DEFORMATIONS OF REPRESENTATIONS OF DISCRETE SUBGROUPS OF SO(3,1)

Michael Kapovich

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1. Introduction

Let M be a closed hyperbolic 3-dimensional orbifold (see [T], [Sc] for definitions), $\rho_0 : \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^3)$ be its holonomy representation. Denote the conjugacy class of ρ_0 by $[\rho_0]$. In this paper we discuss whether for $n = 4$ the point $[\rho_0]$ is isolated in the space

$$R(\pi_1(M), n) = Hom(\pi_1(M), Isom(\mathbb{H}^n))/Isom_+(\mathbb{H}^n)$$

If $[\rho_0]$ is isolated, then the corresponding representation is called *locally rigid.*

A suborbifold Σ in M is said to be a virtual fiber in a fiber bundle over $\mathbb{S}^1$ if M admits a finite-sheeted covering $p : M_0 \to M$ such that M_0 is fibered over a circle and a component of the preimage $p^{-1}\Sigma$ is a fiber of this fibration.

We start with the following

Conjecture 1. The representation ρ_0 is not locally rigid if and only if M contains an incompressible 2-suborbifold which is not a virtual fiber in a fiber bundle over $\mathbb{S}^1$ [Ka 1, Ka 2].

Certainly the case of manifolds is the most interesting and most complicated. The main aim of this paper is to show that Conjecture 1 is not absolutely groundless. First our results deal with reflection orbifolds. In this case we will prove Conjecture 1 and find that $R(\pi_1(M), 4)$ is a smooth manifold of dimension $(f - 4)$, where f is the number of "faces" of the reflection orbifold (Theorem 1). Then we shall consider orbifolds of finite volume and examine the "restriction" map

$$\partial_T : PR(\pi_1(M), 4) \to PR(\pi_1(T), 4)$$

where T is an incompressible Euclidean suborbifold corresponding to a "cusp end" of M; $PR(.)$ mean the space of representations whose restrictions on cyclic parabolic subgroups of $\pi_1(M) \subset Isom(\mathbb{H}^3)$ are induced by conjugations in $Isom(\mathbb{H}^4)$. Under some conditions the image of this map is 1-dimensional (Example 1). The question about deformations of such kind arises naturally if we are trying to construct flat conformal structures on manifolds obtained by gluing of two hyperbolic ones along boundary tori [GLT], [Ka1]. Unfortunately, in the general case, the map ∂_T is zero to the first order (Theorem 3). We prove the "only if" part of Conjecture 1 for infinitely many non-Haken manifolds arising after Dehn surgery on 2-bridge knots (Theorem 2). These are the first examples of closed hyperbolic 3-manifolds M whose fundamental group are locally rigid in $R(\pi_1(M), 4)$. Moreover, we prove that for each hyperbolic 2-bridge knot $K \subset \mathbb{S}^3$ there exists only one conjugacy class of discrete faithful representations of $\pi_1(\mathbb{S}^3 - K)$ into $Isom(\mathbb{H}^4)$ (Section 5).

2. Preliminary geometric results.

In this section we collect several elementary facts about geometry of Euclidean spheres in $\mathbb{S}^3$.

Denote by $\mathcal{C}$ the set of all Euclidean spheres in $\mathbb{S}^3$ of positive radius. Then $\mathcal{C}$ has a natural topology and is a smooth 4-manifold. By $\mathbf{Mob}(\mathbb{S}^n)$ we denote the group of Moebius transformations acting on $\mathbb{S}^n$. We shall suppose that the hyperbolic 3-space $\mathbb{H}^3$ is realized as a unit ball in $\mathbb{R}^3 \subset \overline{\mathbb{R}^3} = \mathbb{S}^3$; thus $Isom(\mathbb{H}^3)$ is the group of Moebius transformations of $\mathbb{S}^3$ which leave $\mathbb{H}^3$ invariant. $\mathbf{Mob}_+(\mathbb{S}^n)$ is the subgroup of orientation-preserving Moebius transformations of $\mathbb{S}^n$.

Lemma 2.1. Let $(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4) \in \mathcal{C}^4$. Then the following trichotomy holds:
either (i) there is a sphere $\Sigma_0 \in \mathcal{C}$ orthogonal to all spheres $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$;
or (ii) $\Sigma_1 \bigcap \Sigma_2 \bigcap \Sigma_3 \bigcap \Sigma_4 \ni p$, where p is a point;
or (iii) spheres $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$ are totally geodesic in some metric of constant positive curvature on $\mathbb{S}^3$.

Corollary 2.1. Let $(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4) \in \mathcal{C}^4$. Denote by τ_j the inversion in the sphere Σ_j , let Γ be the group generated by τ_j . Then the following trichotomy holds:
either (i) Γ is conjugate in $\mathbf{Mob}(\mathbb{S}^3)$ to a subgroup of $\mathrm{Isom}(\mathbb{H}^3)$;
or (ii) Γ is conjugate in $\mathbf{Mob}(\mathbb{S}^3)$ to a subgroup of $\mathrm{Isom}(\mathbb{E}^3)$;
or (iii) Γ is conjugate in $\mathbf{Mob}(\mathbb{S}^3)$ to a subgroup of $\mathrm{Isom}(\mathbb{S}^3)$.

Proof of Lemma 2.1. We present the proof that was suggested to the author by N.Kuiper instead of the original one. Consider the sphere $\mathbb{S}^3$ as a round sphere in the affine space $\mathbb{A}^4 \subset \mathbb{RP}^4$; respectively $\mathbf{Mob}(\mathbb{S}^3) \subset \mathrm{PGL}(5, \mathbb{R})$. Every sphere $\Sigma \in \mathcal{C}$ is the intersection of $\mathbb{S}^3$ with some affine hyperplane $P \subset \mathbb{A}^4, \Sigma = \Sigma(P)$. Denote by $P^* \in \mathbb{RP}^4$ the polar of Σ with respect to $\mathbb{S}^3$ (i.e. such point that tangent cone from P^* to $\mathbb{S}^3$ touches $\mathbb{S}^3$ at Σ). Denote by $\hat{P}$ the closure of P in $\mathbb{RP}^4$.

