On asymptotic cones and quasi–isometry classes of fundamental groups of 3-manifolds

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January 19, 1995

1991 Mathematics Subject Classification: 20F32, 53C21, 57M50.

1 Introduction

Let Γ be a finitely generated group. A finite set of generators $\mathcal{G}$ of Γ determines a Cayley graph $C(\Gamma, \mathcal{G})$. It is a metric space whose quasi-isometry class does not depend on the chosen set of generators $\mathcal{G}$. We are interested in geometric properties of Γ, i.e. quasi-isometry invariants of its Cayley graph. Well-known examples of geometric properties of finitely generated groups include: "finitely presentable", "virtually nilpotent" (Gromov), "virtually abelian" (Gromov, Bridson and Gersten), "word hyperbolic" (Gromov), "being a finite extension of a uniform lattice in $SO(n, 1)$" (Mostow, Tukia, Gabai), "being a finite extension of a nonuniform lattice in a rank 1 symmetric space" (Schwartz), cohomological dimension is a quasi-isometry invariant for fundamental groups of finite aspherical complexes (Gersten).

Quasi-isometries ignore the local geometry. Looking for quasi-isometry invariants we have to understand the large-scale geometry of metric spaces. One aspect of it, namely the asymptotic geometry of finite subsets of distant points in a metric space X is encoded in the geometry of the asymptotic cone of X. This concept has been introduced by Van den Dries and Wilkie [DW] and Gromov [Gr2]. Bi-Lipschitz topological invariants of the asymptotic cone of X are quasi-isometry invariants of X. Papasoglu [Pa] proves that the asymptotic cone of a group satisfying a quadratic isoperimetric inequality is simply connected. The asymptotic cone will be used in [KlL] to prove quasiisometric rigidity of noncompact irreducible symmetric spaces of higher rank.

We study the large-scale geometry of nonpositively curved spaces X. One observes that flats in X are reproduced inside the asymptotic cone, whereas negatively curved subspaces break up into trees. One may think of the asymptotic

*This research was partially supported by the grant SFB 256 "Nichtlineare partielle Differentialgleichungen" and the NSF grant DMS-9306140 (Kapovich).

cone of X as a higher-dimensional analogue of a metric tree. For instance, the asymptotic cone of a higher-rank symmetric space is a generalized affine building [KlL]. We investigate the pattern of flats in the asymptotic cone of certain nonpositively curved spaces of geometric rank one (in the sense of Ballmann, Brin and Eberlein) and obtain non-trivial quasi-isometry invariants.

Metrics of nonpositive curvature appear in abundance in 3-dimensional topology. Thurston proved that atoroidal Haken manifolds are hyperbolic. It is shown in [L] that Haken manifolds with incompressible tori generically admit metrics of nonpositive curvature. In the subsequent paper [KaL1] we show that the fundamental group of every Haken manifold (which is not a Nil- or Sol-manifold) is quasi-isometric to the fundamental group of a 3-manifold of nonpositive curvature.

Due to the geometrization of 3-manifolds we can apply our results about asymptotic cones of nonpositively curved spaces to distinguish quasi-isometry types of fundamental groups of 3-dimensional Haken manifolds. In Theorem 5.1 we prove that if a Haken manifold M_1 contains only hyperbolic components and M_2 is a nonpositively curved manifold which contains a Seifert component with hyperbolic base then $\pi_1(M_1)$ is not quasi-isometric to $\pi_1(M_2)$. Combining this with results of N. Brady, Gersten and Schwartz, one obtains a rough quasi-isometry classification of fundamental groups of Haken manifolds. It follows in particular that the existence of a Seifert (as well as a hyperbolic) component in a Haken manifold is a quasi-isometry invariant of its fundamental group.

The paper is organized as follows. In Section 2 we discuss basic properties of nonpositively curved spaces. We describe a discrete analogon of ruled surfaces in CAT(0)-spaces. In Section 3 we review the concept of ultralimits and asymptotic cones of metric spaces. We use ultralimits to give yet another interpretation of the compactification of representation varieties by actions of groups on trees [Mo], [Be], [Pau]. In Section 4 we study large-scale geometric properties of certain nonpositively curved spaces. We show that fat geodesic triangles in a CAT(0)-space X avoid regions of strictly negative curvature. Assuming that X is negatively curved outside a disjoint union of flats, we deduce geometric and topological properties of the asymptotic cone of X. In particular, distinct embedded 2-discs have at most one point in common. This rules out the possibility that X contains a quasi-isometrically embedded product of the real line and a non-abelian free group. Examples of such CAT(0)-spaces X are given by universal covers of Haken manifolds obtained by gluing hyperbolic components. Another class of examples are universal covers of nonpositively curved manifolds arising from Thurston-Schroeder's cusp-closing construction [Schr]. In Section 5 we apply the results of Section 4 to distinguish quasi-isometry classes of fundamental groups of Haken 3-manifolds.

Acknowledgements. We thank the Mathematical Institute at the University of Bonn for its hospitality during the period when this paper was written. We are grateful to Richard Schwartz and Martin Bridson for remarks concerning the original manuscript of this paper.

2 Preliminaries

2.1 Elementary properties of CAT(0)-spaces

Let X be a complete metric space with metric $d = d_X$. A *geodesic* in X is an isometric embedding $f : I \to X$ of an interval. A *complete geodesic* in X is an isometric embedding $f : \mathbb{R} \to X$. We denote by $[xy]$ a geodesic segment joining points $x, y \in X$, and by $]xy[$ the open segment. An n-dimensional *flat* is an isometric embedding of $\mathbb{R}^n$, $n \geq 2$. X is called a *geodesic space* if any two points can be connected by a geodesic. $\Delta(x, y, z)$ will denote a *geodesic triangle* in X with vertices x, y, z. It is the union of geodesic segments $[xy]$, $[yz]$ and $[zx]$. We define the *inradius* $IR_X(\Delta)$ of a triangle Δ in X to be the infimum of all numbers ρ so that there exists a point in X with distance at most ρ from all sides of Δ.

There is a synthetic way of defining upper curvature bounds for geodesic spaces X via distance comparison. We are only concerned with nonpositive bounds $\kappa \leq 0$. X is said to satisfy the CAT(κ)-*property*, if geodesic triangles in X are not thicker than triangles in the complete simply-connected Riemannian 2-manifold M^2_κ of sectional curvature κ. More precisely, let $\Delta(x, y, z)$ be a triangle in X and choose a triangle $\Delta(x', y', z')$ with the same side lengths in M^2_κ. If p, q are points on $\Delta(x, y, z)$ and p', q' are points on $\Delta(x', y', z')$, which divide corresponding sides in the same ratio, then

$$d(p, q) \leq d(p', q').$$

In fact, it suffices to check this property only in the case when q is a vertex. We say that X has *local upper curvature bound* κ at a subset A if there is a convex subset containing A which satisfies the CAT(κ)-property.

X is a *metric tree* if it satisfies the CAT(κ)-property for arbitrary negative values of κ. In this case all geodesic triangles degenerate to tripods. One can also characterize metric trees as geodesic spaces where any two points can be connected by a unique simple arc (see Lemma 4.7).

We collect a few facts about CAT(0)-spaces, see e.g. [GBS] and [Ba] for details. The CAT(0)-property implies that the distance function is convex. Hence, any two points can be connected by a unique geodesic. In particular, CAT(0)-spaces are contractible. If Y is a convex subset in a CAT(0)-space X, then the *nearest-point-projection* $\pi_Y : X \to Y$ is well-defined and distance-nonincreasing. Two complete geodesic rays $r_1, r_2 : [0, \infty) \to X$ are called asymptotic, if the distance function $t \to d(r_1(t), r_2(t))$ remains bounded. The set $\partial_\infty X$ of equivalence classes of asymptotic rays is called the *ideal boundary* of X.

