map of an equivariant diffeomorphism $h_i : \tilde{V}_i \to \tilde{U}$ (if $y_i = p^{-1}(x) \cap V_i$ then h_i conjugates the action of G_{y_i} on $\tilde{V}_i$ to the action of a subgroup of Γ_x on $\tilde{U}$).

From now on we will assume that the orbifolds under consideration are connected.

The *universal covering* $p : \tilde{O} \to O$ of an orbifold O is the *initial object* in the category of orbifold coverings, i.e. it is a covering such that for any other covering $p' : O' \to O$ there exists a covering $\tilde{p} : \tilde{O} \to O'$ satisfying $p' \circ \tilde{p} = p$. If $p : \tilde{O} \to O$ is the universal covering then the orbifold $\tilde{O}$ is called the *universal covering orbifold* of O.

The group $Deck(p)$ of *deck transformations* of an orbifold covering $p : O' \to O$ is the group of self-diffeomorphisms $h : O' \to O'$ such that $p \circ h = p$. A covering $p : O' \to O$ is called *regular* if $O'/Deck(p) = O$.

The *fundamental group* $\pi_1(O)$ of the orbifold O is the group of deck transformations of its universal covering. Then $O = \tilde{O}/\pi_1(O)$. An alternative definition of the fundamental group based on homotopy-classes of loops in O see in [78, Chapter 13].

Theorem 24 *Each orbifold has a universal covering.*

Definition 16 An orbifold O is called *good* if its universal covering is a manifold. Orbifolds which are not good are called *bad*. An orbifold is called *very good* if is admits a finite-sheeted manifold-covering space.

Example 19 Let $O = M_\Gamma$ be an n-dimensional complex hyperbolic orbifold. Then $\Gamma = \pi_1(O)$ and O is a good orbifold: Its universal covering space is $\mathbb{H}^n_{\mathbb{C}}$. If Γ is finitely generated then, according to Selberg's Lemma, the orbifold O is very good.

Orbifold bundles. Instead of defining orbifold bundles in full generality, I will define these only in the case of compact fibers and connected base, since this will suffice for our purposes:

Definition 17 A smooth orbi-bundle with compact fibers and connected base is a proper submersion $f : \mathcal{O} \to \mathcal{B}$ between orbifolds. Fibers of f are preimages of points under f.

Note that two different fibers need not be isomorphic to each other, but one can prove that they are *commensurable* in the sense that they have a common finite-sheeted orbi-covering.

15 Appendix E: Ends of spaces

Let Z be a locally path-connected, locally compact, Hausdorff topological space. The set of *ends* of Z can be defined as follows (see e.g. [46] for details).

Consider an exhaustion (K_i) of Z by an increasing sequence of compact subsets:

$$K_i \subset K_j, \quad \text{whenever } i \leq j,$$

and

$$\bigcup_{i \in \mathbb{N}} K_i = Z.$$

Set $K_i^c := Z \setminus K_i$. The ends of Z are equivalence classes of decreasing sequences of connected components (C_i) of ${K_i}^c$:

$$C_1 \supset C_2 \supset C_3 \supset \cdots$$

Two sequences (C_i), (C'_j) of components of $({K_i}^c)$, $({K'_j}^c)$ are said to be equivalent if each C_i contains some C'_j and vice-versa. Then the equivalence class of a sequence (C_i) is an *end* e of Z. Each C_i and its closure is called a *neighborhood* of e in Z. The set of ends of Z is denoted $Ends(Z)$. An end e is called *isolated* if it admits a closed 1-ended neighborhood C; such a neighborhood is called *isolating*.

An alternative view-point on the neighborhoods of ends is that there is a natural topology on the union $\hat{Z} = Z \cup Ends(Z)$ which is a compactification of Z and the neighborhoods C of ends e above are intersections of Z with neighborhoods of e in $\hat{Z}$. Then an end e is isolated if and only if it is an isolated point of $\hat{Z}$. A closed neighborhood C of e in Z is isolating if and only if $C \cup \{e\}$ is closed in $\hat{Z}$.

From this definition it is not immediate that the notion of end is independent of the choice of an exhausting sequence (K_i) of compact subsets. The true, but less intuitive, definition of $Ends(Z)$ is by considering the poset (ordered by the inclusion) of all compact subsets $K \Subset Z$. This poset defines the inverse system of sets

$$\{\pi_0(K^c, x) : K \Subset Z\},$$

where the inclusion $K \subset K'$ induces the map

$$\pi_0(Z - K', x') \to \pi_0(Z - K, x'),$$

with $x' \in Z - K' \subset Z - K$. Taking the inverse limit of this system of sets yields $Ends(Z)$ which is, manifestly, a topological invariant. Furthermore, it is an invariant of the proper homotopy type of Z.

