Characterization of covering maps via path-lifting property

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A continuous map between topological \( f : X \to Y \) is said to satisfy the path-lifting property if for any path \( p : [0, 1] \to Y \) and any \( x \in f^{-1}(p(0)) \) there exists a lifting \( \tilde{p} \) of the path \( p \) with the initial value \( x \), i.e. there exists a path \( \tilde{p} \) such that \( f \circ \tilde{p} = p \) and \( \tilde{p}(0) = x \).

Similarly, a smooth map between Riemannian manifolds \( f : X \to Y \) is said to satisfy the rectifiable path-lifting property if the above definition holds for the rectifiable paths \( p(t) \).

Suppose that \( f : X \to Y \) is a local homeomorphism (resp. diffeomorphism) between topological spaces \( X \) and \( Y \) (resp. Riemannian manifolds \( X \) and \( Y \)).

**Lemma 0.1.** \( f \) satisfies the path-lifting (resp. rectifiable path-lifting) property if and only if the following holds: For each continuous (resp. rectifiable) path \( q : [0, T] \to Y \) and each partial lift \( \tilde{q} : [0, t) \to Y \) extends continuously to the point \( t = T \).

**Proof:** The implication \( \Rightarrow \) is clear, we will prove the other implication. We will use the standard arguments of the covering theory: Let \( A \subset [0, 1] \) denote the largest subinterval on which a lift \( \tilde{p} \) of the path \( p \) (with the initial value \( x \)) exists. This subset is nonempty (since \( 0 \in A \)). Suppose that \( A \) is a half-open interval \([0, T), T \leq 1 \). Then, by our assumption the lift \( \tilde{p} \) exists continuously to the point \( T \). Thus \( A = [0, T] \) is a closed interval, it remains to show that \( T = 1 \). Suppose that \( T < 1 \). Let \( U \) denote a neighborhood of \( x := \tilde{p}(T) \) which maps homeomorphically (by \( f \)) onto a neighborhood \( V \) of the point \( y := p(T) \). Then there exists \( 0 < \epsilon < 1 - T \) such that \( p([T, T + \epsilon)) \subset V \) and we define the lift \( \tilde{p} \) on \([T, T + \epsilon)\) by

\[
f^{-1} \circ p : [T, T + \epsilon) \to U.
\]

This contradicts maximality of \( A \). \( \square \)

It is a standard fact of the covering theory that if \( f \) is a covering map then \( f \) satisfies the path-lifting property.

**Theorem 0.2.** Suppose that \( X \) and \( Y \) are connected, semilocally simply-connected (e.g. are manifolds or cell-complexes), resp. Riemannian manifolds and \( f : X \to Y \) is a local homeomorphism (resp. diffeomorphism) which satisfies the path-lifting (resp. rectifiable path-lifting) property. Then \( f \) is a covering map.
\textit{Proof:} Let $\tilde{X}$ denote the universal cover of $X$ and let $g : \tilde{X} \to \tilde{Y}$ denote a lift of $f$. It suffices to show that $g$ is a homeomorphism (resp. diffeomorphism).

\textbf{Lemma 0.3.} $g$ satisfies the path-lifting (resp. rectifiable path-lifting) property.

\textit{Proof:} Let $q : [0, 1] \to \tilde{Y}$ be a (rectifiable) path in $\tilde{Y}$, $p$ be its projection to $X$ and $\tilde{x} \in \tilde{X}$ be such that $g(\tilde{x}) = q(0)$. Let $x$ denote the projection of $\tilde{x}$ to $X$, then $f(x) = p(0)$. Thus there exists a lift $\tilde{p} : [0, 1] \to X$ of the path $p$ with the initial value $x$. Then, since $\tilde{X} \to X$ is a covering, the path $p$ lifts to a path $\tilde{q} : [0, 1] \to \tilde{X}$ such that $\tilde{q}(0) = \tilde{x}$. It is clear from the construction that $\tilde{q}$ is the required lift of the path $q$. \hfill \Box

\textbf{Lemma 0.4.} The mapping $g$ is onto.

\textit{Proof:} Suppose that $g$ is not onto. Then, since $\tilde{Y}$ is connected, there exists a (rectifiable) path $p : [0, 1] \to \tilde{Y}$ so that $p(0) = g(\tilde{x}) \in g(\tilde{X})$ and $p(1) \notin g(\tilde{X})$. Then the path $p$ does not admit a lift with the initial value $\tilde{x}$, which is a contradiction. \hfill \Box

Thus it suffices to show that $g$ is 1-1. We first consider the easier topological setting:

\textbf{Lemma 0.5.} In case $g$ satisfies the path-lifting property, the map $g$ is 1-1.