Let x be a point in the CAT(0)-space X and $r_1, r_2 : [0, \epsilon) \to X$ be geodesic rays emanating from x. The *angle* $\angle_x(r_1, r_2) = \alpha$ between r_1 and r_2 is defined by the formula:

$$2\sin(\frac{\alpha}{2}) = \lim_{t \to 0+} \frac{d(r_1(t), r_2(t))}{t}$$

This limit exists, because the function $t \mapsto d(r_1(t), r_2(t))$ is convex. The definition coincides with the usual one in the case of Riemannian manifolds.

Lemma 2.1 *Let r_1, r_2, r_3 be rays emanating from x. Then the angles between them satisfy the inequality:*

$$\angle_x(r_1, r_2) + \angle_x(r_2, r_3) \geq \angle_x(r_1, r_3)$$

Lemma 2.2 *If the union of the geodesic rays r_1 and r_2 emanating from x is a geodesic with x as interior point, then the angle between r_1 and r_2 equals π.*

Distance comparison in the presence of an upper curvature bound yields angle comparison:

Lemma 2.3 *The angles of a geodesic triangle in a CAT(κ)-space are not greater than the corresponding angles of a comparison triangle in the model space M^2_κ.*

For a geodesic triangle τ in X with angles α, β, γ, we define the *angle deficit* by:

$$\text{deficit}(\tau) := \pi - \alpha - \beta - \gamma$$

Let x, y, z be three points in the CAT(0)-space X. There is a unique geodesic triangle $\Delta(x, y, z)$. Define points x', y', z' by $[xx'] := [xy] \cap [xz]$, $[yy'] := [yz] \cap [yx]$ and $[zz'] := [zx] \cap [zy]$. The triangle $\Delta(x', y', z')$ is called the *open triangle* spanned by x, y, z. $\Delta(x, y, z)$ itself is called *open*, if it coincides with $\Delta(x', y', z')$.

We shall need the following property of CAT(0)-spaces.

Lemma 2.4 *Let $\gamma = [xy] \cup [yz] \cup [zw]$ be a broken geodesic in a CAT(0)-space X such that $[xy] \cup [yz]$ and $[yz] \cup [zw]$ are geodesics. Then γ is a geodesic as well.*

Proof: Suppose that there are points $a \in [xy]$ and $b \in [zw]$ such that $d(a, b) < d(a, y) + d(y, b)$. Consider the comparison triangle $\Delta(a', y', b')$ in the Euclidean plane and the point $z' \in [y'b']$ with $d(y', z') = d(y, z)$. Then $d(a', z') < d(a', y') + d(y', z')$, on the other hand the comparison property implies that $d(a', z') \geq d(a, z) = d(a, y) + d(y, z)$. This contradiction proves the assertion. $\square$

2.2 Nonpositively curved metrics on 3-manifolds

Let M be a compact smooth 3-manifold. A closed smooth surface $S \subset M$ is called *incompressible* if it is 2-sided, has infinite fundamental group and the inclusion $S \subset M$ induces a monomorphism of fundamental groups. A manifold M is said to be *irreducible* if any smooth 2-sphere in the universal cover of M bounds a ball. If M is irreducible and contains a closed incompressible surface then it is called *Haken*. Note that if the boundary of a Haken manifold has zero Euler characteristic then it is incompressible.

Remark 2.5 *Our definition of Haken manifolds is slightly more restrictive than the classical one (see [JS], [J]). However it will suffice for the purposes of this paper.*

Let M be a Haken 3-manifold with boundary of zero Euler characteristic. According to [J],[JS] and [Th] there is a unique finite union T of disjoint incompressible 2-tori and Klein bottles which split M into a collection of hyperbolic and maximal Seifert components. We recall the following results concerning the existence of nonpositively curved metrics on M.

Theorem 2.6 ([L],[LS]) *If M admits a Riemannian metric of nonpositive sectional curvature with totally–geodesic boundary, then T can be isotoped so that $T \cup \partial M$ is totally–geodesic.*

Remark 2.7 *Theorem 2.6 implies that for each component M_j of $M \setminus T$ the universal cover of M_j is convex in the universal cover of M. Hence $\pi_1(M_j)$ is quasi–isometrically embedded into $\pi_1(M)$.*

Theorem 2.8 ([L]) *Suppose that either ∂M is nonempty or $M \setminus T$ has a hyperbolic component. Then M admits a smooth Riemannian metric of nonpositive sectional curvature with totally–geodesic boundary such that T is totally geodesic and the sectional curvature is strictly negative on each hyperbolic component of $M \setminus T$.*

2.3 Straight Fillings

We recall that a ruled surface in a smooth Riemannian manifold is a smooth family of geodesics. It is a classical fact that the intrinsic curvature of a ruled surface is not greater than the curvature of the ambient manifold. The goal of this section is to construct a discrete analogue of filling in geodesic triangles by ruled surfaces.

Let Δ be a non-degenerate triangle in Euclidean plane.

We define a *triangulation* of Δ to be a decomposition of Δ into a finite collection K of Eulidean 2-simplices with disjoint interiors so that the closure of their union equals Δ. Note that our definition differs from the standard one: we allow *interior vertices* on edges of triangles in K.

For a triangulation S of Δ, we denote by S^i the i-skeleton of S. A triangulation T of Δ is called *special* if it can be constructed from the trivial triangulation by the following inductive procedure. There exists a finite sequence of triangulations $\Delta = T_0, \ldots, T_n = T$ of Δ, such that the triangulation T_{k+1} is obtained from T_k by adding a segment σ_k satisfying the properties:

- At least one endpoint of σ_k is contained in T_k^0.
- The intersection of the interior of σ_k with T_k^1 is empty.

Take now a geodesic triangle $\Delta(x, y, z)$ in a CAT(0)-space X. We define a canonical map

$$f : T^1 \to X$$

by mapping Δ to $\Delta(x, y, z)$ and requiring that the restriction of f to every segment σ_k is an affine map. We call such a map f a *straight filling*. We say that a filling is ϵ*-fine* if the image under f of each triangle in T^1 has diameter at most ϵ.

For each triangle δ in T^2, let $\kappa(\delta)$ be the curvature of X at $f(\partial\delta)$. We put a Riemannian metric of constant curvature $\kappa(\delta)$ on δ so that it has geodesic sides and the restriction of f to every side is an isometry. This induces a path metric on Δ which we denote by d_f.

Lemma 2.9 *The map $f : (T^1, d_f|_{T^1}) \to X$ does not increase distances.*

Proof: Suppose that p and q are two points on the boundary of the same triangle δ in T^2. Then the distance comparison inequality implies:

$$d_f(p, q) \geq d(f(p), f(q))$$

The global statement follows immediately. □

We define the angle deficit of the filling f as the sum

$$\operatorname{deficit}(f) := \sum_{\delta \in T^2} \operatorname{deficit}(\delta)$$

Lemma 2.10 *The deficit of the straight filling f is not greater than the angle deficit of the triangle $\Delta(x, y, z)$.*

Proof: The angles of the triangles δ are not smaller than the angles of $f(\delta)$ and the sum of angle deficits is sub-additive with respect to triangulations:

$$\operatorname{deficit}(f) := \sum_{\delta \in T^2} \operatorname{deficit}(\delta) \leq \sum_{\delta \in T^2} \operatorname{deficit}(f(\delta)) \leq \operatorname{deficit}(\Delta(x, y, z))$$

□

Lemma 2.11 *For every interior vertex p in T^0, the sum of the angles adjacent to p is at least 2π. For every vertex p, which is an interior point of a side of Δ, the sum of the angles adjacent to p is at least π.*

Proof: Consider an interior vertex p. There is exactly one segment σ_k which contains p as an interior point. Let ϕ_1 and ϕ_2 be the sums of angles in (Δ, d_f) adjacent to p from two different sides of σ_k. Denote by ψ_i the sums of corresponding angles in X. For each angle α adjacent to p in (Δ, d_f), the corresponding angle in X adjacent to $f(p)$ is not greater than α. Therefore, $\phi_i \geq \psi_i$. By Lemma 2.1 and Lemma 2.2, we conclude that $\phi_i \geq \pi$. The argument for vertices on the boundary is analogous. □

We now compare local curvature bounds of the spaces X and (Δ, d_f).