Then it is easy to see that $\Sigma(P)$ is orthogonal to $\Sigma(Q)$ iff $P^* \in \hat{Q}$ (it is sufficient to consider first the case of $P^* \notin \mathbb{A}^4$ and then apply $\mathbf{Mob}(\mathbb{S}^3) \subset \mathrm{PGL}(5, \mathbb{R})$). Now consider the polars P_j^* corresponding to Σ_j $(j = 1, \dots, 4)$. Let $\hat{P}$ be the extended hypersubspace in $\mathbb{RP}^4$ which passes through these points. Then we have 3 possibilities:
(i) the intersection $P \bigcap \mathbb{S}^3$ is a sphere of positive radius. This is the desired sphere Σ.
(ii) The intersection $P \bigcap \mathbb{S}^3$ is a point $\infty \in \mathbb{S}^3$. Then $P_j \ni \infty$, so we have the case (ii) of Lemma.
(iii) The intersection above is empty. Then $P^* \in \mathrm{int}(\mathbb{S}^3)$ and the point P^* lies in the intersection $P_1 \bigcap P_2 \bigcap P_3 \bigcap P_4$. Applying a projective transformation from $\mathbf{Mob}(\mathbb{S}^3)$ we can map the point P^* in the center of $\mathbb{S}^3$. So, Σ_j are "great spheres" in $\mathbb{S}^3$ which are totally geodesic in elliptic geometry. Lemma is proved. $\square$

Remark 1. Independently a generalization of Lemma 2.2 to higher dimensions was proven in [Lu].

Corollary 2.2. Let l and k be two unlinked Euclidean circles in $\mathbb{S}^3, l \cap k = \emptyset$. Then there exists a Euclidean sphere $\Sigma_0 \subset \mathbb{S}^3$ which is orthogonal to l and k.

Proof. We can realize l and k as intersections $l = \Sigma_1 \cap \Sigma_2$ and $k = \Sigma_3 \cap \Sigma_4$, so that the case (i) of Lemma 2.2 holds for the collection

$$(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4).$$

Then the sphere Σ_0 from Lemma 2.2 is orthogonal to l and k. $\square$

Remark 2. If the sphere Σ_0 (case (i) of Lemma 2.2) is unique (i.e. when Σ_j are not orthogonal to a common circle), then it smoothly depends on

$$(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4) \in \mathcal{C}^4.$$

Notation. If Σ_1, Σ_2 are spheres in $\mathbb{E}^3$ then by

$$0 \le \alpha(\Sigma_1, \Sigma_2) \le \pi$$

we shall denote the (external) angle between them. If $\Sigma_1 \cap \Sigma_2 = \emptyset$ then we put $\alpha(\Sigma_1, \Sigma_2) = 0$.

If $X \subset \mathbb{S}^n$ then $Sp(X)$ will denote the round sphere in $\mathbb{S}^n$ which contains X and has minimal dimension. Uniqueness of this sphere is evident.

3. Compact reflection orbifolds.

Consider a compact convex finite-sided polyhedron Φ in $\mathbb{H}^3$, such that every vertex of Φ belongs to precisely 3 edges. Denote the numbers of vertices, edges and faces of Φ by v, e, f respectively. Then we have: $3v = 2e, 2 = v - e + f$. So $e = 3f - 6$. Let $\Gamma_\Phi \subset Isom(\mathbb{H}^3)$ be the isometry group generated by the reflections in the faces of Φ. Suppose that Γ_Φ is discrete and Φ is a fundamental polyhedron for it. Then we shall identify the polyhedron Φ with the factor–orbifold $\mathbb{H}^3/\Gamma_\Phi$. This orbifold is sufficiently large (i.e. contains an incompressible suborbifold) iff Φ is not a tetrahedron [T]. Each (2-dimensional) face $\Pi_i \subset \partial\Phi$ is contained in the unique round sphere $\Sigma_i^0 = Sp(\Pi_i) \in \mathcal{C}$.

According to the Poincare theorem about fundamental polyhedra for discrete groups, the group Γ_Φ has the presentation

$$\langle \tau_1, ..., \tau_f : (\tau_j \tau_i)^{n_{ij}} = 1 \rangle$$

where $n_{ii} = 1$, $\mathbb{Z} \ni n_{ij} = \pi/\alpha(\Sigma_i^0, \Sigma_k^0)$ (see [Ma]); the elements τ_j are reflections in the faces Π_j of Φ.

Moreover, suppose that we have another configuration

$$(\Sigma_1, ..., \Sigma_f)$$

of spheres in $\mathbb{S}^3$, such that

$$\alpha(\Sigma_i^0, \Sigma_k^0) = \alpha(\Sigma_i, \Sigma_k)$$

for each i, k. Then the subgroup of $SO(4,1)$ generated by the reflections in the spheres Σ_i is isomorphic to Γ_Φ. Thus the problem of deforming the representation ρ_0 of Γ_Φ is equivalent to the problem of deforming the configuration of spheres preserving the angles between the neighboring spheres.

Theorem 1. Near the class $[I]$ of the embedding $id : \Gamma_\Phi \to Isom(\mathbb{H}^4)$ the space $R(\Gamma_\Phi, 4)$ is smooth and has dimension $f - 4$.

3.1. *Proof.* Define $r = (\Sigma_1, ..., \Sigma_f) \in \mathcal{C}^f$. If faces Π_i, Π_k of Φ have a common edge ϵ_j then put $\alpha_j = \alpha(\Sigma_i, \Sigma_k)$. So, we have the map

$$\tilde{\xi_3} = (\alpha_1, \dots \alpha_j \dots, \alpha_e) : \mathcal{C}^f \to \mathbb{R}^e$$

Denote $\alpha_j^0 = \alpha_j(r_0)$. Let $\mathcal{C}_2$ denote the space of spheres orthogonal to the sphere $\partial_\infty \mathbb{H}^3$. Then the groups $\mathbf{Mob}_+(\mathbb{S}^3)$ and $\mathbf{Mob}_+(S^2)$ act on $\mathcal{C}^f$ and $\mathcal{C}_2^f$ respectively. Drop the map $\tilde{\xi_3}$ to the maps $\xi_3 : \mathcal{C}^f / \mathbf{Mob}_+(\mathbb{S}^3) \to \mathbb{R}^e$ and

$$\xi_2 : \mathcal{C}_2^f / \mathbf{Mob}_+(S^2) \to \mathbb{R}^e$$

Hence $[r_0] = (\xi_2)^{-1}(\alpha^0 = (\alpha_1^0, \dots, \alpha_e^0))$. Moreover, the map ξ_2 is an immersion at the point $[r_0]$, since $\mathrm{H}^1(\Gamma_\Phi, so(3,1)) = 0$ [W].

However, $\dim_{[r_0]} \mathcal{C}_2^f/\mathbf{Mob}_+(S^3) = 3f-6 = \dim \mathbb{R}^e$, so the map ξ_2 is also a submersion near $[r_0]$. Hence the map ξ_3 is a submersion near $[r_0]$ too. Thus the variety $(\xi_3)^{-1}(\alpha^0 = (\alpha_1^0, \dots, \alpha_e^0)) \cong R(\Gamma_\Phi, 4)$ is a manifold of dimension $4f - e - 10 = f - 4$ near $[r_0]$. $\square$

Corollary 3.1. The group Γ_Φ is rigid in SO(4, 1) iff the orbifold Φ is not sufficiently large.