In this survey, I adopt the *analyst's viewpoint* on ends of manifolds and conflate isolated ends and their isolating neighborhoods.

16 Appendix F: Generalities on function theory on complex manifolds

For a complex manifold M let O_M denote the ring of holomorphic functions on M. By a *complex manifold with boundary M* I mean a smooth manifold with (possibly empty) boundary ∂M, such that the interior, $\mathrm{int}(M)$, of the manifold M, is equipped with a complex structure, and there exists a smooth embedding $h : M \to X$ to an equidimensional complex manifold X, biholomorphic on $\mathrm{int}(M)$. A holomorphic function on M is a smooth function which admits a holomorphic extension to a neighborhood of M in X.

Suppose that X is a complex manifold and $Y \subset X$ is a codimension 0 smooth submanifold with boundary in X. The submanifold Y is said to be *strictly Levi-convex* if every boundary point of Y admits a neighborhood U in X such that the submanifold with boundary $Y \cap U$ can be written as

$$\{\phi \le 0\},$$

for some smooth submersion $\phi : U \to \mathbb{R}$ satisfying

$$Hess(\phi) > 0.$$

Here $Hess(\phi)$ is the holomorphic Hessian:

$$\left(\frac{\partial^2 \phi}{\partial \bar{z}_i \partial z_j}\right).$$

(Positivity of the Hessian is independent of the local holomorphic coordinates.)

Example 20 If $X = \mathbb{C}^n, Y = \{z \in \mathbb{C}^n : |z| \le 1\}$, then Y is strictly Levi-convex in X: The complex Hessian of the function $\phi(z) = |z|^2 = z \cdot \bar{z}$ is the identity matrix.

Definition 18 A *strongly pseudoconvex manifold M* is a complex manifold with boundary which admits a strictly Levi-convex holomorphic embedding in an equidimensional complex manifold.

Suppose, in addition, that M is compact and $h : M \to X$ is a holomorphic embedding with strictly Levi-convex image Y. Then there exists a strictly Levi-convex submanifold $Y' \subset X$ such that $Y \subset \mathrm{int}(Y')$. Accordingly, M can be biholomorphically embedded in the interior of a compact strongly pseudoconvex manifold M'.

Definition 19 A complex manifold Z is called *holomorphically convex* if for every discrete closed subset $A \subset Z$ there exists a holomorphic function $Z \to \mathbb{C}$ which is proper on A.

Alternatively,[10] one can define holomorphically convex manifolds as follows: For a compact K in a complex manifold M, the *holomorphic convex hull* $\hat{K}_M$ of K in M is

$$\hat{K}_M = \{z \in M : |f(z)| \le \sup_{w \in K} |f(w)|, \forall f \in O_M\}.$$

Then M is holomorphically convex if and only if for every compact $K \subset M$, the hull $\hat{K}_M$ is also compact.

Definition 20 A complex manifold is called *Stein* if it admits a proper holomorphic embedding in $\mathbb{C}^N$ for some N.

Equivalently, M is Stein if and only if it is holomorphically convex and any two distinct points $z, w \in M$ can be separated by a holomorphic function, i.e. there exists $f \in O_M$ such that $f(z) \neq f(w)$. Yet another equivalent definition is: A complex manifold M is Stein if and only if it is strongly pseudoconvex, i.e. it admits an exhaustion by codimension 0 strongly pseudoconvex complex submanifolds with boundary.

In particular:

Theorem 25 *The interior of every compact strongly pseudoconvex manifold Z is holomorphically convex.*

Therefore, by holomorphically embedding such (connected manifold) Z in the interior of another compact strongly pseudoconvex manifold Z' and applying Grauert's theorem to Z', it follows that Z admits nonconstant holomorphic functions.

Kohn and Rossi in [61] proved a certain extension theorem for CR functions defined on the boundary of a complex manifold to holomorphic functions on the entire manifold. I will state only a weak form of their result which will suffice for our purposes.

Theorem 26 (Kohn–Rossi)

Suppose that M is a compact strongly pseudoconvex complex manifold of dimension > 1 which admits at least one nonconstant holomorphic function. Then every holomorphic function on ∂M extends to a holomorphic function on the entire M.

As one of the corollaries of this theorem (Corollary 7.3 of [61]), it follows that if such an M is connected then ∂M is also connected. (If ∂M is disconnected, then one can take a nonconstant locally constant function defined near ∂M: Such a function cannot have a holomorphic extension to M.)