Proposition 2.12 *Suppose that the filling $f : T^1 \to X$ is ϵ-fine. Let p be a point in T^1 so that the ball $B_\epsilon(f(p))$ satisfies the CAT(κ)-property with $\kappa \leq 0$. Then the local curvature of (Δ, d_f) at the point p is bounded from above by κ.*

Proof: The arguments of the proof of Theorem 15 in [Ba] remain valid for singular spaces with piecewise constant curvature. The link condition for (Δ, d_f) is satisfied according to Lemma 2.11. □

Corollary 2.13 *The geodesic space (Δ, d_f) satisfies the CAT(0)-property.*

Proof: According to Theorem 7 in [Ba] it suffices to verify that any two points in (Δ, d_f) are connected by a unique geodesic. Suppose that p, q are points which are connected by two distinct geodesics γ_1 and γ_2. Without loss of generality, we may assume that the interiors of ga_1 and γ_2 are disjoint. Then $\gamma_1 \cup \gamma_2$ bounds a n-gon P which is triangulated by triangles of nonpositive curvature. Using the Gauß-Bonnet formula and Lemma 2.11, we conclude that the sum of angles in P is less than $(n-2)\pi$. On the other hand, P has $n-2$ angles greater or equal to π by Lemma 2.2. □

2.4 Quasi-isometries of metric spaces

Let (X_j, d_j) $(j = 1, 2)$ be a pair of metric spaces. We recall that a map $f : (X_1, d_1) \to (X_2, d_2)$ is a *quasi-isometric embedding* if there are two constants $K > 0$ and C such that

$$K^{-1} d_1(x, y) - C \leq d_2(f(x), f(y)) \leq K d_1(x, y) + C$$

for each $x, y \in X_1$. A map $f_1 : (X_1, d_1) \to (X_2, d_2)$ is a *quasi-isometry* if there are two constants C_1, C_2 and another map $f_2 : (X_2, d_2) \to (X_1, d_1)$ such that both f_1, f_2 are quasi-isometric embeddings and

$$d_1(f_2 f_1(x), x) \leq C_1, d_2(f_1 f_2(y), y) \leq C_2$$

for every $x \in X_1, y \in X_2$. Such spaces X_1, X_2 are called *quasi-isometric.* For example, two geodesic metric spaces which admit cocompact discrete actions by isometries of the same group are quasi-isometric.

A finitely generated group Γ with a fixed finite set of generators carries a canonical metric which is called the word metric. The quasi-isometry class of the word metric does not depend on the generating set.

2.5 Bi-Lipschitz embeddings of Euclidean planes

Lemma 2.14 *Let T be a metric tree and $f : \mathbb{R}^2 \to T \times \mathbb{R}$ be a bi-Lipschitz embedding. Then the image of f is a flat in $T \times \mathbb{R}$.*

Proof: The map f is closed because it is bi-Lipschitz. Consider the projection P of $f(\mathbb{R}^2)$ in the tree T. The set P is a subtree in T. Let $w \in P$ be any point which separates P. Then the line $\{w\} \times \mathbb{R}$ separates $f(\mathbb{R}^2)$ and therefore $f^{-1}((T \setminus \{w\}) \times \mathbb{R})$ is not connected. We denote the intersection $f(\mathbb{R}^2) \cap \{w\} \times \mathbb{R}$ by L. The preimage $f^{-1}(L)$ is closed in $\mathbb{R}^2$. The compact subset $f^{-1}(L) \cup \{\infty\}$ in the one-point compactification $S^2 = \mathbb{R}^2 \cup \{\infty\}$ is homeomorphic to the subset $L \cup \{\infty\}$ in the one-point compactification of the real line $\{w\} \times \mathbb{R}$. Hence by Alexander duality $H^1(L \cup \{\infty\}, \mathbb{Z}) \cong \tilde{H}_0(\mathbb{R}^2 - f^{-1}(L), \mathbb{Z}) \neq 0$, where we use Alexander-Spanier cohomology. Thus $L = \{w\} \times \mathbb{R}$. It follows furthermore that w separates P in exactly two components. Hence P is homeomorphic to an interval. Since f is closed, $f(\mathbb{R}^2) = P \times \mathbb{R}$. P is a complete geodesic in T because f is closed and is a homeomorphism onto its image. $\square$

Corollary 2.15 *The product of a metric tree and $\mathbb{R}$ is not bi-Lipschitz homeomorphic to the product of two metric trees with nontrivial branching.*

Proof: The product of two metric trees with at least 3 ends contains three flats which have exactly one common point. $\square$

3 Ultralimits of metric spaces

Let (X_i) be a sequence of metric spaces which is not precompact in the Gromov-Hausdorff topology. One can describe the limiting behavior of the sequence (X_i) by studying limits of precompact sequences of subspaces $Y_i \subset X_i$. Ultrafilters are an efficient technical device for simultaneously taking limits of all such sequences of subspaces and putting them together to form one object, namely an ultralimit of (X_i). We discuss this concept following Gromov [Gr2].

3.1 Ultrafilters

Let I be an infinite set. A *filter* on I is a nonempty family ω of subsets of I with the properties:

- $\emptyset \notin \omega$.
- If $A \in \omega$ and $A \subset B$, then $B \in \omega$.
- If $A_1, \ldots, A_n \in \omega$, then $A_1 \cap \ldots \cap A_n \in \omega$.

Subsets $A \subset I$ which belong to a filter ω are called ω-large. We say that a property (P) holds for *ω-all i*, if (P) is satisfied for all i in some ω-large set. An *ultrafilter* is a maximal filter. The maximality condition can be rephrased as: for every decomposition $I = A_1 \cup \ldots \cup A_n$ of I into finitely many disjoint subsets, the ultrafilter contains exactly one of these subsets.

For example, for every $i \in I$, we have the *principal* ultrafilter δ_i defined as $\delta_i := \{A \subset I \mid i \in A\}$. An ultrafilter is principal if and only if it contains a finite subset. The interesting ultrafilters are of course the non-principal ones. They cannot be described explicitly but exist by Zorn's lemma: every filter is contained in an ultrafilter. Let $\mathcal{Z}$ be the *Zariski filter* which consists of complements to finite subsets in I. An ultrafilter is a nonprincipal ultrafilter, if and only if it contains $\mathcal{Z}$. For us is not important how ultrafilters look like, but rather how they work: An ultrafilter ω on I assigns a "limit" to every function $f : I \to Y$ with values in a compact space Y. Namely,

$$\omega\text{-}\lim f = \omega\text{-}\lim_i f(i) \in Y$$

is defined to be the unique point $y \in Y$ with the property that for every neighborhood U of y the preimage $f^{-1}U$ is "ω-large". To see the existence of a limit, assume that there is no point $y \in Y$ with this property. Then each point $z \in Y$ possesses a neighborhood U_z such that $f^{-1}U_z \notin \omega$. By compactness, we can cover Y with finitely many of these neighborhoods. It follows that $I \notin \omega$. This contradicts the definition of a filter. Uniqueness of the point y follows, because Y is Hausdorff. Note that if y is an accumulation point of $\{f(i)\}_{i \in I}$ then there is a non-principal ultrafilter ω with ω-lim $f = y$, namely an ultrafilter containing the pullback of the neighborhood basis of y.