3.2. Now describe the basis of $\mathrm{H}^1(\Gamma_\Phi, so(4,1))$. Realize the hyperbolic 3-space $\mathbb{H}^3$ as the upper half-space $\mathbb{R}^3_+ = \{(x_1, x_2, x_3) : x_3 > 0\}$. Pick an arbitrary sphere Σ_i^0 as above , center of Σ_i^0 is a point $(x_1, x_2, 0)$. Then define the family Σ_i^t of spheres with the same radius as Σ_i^0 and center at (x_1, x_2, t) , $(t \in \mathbb{R})$. Let $r_t(i) = (\Sigma_1, ...\Sigma_i^t, ..., \Sigma_f) \in \mathcal{C}^f$. For every sphere Σ_j^0 adjacent to Σ_i^0 the function $\alpha(\Sigma_j^0, \Sigma_i^t)$ has maximum at the point $t = 0$. So,

$$d/dt(\alpha(\Sigma_j^0, \Sigma_i^t))|_{t=0} = 0$$

Therefore, the vector $d/dt(r_t(i))|_{t=0}$ is tangent to $Hom(\Gamma_\Phi, SO(4,1)) \subset \mathcal{C}^f$. By direct calculations it's possible to show that $\{d/dt(r_t(i))|_{t=0} : i = 1, ..., f-4\}$ forms a basis of $\mathrm{H}^1(\Gamma_\Phi, so(4,1))$.

Remark 3. The same construction 3.2 works for hyperbolic reflection orbifolds of arbitrary dimension. Thus, if $\Gamma \subset SO(n,1)$ is any discrete reflection group then $dim\mathrm{H}^1(\Gamma, so(n+1,1)) = max\{f - n - 1, 0\}$ where f is the number of faces of the fundamental polyhedron of Γ.

4. Rigidity of 2-bridge knots.

In this section we will need the following

Lemma 4.1. Suppose that G is a finitely generated group, E a G-modulus, R is an element of G so that $R \cdot \xi = \xi$ for each $\xi \in E$. Denote by $\phi : G \to G/\langle\langle R \rangle\rangle = G'$ the natural projection epimorphism, where $\langle\langle R \rangle\rangle$ is the normal closure of R in G. Let $[x] \in \mathrm{H}^1(G, E)$ be a class such that the restriction of x to $\langle R \rangle$ is zero in $\mathrm{Z}^1(\langle R \rangle, E)$. Then

(i) E is G'-modulus: $g' \cdot \xi = g \cdot \xi$ for any $g \in \phi^{-1}(g')$;

(ii) there is a class $[x'] = \phi_*(x) \in \mathrm{H}^1(G', E)$ such that $x'(g') = x(g)$ for each $g' \in G', g \in \phi^{-1}(g')$.

Proof. Define x' as $x'(g') = x(g)$. Since the action of R on E is trivial, then the definitions of $g' \cdot \xi$ and x' don't depend on the choice of $g \in \phi^{-1}(g')$. $\square$

Let $K \subset \mathbb{S}^3$ be a 2-bridge knot (see [R]). Let (p, q) be a pair of coprime integers. Remove from $\mathbb{S}^3$ an open regular neighborhood $\mathcal{N}(K)$ of the knot K and denote the resulting manifold with boundary by $M_\infty = M(K; \infty)$. We shall consider only hyperbolic 2-bridge knots K, i.e. such that $int(M(K;\infty))$ admits a complete hyperbolic structure. Denote by λ a simple homotopically nontrivial loop on $\partial M(K;\infty)$ such that λ bounds a disc in $\mathcal{N}(K)$; let $\mu \subset \partial M(K;\infty)$ be a simple homotopically nontrivial loop which is homologically trivial in $M(K;\infty)$. Denote by $M_{(p,q)}$ the manifold obtained from M_∞ by attaching a solid torus $\tilde{T}$ along the boundary so that the loop $\lambda^p \mu^q$ bounds a disc in $\tilde{T}$. Suppose that (p, q) are not coprime, and k is their greatest common divisor. Denote by $\tilde{T}(k)$ the orbifold whose underlying set is $D^2 \times \mathbb{S}^1$ and the singular set $\{0\} \times \mathbb{S}^1$ has order k.

Then $M_{(p,q)}$ is the orbifold obtained from M_∞ by attaching $\tilde{T}(k)$ so that the loop $\lambda^{p/k}\mu^{q/k}$ bounds a disc in $\tilde{T}(k)$ with one singular point.

This procedure is called the *generalized Dehn surgery on the knot* K; (p, q) are parameters of the surgery.

Remark 4. This definition is slightly different from the standard one.

Then for all but finite coprime parameters (p, q) of Dehn surgery on K the resulting manifolds are hyperbolic and are not sufficiently large [H-T]. For a group F and representation $\rho : F \to SO(n,1)$ we denote by $Ad_n \circ \rho$ the corresponding adjoint representation on the Lie algebra $so(n,1)$.

Theorem 2. For infinitely many coprime (p,q) the groups $\pi_1 M_{(p,q)}$ are locally rigid in SO(4,1) and moreover $\mathrm{H}^1(\pi_1 M_{(p,q)}, Ad_n) = 0$.

Proof of Theorem 2. Consider the uniformization $M = \mathbb{S}^3 \backslash K = \mathbb{H}^3/\Gamma$. Then $\Gamma = \langle x, y | xw = wy \rangle$, where $w = w(x,y)$, x, y are parabolic elements of $\Gamma \subset PSL(2,\mathbb{C})$. Denote by A the maximal parabolic subgroup of Γ which contains $\langle x \rangle$; $A = \langle x \rangle \oplus \langle z \rangle$ where the elements x, z are represented by the loops λ and μ respectively.

First consider the case of a "singular" Dehn surgery $(r, 0)$ on the knot K such that in the fundamental group of the hyperbolic orbifold $M_{(r,0)}$ the image of x has the order r. Let $\rho_r : \Gamma \to \Gamma_r \cong \pi_1(M_{(r,0)})$ be the holonomy representation; $\mathbb{H}^3/\Gamma_r = M_{(r,0)}$.

Denote by

$$\Sigma = \{\rho_r : \Gamma \to PSL(2,\mathbb{C})\}/PSL(2,\mathbb{C})$$

the collection of conjugacy classes of such representations (where r varies).