Remark 14 If M is Kähler, then Theorem 26 also holds without the assumption on the existence of nonconstant holomorphic functions, see Proposition 4.4 in [73].

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[10] and this is the standard definition.

Theorem 27 (Rossi, [82], Corollary on page 20)

Suppose that M is a compact strongly pseudoconvex complex manifold. Then $\mathrm{int}(M)$ *admits a proper surjective holomorphic map to a Stein space. In particular, if* $\mathrm{int}(M)$ *contains no compact complex subvarieties of positive dimension, then* $\mathrm{int}(M)$ *is Stein.*

I will not define Stein spaces here (strictly speaking, they are not needed for the purpose of this survey), I refer to [43] for various equivalent definitions.

Topology of Stein manifolds and spaces. Every complex n-dimensional Stein space is homotopy-equivalent to an n-dimensional CW complex, see [47, 48]. More precisely (see Theorem 1.1* on page 153 of [40]):

Theorem 28 *Let M be a n-dimensional complex manifold which admits a proper holomorphic map $M \to \mathbb{C}^N$ with fibers of positive codimension. Then M is homotopy-equivalent to an n-dimensional CW complex.*

Corollary 6 *Suppose that M is a connected compact strongly pseudoconvex complex n-manifold with nonempty boundary. Then M is homotopy-equivalent to a CW complex of dimension $2n-2$.*

17 Appendix G (by Mohan Ramachandran): Proof of Theorem 21

Proposition 7 *Let X be a complex manifold of dimension ≥ 2 and let $M \subset X$ be a domain with compact nonempty smooth strongly pseudoconvex boundary. Then every pluriharmonic function on M which vanishes at ∂M, vanishes identically.*

Proof The proof mostly follows that of Proposition 4.4 in [73]. Suppose that $M = \{x \in X : \varphi(x) < 0\}$ for some smooth function φ, which is strictly plurisubharmonic on a neighborhood of ∂M and such that there exists $\epsilon < 0$ such that $\varphi^{-1}([\epsilon, 0])$ is compact and $\varphi|_{\partial M} = 0$. Let $\beta : M \to \mathbb{R}$ be a pluriharmonic function which vanishes at ∂M. Fix $a \in (\epsilon, 0)$, such that φ is strictly plurisubharmonic on $V = \{x \in M : \varphi(x) > a\}$. If β does not vanish identically on a neighborhood of ∂M, we let $b \in \beta(V)$ denote a regular value of β. Thus, $\beta^{-1}(b)$ is disjoint from ∂M. Since $\varphi^{-1}([\epsilon, 0])$ is compact, the restriction of φ to $\beta^{-1}(b)$ has a maximum at some point $x_0 \in V \cap \beta^{-1}(b)$. The holomorphic 1-form $\partial\beta$ determines a (singular) holomorphic foliation on M. Consider the leaf L through x_0 of this holomorphic foliation: This leaf is contained in $\beta^{-1}(b)$ and, hence, the restriction $\varphi|_L$ has a maximum at x_0 contradicting strict plurisubharmonicity of φ. Therefore, β is identically zero near ∂M and, hence, is identically zero.

The next proposition is proven in [72, Theorem 2.6]:

Proposition 8 *Suppose now that M has a complete Kähler metric of bounded geometry,* [11] *∂M is connected and M has at least two ends. Then M admits a nonconstant pluriharmonic function $\beta : M \to \mathbb{R}$ which converges to zero at ∂M.*

By combining the two propositions, we conclude:

Corollary 7 *Suppose that M is a complex manifold of dimension ≥ 2, which admits a holomorphic embedding as a domain with compact nonempty smooth strongly pseudoconvex boundary and which admits a complete Kähler metric of bounded geometry. Then M is 1-ended.*

We can now conclude the proof of Theorem 21: Let $M = M_\Gamma$ be a complex hyperbolic manifold of dimension ≥ 2 and of injectivity radius bounded below. Suppose that $E_0 \subset M$ is a convex end. Let $S_0 \subset \partial\overline{M}$ be the component corresponding to the end E_0. Consider the complex manifold $Y = \check{\Omega}_\Gamma/\Gamma$. Remove from Y all the components of $Y - M$ which are disjoint from S_0 and call the result X. Then M embeds in X as a domain with nonempty smooth strongly pseudoconvex boundary, namely, S_0. Then, by the corollary, M is 1-ended.

[11] i.e. its sectional curvature lies in a finite interval and its injectivity radius is bounded from below

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