3.2 Ultralimits of metric spaces

Let $(X_i)_{i \in I}$ be a family of metric spaces parametrized by an infinite set I. For an ultrafilter ω on I we define the ultralimit

$$X_\omega = \omega\text{-}\lim_i X_i$$

as follows. Let Seq be the space of sequences $(x_i)_{i \in I}$ with $x_i \in X_i$. The distance between two points $(x_i), (y_i) \in Seq$ is given by

$$d_\omega\big((x_i), (y_i)\big) := \omega\text{-}\lim\big(i \mapsto d_{X_i}(x_i, y_i)\big)$$

where we take the ultralimit of the function $i \mapsto d_{X_i}(x_i, y_i)$ with values in the compact set $[0, \infty]$. The function d_ω is a pseudo-distance on Seq with values in $[0, \infty]$. Set

$$(X_\omega, d_\omega) := (Seq, d_\omega)/\sim$$

where we identify points with zero d_ω-distance.

Example 3.1 *Let $X_i = Y$ for all i, where Y is a compact metric space. Then $X_\omega \cong Y$ for all ultrafilters ω.*

The concept of ultralimits extends the notion of Gromov-Hausdorff limits:

Proposition 3.2 *Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of compact metric spaces converging in the Gromov-Hausdorff topology to a compact metric space X. Then $X_\omega \cong X$ for all non-principal ultrafilters ω.*

Proof: Realize the Gromov-Hausdorff convergence in an ambient compact metric space Y, i.e. embed the X_i and X isometrically into Y such that the X_i converge to X with respect to the Hausdorff distance. Then there is a natural isometric embedding

$$X_\omega = \omega\text{-}\lim_i X_i \xrightarrow{\iota} \omega\text{-}\lim_i Y \cong Y$$

Since ω is non-principal, the ω-limit is independent of any finite collection of X_i's and we get:

$$\iota(X_\omega) \subseteq \bigcap_{i_0} \overline{\bigcup_{i\geq i_0} X_i} = X$$

On the other hand $X \subseteq \iota(X_\omega)$, because $\iota\big((x_i)\big) = x$ if (x_i) is a sequence with $x_i \in X_i$ converging in Y to $x \in X$. Hence $\iota(X_\omega) = X$ which proves the claim.
$\square$

If the spaces X_i do not have uniformly bounded diameter, then the ultralimit X_ω decomposes into (generically uncountably many) components consisting of points of mutually finite distance. We can pick out one of these components if the spaces X_i have basepoints x_i^0. The sequence $(x_i^0)_i$ defines a basepoint x_ω^0 in X_ω and we set

$$X_\omega^0 := \{x_\omega \in X_\omega \mid d_\omega(x_\omega, x_\omega^0) < \infty\}$$

Define the *based ultralimit* as

$$\omega\text{-}\lim_i (X_i, x_i^0) := (X_\omega^0, x_\omega^0)$$

Example 3.3 *For every locally compact space Y with a basepoint y_0, we have:*

$$\omega\text{-}\mathit{lim}_i(Y, y_0) \cong (Y, y_0)$$

We observe that some geometric properties pass to ultralimits:

Proposition 3.4 *Let $(X_i, x_i^0)_{i\in I}$ be a sequence of based geodesic spaces and let ω be an ultrafilter. Then X_ω^0 is a geodesic space.*

If the X_i are CAT(κ)–spaces for some $\kappa \leq 0$ then X_ω^0 has the same upper curvature bound κ.

Proof: The ultralimit of geodesic segments in X_i is a geodesic segment in X_ω^0. Therefore X_ω^0 is a geodesic space. It remains to prove that any pair of points $x_\omega = (x_i)$ and $y_\omega = (y_i)$ in X_ω^0 can be joined by a unique geodesic. Suppose that $d_\omega(x_\omega, y_\omega) = s + t$ where $s, t \geq 0$. There are points z_i on the geodesic segments

$[x_i y_i]$ such that for $s_i := d_i(x_i, z_i)$ and $t_i := d_i(z_i, y_i)$ we have $\omega\text{-}\lim s_i = s$ and $\omega\text{-}\lim t_i = t$. Hence $z_\omega := (z_i)$ satisfies $d_\omega(x_\omega, z_\omega) = s$ and $d_\omega(z_\omega, y_\omega) = t$. Suppose that $u_\omega = (u_i)$ is another point with the same property. Consider in the model space M^2_κ comparison triangles $\Delta(x'_i, u'_i, y'_i)$ with the same sidelengths as $\Delta(x_i, u_i, y_i)$. Let z'_i be a division point on $[x'_i y'_i]$ corresponding to z_i on $[x_i y_i]$. Since $\omega\text{-}\lim(d_i(x_i, u_i) + d_i(u_i, y_i) - d_i(y_i, x_i)) = 0$, we have $\omega\text{-}\lim d_i(u_i, z_i) \leq \omega\text{-}\lim d_{M^2_\kappa}(u'_i, z'_i) = 0$ and therefore $u_\omega = z_\omega$. Thus there is a unique point $z_\omega \in X_\omega$ with $d_\omega(x_\omega, z_\omega) = s$ and $d_\omega(z_\omega, y_\omega) = t$. $\square$

Corollary 3.5 *Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of geodesic spaces with upper curvature bounds κ_i tending to $-\infty$. Then for every non-principal ultrafilter ω the ultralimit X_ω is a metric forest, i.e. every component is a metric tree.*

3.3 The asymptotic cone of a metric space

Let X be a metric space and ω be a non-principal ultrafilter on $\mathbb{N}$. The *asymptotic cone* $\mathrm{Cone}_\omega(X)$ of X is defined as the based ultralimit of rescaled copies of X:

$$\mathrm{Cone}_\omega(X) := X^0_\omega, \qquad \text{where} \quad (X^0_\omega, x^0_\omega) = \omega\text{-}\lim_i\left(\frac{1}{i}\cdot X, x^0\right)$$

The limit is independent of the chosen basepoint $x^0 \in X$. The discussion in the previous section implies:

Proposition 3.6

1. *$Cone_\omega(X \times Y) = Cone_\omega(X) \times Cone_\omega(Y)$.*
2. *$Cone_\omega \mathbb{R}^n \cong \mathbb{R}^n$.*
3. *The asymptotic cone of a geodesic space is a geodesic space.*
4. *The asymptotic cone of a CAT(0)-space is CAT(0).*
5. *The asymptotic cone of a space with a negative upper curvature bound is a metric tree by Corollary 3.5.*

Remark 3.7 *For any metric space X the asymptotic cone $Cone_\omega(X)$ is complete [DW].*

Remark 3.8 *Suppose that X admits a cocompact discrete action by a group of isometries. The problem of dependence of the topological type of $Cone_\omega X$ on the ultrafilter ω is open (see [Gr2]).*

To get an idea of the size of the asymptotic cone, note that in the most interesting cases it is homogeneous. We call a metric space X *quasi- homogeneous* if $diam(X/Isom(X))$ is finite.

Proposition 3.9 *Let X be a quasi- homogeneous metric space.*
Then $Cone_\omega(X)$ is a homogeneous metric space for every non-principal ultrafilter ω.