Lemma 4.2. For every $(r,0)$- surgery we have

$$\mathrm{H}^1(\Gamma_r, Ad_4) = 0.$$

Proof. The group Γ_r is generated by two elliptic elements $x_r = \rho_r(x), y_r = \rho_r(y)$, which are conjugate in Γ_r . Consider the group

$$\Gamma_r^* = \langle x | x^r = 1 \rangle * \langle y | y^r = 1 \rangle$$

and the natural projection $\varphi_r : \Gamma_r^* \to \Gamma_r$. Then we have $\mathrm{H}^2(\Gamma_r^*, Ad_4 \circ \varphi_r) = 0$ since Γ_r^* is almost free. Let $[\xi] \in \mathrm{H}^1(\Gamma_r, Ad_4)$, then

$$[\xi^*] = \varphi_r^*[\xi] \in \mathrm{H}^1(\Gamma_r^*, Ad_4)$$

is an integrable infinitesimal deformation. Let $\theta_t : \Gamma_r^* \to SO(4,1)$ be a curve tangent to $[\xi^*]$.

Proposition For every t the group $\theta_t(\Gamma_r^*)$ is conjugate in $SO(4,1)$ to a subgroup of $SO(3,1)$.

Proof. The group $\theta_t(\Gamma_r^*)$ is generated by two elliptic transformations with the fixed-point sets l_t, k_t which are unlinked Euclidean circles in $\mathbb{S}^3$. Then the proposition follows from Corollary 2.3. $\square$

Thus we can find a coboundary

$$\delta_\eta \in B^1(\Gamma_r^*, Ad_4 \circ \varphi_r) = B^1(\Gamma_r, Ad_4)$$

such that $\delta_\eta - \xi \in Z^1(\Gamma_r, Ad_3)$. However,

$$\mathrm{H}^1(\Gamma_r, Ad_3) = 0$$

by Weil's rigidity theorem [W]. So $[\xi] = 0$. This proves Lemma 2.4. $\square$

Corollary 4.2 (i) The restriction map

$$res : \mathrm{H}^1(\Gamma, Ad_4 \circ \rho_r) \to \mathrm{H}^1(\langle x \rangle, Ad_4 \circ \rho_r)$$

is injective.

(ii) The space $\mathrm{H}^1(\Gamma, Ad_4 \circ \rho_r)$ has dimension 2 .

Proof. Consider (i). Let $[\psi] \in Ker(res)$. Then $\psi(x^m) = \beta - Ad_4 \circ \rho_r(x^m)\beta$, where $\beta \in so(4,1)$. Define a cocycles σ, ξ in $\mathrm{Z}^1(\Gamma, Ad_4 \circ \rho_r)$ as $\sigma(g) = \beta - Ad_4 \circ \rho_r(g)\beta$, $\xi(g) = \psi(g) - \sigma(g)$. The cocycle ψ is cohomologous to ξ and the restriction of ξ to $\langle x \rangle$ is identically zero.

Now we can apply Lemma 4.1 to the projection $\rho_r : \Gamma \to \Gamma_r$. Then ξ induces the cocycle $\rho_{r*}(\xi) = \xi' \in Z^1(\Gamma_r, Ad_4) = B^1(\Gamma_r, Ad_4)$ (according to Lemma 4.2). Hence for some $\alpha \in so(4,1)$ we have $\xi'(g') = \alpha - Ad_4(g')(\alpha)$. However, $\xi'(g') = \xi(g) = \alpha - Ad_4 \circ \rho_r(g)(\alpha)$. Therefore,

$$\xi \in B^1(\Gamma, Ad_4 \circ \rho_r)$$

and (i) is proved.

Consider (ii). We have the following diagram

$$\begin{array}{ccccc}
\mathrm{H}^1(\Gamma, Ad_3 \circ \rho_r) & \xrightarrow{res_0} & \mathrm{H}^1(A, Ad_3 \circ \rho_r) & \xrightarrow{res_3} & \mathrm{H}^1(\langle x \rangle, Ad_3 \circ \rho_r) \\
\downarrow \eta & & \downarrow \psi & & \downarrow \vartheta \\
\mathrm{H}^1(\Gamma, Ad_4 \circ \rho_r) & \xrightarrow{res_1} & \mathrm{H}^1(A, Ad_4 \circ \rho_r) & \xrightarrow{res_2} & \mathrm{H}^1(\langle x \rangle, Ad_4 \circ \rho_r)
\end{array}$$

The Abelian group $\rho_r A$ is generated by one hyperbolic and one elliptic element; thus its action on $\mathbb{R}^{3,1}$ has no nonzero fixed vectors and

$$0 = 2dim\mathrm{H}^0(A, \mathbb{R}^{3,1}_{\rho_r}) = \mathrm{H}^1(A, \mathbb{R}^{3,1}_{\rho_r})$$

Therefore, in the exact sequence

$$0 = \mathrm{H}^0(A, \mathbb{R}^{3,1}_{\rho_r}) \to \mathrm{H}^0(A, Ad_3 \circ \rho_r) \to$$

$$\to \mathrm{H}^0(A, Ad_4 \circ \rho_r) \to \mathrm{H}^1(A, \mathbb{R}^{3,1}_{\rho_r}) = 0$$

the homomorphism $\psi : \mathrm{H}^0(A, Ad_3 \circ \rho_r) \to \mathrm{H}^0(A, Ad_4 \circ \rho_r)$ is an isomorphism. On another hand:

$$\vartheta : \mathrm{H}^1(\langle x \rangle, Ad_3 \circ \rho_r) \cong \mathbb{R}^2 \to \mathrm{H}^1(\langle x \rangle, Ad_4 \circ \rho_r) \cong \mathbb{R}^4 \to$$

$$\to \mathrm{H}^1(\langle x \rangle, \mathbb{R}^{3,1}_{\rho_r}) \cong \mathbb{R}^2 \to 0$$

Hence ϑ is a monomorphism with 2-dimensional image.

Recall that for H-modulus F the space F^H is the set of elements of F fixed under the action of H. Now, consider the exact sequence:

$$0 \to \mathbb{R}^2 = \mathrm{H}^1(\langle z \rangle, so(3,1)^{\rho_r(x)}) \to$$

$$\to \mathbb{R}^4 = \mathrm{H}^1(A, Ad_3 \circ \rho_r) \to \mathrm{H}^1(\langle x \rangle, Ad_3 \circ \rho_r)^{\langle z \rangle} \to 0$$

However, y acts trivially on $\mathrm{H}^1(\langle x \rangle, Ad_3 \circ \rho_r)$ because A is Abelian and

$$\mathrm{H}^1(\langle x \rangle, Ad_3 \circ \rho_r) = \mathrm{H}^1(\langle x \rangle, co(2))$$

where $co(2)$ is the Lie algebra of the centralizer of $\rho_r(x)$ and $\rho_r(z)$ in $SO(3,1)$. This follows that the restriction map res_3 is surjective.