\textit{Proof:} We imitate the usual arguments of the covering theory. Suppose that $x, x' \in \tilde{X}$ be distinct points such that $y = g(x) = g(x')$. Let $\alpha : [0, 1] \to \tilde{X}$ be a path connecting $x$ to $x'$. The composition $\beta := g \circ \alpha$ is a loop in $\tilde{Y}$. Hence, since $\tilde{Y}$ is simply-connected, there exists a continuous map

$$H : [0, 1] \times [0, 1] \to \tilde{Y}$$

so that $H(1, s) = y = H(t, 0) = H(t, 1)$ for all $s, t \in [0, 1]$ and $H(t, 0) = \beta(t)$. Our goal is to show that the homotopy $H$ admits a lift $\tilde{H}$ to $\tilde{X}$, which again satisfies:

$$x = \tilde{H}(t, 0), x' = \tilde{H}(t, 1) \text{ for all } t \in [0, 1] \text{ and } H(t, 0) = \alpha(t).$$

This would yield a contradiction since $x \neq x'$. Let $A \subset [0, 1] \times [0, 1] \times [0, 1]$ be a maximal rectangle on which the lift $\tilde{H}$ exists, this rectangle contains the segment $[0, 1] \times \{0\}$ (use $\alpha$ as the lift of $\beta$). By the same covering theory arguments (as in the proof of Lemma 0.1), if the maximal rectangle $A$ is closed then it coincides with $[0, 1] \times [0, 1]$ and we are done. Suppose that $A$ is a half-open rectangle: $A = [0, 1] \times [0, S)$. Let $\tilde{H} : A \to \tilde{X}$ denote the required lift of $H$. Suppose that $H$ does not admit a continuous extension to a point $u := (t, S)$, for some $0 \leq t \leq 1$. This means that there are sequences $z_i, w_i \in A$ convergent to $u$ such that

$$\lim_i \tilde{H}(z_i) = a \neq b = \lim_i \tilde{H}(w_i).$$

Let $\gamma : [0, 1] \to A$ denote the piecewise-linear path in $A$ which connects $z_1$ to $w_1$, $w_1$ to $z_2$, $z_2$ to $w_2$, etc. Since $\lim_i z_i = u = \lim_i w_i$, the path $\gamma$ extends continuously to the point 1, $\gamma(1) = u$. Thus the composition $H \circ \gamma : [0, 1] \to \tilde{Y}$ is a continuous path which has the partial lift

$$\tilde{\gamma} := \tilde{H} \circ \gamma : [0, 1] \to \tilde{X}.$$
However, since \( a \neq b \), the path \( \tilde{\gamma} \) does not extend continuously to the point 1. This contradicts the path-lifting property of \( g \).

We now modify the above arguments in the setting of Riemannian manifolds:

**Lemma 0.6.** In case \( g \) satisfies the rectifiable path-lifting property, the map \( g \) is 1-1.

**Proof:** We follow the proof of Lemma 0.5, modifying it when necessary. We will take \( \alpha \) a smooth curve in \( \tilde{X} \), then \( \beta \) is smooth as well and hence there exists a smooth homotopy \( H \). We again argue that the maximal rectangle \( A \) is closed. Note that if the path \( \gamma : [0, 1] \rightarrow [0, 1] \times [0, 1] \) in the proof of Lemma 0.5 was rectifiable, its image \( H \circ \gamma \) would be rectifiable as well and we would get a contradiction as before. Apriori however \( \gamma \) has infinite length. Note that instead of the original sequences \( z_i \) and \( w_i \) we can freely choose their subsequences: the limits \( a \) and \( b \) would be still different.

We therefore choose subsequences (again denoted \( z_i, w_i \in A \)) such that

\[
d(z_i, u) < 2^{-i-1}, d(w_i, u) < 2^{-i-2}, \forall i.
\]

Then

\[
d(z_i, w_i) + d(w_i, z_{i+1}) < 2^{-i}, \forall i,
\]

and hence the curve \( \gamma \) is rectifiable.

This also concludes the proof of Theorem 0.2.