Proof: The group of sequences of isometries $Isom(X)^{\mathbb{N}}$ acts transitively on the ultralimit ω-$\lim_i(\frac{1}{i}\cdot X)$ which contains $\mathrm{Cone}_\omega(X)$ as a component. $\square$

Lemma 3.10 *Let X be a quasi-homogeneous CAT(-1)–space with uncountable number of ideal boundary points. Then for every nonprincipal ultrafilter ω the asymptotic cone $Cone_\omega(X)$ is a tree with uncountable branching. Every open set in $Cone_\omega(X)$ contains an uncountable discrete subset.*

Proof: Let $x^0 \in X$ be a basepoint and $y, z \in \partial_\infty X$. Denote by γ the geodesic in X with the ideal endpoints z, y. Then $\mathrm{Cone}_\omega([x^0, y[)$ and $\mathrm{Cone}_\omega([x^0, z[)$ are geodesic rays in $\mathrm{Cone}_\omega(X)$ emanating from x^0_ω. Their union is equal to the geodesic $\mathrm{Cone}_\omega\gamma$. This produces uncountably many rays in $\mathrm{Cone}_\omega(X)$ so that any two of them have precisely the basepoint in common. The homogeneity of $\mathrm{Cone}_\omega(X)$ implies the assertion. $\square$

Corollary 3.11 *Let Z be a compact Seifert manifold with hyperbolic base orbifold. Then the space $Cone_\omega(\pi_1(Z))$ is the product of $\mathbb{R}$ and a tree with uncountable branching at every point.*

Proof: Let Γ be the fundamental group of the base orbifold of Z. If Z has non-empty boundary, then $\pi_1(Z)$ virtually splits as the product of $\mathbb{Z}$ and a non-abelian free group. In the case $\partial Z = \emptyset$ it was proven independently by Epstein, Gersten and Mess, that $\pi_1(Z)$ is quasi-isometric to $\mathbb{Z} \times \Gamma$, see [R]. The assertion follows from Lemma 3.10. $\square$

Applications of the asymptotic cone as a quasi-isometry invaraint are based on the following

Proposition 3.12 *Suppose that $f : X \to Y$ is a quasi-isometric embedding. Then for each non-principal ultrafilter ω, f induces a bi-Lipschitz embedding $Cone_\omega(f) : Cone_\omega(X) \to Cone_\omega(Y)$.*

If f is a quasi-isometry then $Cone_\omega(f) : Cone_\omega(X) \to Cone_\omega(Y)$ is a bi-Lipschitz homeomorphism.

We illustrate this property in the following simple case:

Proposition 3.13 *Let X, Y, Z be $CAT(-1)$ spaces which have at least 3 ideal boundary points. Then $\mathbb{R} \times X$ is not quasi-isometric to $Y \times Z$.*

Proof: The spaces $\mathrm{Cone}_\omega(X), \mathrm{Cone}_\omega(Y)$ and $\mathrm{Cone}_\omega(Z)$ are metric trees with at least 3 ends. Therefore by Corollary 2.15, the spaces $\mathrm{Cone}_\omega(Y) \times \mathrm{Cone}_\omega(Z)$ and $\mathrm{Cone}_\omega(X) \times \mathbb{R}$ are not bi-Lipschitz homeomorphic. $\square$

Example 3.14 $\mathbb{H}^p \times \mathbb{H}^q$ *is not quasi-isometric to* $\mathbb{H}^{p+q-1} \times \mathbb{R}$*, where* $p, q \geq 2$.

3.4 Limits of isometric actions on CAT(0)-spaces

In [Mo], Morgan compactifies the space of representations of a finitely generated group Γ into $SO(n,1)$. The ideal points of the compactification are isometric actions of Γ on metric trees. A geometric version of this construction was given in [Be] and [Pau]. In this paragraph, we rephrase their argument in the context of ultralimits and generalize it to the setting of nonpositive curvature.

Let X_n be a sequence of CAT(0)-spaces and $\rho_n : \Gamma \to \mathrm{Isom}(X_n)$ be a sequence of representations. Choose a finite generating set $\mathcal{G}$ of the group Γ. For $x \in X_n$, we denote by $D_n(x)$ the diameter of the set $\rho_n(\mathcal{G})(x)$. Set $D_n := \inf_{x\in X_n} D_n(x)$. We assume that the sequence (ρ_n) diverges in the sense that $\lim_{n\to\infty} D_n = \infty$. Choose points $x_n \in X_n$ such that $D_n(x_n) \leq D_n + 1/n$. For any non-principal ultrafilter ω, there exists a natural isometric action ρ_ω of Γ on the ultralimit of rescaled spaces

$$(X_\omega, x_\omega) := \omega\text{-}\lim_n (D_n^{-1} \cdot X_n, x_n).$$

X_ω is a CAT(0)-space and the action ρ_ω has no global fixed point. If the spaces X_n are CAT(-1), then the limit space X_ω is a metric tree. The tree constructed in [Be] and [Pau] is the minimal invariant subtree. Assume also that the spaces X_n are Hadamard manifolds of uniformly bounded dimension with sectional curvature bounded between two negative constants $-a^2, -1$ and that the representations ρ_n are discrete and faithful. Then the Margulis lemma implies that the action ρ_ω is *small*. This means that the stabilizer of any non-degenerate segment in X_ω is virtually nilpotent.

4 The large-scale geometry of certain CAT(0)-spaces

4.1 Fat triangles in CAT(0)-spaces

Consider a Haken 3-manifold M equipped with a metric of nonpositive curvature as in Theorem 2.8. In this section we will assume that M has at least one hyperbolic component. Let $\epsilon > 0$ be such that the components of $T \cup \partial M$ are 7ϵ-separated. Denote by $\bar{N}$ the 3ϵ-neighborhood of the union of $T \cup \partial M$ and all Seifert components of M. Then there is a negative constant κ such that on every 2ϵ-ball with center outside $\bar{N}$ the sectional curvature is bounded from above by κ. The lift N of $\bar{N}$ to the universal cover $\tilde{M}$ consists of ϵ-separated convex sets N_i. After rescaling, we can assume that $\kappa = -1$.

More generally, we consider a CAT(0)-space X equipped with a collection of disjoint open convex sets N_i which satisfy the property:

$(*)$ There exists $\epsilon > 0$ such that each ball of radius 2ϵ centered at a point x outside $N := \cup_i N_i$ is CAT(-1).

Denote by H the complement of N in X. Consider a geodesic triangle $\Delta(v_1, v_2, v_3)$ in X and choose an ϵ-fine straight filling $f : T^1 \to X$ of this triangle. We denote by Σ the CAT(0)-space (Δ, d_f) constructed in section 2.3. Put $R = R(\epsilon) := 4\epsilon^{-1} + 2\epsilon$. Define C^* to be the set of all points in Σ which have distance not greater than R from two different sides of Δ. The reader may think of C^* as the union of corners of the triangle Σ. (See Figure 1.)

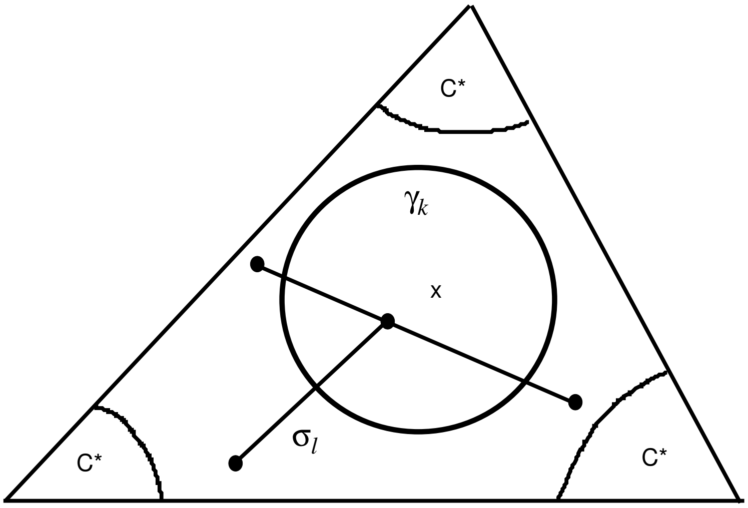

Figure 1: *Corners of the triangle* Σ.