Therefore, $Im(res_2) = Im(res_2 \circ \psi) = Im(\vartheta \circ res_3) = Im(\vartheta) \cong \mathbb{R}^2$. Notice also that $Ker(res_1) = Ker(res_2 \circ res_1) = 0$ (accoridng to (i)) and $Ker(\eta) = 0$ (for instance because $\psi \circ res_0$ is injective). Thus $res_2 \circ res_1$ injects $\mathrm{H}^1(\Gamma, Ad_4 \circ \rho_r)$ into the image of ϑ which is 2-dimensional. Thus,

$$dim\mathrm{H}^1(\Gamma, Ad_4 \circ \rho_r) \leq 2$$

On another hand,

$$\mathrm{H}^1(\Gamma, Ad_4 \circ \rho_r) \supset \eta(\mathrm{H}^1(\Gamma, Ad_3 \circ \rho_r))$$

and thus

$$dim\mathrm{H}^1(\Gamma, Ad_4 \circ \rho_r) \geq 2$$

This implies the second assertion of the Corollary. □.

We continue proof of Theorem 2. The space $R(\Gamma,3)$ has the natural complex structure since we can identify $SO(3,1)_+$ with $PSL(2,\mathbb{C})$. Denote by E the projection to $R(\Gamma,3)$ of the set of representations $\rho_{p,q}:\Gamma\to\Gamma_{p,q}\subset SO(3,1)$ which are the holonomy representations of hyperbolic manifolds $M_{(p,q)}$.

Remark 4. Here and below p and q are coprime integers.

Denote by $R_0(\Gamma,3)$ the connected component of $R(\Gamma,3)$ containing ρ_0.

Lemma 4.3 The set E is Zariski dense (over $\mathbb{R}$) in $R_0(\Gamma,3)$.

Proof. Step 1. The element $\rho_0(x)$ is a parabolic element in $PSL(2,\mathbb{C})$. Take a simply connected neighborhood V of $\rho_0(x)$ in $PSL(2,\mathbb{C})$. For each $g\in PSL(2,\mathbb{C})$ choose a lift $\tilde g$ of g to $SL(2,\mathbb{C})$ and define $\lambda(g)$ to be the ratio of eigenvalues of $\tilde g$ (this is well deifined up to inversion).

We can choose V to be so small that the image of λ does not intersect the set of non-positive real numbers. Then extend the function λ to the orbit of V under the conjugation by $PSL(2,\mathbb{C})$ as:

$$\lambda(hgh^{-1})=\lambda(g)$$

Finally we put

$$u:[\rho]\mapsto\log(\lambda(\rho(x))/2)$$

where we choose the branch of logarithm so that $\log(1)=0$. Then, $u[\rho_0]=0$. The function $u([\rho])$ is well defined up to the multiple ±1 and there is a way to choose this multiple so that function u is a holomorphic embedding of W into $\mathbb{C}$ (see [NZ]). Put $E^*=u(E)$ and

$$U=\{|z|/z, z\in E^*\}$$

The set U is dense on the unit circle (see [NZ]) and the points of E^* accumulate to zero. Therefore, the set E^* cannot lie on any real-analytic subset of $u(W)$.

Step 2. Suppose that E is not Zariski-dense and $E\subset f^{-1}(0)$ for some nontrivial polynomial f. Then $E^*\cap W$ is contained in the real- analytic set $(f\circ u)^{-1}(0)$ which is impossible. □

Corollary 4.3 If E_0 is any finite subset of E, then $E\backslash E_0$ is Zariski dense in $R_0(\Gamma,3)$ over $\mathbb{R}$. □

Lemma 4.4 For infinitely many elements $\rho\in E$

$$\dim \mathrm{H}^1(\Gamma, Ad_4\circ\rho)=2.$$

Proof. Denote by (L,c_ρ) the $so(4,1)$-bundle over M_∞ with the flat connection c_ρ associated with the representation $Ad_n\circ\rho$ (see [JM]). The group cohomology $\mathrm{H}^1(\Gamma,Ad_4\circ\rho)$ can be calculated via simplicial cochains of M_∞ with coefficients in the parallel sections of (L,c_ρ) (see [JM]). Thus the spaces of i-chains $C^i(\mathcal{K}_\rho)$ of the corresponding complex $\mathcal{K}_\rho$ is finite-dimensional. We shall identify $C^i=C^i(\mathcal{K}_\rho)$ for different ρ so that the coboundary operators δ^i_ρ are linear operators between finite-dimensional spaces

$$\delta^i_\rho: C^i\to C^{i+1}$$

which depend algebraically on the parameter ρ.

Suppose now that there exists a finite set $E_0\subset E$ such that

$$\dim \mathrm{H}^1(\Gamma, Ad_4\circ\rho)\geq 3$$

for every $\rho\in E\backslash E_0$.

Denote the dimension of C^i by N_i. The space $Im(\delta^0_\rho)$ has constant dimension Δ_0 since $\mathrm{H}^0(\Gamma, Ad_4 \circ \rho) = 0$ for each $[\rho] \in E$. If $d_\rho = \dim Ker(\delta^1_\rho) \geq 3 + \Delta_0$ for $\rho \in E \backslash E_0$ then

$$Im(\delta^1_\rho) = N_1 - d_\rho \leq N_1 - 3 - \Delta_0.$$

Denote by $\{\mu_s, s = 1, 2, \dots\}$ the complete set of minors of order $(N_1 - 2 - \Delta_0)$ in the matrix δ^1_ρ . Then

$$\mu(\rho) = \sum_{s \geq 1} \mu_s^2(\rho) = 0$$

for every $\rho \in E \backslash E_0$.

Obviously, $\mu(\rho)$ is an algebraic function; however $\mu(\rho_r) \neq 0$ for every $\rho_r \in \Sigma$ since $\dim \mathrm{H}^1(\Gamma, Ad_4 \circ \rho_r) = 2$. So, $E \backslash E_0$ is contained in a proper real-algebraic subset of $R(\Gamma, 3)$ which contradicts to assertion of Corollary 4.3. □

Now we can finish the proof of Theorem 2. We have an infinite subset $F \subset E$ such that $\dim \mathrm{H}^1(\Gamma,\ \mathrm{Ad}_4 \circ \rho) = 2$ for every $\rho \in F$. However $\dim \mathrm{H}^1(\Gamma, Ad_3 \circ \rho) = 2$, so $\dim \mathrm{H}^1(\Gamma_{(p,q)}, Ad_4 \circ \rho_{(p,q)}) = 0$ for $\rho_{(p,q)} \in F$. Theorem 2 is proved. □

5. Deformations of nonuniform lattices

5.1. Let $\Gamma \subset Isom(\mathbb{H}^3)$ is an arbitrary nonuniform lattice (i.e. $vol(\mathbb{H}^3/\Gamma) < \infty$ but $\mathbb{H}^3/\Gamma$ is not compact).