Lemma 4.1 *The set* $f((\Sigma \setminus C^*) \cap T^1)$ *is contained in* N.

Proof: Suppose that x is a point in $(\Sigma \setminus C^*) \cap T^1$. Consider the concentric metric circles γ_k in Σ centered at x with radii $k\epsilon$ for all odd numbers k so that $k\epsilon \leq R - \epsilon$. There are $L := [R/(2\epsilon)]$ such circles. These circles meet at most one side of Δ. Suppose that each circle γ_k contains a point $x_k \in f^{-1}(H) \subset T^1$. The discs $D_\epsilon(x_k) \subset \Sigma$ of radius ϵ centered at x_k are disjoint. Since the filling f is ϵ-fine, every disc $D_\epsilon(x_k)$ is covered by triangles δ_i which are contained in $D_{2\epsilon}(x_k)$. According to Lemma 2.9, every triangle $f(\partial\delta_i)$ is contained in the ball $B_{2\epsilon}(f(x_k))$. By construction of Σ and by the property $(*)$, the curvature $\kappa(\delta_i)$ of the interior of each triangle δ_i is at most -1. For a measurable subset $Y \subseteq \Sigma$, we define the integral

$$\int_Y (-K_\Sigma) dvol := \sum_{\delta \in T^{(2)}} \int_{Y \cap \delta} (-\kappa(\delta)) dvol$$

Using the Gauß-Bonnet formula, we estimate:

$$\text{deficit}(f) = \int_\Sigma (-K_\Sigma) dvol > \sum_k \int_{D_\epsilon(x_k)} (-K_\Sigma) dvol \geq$$

$$\frac{L\pi\epsilon^2}{2} > \frac{(R-2\epsilon)\pi\epsilon^2}{4\epsilon} \geq \pi$$

Here we use the fact that $D_\epsilon(x_k)$ contains a half-disc and hence its area is at least half the area of the Euclidean disc of radius ϵ. On the other hand, it follows from Lemma 2.10 that

$$\text{deficit}(f) \leq \text{deficit}(\Delta(v_1, v_2, v_3)) \leq \pi.$$

This contradiction implies that for at least one circle γ_k, the intersection $\gamma_k \cap T^1$ is entirely contained in $f^{-1}(N)$. Any point on γ_k is at distance at most $\epsilon/2$ from a point in $\gamma_k \cap T^1$. Therefore consecutive points of $\gamma_k \cap T^1$ are at most ϵ apart. Since the convex subsets N_i are ϵ-separated, $f(\gamma_k \cap T^1)$ lies in one component N_i.

We conclude the proof by showing that the convexity of N_i and the straightness of the filling f imply:

$$f(D_{k\epsilon}(x) \cap T^1) \subset N_i$$

We abbreviate $D := D_{k\epsilon}(x)$. The intersection of γ_k with $\partial\Sigma$ is either empty or consists of the endpoints of a subsegment τ of a side of Δ. $f(\tau)$ is contained in N_i, because N_i is convex. Recall that the triangulation T is obtained by successively adding segments σ_l, see Section 2.3. We proceed by induction on l. Suppose that $T^1_{l-1} \cap D \subset f^{-1}(N_i)$. Then

$$\partial(\sigma_l \cap D) \subset (T^1_{l-1} \cap D) \cup (\gamma_k \cap T^1) \subset f^{-1}(N_i).$$

The convexity of N_i implies that $\sigma_l \cap D$ is contained in $f^{-1}(N_i)$. $\square$

We say that the triangle τ in X is *r-fat* if its inradius is strictly greater than r. For every vertex v_i of τ, we define the *r-corner* $C_r(v_i)$ at v_i to be the set of points on τ whose distance from both sides adjacent to v_i is at most r. Note that if τ is r-fat, then the r-corners at its vertices are disjoint. We define the *r-fat part* $\Phi_r(\tau)$ of τ to be $\tau \setminus \cup_i C_r(v_i)$. Recall that $R = R(\epsilon) = 4\epsilon^{-1} + 2\epsilon$.

Proposition 4.2 *Suppose that the triangle $\Delta(v_1, v_2, v_3)$ is $R = R(\epsilon)$-fat. Then the fat part $\Phi_R(\Delta(v_1, v_2, v_3))$ is contained in a single component N_i.*

Proof: Let $f : T^1 \to X$ be an ϵ-straight filling of $\Delta(v_1, v_2, v_3)$. Denote by $C^*_R(v_i)$ the set of points on Σ whose distance in Σ to both sides $[v_i v_{i-1}]$ and $[v_i v_{i+1}]$ is at most R. Then $f(C^*_R(v_i) \cap \partial\Sigma) \subset C_R(v_i)$. The $C^*_R(v_i)$ are convex subsets of Σ and since $\Delta(v_1, v_2, v_3)$ is R-fat by assumption, they are disjoint and intersect at most two sides of $\partial\Sigma$. Thus, their complement $F := \Sigma \setminus \cup_i C_R^*(v_i)$ in Σ is connected. By Lemma 4.1, $F \cap T^1$ is contained in $f^{-1}(N)$. The components N_i are ϵ-separated and the connected set F lies in a $\epsilon/2$-neighborhood of T^1. We conclude that $\Phi_R(\Delta(v_1, v_2, v_3)) \subset f(F \cap T^1)$ is contained in a single component N_i. $\square$

4.2 Asymptotic cones of certain CAT(0)-spaces

We keep the notations and assumptions of Section 4.1. In addition, we require that the sets N_i are 3ϵ-neighborhoods of flats F_i in X.

Pick a non-principal ultrafilter ω. We define $\mathcal{F}$ to be the family of all flats in $\mathrm{Cone}_\omega(X)$ which arise as ultralimits of sequences $(i^{-1} \cdot F_{j(i)})_{i \in \mathbb{N}}$ of flats in the rescaled spaces $i^{-1} \cdot X$.

Proposition 4.3 *The asymptotic cone $Cone_\omega(X)$ satisfies the properties:*

- *(F1) Every open triangle is contained in a flat $F \in \mathcal{F}$.*
- *(F2) Any two flats in $\mathcal{F}$ have at most one point in common.*

Proof: Let $\Delta = \Delta(x, y, z)$ be an open triangle in $\mathrm{Cone}_\omega(X)$. Then Δ is the ultralimit of a sequence of triangles $i^{-1} \cdot \Delta_i$, where $\Delta_i = \Delta(x_i, y_i, z_i)$ are triangles in the original space X. For ω-every i the triangle Δ_i is R-fat, where R is chosen as in Section 4.1. Otherwise, the ultralimit Δ would not be open. By Proposition 4.2, the fat part $\Phi_R(\Delta_i)$ is contained in a set $N_{j(i)}$. Each point w on the side $]xy[$ of Δ corresponds to a sequence of points w_i on $]x_i y_i[$. Since Δ is open, we have:

$$0 < d_\omega(w, [xz] \cup [zy]) = \omega\text{-}\lim \frac{1}{i} \cdot d(w_i, [x_i z_i] \cup [z_i y_i])$$

Hence for ω-every i, w_i does not belong to any R-corner of Δ_i. Therefore, w_i belongs to $N_{j(i)}$ and its distance from the flat $F_{j(i)}$ is at most 3ϵ. We conclude that w lies in the flat $F \in \mathcal{F}$ which arises as the ultralimit of the sequence $(i^{-1} \cdot F_{j(i)})$. This concludes the proof of property (F1).

To verify property (F2), let F and F' be flats in $\mathcal{F}$ which have two distinct points x and y in common. We will show that $F' \subseteq F$. Choose a point z' in F' so that the triangle $\Delta(x, y, z')$ is non-degenerate and pick points u and w on $]xy[$ and $]xz'[$. There is a sequence of flats $(F_{j(i)})$ in X which corresponds to the flat F. Select points $x_i, y_i \in F_{j(i)}$, $z'_i \in X$, $u_i \in]x_i, y_i[$ and $w_i \in]x_i, z'_i[$ so that $(x_i), (y_i), (z'_i), (u_i), (w_i)$ represent the points x, y, z', u, w. For ω-all i, u_i, w_i belong to the fat part $\Phi_R(\Delta(x_i, y_i, z'_i))$. According to Proposition 4.2, the points u_i, w_i belong to the same component $N_{k(i)}$. Since u_i lies on $F_{j(i)}$, $N_{k(i)}$ coincides with $N_{j(i)}$. Hence, w lies on F. We conclude that $z' \in F$, since w was an arbitrary point of $]xz'[$. $\square$

4.3 Special CAT(0)-spaces

In the previous section, we established geometric properties for the asymptotic cone of a CAT(0)-space with isolated flats. The asymptotic cone is a CAT(0)-space itself and now we shall study geometric and topological properties of CAT(0)-spaces Y satisfying the conclusion of Proposition 4.3.

Consider a flat $F \in \mathcal{F}$ and denote by $\pi_F : Y \to F$ the nearest-point-projection onto F.

Lemma 4.4 *Let $\gamma : I \to Y$ be a curve in the complement of F. Then $\pi_F \circ \gamma$ is constant.*

Proof: Assume that $\pi_F \circ \gamma$ is non-constant. Then there exist nearby points p_1 and p_2 on γ with distinct projections $q_i := \pi_F(p_i)$ in F:

$$d(p_i, F) = d(p_i, q_i) > d(p_1, p_2) \qquad (i = 1, 2)$$

The geodesic $[p_1p_2]$ cannot meet F and therefore the piecewise geodesic path $[p_1q_1q_2p_2]$ is not locally minimizing at q_1 or q_2, say at q_1 (see Lemma 2.4). Since $[p_1q_1] \cap [q_1q_2] = \{q_1\}$, the triangle $\Delta(p_1, q_1, q_2)$ spans a non-degenerate open triangle $\Delta(r, q_1, s)$. By property (F1), $\Delta(r, q_1, s)$ lies in a flat F'. Since $F \cap F'$ contains the non-trivial segment $[q_1s]$, F and F' must coincide according to (F2). Thus $[p_1q_1] \cap F$ contains a non-trivial segment $[q_1r]$. This contradicts that $q_1 = \pi_F(p_1)$. $\square$

Lemma 4.5 *Every embedded closed curve $\gamma \subset Y$ is contained in a flat $F \in \mathcal{F}$.*

Proof: Consider the geodesic segment σ joining two distinct points x and y on γ. Since γ is a closed curve, the projection π_σ maps at least two points of γ to an interior point u of σ. Hence there exists a point z on $\gamma \setminus \sigma$ with $\pi_\sigma(z) = u$. Consider a maximal subarc $\alpha \subset \gamma$ containing z with $\pi_\sigma(\alpha) = \{u\}$. At least one of the endpoints of α is different from u, i.e. does not lie on σ. Denote it by z_1. There is a nearby point w on γ whose projection $\pi_\sigma(w) =: v$ is different from u and which satisfies

$$d(w, z_1) < d(u, z_1) = d(\sigma, z_1)$$

As in the proof of Lemma 4.4, we find a flat $F \in \mathcal{F}$ which contains a non-degenerate segment $\sigma' \subset \sigma$.

We proceed by proving that $\pi_F(x) \neq \pi_F(y)$. The intersection $F \cap \sigma$ is a non-degenerate segment $[x'y']$, so that x' lies between x and y'. Consider $x'' := \pi_F(x)$ and suppose that $x'' \neq x'$. Then the piecewise geodesic path $xx'x''$ is not locally minimizing at x'. Since $[xx'] \cap [x'x''] = \{x'\}$, $\Delta(x, x', x'')$ spans a non-degenerate open triangle with vertex x'. As in the proof of Lemma 4.4 we obtain a contradiction. Therefore $\pi_F(x) = x'$ and similarly $\pi_F(y) = y'$. Thus $\pi_F \circ \gamma$ is non-constant.

Suppose now that $\gamma \not\subset F$. Choose a maximal open subarc $\beta \subset \gamma$ in the complement of F. By Lemma 4.4, $\pi_F(\beta)$ is a point $p \in F$. By maximality of β and continuity we conclude that every endpoint of β must coincide with p. Therefore β has at most one endpoint and $\pi_F(\gamma) = \{p\}$. This contradicts $\pi_F(x) \neq \pi_F(y)$. We conclude that γ is contained in F. $\square$

Corollary 4.6 *Every embedded disc in Y of dimension at least 2 is contained in a flat $F \in \mathcal{F}$. In particular, there are no other flats in Y besides the flats $F \in \mathcal{F}$.*

We can use arguments similar to the proof of Lemma 4.4 to show:

Lemma 4.7 *Suppose that T is a metric tree. Then T is a topological tree, i.e. any two points are connected by a unique topologically embedded arc.*

We conclude from Lemma 4.5:

Corollary 4.8 *Suppose that Y is a CAT(0)–space satisfying the conclusion of Proposition 4.3 and all flats in Y have dimension 2. Let T be a tree with nontrivial branching. Then there is no topological embedding $\phi : T \times \mathbb{R} \to Y$.*

Corollary 4.9 *Let Y be a CAT(0)–space satisfying the conclusion of Proposition 4.3. Suppose that T is a metric tree which contains an uncountable discrete subset. Then there is no bi-Lipschitz embedding $\phi : T \times \mathbb{R} \to Y$.*

Proof: Suppose that there is such an embedding ϕ. Lemma 4.5 and property (F2) imply that the image of ϕ is contained in a flat $F \in \mathcal{F}$. We obtain a contradiction, since a flat does not contain uncountable discrete subsets. $\square$

5 Distinction of quasi-isometry classes of 3-manifold groups

The goal of this section is to distinguish quasi-isometry classes of fundamental groups of certain 3-manifolds. Recall that any Haken manifold of zero Euler characteristic can be obtained in a unique way by gluing hyperbolic and maximal Seifert components. In this section we consider only such Haken manifolds.

Theorem 5.1 *Let M_1 be a non-positively curved Haken manifold which has at least one Seifert component with hyperbolic base. Assume that M_2 is a Haken manifold which contains only hyperbolic components. Then the fundamental groups $\pi_1(M_1)$ and $\pi_1(M_2)$ are not quasi-isometric.*

Remark 5.2 *As we shall prove in [KaL1], the condition in Theorem 5.1 that M_1 admits a metric of non-positive curvature is actually obsolete. Namely, we prove that fundamental group of any Haken manifold which is neither* **Sol** *nor* **Nil***, is quasi-isometric to the fundamental group of a 3-manifold of nonpositive curvature.*

Proof: The manifold M_1 contains a Seifert component Z. The universal cover $\tilde{Z}$ of Z is a convex subset in the universal cover $\tilde{M}_1$ according to 2.6. Therefore, the asymptotic cone $\mathrm{Cone}_\omega(\tilde{Z})$ is isometrically embedded in $\mathrm{Cone}_\omega(\tilde{M}_1)$. The asymptotic cone $\mathrm{Cone}_\omega(\tilde{Z})$ is isometric to the product of the real line and a metric tree T with nontrivial branching, see Corollary 3.11. Suppose that there exists a quasi-isometry $\tilde{M}_1 \to \tilde{M}_2$. It induces a homeomorphism $\mathrm{Cone}_\omega(\tilde{M}_1) \to \mathrm{Cone}_\omega(\tilde{M}_2)$. Hence, $\mathbb{R}\times T$ topologically embeds into $\mathrm{Cone}_\omega(\tilde{M}_2)$. The manifold M_2 carries a metric of nonpositive curvature (Theorem 2.8). By Theorem 4.3, $\mathrm{Cone}_\omega(\tilde{M}_2)$ satisfies the properties (F1) and (F2), see the discussion in the beginning of section 4.1. This contradicts Corollary 4.8. $\square$