Definition. Let $\mathrm{PZ}^1(\Gamma, so(4,1))$ be the subspace of cocycles $\xi \in \mathrm{Z}^1(\Gamma, so(4,1))$ such that the restriction of ξ to each cyclic parabolic subgroup $\langle\gamma\rangle \subset \Gamma$ is a coboundary in $\mathrm{Z}^1(\langle\gamma\rangle, so(4,1))$. Then put

$$\mathrm{PH}^1(\Gamma, so(4,1)) = \mathrm{PZ}^1(\Gamma, so(4,1))/\mathrm{B}^1(\Gamma, so(4,1))$$

The space $\mathrm{PH}^1(\Gamma, so(4,1))$ is called the space of "parabolic cohomology classes" and $\mathrm{PZ}^1(\Gamma, so(4,1))$ is the space of "parabolic cocycles".

Theorem 3 For every maximal parabolic subgroup $A \subset \Gamma$ we have

$$res_A : \mathrm{PH}^1(\Gamma, so(4,1)) \to \mathrm{PH}^1(A, so(4,1))$$

is identically zero.

Proof. The space $so(4,1)$ admits the Ad_Γ -invariant decomposition $so(4,1) = so(3,1) \oplus V$, where $V \cong \mathbb{R}^{3,1}$ is the Lorentz vector space [JM]. This splitting induces the natural decomposition

$$\mathrm{PH}^1(\Gamma, so(4,1)) = \mathrm{PH}^1(\Gamma, so(3,1)) \oplus \mathrm{PH}^1(\Gamma, V).$$

However, $\mathrm{PH}^1(\Gamma, so(3,1)) = 0$ by Calabi–Weil–Garland–Raghunathan Rigidity theorem [GR], [R]. Therefore, projections of every $[\xi] \in \mathrm{PH}^1(\Gamma, so(4,1))$ to

$$\mathrm{PH}^1(\Gamma, so(3,1)) \text{ and } \mathrm{PH}^1(A, so(3,1))$$

are zero. However, $\mathrm{PH}^1(A, V) = 0$ since

$$\mathrm{PH}^1(A, so(4,1)) \cong \mathrm{PH}^1(A, so(3,1))$$

So, $res_A([\xi]) = 0$. □

5.2. **Example**. Let $\Gamma \subset SO(3,1)$ be the fundamental group of the complement to any hyperbolic 2-bridge knot (as in the Section 4).

Theorem 4 $\mathrm{PH}^1(\Gamma, so(4,1)) = 0$.

Proof. Suppose that $\xi \in \mathrm{PZ}^1(\Gamma, so(4,1))$ be a nonzero cocycle. Denote by Γ^* the free group generated by x, y. Then, applying the arguments of Lemma 4.2, we construct a smooth family of representations θ_t of Γ^* into $SO(4,1)$ such that:

(i) $\theta_0 = id$;

(ii) $\theta_t(x)$ and $\theta_t(y)$ are all conjugate to x and y for all t ;

(iii) the curve θ_t is tangent to the lift ψ of ξ to Γ^*.

Each transformation $\theta_t(x)$ and $\theta_t(y)$ is conjugate in $SO(4,1)$ to a Euclidean translation. This implies that every 3-dimensional hyperbolic subspace of $\mathbb{H}^4$ which contains 1-dimensional horocycle of $\theta_t(x)$ is invariant under $\theta_t(x)$; the same is true for $\theta_t(y)$. We can assume that $\theta_t(x)$ is a Euclidean translation along a line ℓ in the upper-halfspace model of $\mathbb{H}^4$. Let P be a Euclidean 3-dimensional subspace in $\mathbb{R}^4$ which contains ℓ, horocycle of $\theta_t(y)$ and orthogonal to the absolute of $\mathbb{H}^4$. It follows that $\theta_t(\Gamma)$ has an invariant 3-dimensional hyperbolic hyperplane $P \cap \mathbb{H}^4$. The same arguments as in Lemma 4.2 imply that $\xi \in \mathrm{B}^1(\Gamma, so(4,1))$. $\square$

Remark 7. The arguments of the proof show that if $[\rho] \in PR(\Gamma, 4)$, then the class $[\rho]$ contains a representation ρ_1 with image in $SO(3,1)$. However R.Riley in [Ri] described completely all representations of Γ in $PSL(2,\mathbb{C})$. Any representation which preserves the conjugacy classes of x and z is conjugate to id. Therefore $PR(\Gamma, 4)$ consists of a single point. This implies that if $[\rho] \in R(\Gamma, 4)$ is the conjugacy class of a discrete faithful representation ρ then $[\rho] = [id]$. Indeed, such representation ρ must preserve the conjugacy class of x because $\rho(\langle x \rangle \oplus \langle z \rangle)$ is conjugate to a lattice in $\mathbb{R}^2$.

6. Three examples.

Notation. For any orbifold $\mathcal{O}$ we shall denote by $|\mathcal{O}|$ its underlying set. For a face P of a polyhedra Φ we shall denote by St_P the set off all those faces of Φ which have nonempty intersection with P; $St_P^* = St_P - \{P\}$. We shall suppose that $\mathbb{H}^3$ is realized as a unit ball in $\mathbb{R}^3$.

Bending deformations. The following is not the most general description of the "bending", but it's enough for our aims. Suppose that $G \subset SO(n+1,1)$ is any group which splits as the amalgamated free product $G = G_1 *_J G_2$ so that:

(1) G_1 and G_2 have finite centralizers in $SO(n+1,1)$;

(2) the centralizer Z_J of the group J in $SO(n+1,1)$ is 1-dimensional.

Take a nondegenerate curve θ_t in Z_J which contains 1. Then put G_t to be the group generated by G_1 and $\theta_t G_2 \theta_t^{-1}$. It's easy to see that $G_t = \rho_t(G)$, where $\{\rho_t : t \in [0,1]\}$ is a continuous curve of homomorphisms of G in $SO(n+1,1)$. This curve defines a nontrivial deformation of the identity representation of G in $SO(n+1,1)$. Such deformation is called the "bending in J".

Remark 8. Bending deformations of representations of fundamental groups of hyperbolic manifolds (and orbifolds) of dimension n were constructed by several authors: by W.Thurston [T] for $n = 2$; in the case of certain reflection groups in $\mathbb{H}^3$ – by B.Apanasov and A.Tetenov [AT]; then in infinitesimal form – by J.Lafontane [L]; and later by C.Kuorouniotis [K]. In the most general form (for graphs of groups) this conception is explained by D.Johnson and J.Millson [JM] (see also [G]). There are examples of B.Apanasov [A] of "pea-pod" groups which admit "stamping" deformations; this construction was generalized by S.Tan [Ta]. For further generalizations see also [KM].

6.1. *Example 1.*

Suppose that we are given a finite-sided convex polyhedron $\Omega \subset \mathbb{H}^3$ with the following properties:

(a) for some compact face P of Ω all but one faces of St_P^* are orthogonal to a common geodesic plane $\Pi \subset \mathbb{H}^3$;

(b) among the faces in St_P^* there is a face Q_2 which enters a cusp made by the faces Q_1, Q_2, Q_3, Q_4 where $Q_3, Q_4 \in St_P^*$.

Then Q_2 is orthogonal to Q_3, Q_4. Suppose that Ω is the fundamental polyhedron for the discrete group Γ_Ω generated by reflection in its faces.

Denote by A the group generated by reflections in Q_1, Q_2, Q_3, Q_4.

Consider a nontrivial continuous family $Sp(Q)_2^\epsilon$ of Euclidean spheres ($\epsilon \in [0,1]$) such that:

(i) $Sp(Q)_2^0 = Sp(Q)_2$;

(ii) $Sp(Q)_2^\epsilon$ is orthogonal to $Sp(Q_4), Sp(Q_3)$ and tangent to $Sp(Q_1)$;

(iii) the closed ball in $\mathbb{R}^3$ bounded by $(Sp(Q)_2^\epsilon)$ contains $Sp(Q_2)$; ($\epsilon \in [0,1]$).

Then $\alpha(Sp(P), Sp(Q)_2^\epsilon) \geq \alpha(Sp(P), Sp(Q_2))$. For all sufficiently small values of ϵ there is an elliptic rotation $\varphi^\epsilon \in \mathbf{Mob}(\mathbb{S}^3)$ around the circle $\partial_\infty \Pi$ such that:

$\alpha(Sp(P)^\epsilon, Sp(Q)_2^\epsilon) = \alpha(Sp(P), Sp(Q_2))$, where $\varphi^\epsilon Sp(P) = Sp(P)^\epsilon$.

Define the new configuration of spheres $Sp(\Omega)^\epsilon$, that consists of the same spheres as $Sp(\Omega)$, except of $Sp(P)$ and $Sp(Q_2)$ which are deformed to $Sp(P)^\epsilon$ and $Sp(Q)_2^\epsilon$ respectively. Then $Sp(\Omega)^\epsilon$ has the same combinatorial type as $Sp(\Omega)$ and the same angles between spheres. The group generated by the reflections in the spheres of $Sp(\Omega)^\epsilon$ defines the deformation $\rho_\epsilon : \Gamma_\Omega \to G_\epsilon$ with the following properties:

(a) for sufficiently small values of ϵ the representation ρ_ϵ is discrete and faithful;

(b) projection ∂_A of ρ_ϵ to $Hom(A, SO(4,1))/SO(4,1)$ is a nontrivial path;

(c) $[\rho_\epsilon] \in PR(\Gamma_\Omega, 3)$ (see Introduction).

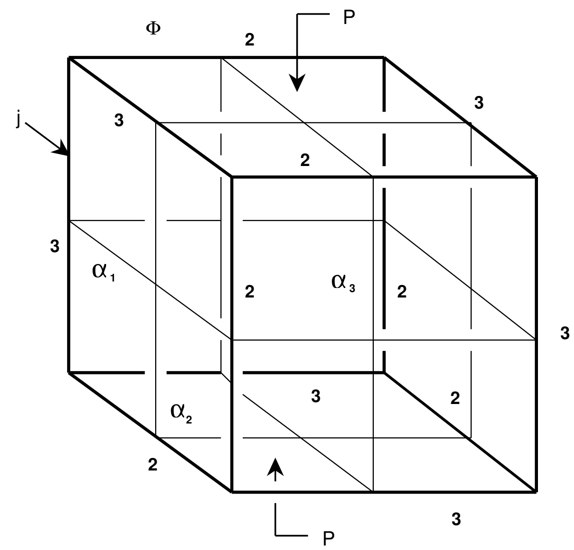

Figure 1:

One can generalize this example, however in general case it is rather difficult to determine : whether or not we obtain nontrivial deformations of cusps.

6.2. *Example 2.* Consider the convex polyhedron Φ in $\mathbb{H}^3$ which is drawn on Figure 1. As usual, if an edge e of Φ is labelled by the integer n then the dihedral angle of Φ at e is π/n. Let G be the group generated by reflections in the faces of Φ. First, find all totally geodesic suborbifolds in Φ . There are only 3 incompressible suborbifolds $\mathcal{D}_i$ in Φ: $\partial|\mathcal{D}_i| = \alpha_j$, $i = 1,2,3$ (see Figure 1).

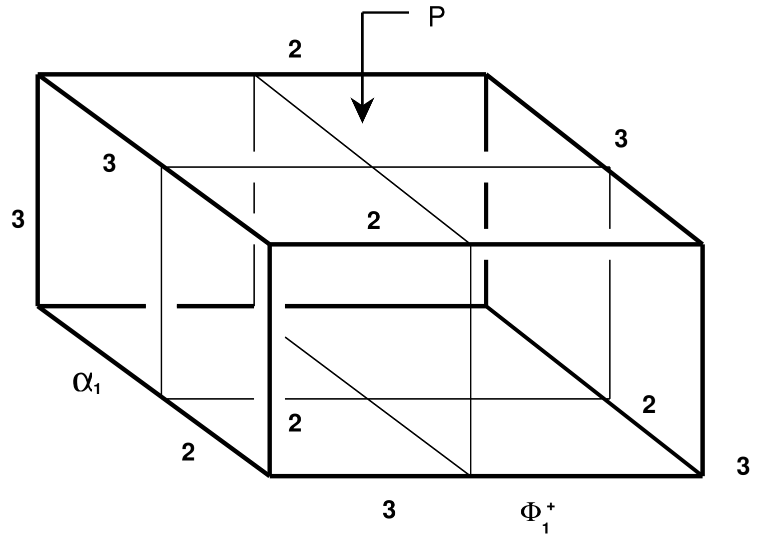

Figure 2:

(i) Suppose $\mathcal{D}_1$ is totally geodesic; then we split Φ along $\mathcal{D}_1$ and consider the "upper half" Φ_1^+ (Figure 2).

Then Φ_1^+ contains an incompressible Euclidean rectangle suborbifold, that is impossible. So, α_1 cannot be the boundary of the underlying set of any totally geodesic suborbifold in Φ.

(ii) Consider $\mathcal{D}_2$. According to Andreev's theorem (see [T]) there exists a convex polyhedron $\Phi_2^+ \subset \mathbb{H}^3$ as on Figure 3.