Theorem 5.3 *Let M be a nonpositively curved Haken 3-manifold with totally-geodesic flat boundary. Assume that M is not flat, not Seifert and not homeomorphic to a closed hyperbolic manifold. Then the asymptotic cone of the universal cover of M contains two flats which have exactly one point in common.*

Proof: Suppose that M contains a hyperbolic component N. By Theorem 2.6, the universal cover $\tilde{N}$ is convex in $\tilde{M}$. Hence, $\mathrm{Cone}_\omega(\tilde{N})$ is isometrically embedded in $\mathrm{Cone}_\omega(\tilde{M})$. Pick two flats F_1 and F_2 in $\partial\tilde{N}$. Then $\mathrm{Cone}_\omega(F_1)$ and $\mathrm{Cone}_\omega(F_2)$ are flats in $\mathrm{Cone}_\omega(\tilde{N})$ which both contain the base point. According to Proposition 4.3 they have exactly one common point.

We are left with the case that M is a graph-manifold. We can find in the universal cover $\tilde{M}$ two convex subsets A_1 and A_2 which are universal covers of Seifert components and whose intersection is a flat F. The sets A_i split off Riemannian factors l_i isometric to the real line. Since M is not Seifert, we may assume that the one-dimensional factors are not parallel in F. Choose flats F_i in A_i different from F and consider the associated flats $\mathrm{Cone}_\omega(F_i)$ in $\mathrm{Cone}_\omega(\tilde{M})$. The intersection of $\mathrm{Cone}_\omega(F_i)$ with $\mathrm{Cone}_\omega(F)$ is a line $\mathrm{Cone}_\omega(l_i)$. The lines $\mathrm{Cone}_\omega(l_i)$ intersect in a single point. Since the intersection of the sets $\mathrm{Cone}_\omega(A_i)$ is precisely $\mathrm{Cone}_\omega(F)$, the flats $\mathrm{Cone}_\omega(F_i)$ have exactly one point in common. $\square$

Theorem 5.1 combined with results of Gromov, Gersten, N. Brady, Schwartz and ourselves leads to a rough classification of quasi-isometry types of fundamental groups of Haken manifolds. We divide Haken 3-manifolds with flat incompressible boundary into the following classes.

1. $\mathcal{H}$: closed hyperbolic 3-manifolds.

2. $\mathcal{CH}$: open hyperbolic 3-manifolds of finite volume.

3. $\mathcal{HH}$: manifolds not contained in $\mathcal{H}, \mathcal{CH}$ which are obtained by gluing hyperbolic components only.

4. $\mathcal{S}$: Seifert manifolds with hyperbolic base-orbifolds.

5. $\mathcal{SS}$: *graph-manifolds*. They are obtained by gluing Seifert manifolds with hyperbolic base and they are not Seifert.

6. $\mathcal{HS}$: manifolds with at least one hyperbolic and Seifert component (with hyperbolic base).

7. Closed Nil-manifolds.

8. Closed Sol-manifolds.

9. Flat manifolds.

Theorem 5.4 *If two 3-manifolds belong to different classes (1–9) then their fundamental groups are not quasi-isometric.*

Proof: The fundamental groups of Nil- and flat manifolds have polynomial growth of degree 4 in the nilpotent and of degree at most 3 in the flat case. Therefore they are not quasi-isometric to each other and to the fundamental groups of all other classes.

The property to be word-hyperbolic is a quasi-isometry invariant [GdH]. Therefore, the fundamental groups of closed hyperbolic manifolds are not quasi-isometric to the fundamental groups of manifolds of all other classes.

Let M be a manifold of the class $\mathcal{CH}$ and Γ be a finitely generated torsion–free group which is quasi-isometric to $\pi_1(M)$. Corollary 4 in the paper of R. Schwartz [Schw2] implies that Γ must be isomorphic to a lattice in $SO(3,1)$ which is commensurable with $\pi_1(M)$. Therefore, if such a group Γ is the fundamental group of a Haken 3-manifold, then Γ belongs to the class $\mathcal{CH}$.

Theorem 5.1 and Remark 5.2 imply that the fundamental groups of the class $\mathcal{HH}$ have different quasi-isometry type from the classes $\mathcal{HS}$, $\mathcal{SS}$ and $\mathcal{S}$.

Gersten introduced in [Ge1] a quasi-isometry invariant notion of divergence of geodesics which measures the rate of growth of diameters of spheres. Using [Br], Gersten [Ge2] shows that fundamental groups of manifolds in the classes $\mathcal{HS}$ and $\mathcal{HH}$ have exponential divergence. In [Ge2] Gersten proves that the fundamental groups of all graph-manifolds fibered over the circle have at most quadratic divergence. On the other hand, [KaL1] implies that the fundamental group of any graph-manifold is quasi-isometric to the fundamental group of a graph-manifold fibered over the circle. This distinguishes the classes $\mathcal{HS}$ and $\mathcal{SS}$. Note that Gersten characterizes closed graph–manifolds as those Haken manifolds whose fundamental groups have precisely quadratic divergence.

To distinguish the fundamental groups of Seifert manifolds and manifolds in $\mathcal{HH}, \mathcal{HS}, \mathcal{SS}$ up to quasi-isometry we observe that their asymptotic cones have different topological properties. Namely, the asymptotic cone of the fundamental group of a Seifert manifold with hyperbolic base splits as a metric product $T \times \mathbb{R}$ where T is a tree with nontrivial branching, see Corollary 3.11. Hence the intersection of bi-Lipschitz embedded Euclidean planes is either empty or contains a line, according to Lemma 2.14. On the other hand, by Theorem 5.3, the asymptotic cones of manifolds in the classes $\mathcal{HH}, \mathcal{HS}, \mathcal{SS}$ contain flats which have precisely one point in common.

To sever the class of *Sol*–manifolds one can use the fact that amenability is a quasi-isometry invariant. The only Haken manifolds with amenable fundamental groups are *Sol*-, *Nil*- and flat manifolds. One may also argue as follows on the level of asymptotic cones. It was shown in [Gr2] that the asymptotic cone of the Lie group *Sol* is not simply-connected. On the other hand, if M is a manifold of nonpositive curvature, then the asymptotic cone of the universal cover $\tilde{M}$ is contractible (see 3.4). $\square$

Remark 5.5 *A theorem of Rieffel [R] distinguishes quasi-isometry classes of fundamental groups of* closed *Seifert manifolds with hyperbolic base from the fundamental groups of all other 3-manifolds.*

Remark 5.6 *Fundamental groups of open and closed aspherical 3-manifolds cannot be quasi-isometric, because they have different cohomological dimension [Ge3].*

Remark 5.7 *The question how to distinguish quasi-isometry types of fundamental groups inside the classes $\mathcal{HS}, \mathcal{SS}$ and $\mathcal{HH}$ remains open. Considerable progress in this direction was achieved by Schwartz [Schw1] who proves that fundamental groups of two open hyperbolic manifolds of finite volume are quasi-isometric iff they are commensurable. We discuss in our consecutive paper [KaL2] the quasi-isometry invariance of the canonical decomposition for (universal covers of) Haken manifolds of zero Euler characteristic.*

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