The face $\mathcal{D}_2 \subset \partial|\Phi_2^+|$ is a rectangle symmetric under rotation θ of order 2 around the axis ℓ . Let $\Phi_2^- = \theta(\Phi_2^+)$; then $\Phi_2^- \cup \Phi_2^+$ is a convex polyhedron isometric to the initial one Φ. So $\mathcal{D}_2$ is a totally geodesic suborbifold.

(iii) The same arguments imply that the orbifold $\mathcal{D}_3$ is totally geodesic.

Thus the orbifold Φ has exactly two totally geodesic 2-dimensional suborbifolds $\mathcal{D}_2, \mathcal{D}_3$. Fundamental groups of $\mathcal{D}_2, \mathcal{D}_3$ have 1-dimensional centralizers in $\mathrm{Isom}(\mathbb{H}^4)$. The corresponding bending deformations β_2, β_3 of the group G in $\mathrm{Isom}(\mathbb{H}^4)$ define classes β_2', β_3' which span the 2-dimensional space $\mathrm{H}^1(G, so(4,1))$.

On another hand, deformation space $R(G,4)$ is a smooth 2-dimensional surface. This shows that bending deformations in intersecting surfaces can span a plane tangent to a smooth surface in the representation variety. This result shows the striking difference between lattices in $Isom(\mathbb{H}^3)$ and $Isom(\mathbb{H}^n)$ $(n > 3)$ since in higher dimensions there are examples [JM] when a linear combination of two bending cocycles is not tangent to any smooth curve in the representation variety.

6.3. *Example 3.* Our next aim is to construct an example of group H which does not admit bending deformations at all, while it is possible to deform H since its fundamental polyhedron has six faces.

Change only one dihedral angle of the polyhedron Φ: instead of the angle $\pi/3$ at the edge j we consider the angle $\pi/5$. Denote the new polyhedron by Ψ.

(i) The same arguments as in Example 2 imply that α_1 cannot be boundary of a totally geodesic suborbifold $\mathcal{D}_1$ of Ψ .

Next notice that both α_2, α_3 don't intersect the edge j .

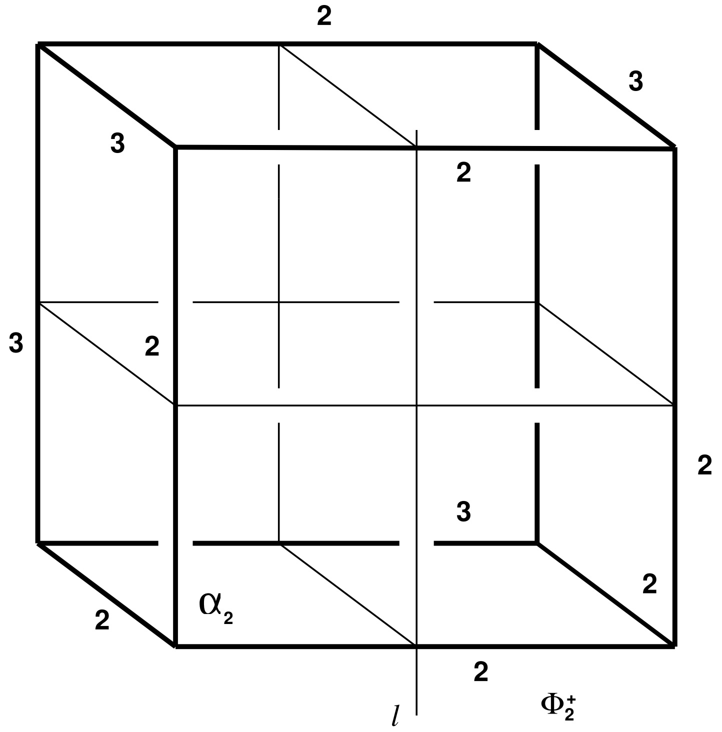

Figure 3:

(ii) Consider the curve $\alpha_2 \subset \partial|\Psi|$ and suppose that $\alpha_2 = \partial|\mathcal{E}_2|, \mathcal{E}_2 \subset \Psi$ is a totally geodesic suborbifold. Then split Ψ along $\mathcal{E}_2$ and obtain two parts : $\Psi_2^+ \supset j$ and Ψ_2^- which does not contain j. The hyperbolic polyhedron Ψ_2^- is isometric to Φ_2^-. Then the rectangles $\mathcal{D}_2$ and $\mathcal{E}_2$ are also isometric. Fix the polyhedron Ψ_2^+, the face $\mathcal{E}_2 \subset \partial|\Psi_2^+|$ and denote the faces in $St^*_{\mathcal{E}_2}$ by $Q_i (i = 1, \dots, 4)$ so that $Q_1 \supset j$. The remaining face in $\partial\Psi_2^+ \backslash St_{\mathcal{E}_2}$ will be denoted by S. Then:

$$S \quad \text{meets } Q_i \;\; (i = 1, 2, 3) \text{ by the angles } \pi/2, \pi/3, \pi/2 \;, \tag{a}$$

$$Sp(S) \quad \text{is orthogonal to } \partial\mathbb{H}^3 . \tag{b}$$

The sphere $Sp(S)$ with the properties (a), (b) is unique up to the reflection τ_2 in $\mathcal{E}_2$ (τ_2 preserves Q_1). Thus, (a) $\&(b) \Leftrightarrow$ the angle between S and Q_1 is equal to $\pi/5$. However, we can consider $Sp(Q_i)$ as spheres which contain faces of the polyhedron Φ_2^+ (since $\mathcal{D}_2$ is isometric to $\mathcal{E}_2$). Let R be the remaining face of $\partial|\Phi_2^+| \backslash St_{\mathcal{D}_2}$. Then R has the same properties (a), (b), however the angle between R and Q_1 is equal to $\pi/3$. This contradiction implies that $\mathcal{E}_2$ cannot be a totally geodesic suborbifold in Ψ .

(iii) The same arguments as above are valid for the curve α_3.

So the curves α_k $(k = 1, 2, 3)$ cannot be boundaries of underlying sets of totally geodesic suborbifolds of Ψ . Hence Ψ does not contain totally geodesic suborbifolds at all.

Let H be the discrete group generated by reflections in faces of Ψ . Then H does not admit bending deformations, however Theorem 1 implies that the deformation space $R(H, 4)$ is 2-dimensional.

6.4. **Conjecture 2**. For every cocompact discrete subgroup $G \subset Isom(\mathbb{H}^3)$ the variety $R(G, 4)$ is smooth at the point $[id]$.

Remark 9. As John Millson explained to me, the first obstruction for deformations in this case is always zero because it belongs to the subspace $\mathrm{H}^2(G, so(3,1))$ of $\mathrm{H}^2(G, so(4,1))$, however $\mathrm{H}^2(G, so(3,1)) \cong \mathrm{H}^1(G, so(3,1)) = 0$.

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